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D.A.B. FOGG

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BICUBIC SPLINE INTERPOLATION FROM DATA IN CONTOUR FORM. (U)

10 D.A.B. Fogg

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SUMMARY

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1. INTRODUCTION

The need for detailed terrain models in tactical simulations involving ground combat has resulted in considerable manual effort being applied to reading terrain heights from contour maps. An increased demand for terrain models has led to the examination of ways of making better use of data graphics equipment such as that used by cartographers(ref.1) and sonic digitizers suitable for map reading.

There is one obstacle in the way of using the above equipment and thereby significantly reducing the manual effort required to construct a terrain model. This is the fact that the programs in use require terrain information at the intersections of the grid lines of a rectangular grid system, but map reading equipment produces data in the form of contours. This could be dealt with by redesigning the tactical simulation programs to use contour information directly, but it is much more difficult to design a simulation which is not based on a rectangular grid and its running time would be considerably greater. The alternative is to change the data, i.e. convert the contour information into grid-point information by a suitable interpolation procedure and leave the program unchanged. The latter seems to be a more satisfactory approach. There are many ways of producing a bivariate interpolation function which is an exact fit to a regular lattice of data points, but an exact fit is rare when the data is irregularly spaced. This is the problem addressed in this Technical Report.

Some techniques are available(ref.2,3) for drawing contours from both regularly and irregularly spaced data, but these shed little light on the problem of converting contour data to grid point data which does not seem to have been dealt with specifically in the literature. Digitized contour data can, of course, be treated as irregularly spaced data, an interpolation being carried out at each grid point using a weighted sum of a number of the surrounding data points. These methods have been discussed in reference 4. They suffer from a number of disadvantages for the type of problem dealt with here. For example they do not make use of the intrinsic regularity of the data. Furthermore, unless "spot heights" (i.e. degenerate contours at the apex of hills or centres of depressions) are included in the data and are given special weighting the resulting interpolation can flatten these areas by an amount which is noticeable. This is especially significant in "line of sight" studies where observers utilize hill tops. In such situations small discrepancies near the top can make large differences in the area of visibility.

The solution presented here attempts to use the additional information contained in the fact that the data, although not on a regular lattice, is not randomly positioned but organised in the form of contours. It will be shown that a knowledge of surface slopes will overcome the problem of flattening peaks and troughs.

The requirement can be summarised as follows: Find a bivariate interpolation procedure which accepts and makes optimum use of contour data to generate terrain values on a regular lattice of points, so that a smooth continuous surface can be fitted to all points.

The uniqueness and especially the minimum curvature properties of bicubic splines(ref.5) made them prime candidates for the solution of this problem. The treatment of bicubic splines usually pre-supposes data points at the intersections of a rectangular mesh. Since data was not available in this form a new method had to be developed. This was done by the procedure described below resulting in an accurate and fast algorithm for producing a terrain surface satisfying all the requirements stated in the summary of the problem given in the previous paragraph.

2. THE DATA BASE

Since contours are continuous and this was to be a digital process the first step was to decide on the data base to be used. This was a selection of (x,y) pairs on the contours, each pair being assigned the value z of the contour on which it lay. The required rectangular, and regular grid system onto which the contour data was to be transferred was superimposed onto the contour map. Each intersection of a contour and a grid line was taken as a data point (see figure 1). If in addition to this information the slope of the terrain was measured in the positive direction at the extremities of the grid lines, it would be possible to fit a unique linear cubic spline to the data points along each x grid-line and each y grid-line. This defined a network of two sets of splines, one set running in a direction at right angles to the other. Each line from one set crossed every line from the other. At the crossing point (x_G, y_G) of splines on the grid-lines $x = x_G$, $y = y_G$ data values $z(x_G), z(y_G)$ of the two splines are not in general identical, (see figure 2). The first real problem was therefore to determine a way of choosing a value of z between $z(x_G)$ and $z(y_G)$ which preserved the continuity and smoothness of both curves while minimizing curvature over the four new splines. This is done in Section 3.

3. A NECESSARY CONDITION FOR THE INTERSECTION OF TWO SPLINES

Linear spline interpolation can be defined as follows(ref.5) -

$$\text{Let } \Delta: \quad a = x_0 < x_1 < \ldots\ldots < x_M = b$$

and a set of real numbers $\{z^i\}$ (i = 0,1, ... M) be given. A function f(x) of class C^2* such that $f(x_i) = z_i$ and which has given slopes $p_0 = f'(x_0)$ and $p_M = f'(x_M)$ at the two end points is referred to as a spline function and is denoted by $S_\Delta(f;x)$. In particular if the interpolating function f(x) is a cubic polynomial the spline is called a linear cubic spline. Let $L(x; x_0, x_1, \ldots x_M)$ denote the linear space of all functions f(x) of class C^2 on the interval (x_0, x_M) which are equal to a cubic polynomial in each of the intervals $[x_{i-1}, x_i]$ i = 1,2, ... M, i.e. piecewise cubic. Then the following theorem holds(ref.6) -

Theorem 1. For each set $\{z^0, z^1, \ldots, z^M, p_0, p_M\}$ of values there exists exactly one $f(x) \in L(x; x_0, \ldots, x_M)$ such that

$$f(x_i) = z^i, \; i = 0,1, \ldots, M, \; f'(x_0) = p_0, \; f'(x_M) = p_M$$

A continuous curve f(x) has a radius of curvature $\rho(x)$ given by the expression -

$$\rho(x) = \frac{(1 + f'^2(x))^{3/2}}{f''(x)}$$

The reciprocal of this expression is the curvature at x. When ρ is large,

* This implies that the second derivative of the function exists and is continuous.

curvature can be approximated by $f''(x)$. Since this can be positive or negative the square of the curvature is often used, as in the following theorem:

Theorem 2. Let Δ and $\{z_i\}$ be defined as before. Then of all the functions $f(x)$ having a constant second derivative on the interval $[a,b]$ and such that $f(x_i) = z_i$ $(i = 0,1, \ldots, M)$; the spline function $S_\Delta(f;x)$ with junction points at the x_i and with $S''_\Delta(f;a) = S''_\Delta(f;b) = 0$, minimizes the integral

$$\int_a^b [f''(x)]^2 \, dx$$

This theorem, due to Holladay(ref.7), is often called the minimum curvature property.

Consider a three dimensional Cartesian coordinate system in which two linear cubic splines are defined. One, $S_{\Delta x}(f;x)$ on the line $y = y_G$ and the other $S_{\Delta y}(g;y)$ on the line $x = x_G$ where,

$$\Delta x: \quad a = x_0 < x_1 < \ldots\ldots < x_M = b \qquad x_i \neq x_G \quad \text{for all } i,$$

$$\Delta y: \quad c = y_0 < y_1 < \ldots\ldots < y_N = d \qquad y_i \neq y_G \quad \text{for all } i,$$

$$S'_{\Delta x}(f;a) = p_0, \; S'_{\Delta x}(f;b) = p_M$$

$$S'_{\Delta y}(g;c) = q_0, \; S'_{\Delta y}(g;d) = q_N$$

Let x_G lie in the segment (x_{u-1}, x_u) and y_G be in the segment (y_{v-1}, y_v). We wish to consider an additional joint in both splines so that the new splines $S_{\Delta x_G}$, $S_{\Delta y_G}$ satisfy,

$$\Delta x_G = a = x_0 < x_1 < \ldots < x_{u-1} < x_G < x_u < \ldots < x_M = b$$

$$\Delta y_G = c = y_0 < y_1 < \ldots < y_{v-1} < y_G < y_v < \ldots < y_N = d$$

with the requirement that -

$$S_{\Delta x_G}(\hat{f};x_G) = S_{\Delta y_G}(\hat{g};y_G) = h$$

and

$$S_{\Delta x_G} \equiv S_{\Delta x} \text{ except on } [x_{u-1}, x_u]$$

$$S_{\Delta y_G} \equiv S_{\Delta y} \text{ except on } [y_{v-1}, y_v]$$

In particular note that

$$S'_{\Delta x_G}(\hat{f}; x_{u-1}) = S'_{\Delta x}(f; x_{u-1})$$

$$S'_{\Delta x_G}(\hat{f}; x_u) = S'_{\Delta x}(f; x_u)$$

$$S'_{\Delta y_G}(\hat{g}; y_{v-1}) = S'_{\Delta y}(g; y_{v-1})$$

$$S'_{\Delta y_G}(\hat{g}; y_v) = S'_{\Delta y}(g; y_v)$$

We now wish to determine four functions -

$$z_1(x) \text{ on } [x_{u-1}; x_G]$$

$$z_2(x) \text{ on } [x_G; x_u]$$

$$z_3(y) \text{ on } [y_{v-1}; y_G]$$

$$z_4(y) \text{ on } [y_G; y_v]$$

Satisfying the relations -

$$\left.\begin{aligned}
z_1'(x_{u-1}) &= S'_{\Delta x_G}(\hat{f}; x_{u-1}) & z_3'(y_{v-1}) &= S'_{\Delta y_G}(\hat{g}; y_{v-1}) \\
z_2'(x_u) &= S'_{\Delta x_G}(\hat{f}; x_u) & z_4'(y_v) &= S'_{\Delta y_G}(\hat{g}; y_v) \\
z_1(x_{u-1}) &= S_{\Delta x_G}(\hat{f}; x_{u-1}) & z_3(y_{v-1}) &= S_{\Delta y_G}(\hat{g}; y_{v-1}) \\
z_2(x_u) &= S_{\Delta x_G}(\hat{f}; x_u) & z_4(y_v) &= S_{\Delta y_G}(\hat{g}; y_v) \\
& z_1(x_G) = z_2(x_G) = z_3(y_G) = z_4(y_G) = h \\
&\text{and} \\
& z_1'(x_G) = z_2'(x_G) \\
& z_3'(y_G) = z_4'(y_G)
\end{aligned}\right\} \quad (1)$$

and such that the functional -

$$v(z_1(x), z_2(x), z_3(y), z_4(y)) = \int_{x_{u-1}}^{x_G} [z_1''(x)]^2\, dx + \int_{x_G}^{x_u} [z_2''(x)]^2\, dx$$

$$+ \int_{y_{v-1}}^{y_G} [z_3''(y)]^2\, dy + \int_{y_G}^{y_v} [z_4''(y)]^2\, dy \quad (2)$$

is minimized.

The latter requirement is in keeping with the minimum curvature property of the previously defined sections of the splines (i.e. outside the segments (x_{u-1}, x_u), (y_{v-1}, y_v)).

The problem is one of determining an extremum, i.e. a functional which minimizes the overall curvature in four functions, each of which is fixed and has a fixed slope at one point and which meet at the point (x_G, y_G, h) (h arbitrary) with the added provision of continuity of the first derivative in both the x and y directions in the neighbourhood of (x_G, y_G, h).

A necessary condition for an extremum can be obtained from the "Calculus of Variations" (see reference 8). A set of curves $C = (z_1(x), z_2(x), z_3(y), z_4(y))$ providing an extremum must satisfy the Euler-Poisson equation which for functionals of the form -

$$v(z(t)) = \int_{t_0}^{t_1} F(t, z, z', \ldots, z^{(n)})\, dt$$

is the nth order differential equation -

$$F_z - \frac{d}{dt} F_{z'} + \frac{d^2}{dx^2} F_{z''} + \ldots + (-1)^n \frac{d^n}{dx^n} F_{z^{(n)}} = 0 \qquad (3)$$

where F_p denotes the partial derivative of F with respect to p.

For the functional (2) the set C must satisfy the four equations -

$$\left.\begin{aligned} F_{z_i} - \frac{d}{dx} F_{z_i'} + \frac{d^2}{dx^2} F_{z_i''} &= 0 \quad (i = 1,2) \\ F_{z_i} - \frac{d}{dy} F_{z_i'} + \frac{d^2}{dy^2} F_{z_i''} &= 0 \quad (i = 3,4) \end{aligned}\right\} \qquad (4)$$

obtained by repeated use of equation (3). The general solutions of these fourth order differential equations are the four functions -

$$z_i = z_i(x, C_{i1}, C_{i2}, C_{i3}, C_{i4}) \quad (i = 1,2)$$

$$z_i = z_i(y, C_{i1}, C_{i2}, C_{i3}, C_{i4}) \quad (i = 3,4)$$

each of which involves four arbitrary constants. These will be obtained from the boundary conditions and the fundamental necessary condition for an extremum, $\delta v = 0$.

Since the F are functions of z_i'' alone, we have -

$$F_{z_i} = 0 \text{ and } F_{z_i'} = 0 \quad (i = 1, \ldots, 4)$$

so that the equations (4) become -

$$\frac{d^2}{dx^2} F_{z_i''} = 0 \; (i = 1,2) \qquad \frac{d^2}{dy^2} F_{z_i''} = 0 \; (i = 3,4)$$

or

$$z_i^{(4)}(x) = 0 \; (i = 1,2) \qquad z_i^{(4)}(y) = 0 \; (i = 3,4)$$

The solutions are therefore the cubic polynomials -

$$z_i(x) = \sum_{j=0}^{3} a_{ij} x^j \quad (i = 1,2)$$

$$z_i(y) = \sum_{j=0}^{3} a_{ij} y^j \quad (i = 3,4)$$

The equation (1) provide only 13 independent relations, therefore 3 additional conditions are required to solve for the sixteen coefficients.

The four z_i terminate or begin on the line $x = x_G$, $y = y_G$. By considering the arbitrary variations δz_G, $\delta z'_{xG}$, $\delta z'_{yG}$ and the fundamental necessary condition for an extremum, namely $\delta v = 0$, (see Appendix I) the following relations can be derived -

$$\left.\begin{aligned}
\frac{d}{dx} F_{z_1''}\Big|_{x = x_G^-} - \frac{d}{dx} F_{z_2''}\Big|_{x = x_G^+} + \frac{d}{dy} F_{z_3''}\Big|_{y = y_G^-} - \frac{d}{dy} F_{z_4''}\Big|_{y = y_G^+} &= 0 \\
F_{z_1''}\Big|_{x = x_G^-} - F_{z_2''}\Big|_{x = x_G^+} &= 0 \\
F_{z_3''}\Big|_{y = y_G^-} - F_{z_4''}\Big|_{y = y_G^+} &= 0
\end{aligned}\right\} \quad (5)$$

Since $F_{z_i''} = 2z_i''$ and therefore $\frac{d}{dx} F_{z_i''} = 2z_i'''$, the equations (5) can be written as -

$$\left.\begin{aligned}
z_1'''(x_G) - z_2'''(x_G) + z_3'''(y_G) - z_4'''(y_G) &= 0 \\
z_1''(x_G) - z_2''(x_G) &= 0 \\
z_3''(y_G) - z_4''(y_G) &= 0
\end{aligned}\right\} \quad (6)$$

or since

$$z_i''(x) = 2a_{i2} + 6a_{i3}\,x, \quad z_i'''(x) = 6a_{i3} \qquad (i = 1,2)$$

and

$$z_i''(y) = 2a_{i2} + 6a_{i3}\,y, \quad z_i'''(y) = 6a_{i3} \qquad (i = 3,4)$$

these can be written as,

$$\left.\begin{array}{l} a_{13} - a_{23} + a_{33} - a_{43} = 0 \\ 2a_{12} + 6a_{13}\,x_G - 2a_{22} - 6a_{23}\,x_G = 0 \\ 2a_{32} + 6a_{33}\,y_G - 2a_{42} - 6a_{43}\,y_G = 0 \end{array}\right\} \qquad (7)$$

These provide the remaining 3 relations required to uniquely determine the sixteen coefficients a_{ij}. These can readily be shown (after arranging the a_{ij} for convenience in the subsequent reduction) to satisfy -

$$[C]\mathbf{a} = \mathbf{b}$$

where -

$$\mathbf{a}' = [a_{10}, a_{20}, a_{30}, a_{40}, a_{11}, a_{21}, a_{31}, a_{41}, a_{12}, a_{22}, a_{32}, a_{33}, a_{13}, a_{23}, a_{42}, a_{43}]$$

$$\mathbf{b}' = [z_1(x_{u-1}), z_2(x_u), z_3(y_{v-1}), z_4(y_v), z_1'(x_{u-1}), z_2'(x_u), z_3'(y_{v-1}), z_4'(y_v), 0, \ldots, 0]$$

and

$$[C] = \left[\begin{array}{cccccccccccccccc}
1 & & & & x_{u-1} & & & & x_{u-1}^2 & & & & x_{u-1}^3 & & & \\
 & 1 & & & & x_u & & & & x_u^2 & & & & x_u^3 & & \\
 & & 1 & & & & y_{v-1} & & & & y_{v-1}^2 & y_{v-1}^3 & & & & \\
 & & & 1 & & & & y_v & & & & & & & y_v^2 & y_v^3 \\
 & & & & 1 & & & & 2x_{u-1} & & & & 3x_{u-1}^2 & & & \\
 & & & & & 1 & & & & 2x_u & & & & 3x_u^2 & & \\
 & & & & & & 1 & & & & 2y_{v-1} & 3y_{v-1}^2 & & & & \\
 & & & & & & & 1 & & & & & & & 2y_v & 3y_v^2 \\
1 & & & -1 & x_G & & & -y_G & x_G^2 & & & & x_G^3 & & -y_G^2 & -y_G^3 \\
 & 1 & & -1 & & x_G & & -y_G & & x_G^2 & & & & x_G^3 & -y_G^2 & -y_G^3 \\
 & & 1 & -1 & & & y_G & -y_G & & & y_G^2 & y_G^3 & & & -y_G^2 & -y_G^3 \\
 & & & & & & & & & & & 1 & 1 & -1 & & -1 \\
 & & & & 1 & -1 & & & 2x_G & -2x_G & & & 3x_G^2 & -3x_G^2 & & \\
 & & & & & & -1 & -1 & & & 2y_G & 3y_G^2 & & & -2y_G & -3y_G^2 \\
 & & & & & & & & 2 & -2 & & & 6x_G & -6x_G & & \\
 & & & & & & & & & & 2 & 6y_G & & & -2 & -6y_G
\end{array}\right]$$

This matrix was then reduced to triangular form by elementary row operations (see Appendix II), the same operations being applied to $\underset{\sim}{b}$. Thus a set of relations defining the a_{ij} could be written down. These are shown in Appendix III, together with a program for determining the values of the a_{ij} for any given boundary values.

4. BICUBIC SPLINE INTERPOLATION FROM CONTOUR DATA

The primary aim of this Technical Report was to Establish the result given in Section 3. There are a number of ways of making use of this result to define a surface which is representative of the surface described by the original contours. To make best use of the properties engendered by the process used to derive the lattice point data, piecewise bicubic interpolation should be used. This is described below.

4.1 Use of the lattice-point data alone

Consider a rectangular area on which a data base has been established as in Section 2 and on which the value of z has been determined by the method of Section 3, for each of the lattice points (x_i,y_j) $i = 1, \ldots, I - 1$, $j = 1, \ldots, J - 1$. And furthermore that there be given values -

$$\left.\begin{array}{llll} & z(x_i,y_j) & i = 0,I, & j = 0, \ldots, J \\ & & j = 0,J, & i = 1, \ldots, I \\ p_{ij} = & z_x(x_i,y_j) & i = 0,I, & j = 0, \ldots, J \\ q_{ij} = & z_y(x_i,y_j) & j = 0,J, & i = 0, \ldots, I \\ s_{ij} = & z_{xy}(x_i,y_j) & i = 0,I, & j = 0,J \end{array}\right\} \quad (8)$$

(See figure 3 which displays the position of the data graphically). Then it has been shown(ref.6) that given the above values, the following theorem holds:

Theorem 3. There exists exactly one piecewise bicubic function z(x,y) of the form -

$$z(x,y) = \sum_{m=0}^{I+2} \sum_{n=0}^{J+2} \beta_{mn} \varphi_m(x) \psi_n(y)$$

where the φ_m and ψ_n are piecewise cubic and of class C^2 on R: $x_0 \leqslant x \leqslant x_I$, $y_0 \leqslant y \leqslant y_J$.

This theorem establishes the existence and uniqueness of the desired function. Furthermore an efficient computational scheme for its evaluation has been given in the same paper. Within a given rectangle R_{ij}: $x_{i-1} \leqslant x \leqslant x_i$; $y_{j-1} \leqslant y \leqslant y_j$, the interpolating function z(x,y) equals a bicubic polynomial,

$$C_{ij}(x,y) = \sum_{m,n=0}^{3} \gamma_{mn}^{ij}(x - x_{i-1})^m (y - y_{j-1})^n \quad (9)$$

The problem has been reduced to the easily handled case of a regular mesh at the cost of being removed one step further from the original data since the values $z(x,y)$ forming the data base of Section 2 are no longer involved explicitly.

4.2 Retaining the data base

In order to fit a surface to both the data base points and the lattice points it would be necessary to divide the X-Y plane over the region of interest into rectangular areas by connecting the data base points $\{(x_i,y_j),\ y_j = y_n\}$ on grid line $y = y_n$ to adjoining grid lines $y = y_{n-1}$ and $y = y_{n+1}$ by means of perpendiculars, and likewise the data base points $\{(x_i,y_j),\ x_i = x_m\}$, on grid line $x = x_m$ with adjoining grid lines $x = x_{m-1}$, $x = x_{m+1}$. The values of z and either p or q at the corners of the resulting rectangles could be determined and hence linear cubic splines for these connecting lines derived. Also the values of z and p or q at the intersections of such connecting lines could be obtained. Some means would then have to be devised to obtain the s_{ij} at the corners of each of the rectangles formed by this process. Then within each such rectangle a bicubic spline of the form given by equation (9) would be used as the interpolating function. The increased effort which would be involved (if in fact a suitable method for determining the s_{ij} could be devised) does not seem warranted by the expected increase in accuracy.

5. DISCUSSION AND COMPARISON WITH INVERSE DISTANCE WEIGHTED INTERPOLATION

The existing approaches to the problem of interpolating from irregularly spaced data have been summarized in reference 4. The most successful one is based on assigning a weighted sum of local values to the point at which interpolation is required. The weighting function may be a pure inverse distance function, where in the neighbourhood of a data point the interpolated value is computed from the difference of two almost equal numbers, which results in significant computational error. Another disadvantage for terrain modelling work is the fact that the method imposes zero directional derivatives at each data point; a suitable choice of exponent would alleviate this problem to some extent. It has been discovered empirically that a value of 2 for the exponent is best. Modifications to improve the weighting function include:

(a) adding a small constant to the denominator to improve the performance near data points,

(b) using an exponential form to improve smoothness while using only a small number of local data points,

(c) including direction in the weighting function, and

(d) (of particular relevance to the current application) attempting to correct the undesirable property of zero directional derivatives at the data points by the introduction of slope into the weighting function.

The process described in reference 4 for implementing method (d) was somewhat arbitary and considering the importance of slope in "line of sight" work especially at hill tops where it is critical, it was considered that a new approach was required, and this has been provided in Section 3.

To illustrate the difference between various methods, an example taken from reference 3 of an analytic surface representing "a steep hill rising from a plain" defined by the funciton -

$$z = e^{-\{(x-5)^2 + (y-5)^2\}}$$

was used. Contours at intervals of 0.2 were calculated with a data base extracted using the technique described in Section 2. The relevant calculations for the bicubic spline method are described in Appendix IV.2, and the results are illustrated in figure 4.

Two interpolation procedures based on inverse distance weighting functions were also used. The first was the simple case with $w(d) = d^{-2}$ given by(ref.4).

$$Z(P)\begin{cases} = \left\{\sum_{i=1}^{N} d_i^{-2} z^i\right\} \Big/ \sum_{i=1}^{N} d_i^{-2} & \text{if } d_i \neq 0 \text{ for all N data points} \\ = z_i & \text{if } d_i = 0 \end{cases}$$

where P is the point at which interpolation is required

z^i i = 1, ..., N are data points

d_i is the distance between P and the i^{th} data point

N is the number of data points

This method is poor for the current application since it produces a dip in place of a crest at the critical point - the apex of the hill. This flattening of hill tops and valley floors which is characteristic of weighted inverse distance smoothing is alleviated to some extent by incorporating two quantities A_i and B_i which represent the desired slopes in the x and y directions at the i^{th} data point and are weighted averages of the divided differences of z about that point. They are given by the expressions (see reference 4) -

$$A_i = \left[\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{d_i^{-2}(z^j - z^i)(x_j - x_i)}{d[D_j, D_i]^2}\right] \Big/ \sum_{\substack{j=1\\ j\neq i}}^{N} d_i^{-2}$$

$$B_i = \left[\sum_{\substack{i=1\\ j\neq i}}^{N} \frac{d_i^{-2}(z^j - z^i)(y_j - y_i)}{d[D_j, D_i]^2}\right] \Big/ \sum_{\substack{j=1\\ j\neq i}}^{N} d_i^{-2}$$

where $d(D_j, D_i)$ is the distance between the i^{th} and j^{th} data points. The interpolating function then becomes -

$$Z(P)\begin{cases} = \left\{\sum_{i=1}^{N} d_i^{-2}(z^i + \Delta z^i)\right\} \Big/ \sum_{i=1}^{N} d_i^{-2} & \textit{If } d_i \neq 0 \textit{ for all u data points} \\ = z^i & \text{If } d_i = 0 \end{cases}$$

whore

$$\Delta z^i = [A_i(x - x_i) + B_i(y - y_i)] \left[\frac{V}{V + d_i}\right]$$

and

$$V = 0.1[\max\{z^i\} - \min\{z^i\}]/[\max\{A_i^2 + B_i^2\}]^{\frac{1}{2}}$$

This method removes the dip produced by the previous function and achieves a value of 0.901 at the apex. The curve fitted by the bicubic splines method however achieves the value 0.968 giving a better approximation to the desired curve with a value of 1.0 at the apex. The data used and the values of the interpolation functions at selected points are shown in Table 2.

6. CONCLUSIONS

A method has been developed which enables more efficient use to be made of the data from automatic map reading instruments. This can now be processed by the means outlined in this Technical Report into a form which can be used directly in existing terrain and vegetation models.

The technique for determining the minimum curvature solution for the intersection of two splines has more general applicability. For example it would provide a good method of estimating the values of missing data points in a regular mesh data set.

NOTATION

a_{ij}	$i = 1, 2, \ldots, 4$, $j = 0,1, \ldots, 3$. The j^{th} coefficient in the i^{th} of the four cubic functions.
$\underset{\sim}{a}$	the vector $(a_{10}, a_{20}, a_{30}, a_{40}, a_{11}, a_{21}, a_{31}, a_{41}, a_{12}, a_{22}, a_{32}, a_{33}, a_{13}, a_{23}, a_{42}, a_{43})$
$\underset{\sim}{b}$	the vector $(z_1(x_{u-1}), z_2(x_u), z_3(y_{v-1}), z_4(y_v), z_1'(x_{u-1}), z_2'(x_u)$ $z_3'(y_{v-1}), z_4'(y_v), 0, 0, \ldots, 0)$
F_{z_i}	$\dfrac{\partial F(x_1\ z_i,\ z_i\ z_i)}{\partial\ z_i}$ $i = 1,2$ $\dfrac{\partial F(y_1\ z_i,\ z_i\ z_i)}{\partial\ z_i}$ $i = 3,4$
$F_{z_i'}, F_{z_i''}$	similar definitions to those for F_{z_i}
$H_{i,j}$	interchange the i^{th} j^{th} rows
$H_i(k)$	multiply every element in the i_{th} row by a non-zero scalar k
$H_{i,j}(k)$	add to the elements of the i^{th} row k (a scalar) times the corresponding elements of the j^{th} row
p_i	$z'(x_i)$
p_{ij}	$\dfrac{\partial z(x_i, y_i)}{\partial x}$
q_j	$z'(y_i)$
q_{ij}	$\dfrac{\partial z(x_i, y_j)}{\partial y}$
R_{ijk}	an element in row i, column j as modified at operation number k
$R_i\ R_k$	the term in $\underset{\sim}{b}$ corresponding to R_{ijk}
s_{ij}	$\dfrac{\partial z(x_i, y_j)}{\partial x\ \partial y}$
$\left.\begin{matrix} x \\ y \\ z \end{matrix}\right\}$	respectively refer to the three Cartesian coordinate axes
$\left.\begin{matrix} x_i \\ y_i \end{matrix}\right\}$	particular values of x and y
(x_G, y_G, z_G)	the coordinates of the inserted joint
x_{u-1}, x_u	the segment containing the value x_G

y_{v-1}, y_v	the segment containing the value y_G
$z(x_i, y_i)$	the value of z at the point (x_i, y_i)
z^i	a particular value $z(x_i)$, or $z(x_i, y_j)$ for a fixed j
z^j	a particular value $z(y_i)$, or $z(x_i, y_j)$ for a fixed i
z_i	$\left.\begin{array}{ll} z_i(x) & i = 1,2 \\ z_i(y) & i = 3,4 \end{array}\right\}$ The four functions to be optimized with respect to curvature
z_k^i	the value $z_k(x_i)$ $k = 1,2$ or $z_k(y_i)$ $k = 3,4$

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APPENDIX I

A PARTICULAR CASE OF THE FUNDAMENTAL NECESSARY CONDITION FOR AN EXTREMUM

An increment Δv of a functional of the form $v(z(x))$ can be written as -

$$\Delta v = v(z(x) + \delta z) - v(z(x))$$

If it is possible to resolve this into two components so that -

$$\Delta v = L(z(x), \delta z) + M(z(x), \delta z) \max |\delta z|$$

where $L(z(x), \delta z)$ is a linear functional in δz

$\max |\delta z|$ is the maximum value of δz

and

$M(z(x), \delta z) \to 0$ whenever $\max |\delta z| \to 0$

then $L(z(x), \delta z)$ is called the variation of the functional and is denoted by δv. If the variation of a functional exists and if v takes on a minimum or a maximum along $z = z_0(x)$ then $\delta v = 0$ along $z = z_0(x)$ (ref.8). This is the fundamental necessary condition for an extremum and is readily extended to functionals involving several independent functions.

The functional under consideration in the main body of the text is

$$v(z_1(x), z_2(x), z_3(y), z_4(y)) = \int_{x_{u-1}}^{x_G} [z_1''(x)]^2 \, dx + \int_{x_G}^{x_u} [z_2''(x)]^2 \, dx$$

$$+ \int_{y_{v-1}}^{y_G} [z_3''(y)]^2 \, dy + \int_{y_G}^{y_v} [z_4''(y)]^2 \, dy \qquad \text{(I.1)}$$

It can be seen that it consists of the sum of four functionals of the form -

$$v(z(t)) = \int_{t=t_0}^{t=t_1} F(t, z, z', z'') \, dt$$

For functionals of this form with one fixed and one moveable boundary point, the fundamental necessary condition for an extremum becomes (see reference 8) -

$$\left[F - z' F_{z'} - z'' F_{z''} + z' \frac{d}{dt} F_{z''}\right]_{t=t_1} \delta t_1 + \left[F_{z'} - \frac{d}{dt} F_{z''}\right]_{t=t_1} \delta z_1 + \left[F_{z''}\right]_{t=t_1} \delta z_1' = 0$$

(I.2)

Now using the linearity of the variation of a functional and interchanging the limits of integration, the variation δv of equation (I.1) can be written as -

$$\Delta v = \Delta\int_{x_{u-1}}^{x_G+\delta x} [z_1''(x)]^2\,dx - \Delta\int_{x_u}^{x_G+\delta x} [z_2''(x)]^2\,dx + \Delta\int_{y_{v-1}}^{y_G+\delta y} [z_3''(y)]^2\,dy$$

$$- \Delta\int_{y_v}^{y_G+\delta y} [z_4''(y)]^2\,dy$$

so that, using equation (I.2)

$$\Delta v = \sum_{i=1}^{2} (-1)^{i+1}\left\{\left[F - z_i' F_{z_i'} - z_i'' F_{z_i''} + z_i' \frac{d}{dx} F_{z_i''}\right]_{x=x_G^*} \delta x_G\right.$$

$$\left. + \left[F_{z_i'} - \frac{d}{dx} F_{z_i''}\right]_{x=x_G^*} \delta z_G + [F_{z_i''}]_{x=x_G^*}\, \delta z'_{x_G}\right\}$$

$$+ \sum_{i=3}^{4} (-1)^{i+1}\left\{\left[F - z_i' F_{z_i'} - z_i'' F_{z_i''} + z_i' \frac{d}{dy} F_{z_i''}\right]_{y=y_G^*} \delta y_G\right.$$

$$\left. + \left[F_{z_i'} - \frac{d}{dy} F_{z_i''}\right]_{y=y_G^*} \delta z_G + [F_{z_i''}]_{y=y_G^*}\, \delta z'_{y_G}\right\}$$

Now δx_G and δy_G are zero but δz_G, $\delta z'_{x_G}$ (the derivative of the variation with respect to x) and $\delta z'_{y_G}$ (the derivative of the variation with respect to y) are all arbitrary so that their coefficients must be identically zero. Noting that $F_{z_i'}$ i = 1,2,3,4, are zero this can be written as -

$$\frac{d}{dx} F_{z_1''}\Big|_{x=x_G^-} - \frac{d}{dx} F_{z_2''}\Big|_{x=x_G^+} + \frac{d}{dy} F_{z_3''}\Big|_{y=y_G^-} - \frac{d}{dy} F_{z_4''}\Big|_{y=y_G^+} = 0 \qquad \text{(I.3)}$$

$$F_{z_1''}\Big|_{x=x_G^-} - F_{z_2''}\Big|_{x=x_G^+} = 0 \qquad \text{(I.4)}$$

* Indicates derivative from either left or right as required

$$F_{z_3}'' \Big|_{y=y_G^-} - F_{z_4}'' \Big|_{y=y_G^+} = 0 \tag{I.5}$$

These are the forms used in the text.

APPENDIX II

THE REDUCTION OF [C] TO TRIANGULAR FORM

The columns of the matrix [C] and the corresponding terms of the vectors $\mathbf{a}$ and $\mathbf{b}$ were initially arranged to facilitate the reduction described in this Appendix. The notation used for elementary row operations is as follows:

$H_{i,j}$	interchange the i^{th} and j^{th} rows
$H_i(k)$	multiply every element in the i^{th} row by a non-zero scalar k
$H_{i,j}(k)$	add to the elements of the i^{th} row k (a scalar) times the corresponding elements of the j^{th} row.

An element in row i, column j, as modified at operation number k (below) is denoted by R_{ijk} and the corresponding term in $\mathbf{b}$ by $R_i\ R_k$.

The following list shows the operation followed by the changed row and the corresponding element of b -

(1) $H_{9,1}(-1)$: $0,0,0,-1,x_G-x_1,\ 0,0,-y_G,\ x_G^2-x_1^2,\ 0,0,0,\ x_G^3-x_1^3,\ 0,\ -y_G^2,\ -y_G^3;\quad -z_1$

(2) $H_{10,2}(-1)$: $0,0,0,-1,0,\ x_G-x_2,0,\ -y_G,\ 0,\ x_G^2-x_2^2,\ 0,0,0,\ x_G^3-x_2^3,\ -y_G^2,\ -y_G^3;\quad -z_2$

(3) $H_{11,3}(-1)$: $0,0,0,-1,0,0,y_G-y_1,\ -y_G,\ 0,0,y_G^2-y_1^2,\ y_G^3-y_1^3,0,0,-y_G^2,\ -y_G^3;\quad -z_3$

(4) $H_{9,4}(1)$: $4 \times$ "0", $x_G-x_1,\ 0,0,\ y_2-y_G,x_G^2-x_1^2,0,0,0,x_G^3-x_1^3,0,y_2^2-y_G^2,\ y_2^3-y_G^3;\quad z_4-z_1$

(5) $H_{10,4}(1)$: $5 \times$ "0", $x_G-x_2,\ 0,\ y_2-y_G\ 0,\ x_G^2-x_2^2,\ 0,0,0,\ x_G^3-x_2^3,\ y_2^2-y_G^2,\ y_2^3-y_G^3;\quad z_4-z_2$

(6) $H_{11,4}(1)$: $6 \times$ "0", $y_G-y_1,\ y_2-y_G,0,0,y_G^2-y_1^2,\ y_G^3-y_1^3,0,0,y_2^2-y_G^2,\ y_2^3-y_G^3;\quad z_4-z_3$

(7) $H_{1,5}(-x_1)$: $1,7 \times$ "0", $-x_1^2,\ 0,0,0,\ -2x_1^3,\ 0,0,0,;\quad z_1-x_1z_1'$

(8) $H_{9,5}(x_1-x_G)$: $7 \times$ "0", $y_2-y_G,\ (x_G-x_1)^2,\ 0,0,0,x_G^3-x_1^3-(x_G-x_1)^3\ x_1^2,\ 0,\ y_2^2-y_G^2,\ y_2^3-y_G^3;\quad z_4-z_1-(x_G-x_1)z_1'$

(9) $H_{13,5}(-1)$: $5 \times$ "0", $-1,0,0,2(x_G-x_1),-2x_G,0,0,3(x_G^2-x_1^2),-3x_G^2,0,0;\quad -z_1'$

(10) $H_{10,6}(x_2-x_G)$: $7 \times$ "0", $y_2-y_G,0,(x_G-x_2)^2,0,0,0,x_G^3-x_2^3-(x_G-x_2)3x_2^2,\ y_2^2-y_G^2,\ y_2^3-y_G^3;\quad z_4-z_2-(x_G-x_2)z_2'$

(11) $H_{2,6}(-x_2)$: $0,1,7 \times$ "0", $-x_2^2,0,0,0,-2x_2^3,\ 0,0;\quad z_2-x_2z_2'$

(12) $H_{13,6}(1)$: $8 \times$ "0", $2(x_G-x_1),\ 2(x_2-x_G),0,0,3(x_G^2-x_1^2),\ 3(x_2^2-x_G^2),0,0;\quad z_2'-z_1'$

(13) $H_{3,7}(-y_1)$: $0,0,1,7 \times \text{"0"}, -y_1^2, -2y_1^3, 4 \times \text{"0"};\ z_3 - y_1 z_3'$

(14) $H_{11,7}(y_1 - y_G)$: $7 \times \text{"0"}, y_2 - y_G, 0, 0, (y_G - y_1)^2, y_G^3 - y_1^3 - (y_G - y_1)3y_1^2, 0, 0,$
$(y_2^2 - y_G^2), y_2^3 - y_G^3;\ z_4 - z_3 - (y_G - y_1)z_3'$

(15) $H_{14,7}(-1)$: $7 \times \text{"0"}, -1, 0, 0, 2(y_G - y_1), 3(y_G^2 - y_1^2), 0, 0, -2y_G, -3y_G^2;\ -z_3'$

(16) $H_{4,8}(-y_2)$: $0,0,0,1,10 \times \text{"0"}, -y_2^2, -2y_2^3;\ z_4 - y_2 z_4'$

(17) $H_{14,8}(1)$: $10 \times \text{"0"}, 2(y_G - y_1), 3(y_G^2 - y_1^2), 0, 0, 2(y_2 - y_G), 3(y_2^2 - y_G^2);$
$z_4' - z_3'$

(18) $H_{11,8}(y_G - y_2)$: $10 \times \text{"0"}, (y_G - y_1)^2, y_G^3 - y_1^3 - (y_G - y_1)3y_1^2, 0, 0, -(y_G - y_2)^2,$
$y_2^3 - y_G^3 - (y_2 - y_G)3y_2^2;\ z_4 - z_3 - (y_G - y_1)z_3' - (y_2 - y_G)\ z_4'$

(19) $H_{10,8}(y_G - y_2)$: $9 \times \text{"0"}, (x_G - x_2)^2, 0, 0, 0, x_G^3 - x_2^3 - (x_G - x_2)\ 3x_2^2, -(y_G - y_2)^2,$
$y_2^3 - y_G^3 - (y_2 - y_G)3y_2^2;\ z_4 - z_2 - (x_G - x_2)z_2' - (y_2 - y_G)z_4'$

(20) $H_{9,8}(y_G - y_2)$: $8 \times \text{"0"}, (x_G - x_1)^2, 0, 0, 0, x_G^3 - x_1^3 - (x_G - x_1)3x_1^2, 0, -(y_G - y_2)^2,$
$y_2^3 - y_G^3 - (y_2 - y_G)3y_2^2;\ z_4 - z_1 - (x_G - x_1)z_1' - (y_2 - y_G)\ z_4'$

(21) $H_{15}(\tfrac{1}{2})$: $8 \times \text{"0"}, 1, -1, 0, 0, 3x_G, -3x_G, 0, 0;\ 0$

(22) $H_{16}(\tfrac{1}{2})$: $10 \times \text{"0"}, 1, 3y_G, 0, 0, -1, -3y_G;\ 0$

(23) $H_{9,15}$: (interchange 9 and 15)

(24) $H_{13,9}(2(x_1 - x_G))$: $9 \times \text{"0"}, 2(x_2 - x_1), 0, 0, -3(x_1 - x_G)^2, 3(x_2^2 - x_G^2) + 6(x_G - x_1)x_G,$
$0, 0;\ z_2' - z_1'$

(25) $H_{15,9}(-(x_G - x_1)^2)$: $9 \times \text{"0"}, (x_G - x_1)^2, 0, 0, 2(x_1 - x_G)^3, (x_G - x_1)^2\ 3x_G,$
$(y_2^2 - y_G^2) - (y_2 - y_G)2y_2, y_2^3 - y_G^3 - (y_2 - y_G)3y_2^2;$
$z_4 - z_1 + (x_G - x_1)z_1' - (y_2 - y_G)\ z_4'$

(26) $H_{10}(1./(x_G - x_2)^2)$: $9 \times \text{"0"}, 1, 0, 0, 0, (x_G^3 - x_2^3 - (x_G - x_2)3x_2^2)/(x_G - x_2)^2,$
$((y_2^2 - y_G^2) - (y_2 - y_G)2y_2)/(x_G - x_2)^2, ((y_2^3 - y_G^3) - (y_2 - y_G)3y_2^2)/$
$(x_G - x_2)^2;\ (z_4 - z_2 - (x_G - x_2)z_2' - (y_2 - y_G)z_4')/(x_G - x_2)^2$

(27) $H_{13,10}(2(x_1 - x_2))$: $12 \times \text{"0"}, -3(x_1 - x_G)^2, 3x_2^2 - 3x_G^2 + 6(x_G - x_1)x_G - 2(x_2 - x_1)$
R101426, $-2(x_2 - x_1)$R101526, $-2(x_2 - x_1)$R101626; $z_2' - z_1'$
$-2(x_2 - x_1)$ R10R26

(28) $H_{15,10}(-(x_G - x_1)^2)$: $12 \times \text{"0"}, 2(x_1 - x_G)^3, (x_G - x_1)^2\ 3x_G - (x_G - x_1)^2$ R101426,
$(y_2^2 - y_G^2) - (y_2 - y_G)2y_2 - (x_G - x_1)^2$R101526, $y_2^3 - y_G^3 - (y_2 - y_G)3y_2^2$
$-(x_G - x_1)^2$R101626; $z_4 - z_1 - (x_G - x_1)z_1' - (y_2 - y_G)z_4' - (x_G - x_1)^2$R10R26

(29) $H_{11,16}$: (interchange 16 and 11)

(30) $H_{14,11}(-2(y_G - y_1))$: $11 \times \text{"0"}, -3(y_G - y_1)^2, 0, 0, 2(y_2 - y_1), 3y_2^2 - 3y_G^2 + 2(y_G - y_1)$
$3y_G;\ z_4' - z_3'$

(31) $H_{16\ 11}(-(y_G-y_1)^2)$: 11 x "0", $2(y_1-y_G)^3, 0, 0, y_2^2-y_G-2y_2(y_2-y_G)+(y_G-y_1)^2$, $y_2^3-y_G^3-(y_2-y_G)3y_2^2+(y_G-y_1)^2 3y_G$; $z_4-z_3-(y_G-y_1)z_3' -(y_2-y_G)z_4'$

(32) $H_{14,12}(3(y_G-y_1)^2)$: 12 x "0", $3(y_G-y_1)^2, -3(y_G-y_1)^2, 2(y_2-y_1), 3(y_2^2-y_G^2) + 6(y_G-y_1)y_G - 3(y_G-y_1)^2$; $z_4'-z_3'$

(33) $H_{16,12}(-2y_1-y_G)^3)$: 12 x "0", $-2(y_1-y_G)^3, 2(y_1-y_G)^3, (y_G-y_1)^2-(y_G-y_2)^2$, $2y_G^3-2y_2^3+3y_2^2\ y_G+3y_1y_G(y_1-2y_G)+2(y_1-y_G)^3$; $z_4-z_3-(y_G-y_1)z_3' - (y_2-y_G)z_4'$

(34) $H_{13}(-1./3(x_1-x_G)^2)$: 12 x "0", 1, $(3x_2^2-3x_G^2+6(x_G-x_1)x_G-2(x_2-x_1)$ $R101426/-3(x_1-x_G)^2, -2(x_2-x_1)R101526/-3(x_1-x_G)^2$, $-2(x_2-x_1)R101626/-3(x_1-x_G)^2$; $[z_2'-z_1'-2(x_2-x_1)R10R26] /-3(x_1-x_G)^2$

(35) $H_{14,13}(-3(y_G-y_1)^2)$: 13 x "0", $-3(y_G-y_1)^2(1+R131434)$, $2y_2-2y_1-3(y_G-y_1)^2$ $R131534$, $3y_2^2-3y_G^2+6(y_G-y_1)y_G-3(y_G-y_1)^2$ $(1+R131634)$; $z_4'-z_3'-3(y_G-y_1)^2\ R13R34$

(36) $H_{15,13}(-2(x_1-x_G)^3)$: 13 x "0", $(x_G-x_1)^2\ 3x_G-(x_G-x_1)^2R101426 - 2(x_1-x_G)^3$ $R131434$, $(y_2^2-y_G^2)-(y_2-y_G)2y_2-(x_G-x_1)^2R101526-2(x_1-x_G)^3$ $R131534$, $(y_2^3-y_G^3)-(y_2-y_G)3y_2^2-(x_G-x_1)^2R101626$ $-2(x_1-x_G)^3\ R131634$; $z_4-z_1-(x_G-x_1)z_1'-(y_2-y_G)z_4'-(x_G-x_1)^2$ $R10R26 - 2(x_1-x_G)^3\ R13R34$

(37) $H_{16,13}(2(y_1-y_G)^3)$: 13 x "0", $2(y_1-y_G)^3(1+R131434)$, $(y_G-y_1)^2-(y_G-y_2)^2 +$ $2(y_1-y_G)^3\ R131534, 2y_G^3-2y_2^3+3y_2^2\ y_G+3y_1\ y_G(y_1-2y_G) +$ $2(y_1-y_G)^3\ (1+R131634)$; $z_4-z_3-(y_G-y_1)z_3'-(y_2-y_G)$ $z_4'+2(y_1-y_G)^3\ R13R34$

(38) $H_{14}(1./-3(y_G-y_1)^2(1+R131434)^*)$: 13 x "0", $[2y_2-2y_1-3(y_G-y_1)^2\ R131534]\ F38$, $[3y_2^2-3y_G^2+6(y_G-y_1)y_G-3(y_G-y_1)^2(1+R131634)]$ $F38$; $[z_4'-z_3'-3(y_G-y_1)^2R13R34]\ F38$

* Factor referred to as F38

(39) $H_{15,14}(-R151436)$: 14 x "0", R151536-R151436.R141538, R151636-R151436. R141638; R15R36-R151436.R14R38

(40) $H_{16,14}(-R161437)$: 14 x "0", R161537-R161437.R141538,R161637-R161437. R141638; R16R37-R161437.R14R38

(41) $H_{15}(1./R151539)$: 14 x "0",1, [R151636-R151436.R141638]/R151539; [R15R36-R151436.R14R38]/R151539

(42) $H_{16,15}(-R161540)$: 15 x "0", R161637-R161437.R141638-R161540.R151641; R16R37-R161437.R14R38-R161540.R15R41

(43) $H_{16}(1./[R161637-R161437.R141638-R161540.R151641])$

This sequence of operations reduces [C] to the partioned form -

$$\left[\begin{array}{c|c} \underset{8\times 8}{[I]} & \\ \cline{1-1} \underset{8\times 8}{[0]} & \underset{16\times 8}{[D]} \end{array}\right]$$

Where [I] is a diagonal matrix, [0] is a zero matrix and [D] is given by,

$$[D] = \begin{bmatrix}
-x_1^2 & & & & -2x_1^3 & & & \\
 & -x_2^2 & & & & -2x_2^3 & & \\
 & & -y_1^2 & -2y_1^3 & & & & \\
 & & & & & & -y_2^2 & -2y_2^3 \\
2x_1 & & & & 3x_1^2 & & & \\
 & 2x_2 & & & & 3x_2^2 & & \\
 & & 2y_1 & 3y_1^2 & & & & \\
 & & & & & & 2y_2 & 3y_2^2 \\
1 & -1 & & & 3x_G & -3x_G & & \\
 & 1 & & & & t_1 & t_2 & t_3 \\
 & & 1 & 3y_G & & & -1 & -3y_G \\
 & & & 1 & 1 & -1 & & -1 \\
 & & & & 1 & t_4 & t_5 & t_6 \\
 & & & & & 1 & t_7 & t_8 \\
 & & & & & & 1 & t_9 \\
 & & & & & & & 1
\end{bmatrix}$$

The terms t_1, t_2 ..., t_9 are given by -

$$t_1 = [x_G^3 - x_2^3 - (x_G - x_2)3x_2^2]/(x_G - x_2)^2 = x_G + 2x_2 = R101426$$

$$t_2 = [y_2^2 - y_G^2 - (y_2 - y_G)2y_2]/(x_G - x_2)^2 = -(y_G - y_2)^2/(x_G - x_2)^2 = R101526$$

$$t_3 = [y_2^3 - y_G^3 - (y_2 - y_G)3y_2^2]/(x_G - x_2)^2 = R101626$$

$$t_4 = [3x_2^2 - 3x_G^2 + 6(x_G - x_1)x_G - 2(x_2 - x_1)\ R101426]/(3(x_1 - x_G)^2) = R131434$$

$$t_5 = -2(x_2 - x_1)R101526/(-3(x_1 - x_G)^2) = R131534$$

$$t_6 = -2(x_2 - x_1)R101626/(-3(x_1 - x_G)^2) = R131634$$

$$t_7 = [2(y_2 - y_1) - 3(y_G - y_1)^2\ R131534]/[-3(y_G - y_1)^2(1 + R131434)]$$

$$t_8 = [3(y_2^2 - y_G^2) + 6(y_G - y_1)y_G - 3(y_G - y_1)^2(1 + R131634)]/[-3(y_G - y_1)^2(1 + R131434)]$$

$$t_9 = (R151636 - R151436.R141638)/R151539$$

The vector $\underset{\sim}{b}$ becomes -

$$\begin{bmatrix} z_1 - x_1 \quad z_1' \\ z_2 - x_2 \quad z_2' \\ z_3 - y_1 \quad z_3' \\ z_4 - y_2 \quad z_4' \\ z_1' \\ z_2' \\ z_3' \\ z_4' \\ 0 \\ [z_4 - z_2 - (x_G - x_2) z_2' - (y_2 - y_G) z_4'] / (x_G - x_2)^2 \\ 0 \\ 0 \\ [z_2' - z_1' - 2(x_2 - x_1) R10R26] / [-3(x_1 - x_G)^2] \\ [z_4' - z_3' - 3(y_G - y_1)^2 R13R34] / [-3(y_G - y_1)^2 (1 + R131434)] \\ (R15R36 - R151436.R14R38)/R151539 \\ (R16R37 - R161437.R14R38 - R161540.R15R41)/R161642 \end{bmatrix}$$

A table showing the operations which affect each row in the order in which they were used (last on the right) is given below:

Row	Operation numbers
1	7
2	11
3	13
4	16
5-8	Not changed
9	1,4,8,20,23
10	2,5,10,19,26
11	3,6,14,18,29
12	Not changed
13	9,12,24,27,34
14	15,17,30,32,35,38
15	21,23,25,28,36,39,41
16	22,29,31,33,37,40,42,43

APPENDIX III

THE COMPUTING ALGORITHM AND PROGRAMME FOR THE INTERSECTION OF TWO SPLINES

From the reduction of [C] and $\underset{\sim}{b}$ given in Appendix II the relations defining the sixteen components a_{ij} of the vector $\underset{\sim}{a}$ in the equation $[C]\ \underset{\sim}{a} = \underset{\sim}{b}$, Section 3 of the main text, can be written down. These are, in order of calculation, and using the row element terminology introduced in Appendix II.

$$a_{43} = [\text{R16R37-R161437.R14R38-R161540.R15R41}] / [\text{R161637-R161437.R141638-R161540.R151641}]$$

$$a_{42} = \text{R15R41} - t_9\ a_{43}$$

$$a_{23} = \text{R14R38} - t_8\ a_{43} - t_7\ a_{42}$$

$$a_{13} = \text{R13R34} - t_6\ a_{43} - t_5\ a_{42} - t_4\ a_{23}$$

$$a_{33} = a_{23} + a_{43} - a_{13}$$

$$a_{32} = 3y_G\ a_{43} + a_{42} - 3y_G\ a_{33}$$

$$a_{22} = \text{R10R26} - t_3\ a_{43} - t_2\ a_{42} - t_1\ a_{23}$$

$$a_{12} = 3x_G\ a_{23} - 3x_G\ a_{13} + a_{22}$$

where the t_i terms are defined in Appendix II.

Writing p_i, i = 1,2,3,4 for the variables x_1, x_2, y_1, y_2 respectively

$$a_{i1} = z'_i - 2p_i\ a_{i2} - 3p_i^2\ a_{i3} \qquad \text{for } i = 1,2,3,4$$

$$a_{i0} = z_i - p_i\ z'_i + 2p_i^3\ a_{i3} + p_i^2\ a_{i2} \qquad \text{for } i = 1,2,3,4$$

During the reduction of [C] to triangular form certain restrictions were introduced by the fact that the scalars k in the elementary row operations $H_i(k)$, $H_{ij}(k)$ were required to be non-zero so that we have -

(7) $x_1 \neq 0$

(8) $x_1 \neq x_G$

(10) $x_2 \neq x_G$

(11) $x_2 \neq 0$

(13) $y_1 \neq 0$

(14) $y_1 \neq y_G$

(16) $y_2 \neq 0$

(18) $y_G \neq y_2$

(27) $x_1 \neq x_2$

(38) R131434 $\neq -1$

(39) R151436 $\neq$ 0

(40) R161437 $\neq$ 0

(41) R151539 $\neq$ 0

(42) R161540 $\neq$ 0

and from the calculation of a_{43}

R161637 - R161437.R141638-R161540.R151641 $\neq$ 0

The basic computing routine ROUTINE 1 incorporates checks for the above conditions.

```
      SUBROUTINE RTINE1(X1,X2,Y1,Y2,XG,YG,Z1,Z1D,Z2,Z2D,Z3,Z3D,Z4,Z4D,
     1 NFLAG)
C-----------------------------------------------------------------------
C   THIS SUBROUTINE ACCEPTS BOUNDARY VALUES XI,YI,ZI AND A PARTIAL
C   DERIVATIVE OF Z(XI,YI) I=1,..4  AND FINDS INTERPOLATION FUNCTIONS
C   TO DETERMINE THE VALUE OF Z AT THE POINT (XG,YG).
C
C          ERROR FLAGS:
C       NFLAG    ERROR
C         1    X1=0
C         2     X1=XG
C         3     X2=XG
C         4     X2=0
C         5     Y1=0
C         6     Y1=YG
C         7     Y2=0
C         8     Y2=YG
C         9     X1=X2
C        10     R131434=1
C        11     R151436=0
C        12     R161437=0
C        13     R151339=0
C        14     R161540=0
C        14     R161637-R161437.R141638-R161540.R151641=0
C
C   DIF IS THE MAXIMUM DIFFERENCE BETWEEN THE THEORETICALLY EQUIVALENT
C   VALUES OF ZG1,ZG2,ZG3 AND ZG4.
C-----------------------------------------------------------------------
C
      IMPLICIT REAL*8(A-H,O-Z)
C
C---COMPUTE POWERS TO SAVE REPEATED CALCULATION OF THESE VALUES------
      X12=X1**2
      X22=X2**2
      XG2=XG**2
      X13=X12*X1
      X23=X22*X2
      XG3=XG2*XG
      Y12=Y1**2
      Y22=Y2**2
      YG2=YG**2
      Y13=Y12*Y1
      Y23=Y22*Y2
      YG3=YG2*YG
C
C---TERMS REQUIRED FOR LAST STAGES OF TRIANGULAR REDUCTION-----------
      F26=1./(XG-X2)**2
      R01426=XG+2.*X2
      R01526=(Y22-YG2-(Y2-YG)*2.*Y2)*F26
      R01626=(Y23-YG3-(Y2-YG)*3.*Y22)*F26
      ROR26=(Z4-Z2-(XG-X2)*Z2D-(Y2-YG)*Z4D)*F26
      F34=1./(-3.*(X1-XG)**2)
      R31434=(3.*(X22-XG2)+2.*(XG-X1)*3.*XG-2.*(X2-X1)*R01426)*F34
```

```
      R31534=-R01526*2.*(X2-X1)*F34
      R31634=-R01626*2.*(X2-X1)*F34
      R3R34=(Z2D-Z1D-2.*(X2-X1)*R0R26)*F34
      R51436=(XG-X1)**2*3.*XG-(XG-X1)**2*R01426-
     12.*(X1-XG)**3*R31434
      R51536=(Y22-YG2)-(Y2-YG)*2.*Y2-(XG-X1)**2*R01526
     1-2.*(X1-XG)**3*R31534
      R51636=Y23-YG3-(Y2-YG)*3.*Y22-(XG-X1)**2*R01626
     1-2.*(X1-XG)**3*R31634
      R5R36=-Z1+Z4-(XG-X1)*Z1D-(Y2-YG)*Z4D-(XG-X1)**2*R0R26
     1-2.*(X1-XG)**3*R3R34
      R61437=2.*(Y1-YG)**3*(R31434+1.)
      R61537=(YG-Y1)**2-(YG-Y2)**2+2.*(Y1-YG)**3*R31534
      R61637=2.*YG3-2.*Y23+3.*Y22*YG+3.*Y1*YG*(Y1-2.*YG)+2.*(Y1-YG)**3
     1+2.*(Y1-YG)**3*R31634
      R6R37=Z4-Z3-(YG-Y1)*Z3D-(Y2-YG)*Z4D+2.*(Y1-YG)**3*R3R34
      F38=1./(3.*(YG-Y1)**2*(-1.-R31434))
      R41538=(2.*Y2-2.*Y1-3.*(YG-Y1)**2*R31534)*F38
      R41638=(3.*Y22-3.*YG2+6.*YG*(YG-Y1)-3.*(YG-Y1)**2*(R31634+1.))*F38
      R4R38=(Z4D-Z3D-3.*(YG-Y1)**2*R3R34)*F38
      R51539=R51536-R51436*R41538
      R61540=R61537-R61437*R41538
      R51641=(R51636-R51436*R41638)/R51539
      R5R41=(R5R36-R51436*R4R38)/R51539
C
C----ZERO DIVIDE CHECKS---------------------------------------------
      NFLAG=1
      IF(X1.EQ.0.)GOTO100
      NFLAG=NFLAG+1
      IF(X1-XG.EQ.0.)GOTO100
      NFLAG=NFLAG+1
      IF((X2-XG).EQ.0.)GOTO100
      NFLAG=NFLAG+1
      IF(X2.EQ.0.)GOTO100
      NFLAG=NFLAG+1
      IF(Y1.EQ.0.)GOTO100
      NFLAG=NFLAG+1
      IF((Y1-YG).EQ.0.)GOTO100
      NFLAG=NFLAG+1
      IF(Y2.EQ.0.)GOTO100
      NFLAG=NFLAG+1
      IF((Y2-YG).EQ.0.)GOTO100
      NFLAG=NFLAG+1
      IF(R31434.EQ.-1.)GOTO100
      NFLAG=NFLAG+1
      IF(R51436.EQ.0.)GOTO100
      NFLAG=NFLAG+1
      IF(R61437.EQ.0.)GOTO100
      NFLAG=NFLAG+1
      IF(R51539.EQ.0.)GOTO100
      NFLAG=NFLAG+1
      IF(R61540.EQ.0.)GOTO100
```

```
      NFLAG=NFLAG+1
      IF(R61637-R61437*R41638-R61540*R51641.EQ.0.)GOTO100
      GOTO10
  100 WRITE(3,19)NFLAG
      RETURN
   19 FORMAT(I6)
C
C------COMPUTE COEFFICIENTS A(I,J)------------------------------------
   10 A43=(R6R37-R61437*R4R38-R61540*R5R41)/
     1(R61637-R61437*R41638-R61540*R51641)
      A42=R5R41-R51641*A43
      A23=R4R38-R41638*A43-R41538*A42
      A13=R3R34-R31634*A43-R31534*A42-R31434*A23
      A33=-A13+A23+A43
      A32=-3.*YG*A33+A42+3.*YG*A43
      A22=R0R26-R01626*A43-R01526*A42-R01426*A23
      A12=A22-3.*XG*A13+3.*XG*A23
      A11=Z1D-2.*X1*A12-3.*X12*A13
      A21=Z2D-2.*X2*A22-3.*X22*A23
      A31=Z3D-2.*Y1*A32-3.*Y12*A33
      A41=Z4D-2.*Y2*A42-3.*Y22*A43
      A10=Z1-X1*Z1D+2.*X13*A13+X12*A12
      A20=Z2-X2*Z2D+2.*X23*A23+X22*A22
      A30=Z3-Y1*Z3D+2.*Y13*A33+Y12*A32
C
C------COMPUTE VALUES OF HEIGHT AT (XG,YG)------------------------
      ZG1=A10+A11*XG+A12*XG2+A13*XG3
      ZG2=A20+A21*XG+A22*XG2+A23*XG3
      ZG3=A30+A31*YG+A32*YG2+A33*YG3
      ZG4=A40+A41*YG+A42*YG2+A43*YG3
C
C------FIND MAXIMUM DIFFERENCE BETWEEN THE ZGI--------------------
      D1=DABS(ZG1-ZG2)
      D2=DABS(ZG1-ZG3)
      D3=DABS(ZG1-ZG4)
      D4=DABS(ZG2-ZG3)
      D5=DABS(ZG2-ZG4)
      D6=DABS(ZG3-ZG4)
      DIF=DMAX1(D1,D2,D3,D4,D5,D6)
C
C------OUTPUT---------------------------------------------------------
      WRITE(3,12)A10,A11,A12,A13,
     1           A20,A21,A22,A23,
     1           A30,A31,A32,A33,
     1           A40,A41,A42,A43
      WRITE(3,12)XG,YG
   12 FORMAT(4F15.4)
      WRITE(3,12)ZG1,ZG2,ZG3,ZG4,DIF
      RETURN
      END
```

APPENDIX IV

AN EXAMPLE

IV.1 An example: the intersection of two splines

Consider the two splines $S_{\Delta x_G}(f;x)$, $S_{\Delta y_G}(g;y)$ defined in Section 3 of the text. Given the values -

$$(x_{u-1}, y_G, z_1(x_{u-1}), z_1'(x_{u-1}+)) = (10.1,\ 11,\ 10,\ 0.2)$$

$$(x_u,\ y_G,\ z_2(x_u),\ z_2'(x_u-)) = (11.5,\ 11,\ 12.1,\ -0.2)$$

$$(y_{v-1}, x_G, z_3(y_{v-1}), z_3'(y_{v-1}+)) = (10.3,\ 11,\ 12.2,\ 0.4)$$

$$(y_v, x_G, z_4(y_v), z_4'(y_v-)) = (11.4,\ 11,\ 11.6,\ -0.11)$$

$$(x_G, y_G) = (11.,11.)$$

we wish to find the functions $z_1(x)$, $z_2(x)$, $z_3(y)$, $z_4(y)$ satisfying the conditions (1) in Section 3 of the text. The situation is depicted graphically in figure 2.

ROUTINE 1 is used to evaluate the coefficients a_{ij} and hence the values of z_G calculated from each of the four cubic equations z_1, z_2, z_3 and z_4. Also the coefficients a_{ij} are used to compute the product $[C]\ \underset{\sim}{a}$ and the resultant vector $\underset{\sim}{b}$ is compared with the expected vector $\underset{\sim}{b}$. These values, calculated using an IBM 370/168 computer in both single and double precision, are shown in Table 1. The derivatives $z_1'(x)|_{x=x_G-}$, $z_2'(x)|_{x=x_G+}$, $z_3'(y)|_{y=y_G-}$, $z_4'(y)|_{y=y_G+}$, are also tabulated. In general the results show that double precision adequately satisfies all the conditions (1) and single precision gives a good approximation which may be adequate for some purposes.

The functions z_i $i = 1,2,3,4$ are plotted in figure 4 using the double precision results. The unbroken lines represent the sections of the cubics in the required regions $[x_{u-1}, x_u]$, $[y_{v-1}, y_v]$ and the dotted curves are included for interest and show these functions either side of their required ranges.

IV.2 An example: the interpolation at a local extremum

The function described in reference 3 as a "steep hill rising from a plain" given by the expression -

$$z(x,y) = e^{-\{(x-5)^2+(y-5)^2\}} \qquad \text{(IV.1)}$$

was taken as an example of a local maximum to illustrate the performance of the form of interpolation, recommended in this Technical Report.

The function was considered to be defined by contours at intervals of 0.2, without "spot heights", (i.e. no degenerate contour at the point (5., 5., 1.)). The sixteen relevant points in the data base, defined as specified in Section 2 are shown below -

3.731364	4.042769	4.285279	4.527619	5.472381	5.714721	5.97231	6.268636
5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
0.2	0.4	0.6	0.8	0.8	0.6	0.4	0.2

These values are used to define the two splines $S_\Delta(f;x_G)$, $S_\Delta(g;y_G)$ from which the values $z_1'(x_{u-1})$, $z_2'(x_u)$, $z_3'(y_{v-1})$, $z_4'(y_v)$, can be obtained. In addition to the data in the above table the values p_i, $i = 0,m$, q_{ij}, $j = 0,M$ are needed. These can be obtained from equation (IV.1) noting that $z(x,y)$ is symmetrical about the line $x = 5$, $y = 5$.

$$\left.\frac{\partial z_1}{\partial x}\right|_{x=3.731364} = \left.\frac{\partial z_3}{\partial y}\right|_{y=3.731364} = 0.507454$$

$$\left.\frac{\partial z_2}{\partial x}\right|_{x=6.268636} = \left.\frac{\partial z_4}{\partial y}\right|_{y=6.268636} = -0.507454$$

The $z^i{}_k$ (for $k = 1,2,3,4$) are determined by using the following relationship cited in reference 6 -

$$\Delta x_i\ p_{i-1} + 2(\Delta x_i + \Delta x_{i-1})p_i + \Delta x_{i-1}\ p_{i+1} = 3\left[\Delta x_{i-1}\frac{\Delta z_k^i}{\Delta x_i} + \Delta x_i\frac{\Delta z_k^{i-1}}{\Delta x_{i-1}}\right] \quad i=1,..,M-1 \tag{IV.2}$$

where

$$\Delta x_i = x_{i+1} - x_i \text{ and } \Delta z_k^i = z_i^{i+1} - z_k^i$$

Using the notation $t_i = 2(\Delta x_i + \Delta x_{i-1})$, $u_i = \Delta x_i$ the above relation can be written as -

$$\begin{bmatrix} t_1 & u_0 & 0 & 0 & 0 & 0 \\ u_2 & t_2 & u_1 & 0 & 0 & 0 \\ 0 & u_3 & t_3 & u_2 & 0 & 0 \\ 0 & 0 & u_4 & t_4 & u_3 & 0 \\ 0 & 0 & 0 & u_5 & t_5 & u_4 \\ 0 & 0 & 0 & 0 & u_6 & t_6 \end{bmatrix} \begin{bmatrix} p_1 \\ . \\ . \\ . \\ . \\ p_6 \end{bmatrix} = \begin{bmatrix} \beta_1 - \Delta x_1\ p_0 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 - \Delta x_5\ p_7 \end{bmatrix}$$

where the β_i are given by the right hand side of equation (IV.1).

The sequence of elementary row operations $H_1(t_1)$, $(H_{i,i-1}(-u_i)$, $H_i(R_i))$ $i = 2, \ldots, 6$, reduces the matrix to triangular form, so that -

$$\begin{bmatrix} 1 & u_0/R_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & u_1/R_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & u_2/R_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & u_3/R_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & u_4/R_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_6 \end{bmatrix} = \begin{bmatrix} \beta'_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \beta'_6 \end{bmatrix}$$

where

$$R_1 = t_1$$

$$R_i = t_i - u_i u_{i-2}/R_{i-1} \qquad i = 2, \ldots, 6$$

and

$$\beta'_1 = (\beta_1 - \Delta x_1 \; p_0)/R_1$$

$$\beta'_i = (\beta_i - u_i \; \beta'_{i-1})/R_i \qquad i = 2, \ldots, 5$$

$$\beta'_6 = ((\beta_6 - \Delta x_5 \; p_7) - u_6 \; \beta'_5)/R_6$$

The p_i can therefore be obtained from the sequence -

$$p_6 = \beta'_6$$

$$p_i = \beta'_i - u_{i-1} \; p_{i+1}/R_i \qquad i = 5, \ldots, 1$$

This enables the z_i to be determined noting that $q_i = p_i$ because of the symmetry of the function,

$$z'_1(x_{u-1}) = p_3 \qquad z'_2(x_u) = p_4$$

$$z'_3(y_{v-1}) = q_3 = p_3 \qquad z'_4(y_v) = q_4 = p_4$$

The other required quantities, which can be obtained directly from the data base are -

$$(x_{u-1}, y_G, z_1(x_{u-1})) = (4.527619, 5., 0.8)$$

$$(x_u, y_G, z_2(x_u)) = (5.472381, 5., 0.8)$$

$$(x_G, y_{v-1}, z_3(y_{v-1})) = (5., 4.527619, 0.8)$$

$$(x_G, y_v, z_4(y_v)) = (5., 5.472381, 0.8)$$

This information was used in ROUTINE 1 (see Appendix III). The value obtained for z(5., 5.) was 0.9682. The curve (IV.1) together with the interpolating function is shown plotted in figure 5. The derivatives of both functions are also shown in the same figure.

TABLE 1. DATA FOR THE INTERSECTION OF THE TWO SPLINES OF EXAMPLE IV.1

a_{ij}	Single* precision	Double* precision	Difference	Difference (%)
a_{10}	2855.74	2853.87822	-1.9	-0.1
a_{11}	-804.72	-804.18803	0.5	0.5
a_{12}	75.641	75.59007	-0.05	-0.1
a_{13}	-2.3626	-2.3609826	-0.002	-0.1
a_{20}	-942.69	-937.03649	5.7	0.6
a_{21}	231.21	229.69779	-1.5	-0.7
a_{22}	-18.535	-18.399545	0.14	0.8
a_{23}	0.49121	0.48718773	-0.004	-0.8
a_{30}	-2487.51	-2482.5837	4.9	0.2
a_{31}	697.28	695.90655	-1.4	-0.2
a_{32}	-64.748	-64.619135	0.13	0.2
a_{33}	2.0012	1.9971962	-0.004	-0.2
a_{40}	1310.9	1308.3311	-2.6	0.2
a_{41}	-338.65	-337.979	0.67	0.2
a_{42}	29.4284	29.370486	-0.06	0.2
a_{43}	-0.852638	-0.85097414	0.002	0.2
z_1	11.7400	11.741209563968	0.001	0.01
z_2	11.7371	11.741209563955	0.004	0.04
z_3	11.7314	11.741209563956	0.01	0.08
z_4	11.7454	11.741209563958	-0.004	-0.03
mean z	11.7385	11.741209563957	0.0027	0.02
var z	0.0058	0	0.0058	-

	Single precision	Difference	Double precision	Difference
z_1 (x-)	1.75721	0.00121	1.7569590	0.0000004
z_2 (x+)	1.7560		1.7569586	
z_3 (y-1)	-0.73422	-0.00075	-0.7322041	0.0000007
z_3 (y+)	-0.73347		-0.7322034	

TABLE 1(CONTD.).

b	b_1 single precision	Error $b - b_1$	b_2 double precision	Error $b - b_2$
10.0	10.0024	-0.0024	10.0	0
12.10	12.0952	0.0048	12.10	0
12.20	12.1890	0.0110	12.20	0
11.60	11.6040	-0.0040	11.60	0
0.20	0.2002	-0.0002	0.20	0
-0.20	-0.2002	0.0002	-0.20	0
0.40	0.3999	0.0001	0.40	0
-0.110	-0.1099	-0.0001	-0.110	0
0	-0.0056	-0.0056	0	0
0	-0.0083	-0.0083	0	0
0	-0.0142	-0.0142	0	0
0	-0.7229	-0.7229	0	0
0	0.0005	0.0005	0	0
0	-0.0007	-0.0007	0	0
0	0.0000	0.0000	0	0
0	0.0000	0.0000	0	0

* The IBM 370/168 computer uses a 32 bit word on single precision

TABLE 2. DATA AND FUNCTION VALUES FOR EXAMPLE IV.2

Data base for interpolation examples

x	y	z
4.2852	5.0000	0.6000
4.5276	5.0000	0.8000
5.4723	5.0000	0.8000
5.7147	5.0000	0.6000
5.0000	4.2852	0.6000
5.0000	4.5276	0.8000
5.0000	5.4723	0.8000
5.0000	5.7147	0.6000
5.5053	4.4946	0.6000
5.5053	5.5053	0.6000
4.4946	4.4946	0.6000
4.4946	5.5053	0.6000

Simple inverse distance function

x	y	z
5.0000	5.0	0.7067
5.1500	5.0	0.7108
5.2000	5.0	0.7156
5.3000	5.0	0.7373
5.4000	5.0	0.7789
5.6000	5.0	0.6856

Inverse distance function with gradients

x	y	z
5.0000	5.0	0.9013
5.1500	5.0	0.8898
5.3000	5.0	0.8563
5.6000	5.0	0.7099

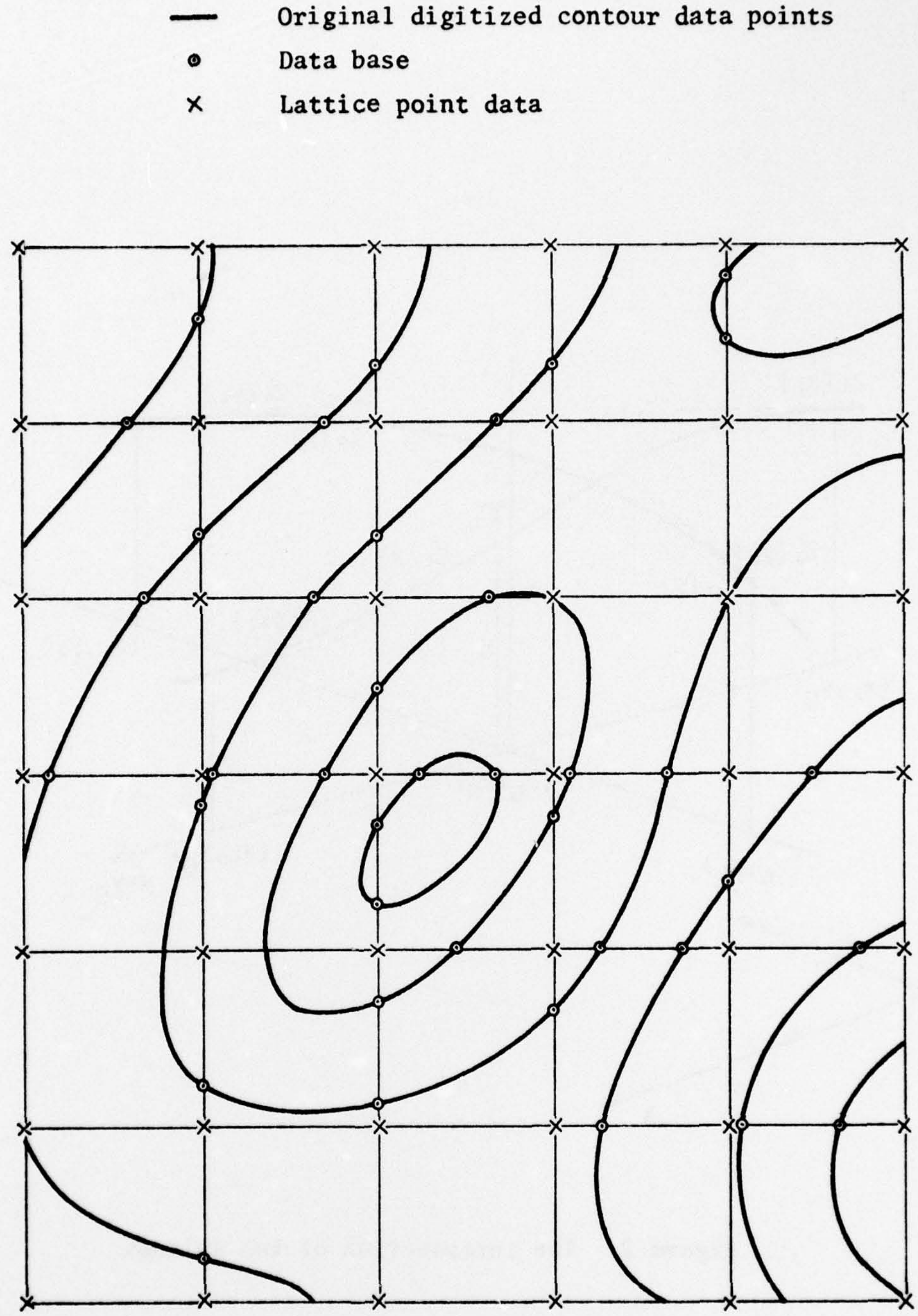

Figure 1. Terrain data types

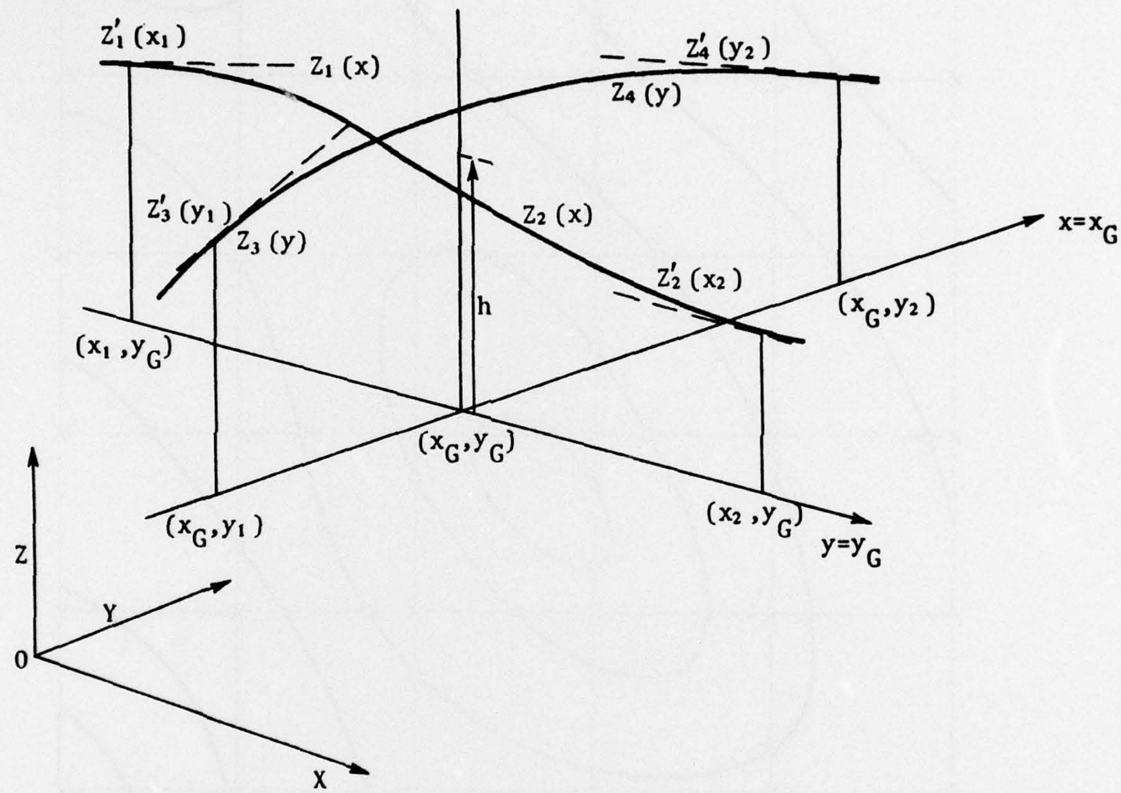

Figure 2. The intersection of two splines

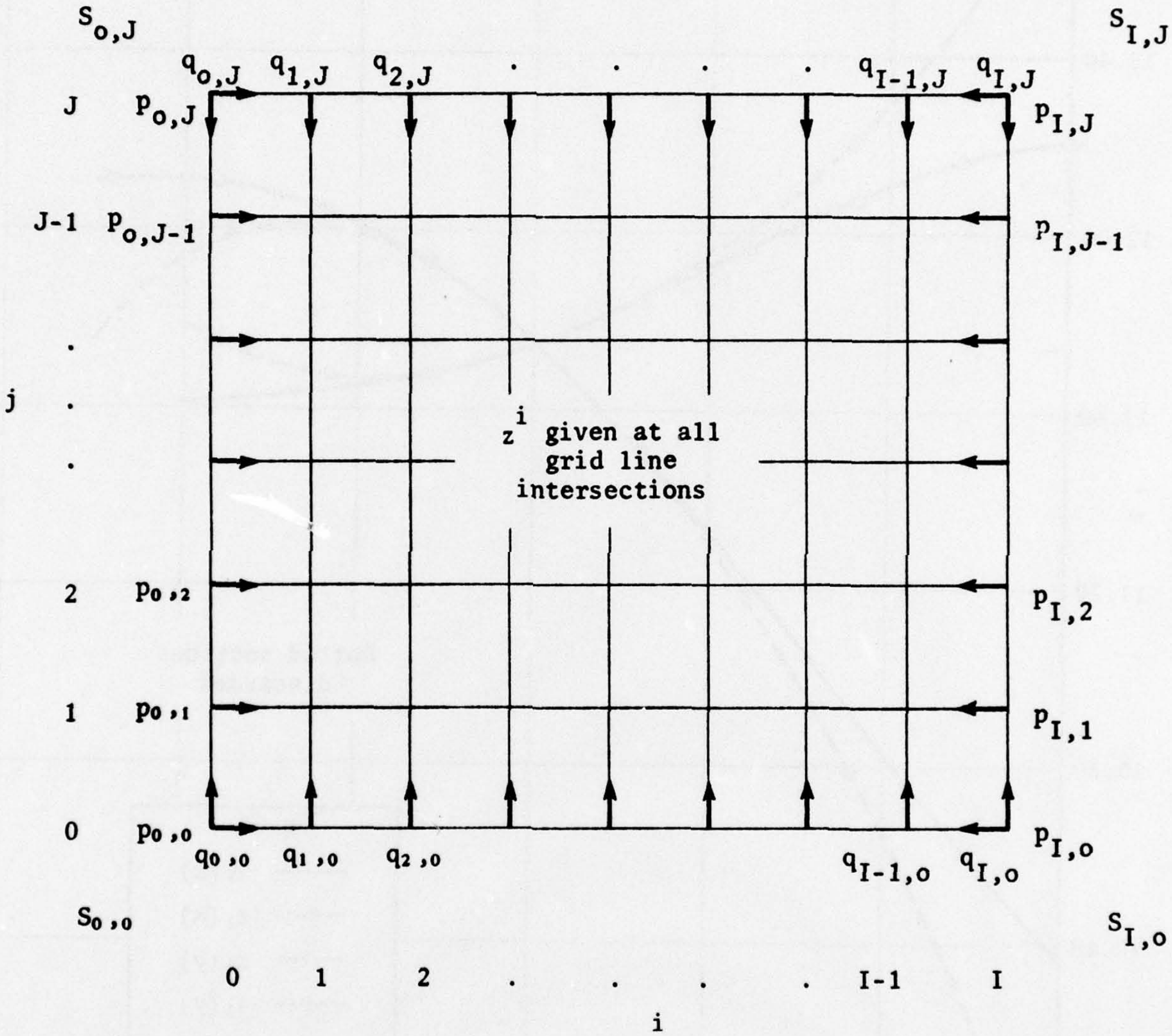

Figure 3. The data required for bicubic spline interpolation on a rectangular lattice

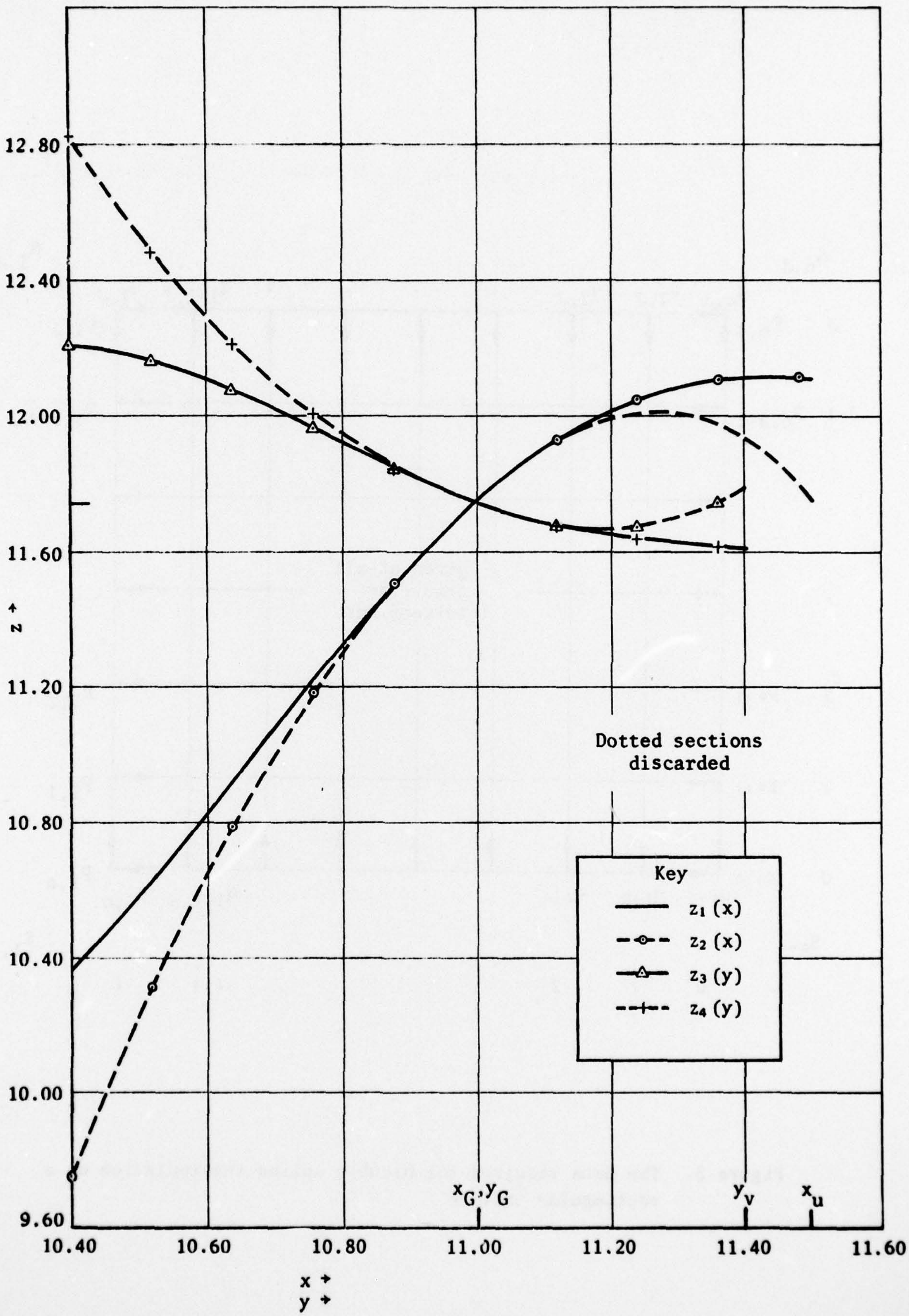

Figure 4. The interpolating functions for example IV.1

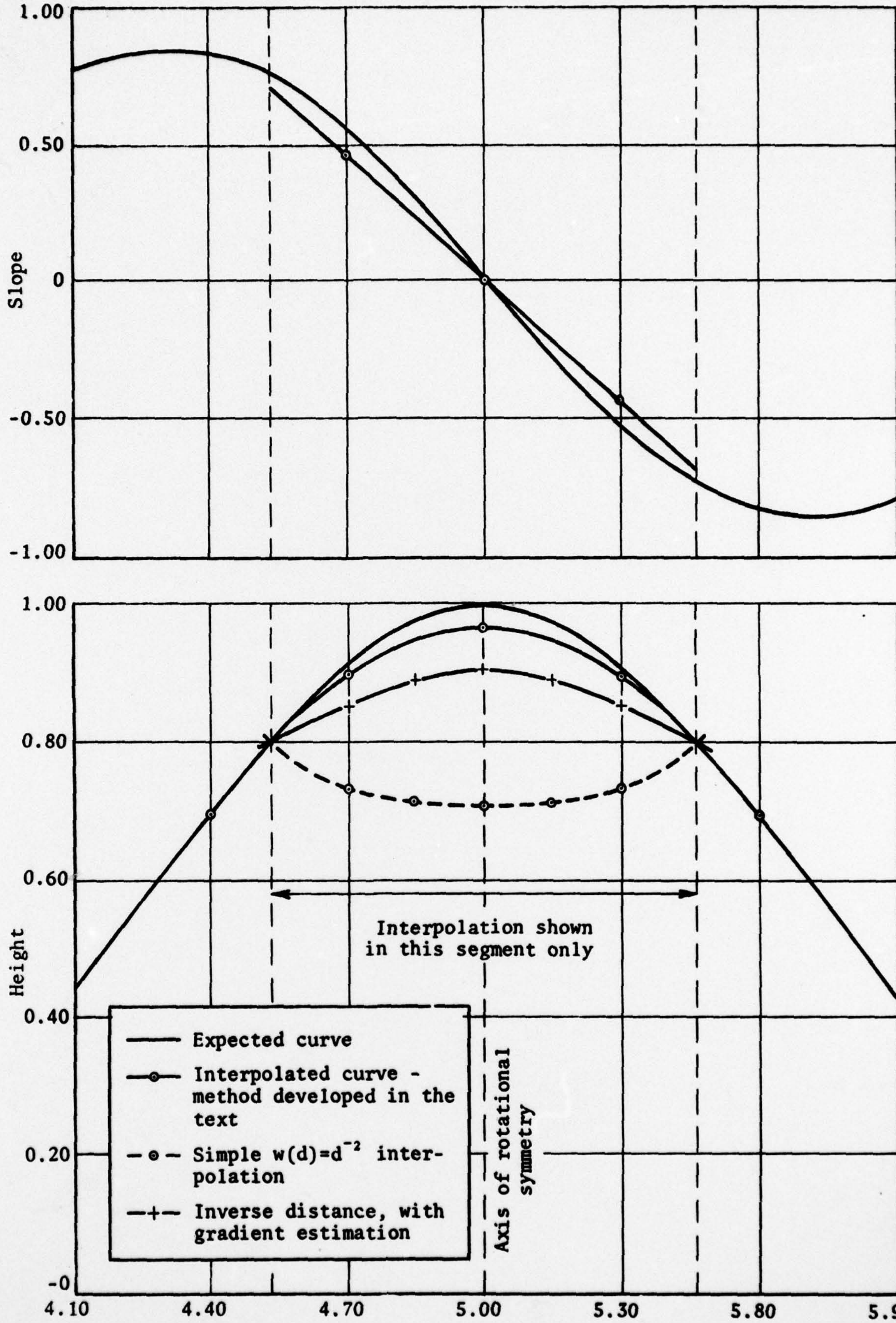

Figure 5. The interpolating functions for example IV.2

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