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(6) # SYSTEMS DEFENSE GAMES: COLONEL BLOTTO, COMMAND AND CONTROL

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SYSTEMS DEFENSE GAMES:

COLONEL BLOTTO, COMMAND AND CONTROL*

by

Martin Shubik and Robert James Weber

1. *COLONEL BLOTTO GAMES*

The first example of what is usually referred to as a Colonel Blotto game appears to have been given by Borel:** a defender is defending three points against an aggressor, and the sides have equal forces.

The objective of the aggressor can be formulated either as:

(i) Maximize the expected number of points captured,

or (ii) maximize the expectation that a majority of points are captured.

For three targets and equal forces these objectives are essentially the same.

Games involving the first type of objective were generalized by Tukey and several others*** to a class of assignment games with military applications known in the literature as Colonel Blotto games. Quoting

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**Borel (1938).

***Tukey (1949), Blackett (1958), Beale and Heselden (1962), Dresher (1961), Gross (1950).

Beale and Heselden:

> A Blotto game is a zero-sum game involving 2 players, who may be called A and B, and K independent battlefields (which may, in particular, represent target areas). A has M units of force to distribute between the battlefields, and B has N units. Each player must distribute these forces between these battlefields, once for all, and without knowing his opponent's distribution. Then if A sends x_k units and B y_k units to the k^{th} battlefield, there is a payoff of $P_k(x_k, y_k)$ from B to A at this battlefield; and the payoff for the game as a whole is simply the sum of the payoffs at the individual battlefields.*

In this paper we consider a further generalization which is of importance to a class of military problems. Specifically we wish to take into account the possibility that there exists a complementarity among the posts being defended, i.e. the "score" is not determined merely by adding up individual target values but is determined by considering the worth of capturing or "neutralizing" various configurations of targets. Our generalization includes the classical Blotto games as well as games involving objectives of the second type (ii).

We consider the possibility that the defending forces may be of different size than the attacking forces. The minimum defense force requirement for a guaranteed defense can be calculated (if such a defense is possible). If the defending forces are less than this minimum then our concern is with the level of expected success of the defenders.

By considering complementarity among targets we are in a position to model networks and network failure. Given the redundancy in systems such as telephone and other communication systems (for example, early warning networks and command and control systems or electrical power grids),

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*Beale and Heselden (1962), p. 65.

it is natural to consider how many components can be knocked out before the system can no longer perform its function. Furthermore we may wish to consider cost tradeoffs between built in redundancy and defense costs.

If one or even a few nodes of a network are inactivated messages may be rerouted or power redirected. Beyond some critical level however the system is no longer viable. Although in many instances systems degrade in a continuous manner, for many purposes it is sufficient to consider two states corresponding to "on" or "off"; that is, to functioning or not functioning at an acceptable level. For example a minimal size for a defensive second strike force may have been selected in advance and even though some retaliation might be feasible with fewer weapons than the minimum level selected, regarding the overall system as merely having two states may be an adequate approximation for the purposes at hand. We consider the general case but investigate the more special case as well. Surprisingly it provides mathematical links among military, voting and circuit design problems.*

2. *SYSTEMS PERFORMANCE AND THE CHARACTERISTIC FUNCTION*

An n-person game in coalitional form is described by a *characteristic function* $v(\cdot)$ defined on all subsets of the set of all players N. If one is considering networks or battlefields or key targets, then the $v(S)$ may be interpreted as the value remaining in the system if only the set of nodes S is held. (In traditional cooperative game theory it is frequently assumed that the characteristic function is superadditive; i.e. if S and T are disjoint, then $v(S) + v(T) \leq v(S \cup T)$. However,

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*Beale and Heselden (1962), Young (1977), Dubey and Shapley (1977).

in a competitive context this assumption may not be reasonable. For example if one is protecting a network of Doomsday devices, the characteristic function may assign a value of 1 to every nonempty set.)

The $v(S)$ reflect in an extremely general way the many types of complementarity which can exist among the various combinations of points in the network.

2.1. Solutions to Games in Cooperative Form

There are many different solutions which have been suggested by game theorists for games in coalitional form. They all reflect various aspects of dealings among cooperative players with different goals. Here we note the *value solutions* and the *nucleolus* which can be given natural interpretations in terms of a military problem of defending a system with n nodes. In order to give this interpretation in detail we must reformulate the original n-person game in coalitional form as a two-person noncooperative game. We do this in Section 3. Prior to doing this the cooperative solutions are defined and illustrated.

The *Shapley value*[*] awards to each individual his expected marginal worth on the assumption that all individuals enter all coalitions in a completely random order. The amount assigned to an individual i may be described as

$$(1) \qquad \phi_i = \frac{1}{n!} \sum_{S \subseteq N \setminus i} s!(n-s-1)![v(S \cup i) - v(S)]$$

Consider the 3-person game with a characteristic function as follows[**]

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[*]Shapley (1959).

[**]The notation $v(ij)$ stands for the worth of the set consisting of i and j. We shall at times omit the braces from one-element sets.

$$v(1) = v(2) = v(3) = 0$$

$$v(12) = 1 \qquad v(13) = 2\ , \quad v(23) = 3$$

$$v(123) = 4$$

A simple calculation gives the Shapley value of this game as $\phi_1 = 5/6$, $\phi_2 = 8/6$ and $\phi_3 = 11/6$.

A different value solution originally applicable only to voting games was suggested by Banzhaf.* Here an individual i may be regarded as a "switch" with on-off probabilities of 50:50. We then use the same type of marginal consideration as before, obtaining

$$\beta_i = \sum_{S \subset N \setminus i} \frac{1}{2^{n-1}} [v(S \cup i) - v(S)] \ . \tag{2a}$$

The β_i will not necessarily sum to $v(N)$ in this formulation, hence if we wish we can define a "normalized Banzhaf value" as

$$\beta_i' = \frac{\beta_i}{\sum_{j=1}^{n} \beta_j} v(N) \ . \tag{2b}$$

Applying these formulae to the example above we obtain for the unnormalized values $\beta = (1,\ 3/2,\ 2)$, or $\beta' = (8/9,\ 12/9,\ 16/9)$.

Instead of regarding the probabilities that an individual will be "on" or "off" as 50:50 we could consider them more generally as given by t and $1-t$ where $0 < t < 1$. A general class of values has been considered with:

$$d_i = \sum_{S \subset N \setminus i} t^s (1-t)^{n-s-1} [v(S \cup i) - v(S)] \ .$$

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*Banzhaf (1965).

The Shapley value is simply the unweighted average of all of these "t-values."

Dubey and Weber have shown that there is a whole class of (not necessarily symmetric) value solutions* which includes both the Shapley and Banzhaf values as special cases; the solutions differ from each other in the weights or probabilities placed upon the formation of the different coalitions. Why one should choose one set of weights over another appears to be a problem better answered by the needs and reality of a specific model than one amenable to purely *a priori* considerations.

The *nucleolus* is essentially the center of gravity of the core of a game, if a core already exists, or it is the point at which the core first appears if a coreless game is appropriately modified. In order to make this statement more precise the *excess* of a coalition S, when viewing a prospective payoff vector $a = (a_1, \ldots, a_n)$, is defined as

$$e_S(a) = v(S) - \sum_{i \in S} a_i . \tag{4}$$

The excess is a measure of how much more (or less) a coalition S can claim for itself in comparison with what S obtains at the specific imputation a.

The nucleolus is the imputation at which the maximum excess of any coalition is minimized. (Nonuniqueness is resolved by successive minimization of the nonmaximal excesses.)

For the game above, the imputation $a = (a_1, a_2, a_3)$ that minimizes the maximum excess is the point $(2/4, 5/4, 9/4)$, for which

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*Dubey and Weber (1977). In Dubey, Neyman and Weber (1978), it is shown that the additional requirement of symmetry yields precisely the family of all weighted averages of t-values.

$$e_{12}(a) = 1\tfrac{3}{4} - 1 = \tfrac{3}{4}$$

$$e_{13}(a) = 2\tfrac{3}{4} - 2 = \tfrac{3}{4}$$

$$e_{23}(a) = 3\tfrac{1}{2} - 3 = \tfrac{1}{2} .$$

In the remainder of this paper we will not deal with the nucleolus. However in slightly different models than the ones used here it plays an important role.*

The solution concepts above appear to offer different ways for assigning values or worths to the components of the game, but we have given no indication of how to link these valuation schemes with competition or conflict. We do this in Section 2.2.

2.2. *The Noncooperative Game*

We recast the game given in characteristic function form as though it were a two-person zero-sum game played between two opponents, a defender and an attacker. The n players in the original game are regarded as nodes or individual targets in a network that the defender is trying to protect and the attacker is trying to destroy.

There are several different models of combat at a single target that we can choose. The validity of different models of combat undoubtedly depends directly upon the type of target and the nature of attacking and defending forces. Specific mathematical forms to describe the battle outcome at a single target are discussed in Section 3.

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*Shubik and Young (1978).

Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be the assignments of forces of the defender and attacker to the n targets and let $f_i(x_i, y_i)$ be the function (as yet unspecified) which indicates the outcome of the battle at point i. A natural interpretation which we take at this time is that it specifies the probability that the defender retains point i.

Assume that the goal of the defender is to maximize the (expected) effectiveness of the surviving configuration of targets. The probability that the targets in the set S survive, while all others are destroyed, is $\prod_{i \in S} f_i(x_i, y_i) \prod_{j \notin S} [1 - f_j(x_j, y_j)]$. Therefore, the expected effectiveness of the surviving collection is

$$H(x, y) = \sum_{S \subseteq N} \left[\prod_{i \in S} f_i(x_i, y_i) \prod_{j \notin S} [1 - f_j(x_j, y_j)] \right] v(S).$$

Let A and B be the respective amounts of strategic resources (for example, troops or ballistic/anti-ballistic missiles) held by the defender and the attacker. If we assume that the interests of the attacker are directly opposed to those of the defender, then we have at hand a two-person zero-sum game. The defender may choose any allocation $x = (x_1, \ldots, x_n)$ of resources, subject to the constraint that $\sum x_i = A$. Similarly, the attacker may choose any allocation $y = (y_1, \ldots, y_n)$ for which $\sum y_i = B$. The payoff (to the defender) is $H(x, y)$.

If we suspend the interpretation of the functions f_i as indicating probabilities, we find that this competitive game directly generalizes the traditional Colonel Blotto games, as described in the first section of this paper. Assume that the underlying characteristic function is additive, so that $v(S) = \sum_{k \in S} v(k)$ for all $S \subseteq N$. Then

$$D(x,y) = \sum_{k=1}^{n} f_k(x_k, y_k) \cdot v(k)$$

By identifying $F_k(x_k, y_k)$ with $f_k(x_k, y_k) \cdot v(k)$ (for example, by taking $F_k = f_k$ and $v(k) = 1$ for all $k \in N$), we may obtain any classical Blotto game we desire.

Prior to investigating the two-person zero-sum game, it is desirable to describe some models for individual battle outcomes. These are critical for calculating the probability of the capture or destruction of an individual target. It is this tactical information which is needed as a basis for overall strategic command decisions concerning allocation of forces.

3. BATTLE MODELS

It may well be reasonable to state that the probability that a target j is captured or destroyed is a function $f_j(x_j, y_j)$ of the resources expended in attack and defense by the two sides. The actual appearance of this function is an empirical question which depends upon target type, force mix, doctrine used, morale and many other factors which cannot be stated in closed form.

A listing of the various battle models which have been considered together with a critical evaluation of their validity is beyond the scope of this paper. Such a study would be of considerable worth but does not appear to be available. Even Napoleon's dictum that God is on the side of the strongest battalion does not appear to be borne out when the statistics of the size of forces of victors and losers of major battles are compared.*

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*See Dupuy (1977), p. 89.

For the purposes of this paper we consider a simplified class of models where the attacker and defender have homogeneous resources; hence, force mix problems are set aside.

In particular we consider

$$(5) \qquad f(x,y) = \frac{\gamma x^m}{\gamma x^m + (1-\gamma)y^m} \quad \text{unless} \quad x = 0\,, \quad y = 0$$

$$= \gamma \qquad \text{if} \quad x = 0\,, \quad y = 0\,.$$

γ may be interpreted as an indicator of the natural defensibility of the target. If $x = y$, then $f(x,y) = \gamma$.

m reflects the importance of the difference in size between the attacking and defending forces.

The homogeneity of the function f allows us to concern ourselves with the ratio $k = A/B$ of defending to attacking forces, rather than with the specific amounts A and B.

Surprisingly, at one extreme in the class of mechanisms suggested by (5) we have a mathematical analogy between an economic market and a kill or capture probability. At the other extreme the Colonel Blotto capture conditions appear, and we observe a mathematical analogy between combat and a peculiar auction known as "the dollar auction."* This auction serves to illustrate problems in escalation. It is discussed further below.

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*Shubik (1971).

3.1. *Proportional Bid and Kill Probabilities*

If we set $\gamma = 1$ and $m = 1$ then (5) becomes:

$$f(x,y) = \frac{x}{x+y} . \tag{6}$$

This condition can be stated as "you get in proportion to what you pay."

Suppose for example that there is a single target. Then in a military context (6) gives the probability that the defender wins. Alternatively we may consider an economic context as follows: Imagine that instead of a single target there is a single good to be sold. Furthermore consider that the good is divisible (for instance a thousand gallons of gasoline). Interpret the x and y as sums of money bid for the good. Then $x+y$ may be regarded as an overall price, and the price divided into the amount of money bid by an individual indicates the proportion of the good that individual receives.*

3.2. *Colonel Blotto Models*

Set $\gamma = 1/2$ and let $m \to \infty$. Then (5) becomes

$$f(x,y) = \begin{cases} 1 & \text{if } x > y \\ 1/2 & \text{if } x = y \\ 0 & \text{if } x < y \end{cases} . \tag{7}$$

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*Shubik (1973), Shapley and Shubik (1977).

The function in (7) presents the crudest form of "superior forces" model. It states that superior forces will win with certainty.

If one imagines that the forces on each side are essentially integral (for example the defender has m battalions and the attacker has n battalions and an assignment of forces must always be made in integral units) then the force-allocation game can be solved as a matrix game. If one allows for a continuous distribution of forces, all except a few highly special cases are difficult to analyze fully.*

A natural question to ask is whether the finite models show nice limiting behavior as the grid is made finer. That is, suppose we allow the splitting of battalions into brigades, or even into individual troops: as we consider closer and closer approximates to continuous distributions of forces do the solutions behave in a regular way?

The original Blotto games use the battle condition reflected in (7) together with the further simplification that the values of the targets are independent. This is equivalent to stating that $v(\cdot)$ is additive; that is,

$$v(S) = \sum_{i \in S} v(i) \quad \text{for all} \quad S \subset N .$$

Here by having a general characteristic function $v(\cdot)$ and using the battle conditions of (7) we describe a much more general class of Blotto games. Unfortunately, if resources of the defender and attacker are the same or even close, in general there are no pure strategy solutions

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*Gross (1950), Beale and Heselden (1962).

to these games. The existence of pure strategies, as is shown elsewhere,* will depend upon a relationship between the relative size of forces k and the exponent m in (5). In particular, as has already been noted by Peyton Young (in a different and more specialized context), if k is large enough then the Colonel Blotto game will have a pure strategy solution.

It is clear that whenever the relative size of the defending force to the attacking force is such that the defender can guarantee the allocation of superior forces to the defense of all n targets there will be a pure strategy solution. This is correct but trivial and suggests that a better model is called for. In particular a natural extension of the model which is discussed in Section 5 relates the cost of the defending forces to the value of the targets defended. In the formulation above the forces are given and their cost is not calculated in the payoffs.

We noted at the start of Section 3 that the Blotto game formulation could be related to a peculiar form of market. The analogy is not as far-fetched as it may seem at first glance. In a normal price market, individuals commit resources in the form of money and they receive goods in proportion to the amounts bid. In an auction market individuals commit resources in the form of promises to pay; the individual who wins must provide the money bid and obtains the prize, while those who lose make no payments. In a military engagement both sides commit their resources and, although only one side gets the prize of victory, both must pay.

The dollar auction is an elementary game in which someone auctions

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*Shubik and Weber (1978).

off a dollar. Bids are sequentially accepted (in units, for example, of five cents). When no bid is entered in a fixed interval of time the game ends, and the dollar is given to the highest bidder in exchange for his bid. There is, however, the additional rule that the second highest bidder must also pay the auctioneer the amount of his bid, and obtains nothing. When this game is played with open sequential bids it provides a classical example of escalation. Suppose, for example, that A has bid \$1 and B has bid 95¢. B may decide to bid \$1.05 in order to cut his losses to 5¢. Using the same reasoning A may then raise his bid to \$1.10, and so forth.*

If we consider a similar game, played with both individuals making single simultaneous bids, then the relation to the Colonel Blotto game emerges. Suppose each of the two players has \$2. We use as the payoff functions

$$P_1(x,y) = \begin{cases} 1-x & \text{if } x > y \\ \frac{1}{2} - x & \text{if } x = y \\ -x & \text{if } x < y \end{cases} \tag{8}$$

and

$$P_2(x,y) = \begin{cases} 1-y & \text{if } y > x \\ \frac{1}{2} - y & \text{if } y = x \\ -y & \text{if } y < x \,. \end{cases}$$

*Shubik (1971).

This formulation implicitly assumes that there is a direct and simple relationship between resources committed and their costs. This is clearly true in the dollar auction. In a battle however, one might attribute some value to victory, but there is a difficult problem in casting the value of victory, the resources committed and the costs of the resources in commensurate units. We return to this problem in Section 4.

A relationship between auctions and Blotto games has been remarked upon before by Sakagushi.*

3.3. *A Comment on Conflict Models*

Zero-sum games can be qualitatively classified, according to whether they have pure-strategy optimal solutions, or require the use of randomization for optimal play. Pure-strategy solutions to a competitive defender/attacker game are closely related to the t-values of the underlying characteristic function game.

Specifically, assume that the same outcome function $f(\cdot,\cdot)$ describes the situation at all n targets (battlefields), and further assume that f is homogeneous of degree zero (so that $f(x,y) = f(\alpha x, \alpha y)$ for all $\alpha > 0$). Let the initial resources of the opposing sides be A and B, respectively. Then, if both sides have optimal pure strategies, these strategies must be resource allocations proportional to the $f(A,B)$-value of the underlying game.

Furthermore, let f have the form $f(x,y) = \gamma x^m/(\gamma x^m + (1-\gamma)y^m)$. Then, for all sufficiently small values of m, the allocations proportional to the $f(A,B)$-value are indeed optimal. (Note that small values

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*Sakagushi (1962).

of m correspond to outcome functions which are relatively insensitive to small differences in opposing allocations at a target. It is not unreasonably to expect such a situation to occur.)

Further details concerning these results are presented elsewhere.*

4. *THE COSTS OF SYSTEMS DEFENSE*

"What price freedom?" is a saying that is difficult to operationalize for political philosophers, for Department of Defense budget proposers, or for economists.

A model that links the value and the cost of defense is presented here and a different model is also noted in Section 5.

Here we consider the value of defense in relationship to its costs. In Section 5 we take the costs of defense as given but consider the possibility of trade-offs between systems design and the defensibility of a system. (From the point of view of modelling the process of defense the model here is far less satisfactory than that in Section 5.)

At a high level of abstraction we can consider four major factors in the description of the defense of a system:

(1) The military or societal "worth" of defense;

(2) The type of forces, quantity of forces, and force-structure used in defense;

(3) The cost of the forces;

(4) The "hardness" or "defensive strength" of individual targets.

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*Shubik and Weber (1978).

The models in Section 3 essentially avoid the problems of comparing value and cost by portraying value in the characteristic function and specifying the available attack and defense forces. Thus the military resources enter only as boundary conditons on a force assignment problem, rather than as resources whose costs must be taken into account in the payoffs. By using this formulation there is no need to compare value and cost.

In economic markets involving bidding or prices the mechanism is explicitly designed to include value and cost in the payoffs. If there is some item selling at price p and an individual buys x units of it, paying in some other commodity of which he has a supply M, then his payoff is given by:

$$\phi(x,\ M - px) \ .$$

If we were to regard M as a money which the individual values more or less at a constant worth we might write his payoff as:

$$\phi(x) + M - px \ .$$

We can easily modify the games of Section 3 to include costs in the following manner. The defender and attacker first each select force levels k_1 and k_2, incurring costs of $c_1(k_1)$ and $c_2(k_2)$. They then each assign forces and the payoffs are given by:

$$(9) \qquad \begin{aligned} P_1 &= v(S) - c_1(k_1) \\ P_2 &= v(S^c) - c_2(k_2) \end{aligned} \qquad \text{for all } S \ .$$

where the $v(S)$ is the worth (in monetary units) of the set S of targets

defended successfully. (In analogy with our earlier zero-sum model, we could alternatively define $P_2 = -v(S) - c_2(k_2)$; if v is constant-sum, these two approaches are equivalent.) This is a two-stage nonconstant-sum game.

The fact that the above game formulates well as a two stage process should call attention to the possibility that in actuality the two stages are separate, in both time and bureaucratic control. The problem in a defense department in dealing with the government as a whole is to select k_i , incurring the budgetary expense $c_i(k_i)$. The problem of the commander, having been presented with forces k_i , is to allocate these forces wisely.

From the viewpoint of analysis it thus seems to be reasonable to regard the models of Section 3 as worth pursuing at the level of command and control but to consider the type of model suggested by (9) as a different level of decisionmaking which involves deep problems in the modelling of defense budgeting.*

5. *THE HARDENING OF TARGETS*

In order to illustrate the preceding considerations, we analyze a simple example. Assume that the defender seeks to protect three sites, at each of which several anti-ballistic missiles are siloed. If the attacker destroys any two (or all three) of the targets, the overall defensive system will collapse. The first site houses fewer missiles than the second, which in turn houses fewer than the third; although any two surviving sites will yield an adequate system, the survival of all three provides even greater security. We model this situation with a charac-

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*Hitch and McKean (1960).

teristic function v , which satisfies $v(123) = 4$; $v(12) = 1$, $v(13) = 2$, $v(23) = 3$; $v(S) = 0$ if $|S| \leq 1$.

Assume that the attacker and defender possess comparable amounts of strategic resources; say, $A = B = 1$. Let the outcome of conflict at site k be represented by the function $p_k(x,y) = \gamma_k x^m/(\gamma_k x^m + (1-\gamma_k)y^m)$, for some relatively small value of m (that is, assume that equal forces engaged at site k will yield a result favorable to the defender with probability γ_k , and further assume that small differences in troop assignments lead to only small changes in this probability). The parameter γ_k indicates the "hardness" of the target at site k --its natural strength against attack. The optimal allocation of strategic forces by each side will be proportional to the $(\gamma_1, \gamma_2, \gamma_3)$-value of the game v . Hence, this allocation will be proportional to the vector

$$\beta = (\gamma_2 + 2\gamma_3 - 2\gamma_2\gamma_3,\ \gamma_1 + 3\gamma_3 - 2\gamma_1\gamma_3,\ 2\gamma_1 + 3\gamma_2 - 2\gamma_1\gamma_2)$$

In particular, if we initially have $\gamma_1 = \gamma_2 = \gamma_3 = 1/2$, the optimal allocation for each side is (2/9, 3/9, 4/9) .

Now, assume that additional capital is available to the defender, which may be used to harden any of the targets. Indeed, assume that an investment of Δc_k units of capital at site k will yield an increase of $(1-\gamma_k)\cdot\Delta c_k$ in the hardness of target k ; that is, $\partial\gamma_k/\partial c_k = (1-\gamma_k)$. A natural question is how best to invest the additional capital.

Assume that the defender allocates his forces according to $x = (x_1, x_2, x_3)$, while the attacker's deployment is $y = (y_1, y_2, y_3)$. Then the value of the outcome of the competitive game, to the defender, is

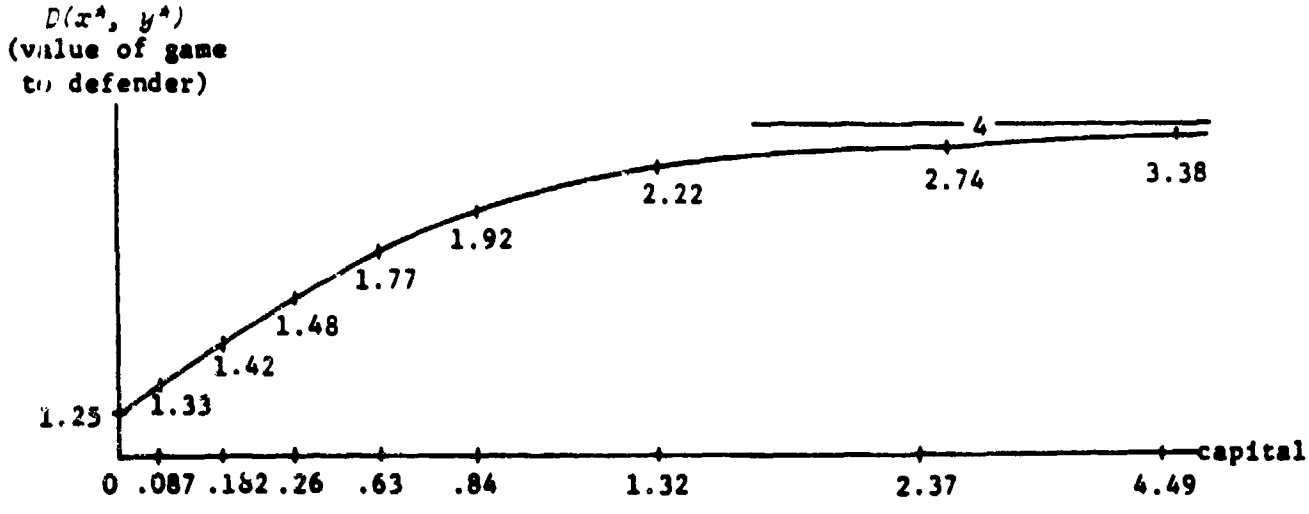
$$D(x,y) = p_1p_2 + 2p_1p_3 + 3p_2p_3 - 2p_1p_2p_3 ,$$

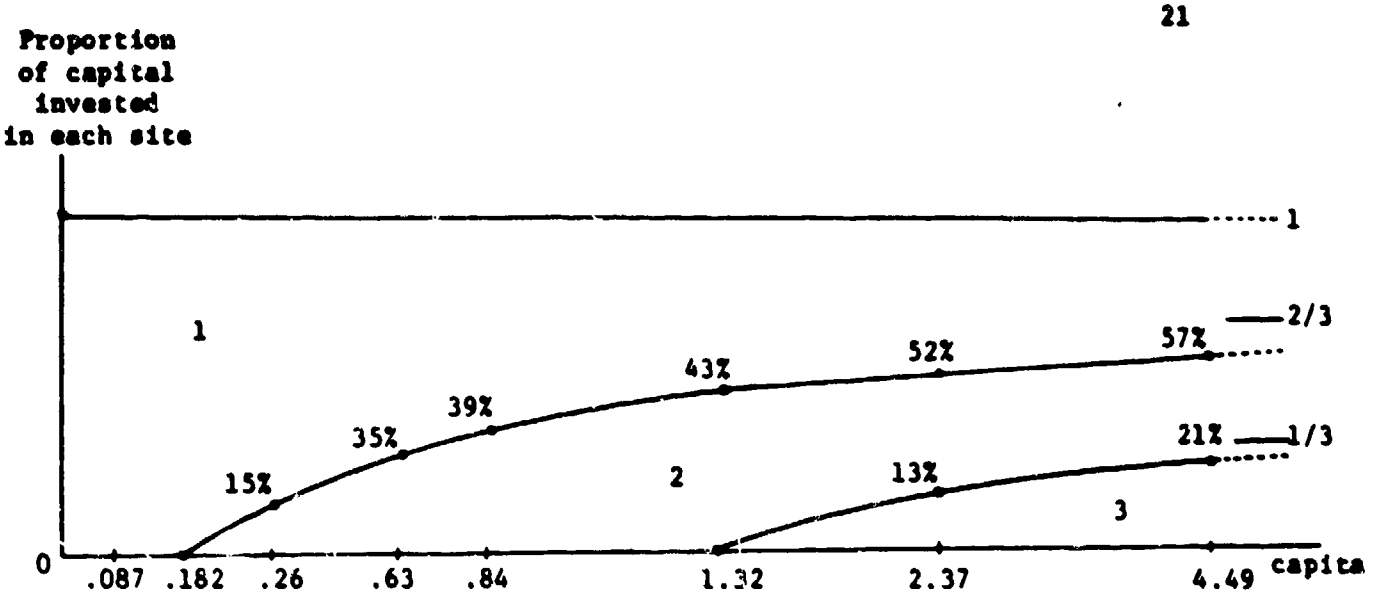
where each p_k is evaluated at (x_k, y_k) . The optimal strategies are $x^* = y^* = \beta/\sum\beta_i$. Therefore, the rate of gain from investment in the hardening of target k is

$$\frac{\partial D}{\partial c_k}(x^*, y^*) = \frac{\partial D}{\partial p_k}(x^*, y^*)\cdot\frac{\partial p_k}{\partial \gamma_k}(x^*, y^*)\cdot\frac{\partial \gamma_k}{\partial c_k}$$

$$= (\beta_k/\sum\beta_i)\cdot 1\cdot(1-\gamma_k) .$$

The best investment is in the target (or targets) for which this expression is maximized. But the expression varies with the parameters γ_1 , γ_2 , and γ_3 . Hence, if we begin with all γ_k equal, it is best to initially invest in work at the site for which β_k is maximal; this changes β as well as γ_k , after which we can determine the next target for further investment. Beginning with $\gamma_1 = \gamma_2 = \gamma_3 = 1/2$, we obtain the results indicated in the figures. (As the available capital increases without limit, the value of $D(x^*, y^*)$ approaches 4, and the three sites attract nearly equal proportions of the capital.)





This model is presented merely as a simple suggestive example of the type of computation which, although not easy, appears to be feasible and relevant to studying tradeoffs in defense, in hardening of targets, and in redundancy in systems.

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