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Are all non-linear systems (approx.) bilinear?

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There is a rumour going around in mathematical system theory circles that all non-linear systems are bilinear or nearly so. This note examines the case for such an assertion and finds it wanting and en passant, offers some comments on the current proliferation of mathematical literature on system theory, in the tradition of one of Mortensen's book reviews.

## Introduction

There is now a large theory of 'bilinear' systems [2] following the initial evangelical work of Mohler [1]. A 'bilinear' system is characterised by a state-equation of the form:

$$\dot{x} = (A + Bu)x$$

with output $v(\cdot)$ (assumed one-dimensional here) given by:

$$v = Cx,$$

where the control $u(\cdot)$ appears "linearly", and is the defining 'bilinear' feature since

$$Bux = B(u,x)$$

where $B(\ ,\ )$ is a 'bilinear form' in the control and state variables. Such a system has the remarkable property that the 'product' of two bilinear systems is bilinear. In other words let

$$v_1 = C_1 x_1 \qquad \dot{x}_1 = (A_1 + B_1 u)x_1$$

$$v_2 = C_2 x_2 \qquad \dot{x}_2 = (A_2 + B_2 u)x_2$$

Then

$$v_1 v_2 = (C_1 x_1)(C_2 x_2)$$

can be expressed as:

$$= C_3 x_3$$

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where

$$\dot{x}_3 = (A_3 + B_3 u)x_3.$$

We can clearly take $x_1, x_2$ to have the same dimensions, and in the trivial case where the dimension is one for all vectors, we have

$$v_1 v_2 = C_1 \, x_1 x_2 C_2$$

and hence taking

$$x_3 = x_1 x_2$$

and

$$A_3 = A_1 + A_2, \; B_3 = B_1 + B_2$$

$$C_3 = C_1 C_2$$

we have our bilinear system. Except for a little algebra, the extension to the general case is straightforward. See [3].

There are of course many eminently 'practical' systems which are not bilinear -- here is one example from aerospace: of (planar) rocket flight in a resisting medium (that has been around for many a moon):

$$\dot{h}(t) - v(t) \sin \gamma(t) = 0$$

$$\dot{v}(t) + g \sin \gamma(t) - f_1(h(t), v(t))$$

$$+ f_2(h(t), v(t)) \, u^2(t) = 0$$

$$v(t) \dot{\gamma}(t) + g \cos \gamma(t) + u(t) = 0$$

where h(t) is a vertical coordinate, v(t) the magnitude of the velocity
vector, γ(t) the flight angle (inclination of the flight path with respect
to the horizon), u(t) the lift, and g the acceleration due to gravity. The
lift program u(t) is taken as the control.

Hence it is clearly illusory ("Maya") to claim that all non-linear
systems are bilinear even in the "Real World". But they are "nearly" so!,
shout the bilinear-enthusiasts. Let us examine the basis for their claim.
This would appear to be mainly the work of H. Sussman [3, and M. Fliess
see reference therein]. His result is that all non-linear systems can be
approximated as closely as we wish by bilinear systems. Sounds good -- until
we examine the result in detail, and notice the catch in the 'fine print'
(figuratively speaking of course). Let it be made perfectly clear at the
outset that as a mathematical result it is absolutely correct. Only the
disciples have gone overboard in their 'interpretation'!

Fix the initial state of the non-linear system once and for all at the
fixed initial time zero, say. Then we get an input-output map:

$$v(t) = F(t; u(s), 0 \leq s \leq t) \quad t \geq 0$$

Fix a time-interval, finite, $0 \leq t \leq T \leq \infty$ say. Then Sussman's result is
given $\varepsilon > 0$ we can find a bilinear system

$$\dot{x} = (A + Bu)x$$

$$v_b = Cx$$

such that

$$\sup_{0 \leq t \leq T} |v(t) - v_b(t)| < \varepsilon$$

for all bounded inputs (with the same bound) <u>provided</u> (and here is the catch) that the non-linear system has the following "continuity" property [P] with respect to inputs: Let $u_n(\cdot)$ be bounded in $[0,T)$ and let $u_n(\cdot)$ converge weakly to $u(\cdot)$ over $L_2[0,T]$. Then if $v_n(\cdot)$, $v(\cdot)$ denote the corresponding outputs,

$$\sup_{0 \leq t \leq T} |v_n(t) - v(t)| \to 0.$$

This condition is essential in order to be able to apply the Stone-Weierstrass theorem. On the other hand [H. Fattorini has the credit for this observation] this condition makes the system <u>almost</u> bilinear <u>already</u>! For example, suppose a non-linear system is defined by

$$\dot{x} = f(x,u)$$

$$v = Cx$$

where $f(\cdot,\cdot)$ is say continuous in both variables. Then, the imposed condition [P] will make $f(x,u)$ linear in $u(\cdot)$! This should be fairly familiar to control theorists who have looked at existence theorems or "chattering controls". Thus we can produce a sequence of chattering controls chattering between any two values $\bar{u}_1$, $\bar{u}_2$ say such that $u_n(\cdot)$ converges weakly to any convex combination $\theta\bar{u}_1 + (1 - \theta)\bar{u}_2$. We know that if limit $x_n(t)$ is denoted $x_0(t)$,

$$x_0(t) = \theta\int_0^t f(x_0,\bar{u}_1)ds + (1 - \theta)$$

$$\int_0^t f(x_0,\bar{u}_2)ds + x(0)$$

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which by assumption must equal:

$$\int_0^t f(x_0, \theta\bar{u}_1 + (1 - \theta)\bar{u}_2)ds + x(0)$$

or,

$$f(x_0(t), \bar{u}_1 + (1 - \theta)\bar{u}_2)$$

$$= \theta f(x_0(t),\bar{u}_1) + (1 - \theta) f(x_0(t),\bar{u}_2).$$

It is an easy step to deduce from this $f(x,u)$ must be linear in u!  The implication of this is clear.  The condition [P] -- whatever the mathematical reason for its inclusion -- makes the non-linear system already bilinear -- or nearly so!  Indeed one can impose many other conditions [see [4] for example] which will make the state-equation linear in $u(\cdot)$ and this linearity is of course the crucial assumption concerning the system. More precisely, the Sussman result [3] says that non-linear systems with state equations of the form:

$$\dot{x} = f(x) + g(x)u$$

(with some technical restrictions on $f(\cdot)$ and $g(\cdot)$) can be "approximated" for each fixed initial condition by bilinear systems of the form:

$$\dot{x} = (A + Bu)x$$

This result can hardly be taken to provide the basis for the statement that all non-linear systems are bilinear - or nearly so.

The nature of the approximation offered is also impractical.  Take the case $u \equiv 0$; then for the non-linear system output then we can take

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any continuous function. Putting u ≡ 0 in the bilinear approximation yields
a homogeneous linear equation. Thus we are no more and no less than
approximating a continuous function by the solution of a linear equation
with appropriate initial conditions -- by "exponentials". The approximation
of course has little significance for structural questions such as
"controllability" etc. And even less for optimal control problems.

Thus whatever the undisputed merits as an "Applied Mathematics"
result, it has little to do with what the word "system-approximation" can
conjure up in an engineering sense. A mathematical theorem (Stone-
Weierstrass) has been applied to yield a result in "system theory" by
tacking on mathematical assumptions which all but removed any practical
significance from the result. This would appear to be typical of the
current proliferation of so-called "mathematical system theory" producing
mathematical theorems purporting to be about physical systems. A theory is
evolved starting from a "physical motivation" but no attempt is made to
close the loop to see whether indeed the theory offers any solution
to the problems motivating it. Of course the consolation is that there is
always the hope that it may someday, and who knows, to a different more
important problem!

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