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BIB DESIGNS WITH VARIABLE SUPPORT SIZES WHEN BLOCKS ARE OF SIZE--ETC(U)
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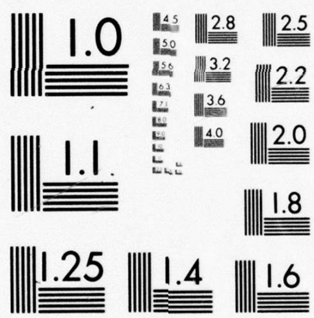

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We say a BIB design based on v treatments, b blocks each of size k has support size b^* if exactly b^* of the b blocks are distinct. BIB designs with $b^* < b$ have interesting applications in design of experiments and finite population sampling.

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20. Abstract

as explained in detail in Foody and Hedayat (1977). A method called "trade off" is introduced which is very powerful for the construction of BIB designs with $b^* < b$. This method is applied to the case of $v = 7$, $k = 3$. For this family of designs b must be a multiple of 7 and the results are:

1) If $b = 7$, then there is a unique design which has $b^* = 7$. This is a well know result.
2) If $b = 14$, then the only possible designs are those with $b^* = 11,13,14$ which are exhibited in Table 1.
3) If $b = 21$, then no design can exist with $b^* = 8,9,10,12$. We conjecture that there is also no design with $b^* = 16$. For all other cases we have exhibited a design in Table 2.
4) If $b = 28$, then no design can exist with $b^* = 8,9,10,12$ and 27. We conjecture that there is also no design with $b^* = 16$. For all other cases we have exhibited a design in Table 3.
5) If $b = 35$, then no design can exist with $b^* = 8,9,10,12$, 30,32,33, and 34. We conjecture that there is also no design with $b^* = 16$. For all other cases we have exhibited a design in Table 4.
6) For $b^* = 30,32,33$ we have found designs with minimum number of blocks, i.e., $b = 42$, see Table 6.
7) For $b^* = 34$ no design can be constructed utilizing 42 blocks. The existence of a design with $b = 49$ and $b^* = 34$ is doubtful. We have found a design with $b^* = 34$ and $b = 56$, see page 21.

(2)

BIB Designs With Variable Support Sizes When Blocks are of Size Three

by

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July, 1977
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BIB Designs With Variable Support Sizes When Blocks are of Size Three

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1. Background and Motivation. Suppose one wants to run an experiment testing and evaluating $v = 7$ treatments based on b blocks each of size $k = 3$. Under the usual homoscedastic linear additive model for the measurements it is known that the best possible design under any reasonable statistical criterion is a BIB design. But when $v = 7$, $k = 3$ for a BIB design to exist it is necessary that b be a multiple of 7. If $b < 7$ or b is not a multiple of 7 then it is sad to report that the existing literature is of no help to the experimenter. If b is a multiple of 7 then the existing literature provides the following solution. For $b = 7$ there is a BIB design. One such example is

1	2	4		5	6	1
2	3	5		6	7	2
3	4	6		7	1	3 .
4	5	7				

If $b = t7$ then by taking t copies of the above design one would obtain the necessary design. In particular, if $b = 35$ then there are two choices, viz., 5 copies of the above design or taking $\binom{7}{3} = 35$ possible blocks of size 3 based on 7 treatments. Note that a design based on t copies of the above design consists of (supported by) seven distinct blocks only. This might concern the experimenter who is not sure about one or more of the mixture of three treatments listed in the above design. If it is not possible to avoid these mixtures by relabeling the treatments then insisting on BIB designs the only course of action left is to search for BIB designs with more than t distinct blocks. This leads us to the following problem:

Problem: For $v = 7$ treatments is it possible to construct BIB designs based on $b = t7$ blocks each of size $k = 3$ which are supported by $7 \leq b^* \leq t7$ distinct blocks, $t = 2,3,4,5$?

Note that we do not have to consider cases where $t \geq 6$ since in our setting we can have at most $\binom{7}{3} = 5(7)$ distinct blocks. As we shall see later, fortunately the answer is essentially yes. There are few cases which no such designs can be constructed. In all other cases where the answer is positive we have given at least one such design. To solve the problem we have heavily relied on a method called "trade off" which is introduced and studied in Section 2.

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2. The Method of Trade Off.

Let B_1 and B_2 be two collections of n distinct blocks each of size k whose elements belong to a set Ω. Let $\lambda_{ij}^{(1)}$ and $\lambda_{ij}^{(2)}$ be the number of blocks which contain the pair (i,j), $i,j \in \Omega$, in B_1 and B_2 respectively. Then we say B_1 and B_2 are equivalent for covering pairs if $\lambda_{ij}^{(1)} = \lambda_{ij}^{(2)}$ for all $i,j \in \Omega$. We shall use the notation

$B_1 \overset{2}{\approx} B_2$ to indicate that B_1 and B_2 are equivalent for covering pairs. For example, the following two collections of blocks are equivalent for covering pairs.

$$\begin{matrix} 1 & 2 & 3 \\ 1 & 4 & 6 \\ 2 & 5 & 6 \\ 3 & 4 & 5 \end{matrix} \quad \overset{2}{\approx} \quad \begin{matrix} 1 & 3 & 4 \\ 1 & 2 & 6 \\ 2 & 3 & 5 \\ 4 & 5 & 6. \end{matrix}$$

Two immediate and important problems related to the concept of equivalence for covering pairs are:

(i) If $B_1 \overset{2}{\approx} B_2$ then what do we know about n, i.e., the cordinality of B_1, $B_1 \cap B_2 = \phi$?

(ii) For given k and an admissiable n how to construct B_1 and B_2 such that $B_1 \overset{2}{\approx} B_2$ and $B_1 \cap B_2 = \phi$.

For arbitrary k both problems are very difficult. Here we give a solution when $k = 3$. In regard to n we have the following result.

Proposition 2.1. If $k = 3$ and $B_1 \overset{2}{\approx} B_2$, $B_1 \cap B_2 = \phi$ then $n \neq 1,2,3,5$.

The fact that n cannot be 1,2 is straightforward. The case of $n = 3$ and 5 can be settled by a counting argument. Also for $n = 5$ an argument depends on the theory of Euler's triangulation of a compact manifold can be given which will be reported elsewhere.

In regard to the construction of B_1 and B_2 when $n \neq 1,2,3,5$ we have:

Proposition 2.2. If $k = 3$ then there exist B_1 and B_2 with $B_1 \overset{2}{\approx} B_2$ for all $n \neq 1,2,3,5$.

Proof: It suffices to construct such B_1 and B_2 for $n = 4,6,7,9$. We have already constructed B_1 and B_2 for $n = 4$. Examples for $n = 6,7$ and 9 are exhibited below.

$$B_1 = \begin{matrix} 1 & 2 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 6 \\ 2 & 3 & 6 \\ 3 & 5 & 7 \\ 5 & 6 & 7 \end{matrix}, \qquad B_2 = \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 6 \\ 1 & 5 & 7 \\ 2 & 5 & 7 \\ 3 & 5 & 6 \\ 3 & 6 & 7 \end{matrix}, \quad n = 6$$

$$B_1 = \begin{matrix} 1 & 2 & 6 \\ 1 & 3 & 4 \\ 1 & 5 & 7 \\ 2 & 3 & 5 \\ 2 & 4 & 7 \\ 3 & 6 & 7 \\ 4 & 5 & 6 \end{matrix}, \qquad B_2 = \begin{matrix} 1 & 2 & 7 \\ 1 & 3 & 6 \\ 1 & 4 & 5 \\ 2 & 3 & 4 \\ 2 & 5 & 6 \\ 3 & 5 & 7 \\ 4 & 6 & 7 \end{matrix}, \quad n = 7$$

$$B_1 = \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & 6 \\ 1 & 3 & 7 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \\ 4 & 7 & 9 \\ 1 & 5 & 8 \\ 2 & 8 & 9 \\ 3 & 5 & 9 \end{array} , \qquad B_2 = \begin{array}{ccc} 2 & 4 & 6 \\ 3 & 6 & 7 \\ 1 & 4 & 7 \\ 5 & 6 & 8 \\ 7 & 8 & 9 \\ 5 & 4 & 9 \\ 1 & 2 & 8 \\ 2 & 3 & 9 \\ 1 & 3 & 5 \end{array} , \quad n = 9 .$$

An easy technique for constructing $B_1 \overset{2}{\approx} B_2$ when $n = 4,6$ and 6 are given here.

For $n = 4$: Draw the following figures:

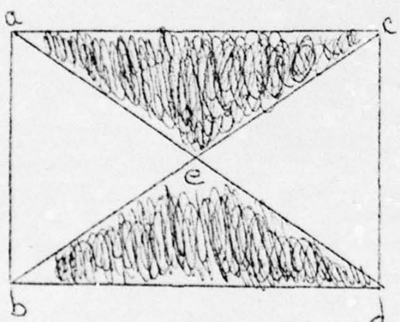


Fig. 1

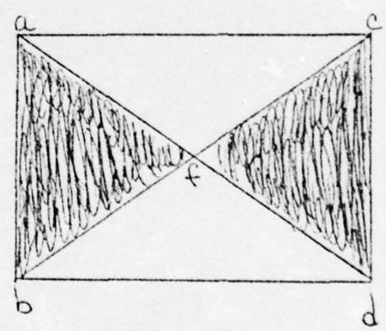


Fig. 2

Note that 6 distinct letters are used in naming vertices. In labeling vertices Fig. 1 and Fig. 2 are identical except for the labels of the center vertices. Form B_1 and B_2 from the vertices of the shaded and unshaded triangles respectively.

$$B_1 = \begin{array}{ccc} a & c & e \\ b & e & d \\ a & b & f \\ f & c & d \end{array} , \qquad B_2 = \begin{array}{ccc} a & b & e \\ c & e & d \\ a & c & f \\ b & f & d \end{array}$$

Now $B_1 \overset{2}{\approx} B_2$ and one can replace a to d by any arbitrary 6 numbers. It is easy to argue that whenever $B_1 \overset{2}{\approx} B_2$ with 4 blocks each then they have necessarily come from such two figures, i.e., that is the only way to construct $B_1 \overset{2}{\approx} B_2$ with $n = 4$ blocks each.

<u>For $n = 6$</u>: Draw the following figure:

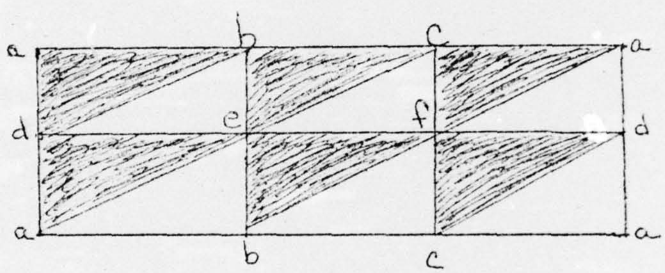
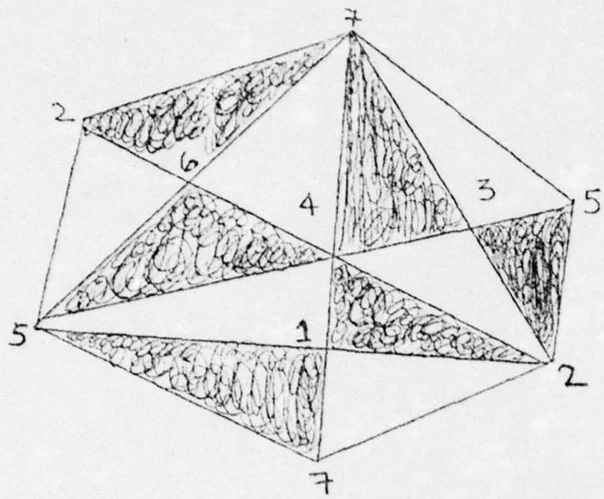


Fig. 3

Let blocks of B_1 be the vertices of shaded triangles. Similarly form B_2 from the unshaded triangles.

$$B_1 = \begin{array}{ccc} a & b & d \\ b & c & e \\ c & f & a \\ d & e & a \\ e & f & b \\ f & d & c \end{array} , \qquad B_2 = \begin{array}{ccc} d & b & e \\ c & e & f \\ a & f & d \\ e & a & b \\ f & b & c \\ d & c & a \end{array}$$

By noting the way we have labeled the vertices it is easy to argue that $B_1 \overset{2}{\approx} B_2$.



$B_1 \overset{2}{\approx} B_2$ with $O(B_i) = 6$ based on 7 distinct numbers.

<u>For n = 9</u>: Draw the following figure:

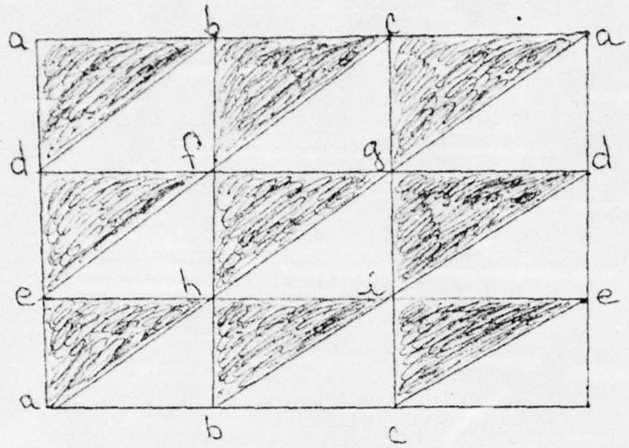


Fig. 4

Form B_1 and B_2 from the vertices of shaded and unshaded triangles respectively.

$$B_1 = \begin{array}{ccc} a & b & d \\ c & b & f \\ c & a & g \\ d & f & e \\ f & g & h \\ g & d & i \\ e & h & a \\ h & i & b \\ i & e & c \end{array}, \qquad B_2 = \begin{array}{ccc} b & d & f \\ c & f & g \\ a & g & d \\ f & e & h \\ g & h & i \\ d & i & e \\ h & a & b \\ i & b & c \\ e & c & a \end{array}$$

Again by the way we have labeld the vertices it is easy to see that $B_1 \overset{2}{\approx} B_2$.

Before we give an application of the concept introduced we need a formal definition for the support of a BIB design.

Let d be a BIB design with parameters v,b,r,k,λ based on Ω consisting of v distinct elements. Note that the definition of a BIB design does not require that its b blocks be distinct. Following Foody and Hedayat (1977) we define:

Definition 2.1. The support of a BIB design, d, is the collection of distinct blocks in d, denoted by d^*. The number of elements in d^* is denoted by b^* and called the support size of d.

A BIB design with parameters v,b,r,k,λ whose support size is b^* is denoted by $BIB(v,b,r,k,\lambda|b^*)$.

We are now ready to indicate how the stated concepts and results could be utilized for the purpose of increasing or decreasing the support size of a BIB design. Suppose d is a $BIB(v,b,r,k,\lambda|b^*)$ with support d^*. Further, suppose that we can find a collection of n distinct blocks, say B_1, in d^* such that there is a $B_2 \overset{2}{\approx} B_1$. Let the cardinality of $B_2 \cap d^*$ be m. Then by replacing B_1 by B_2 (trade off) we obtain a $BIB(v,b,r,k,\lambda|b^{**})$ with

$$b^{**} = b^* - \left[\sum_{i=1}^{n} \chi(i) - (n-m)\right]$$

where

$$\chi(i) = 1 \text{ if } f_i = 1$$
$$= 0 \text{ if } f_i > 1$$

with f_i being the number of copies of the i-th block of B_1

in d_1. Depending on the choice of B_1 the value of b^{**} could be greater, equal or less than b^*. We shall now give three examples with $b^{**} < b^*$, $b^{**} = b^*$ and $b^{**} > b^*$.

Example 2.1. Consider the following BIB(7,14,6,3,2|14)

$$d = \begin{matrix} 1 & 2 & 3 & \quad & 1 & 2 & 5 \\ 1 & 4 & 7 & & 1 & 3 & 7 \\ 1 & 5 & 6 & & 1 & 4 & 6 \\ 2 & 4 & 6 & & 2 & 3 & 4 \\ 2 & 5 & 7 & & 2 & 6 & 7 \\ 3 & 4 & 5 & & 3 & 5 & 6 \\ 3 & 6 & 7 & & 4 & 5 & 7 \end{matrix}$$

Let

$$B_1 = \begin{matrix} 1 & 2 & 3 \\ 1 & 4 & 7 \\ 2 & 4 & 6 \\ 3 & 6 & 7 \end{matrix}$$

Then

$$B_2 = \begin{matrix} 1 & 2 & 4 \\ 1 & 3 & 7 \\ 2 & 3 & 6 \\ 4 & 6 & 7 \end{matrix}$$

In this example $d^* = d$. $B_2 \cap d^* = \{137\}$ and thus by trading off B_1 with B_2 we obtain the following design

1 5 6		1 2 5	
2 5 7		1 3 7	*
3 4 5		1 4 6	
1 2 4		2 3 4	.
1 3 7	*	2 6 7	
2 3 6		3 5 6	
4 6 7		4 5 7	

which is a BIB(7,14,6,3,2|13) and thus $b^{**} = 13 < b^* = 14$. Note that the resulting design has two copies of {1,3,7}.

Example 2.2. Let d be the following BIB(7,14,6,3,2|11).

$$d = \begin{matrix} 1&2&4 && 1&2&3 \\ 1&3&7 && 1&4&7 \\ 2&3&5 && 2&4&5 \\ 4&5&7 && 3&5&7 \\ 1&5&6 && 1&5&6 \\ 2&6&7 && 2&6&7 \\ 3&4&6 && 3&4&6 \end{matrix}$$

If we now select B_1 to be

$$B_1 = \begin{matrix} 2&3&5 \\ 2&6&7 \\ 3&4&6 \\ 4&5&7 \end{matrix},$$

then

$$B_2 = \begin{matrix} 2&3&6 \\ 2&5&7 \\ 3&4&5 \\ 4&6&7 \end{matrix}.$$

For this choice of B_1 we have $B_2 \cap d^* = \phi$ and $\sum_{i=1}^{4} X(i) = 4$

and hence

$$b^{**} = 11 - [4-(4-0)] = 11 = b^{*}.$$

Example 2.3. Consider the following BIB(7,14,6,3,2|11)

$$d = \begin{matrix} 1 & 2 & 4 & & 1 & 2 & 3 \\ 1 & 3 & 6 & & 1 & 4 & 6 \\ 2 & 3 & 5 & & 2 & 4 & 5 \\ 4 & 5 & 6 & & 3 & 5 & 6 \\ 1 & 5 & 7 & & 1 & 5 & 7 \\ 2 & 6 & 7 & & 2 & 6 & 7 \\ 3 & 4 & 7 & & 3 & 4 & 7 \end{matrix}$$

For the choice of

$$B_1 = \begin{matrix} 1 & 2 & 4 \\ 1 & 5 & 7 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{matrix}$$

we obtain

$$B_2 = \begin{matrix} 1 & 2 & 5 \\ 1 & 4 & 7 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{matrix} .$$

Thus by trading off B_1 with B_2 in d we shall obtain

$$\begin{matrix} 1 & 2 & 5 & & 1 & 2 & 3 \\ 1 & 3 & 6 & & 1 & 4 & 6 \\ 1 & 4 & 7 & & 2 & 4 & 5 \\ 4 & 5 & 6 & & 3 & 5 & 6 \\ 2 & 3 & 4 & & 1 & 5 & 7 \\ 2 & 6 & 7 & & 2 & 6 & 7 \\ 3 & 5 & 7 & & 3 & 4 & 7 \end{matrix}$$

which is a BIB(7,14,6,3,2,|13) and thus $b^{**} = 13 > b^{*} = 11$.

3. BIB Designs With $v = 7$ And $k = 3$. Using the relations $rv = bk$ and $\lambda(v-1) = r(k-1)$ which hold in any BIB(v,b,r,k,λ), one can see that if $v = 7$ and $k = 3$ then b must be a multiple of seven. Thus it is theoretically interesting and practically useful to investigate the existence and construction of BIB designs with all possible support sizes when $b = 14, 21, 28$ and 35. Using the results in Section 7 of Foody and Hedayat (1977), Theorem 3.2 of van Lint and Ryser (1972) and results of Pesotchinsky (1977) it can be argued that there is no BIB$(7,b,r,3,\lambda)$ based on exactly 8,9,10,12 distinct blocks (support) no matter what the total number of blocks, b, is. When $b = 35$ there are also no designs based on exactly 30,32,33,34 distinct blocks. These latter conclusions follow directly from Proposition 2.1. When $b = 28$ there is no design based on exactly 27 distinct blocks (see Theorem 3.1). It *seems* no matter what the total number of blocks, b, is, one cannot construct a BIB design which is supported on exactly 16 distinct blocks if $v = 7$ and $k = 3$. In all other cases one such design is given in Tables 1, 2, 3 and 4. If we allow b to be greater than 35, then it is possible to construct BIB designs based on 30, 32, 33, and 34 distinct blocks (see Section 4).

Theorem 3.1. _If_ $v = 7$, $k = 3$ _then there is no design with_ $b = 28$ _which is supported on exactly_ 27 _distinct blocks_.

Proof: The proof is by contradiction. Let $\Omega = \{1,2,\ldots,7\}$. Assume that such a design exists. Say, the eight blocks missing from the design are $B_1, B_2,\ldots,B_8$ and the unique repeated block is $B_9 = \{1,2,3\}$. Since $\lambda = bk(k-1)/v(v-1) = 4$, every pair is covered by 4 blocks in the design. On the other hand, every pair is covered by 5 blocks in the complete design (i.e., taking all of $\binom{7}{3} = 35$ blocks). Thus $B_1,B_2,\ldots,B_8$ cover all the $\binom{7}{2} = 21$ pairs. The free pairs $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ are doubly covered, and all other pairs are singly covered. We may then assume that $B_1 = \{1,2,u\}$, $B_2 = \{1,2,v\}$, $B_3 = \{1,3,w\}$, $B_4 = \{1,3,x\}$, $B_5 = \{2,3,y\}$, and $B_6 = \{2,3,z\}$. But the above covering properties of these 8 blocks imply that u, v, w, x, y, z are distinct elements from the 4-element set $\{4,5,6,7\}$. This is a contradiction.

Table 1
BIB Designs With v = 7 and k = 3
All Possible Support Sizes When b = 14

b*	7	8	9	10	11	12	13	14
123	-	DOES NOT EXIST	DOES NOT EXIST	DOES NOT EXIST	-	DOES NOT EXIST	-	-
124	2				1		-	1
125	-				-		1	-
126	-				-		-	1
127	-				1		1	-
134	-				-		-	1
135	2				-		-	1
136	-				2		2	-
137	-				-		-	-
145	-				1		1	-
146	-				-		-	-
147	-				-		1	-
156	-				-		-	-
157	-				1		-	1
167	2				-		-	1
234	-				1		1	-
235	-				-		-	1
236	-				-		-	-
237	2				1		1	1
245	-				-		-	-
246	-				-		1	-
247	-				-		-	1
256	2				2		1	1
257	-				-		-	-
267	-				-		-	-
345	-				1		1	-
346	2				-		-	1
347	-				-		-	-
356	-				-		-	-
357	-				1		1	-
367	-				-		-	1
456	-				-		-	1
457	2				-		-	1
467	-				2		1	-
567	-				-		1	-

Table 2
BIB Designs With v = 7 and k = 3
All Possible Support Sizes When b = 21

b*	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
123	-	DOES NOT EXIST	DOES NOT EXIST	DOES NOT EXIST	2	DOES NOT EXIST	2	-	-	ITS EXISTENCE IS DOUBTFUL	-	1	-	1	1
124	-				1		1	1	-		-	1	3	1	1
125	-				-		-	-	1		1	1	-	1	1
126	3				-		-	2	-		-	-	-	-	-
127	-				-		-	-	2		2	-	-	-	-
134	3				-		-	2	2		1	-	-	-	-
135	-				-		1	-	-		-	1	1	1	-
136	-				1		-	-	1		2	1	1	-	1
137	-				-		-	1	-		-	-	1	1	1
145	-				-		2	-	1		1	1	-	1	1
146	-				2		-	-	-		-	-	-	1	-
147	-				-		-	-	-		1	1	-	-	1
156	-				-		-	1	1		1	-	1	-	1
157	3				3		-	2	-		-	-	1	-	-
167	-				-		3	-	1		-	2	1	2	1
234	-				-		-	-	1		1	1	-	1	1
235	-				1		-	1	-		-	-	1	-	-
236	-				-		1	-	2		1	1	1	1	-
237	3				-		-	2	-		1	-	1	-	1
245	3				2		-	2	1		1	-	-	-	-
246	-				-		-	-	-		1	-	-	-	1
247	-				-		2	-	1		-	1	-	1	-
256	-				-		2	-	1		1	1	1	1	1
257	-				-		1	-	-		-	1	1	1	1
267	-				3		-	1	-		-	1	1	1	1
345	-				-		-	-	-		1	-	1	-	1
346	-				-		2	1	-		-	1	1	1	1
347	-				3		1	-	-		-	1	1	1	-
356	3				2		-	2	-		-	-	-	1	1
357	-				-		2	-	3		2	2	-	1	1
367	-				-		-	-	-		-	-	-	-	-
456	-				1		1	-	1		-	2	1	1	-
457	-				-		-	1	-		-	-	1	1	1
467	3				-		-	2	2		2	-	1	-	1
567	-				-		-	-	-		1	-	-	-	-

Table 3
BIB Designs With $v = 7$ and $k = 3$
All Possible Support Sizes When $b = 28$

b*	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
123	-	DOES NOT EXIST	DOES NOT EXIST	DOES NOT EXIST	2	DOES NOT EXIST	-	3	-	ITS EXISTENCE IS DOUBTFUL	1	-	1	2	1	-	-	-	-	1	DOES NOT EXIST	1
124	4				2		-	-	-		-	-	1	1	1	-	3	2	2	1		1
125	-				-		4	1	2		3	3	2	1	2	2	-	1	1	1		1
126	-				-		-	-	-		-	-	-	-	-	-	1	-	-	-		-
127	-				-		-	-	2		-	1	-	-	-	2	-	1	1	1		1
134	-				-		1	1	3		1	1	1	-	-	1	1	-	-	-		-
135	4				-		-	-	-		-	-	1	-	1	1	1	2	2	1		1
136	-				2		3	-	1		2	3	2	2	2	2	1	1	1	1		1
137	-				-		-	-	-		-	-	-	-	1	-	1	1	1	1		1
145	-				-		-	-	1		-	1	-	1	-	1	-	-	-	-		1
146	-				2		-	-	-		-	-	-	-	1	1	-	1	1	2		1
147	-				-		3	3	-		3	2	2	2	2	1	-	1	1	1		1
156	-				-		-	3	1		1	-	-	1	-	-	1	1	1	1		1
157	-				4		-	-	-		-	-	-	1	-	-	2	-	-	1		-
167	4				-		1	1	2		1	1	2	1	1	1	1	1	1	-		1
234	-				-		-	-	1		-	1	1	1	1	1	-	1	2	1		1
235	-				2		-	-	-		-	-	1	-	1	-	2	1	-	-		-
236	4				-		1	1	3		1	1	-	-	-	2	1	1	1	1		1
237	-				-		3	-	-		2	2	2	1	2	1	1	1	1	1		1
245	-				2		-	-	1		-	-	-	1	-	1	-	1	1	1		1
246	-				-		3	3	-		3	2	1	-	1	1	-	1	-	1		1
247	-				-		1	1	2		1	1	-	1	-	1	1	-	-	-		-
256	-				-		-	-	1		-	1	2	2	2	1	1	-	1	1		1
257	4				-		-	3	-		1	-	-	-	-	-	1	1	1	1		1
267	-				4		-	-	-		-	-	2	2	2	-	1	1	1	1		1
345	-				-		3	3	-		3	2	-	1	1	1	1	1	1	2		1
346	-				-		-	-	-		-	-	1	2	1	-	1	1	1	-		1
347	4				4		-	-	-		-	-	-	-	-	1	1	2	1	1		1
356	-				2		-	-	-		-	-	1	-	1	-	-	-	-	1		1
357	-				-		1	1	4		1	2	1	3	-	2	-	-	1	-		1
367	-				-		-	3	-		1	-	-	-	-	-	1	-	-	1		-
456	4				2		1	1	2		1	1	2	1	1	1	2	1	1	-		-
457	-				-		-	-	-		-	-	2	-	2	-	1	1	1	1		1
467	-				-		-	-	2		-	1	-	1	-	1	1	-	1	1		1
567	-				-		3	-	-		2	2	-	-	1	2	-	2	1	1		1

Table 4
BIB Designs With v = 7 and k = 3
All Possible Support Sizes When b = 35

b*	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
123	5				5		-	-	5		-	-	-	-	4	-	3	1	-	-	1	2	-		1				1
124	-				-		-	1	-		2	1	3	3	1	-	2	1	2	2	2	2	1		1				1
125	-				-		1	-	-		-	1	-	1	-	3	-	1	1	1	-	1	1		1				1
126	-				-		4	4	-		3	3	1	1	-	1	-	2	2	2	1	-	2		2				1
127	-				-		-	-	-		-	-	1	-	-	1	-	-	-	-	1	-	1		-				1
134	-				-		-	4	-		3	-	-	-	-	1	-	-	2	1	1	1	1		1				1
135	-				-		-	1	-		1	-	-	2	1	-	2	1	1	1	-	1	1		1				1
136	-				-		-	-	-		1	1	4	3	-	2	-	-	1	2	2	1	2		1				1
137	-				-		5	-	-		-	4	1	-	-	2	-	3	1	1	1	-	1		1				1
145	-				-		4	-	-		-	3	2	-	-	2	-	2	1	1	1	-	2		1				1
146	5				4		1	-	4		-	1	-	-	3	1	2	1	-	-	1	1	-		1				1
147	-				1		-	-	1		-	-	-	2	1	1	1	1	-	1	-	1	1		1				1
156	-				1		-	-	1		-	-	-	-	1	-	1	1	-	-	1	1	-		-				1
157	5				4		-	4	4		4	1	3	2	3	-	2	-	2	2	3	2	1		2				1
167	-				-		-	1	-		1	-	-	1	1	1	2	1	2	1	-	2	1		1				1
234	-				-		1	-	-		-	1	2	2	-	1	-	3	1	2	1	-	2		1				1
235	-				-		4	4	-		4	4	1	2	-	2	-	1	2	2	1	-	2		1				1
236	-				-		-	-	-		-	-	-	-	1	-	1	-	-	-	1	1	-		-				1
237	-	DOES NOT EXIST	DOES NOT EXIST	DOES NOT EXIST	-	DOES NOT EXIST	-	1	-	ITS EXISTENCE IS DOUBTFUL	1	-	2	1	-	2	1	-	2	1	1	2	1	DOES NOT EXIST	2	DOES NOT EXIST	DOES NOT EXIST	DOES NOT EXIST	1
245	5				5		-	-	4		-	-	-	-	3	-	2	-	-	-	1	1	-		1				1
246	-				-		-	-	-		-	-	-	-	-	3	-	-	1	-	-	1	1		1				1
247	-				-		4	4	1		3	3	-	-	1	1	1	1	1	1	1	1	1		1				1
256	-				-		-	1	1		1	-	3	1	1	-	2	1	1	1	2	2	1		1				1
257	-				-		-	-	-		-	-	1	1	1	-	1	2	1	1	1	1	1		1				1
267	5				5		1	-	4		1	2	1	3	3	1	2	2	1	2	1	1	1		1				1
345	-				-		-	-	1		-	-	2	-	1	1	1	-	-	-	2	2	-		1				1
346	-				1		4	1	1		1	3	-	1	1	1	2	2	1	1	-	1	1		1				1
347	5				4		-	-	3		1	1	1	2	3	1	2	-	1	1	1	1	1		1				1
356	5				4		1	-	3		-	1	1	-	2	2	1	2	2	1	1	1	1		2				1
357	-				1		-	-	1		-	-	1	1	1	-	1	1	-	1	1	1	1		-				1
367	-				-		-	4	1		3	-	-	1	1	-	1	1	1	1	1	1	1		1				1
456	-				-		-	4	-		4	1	1	4	1	-	1	1	2	3	1	1	2		1				1
457	-				-		1	1	-		1	1	-	1	-	2	1	2	2	1	-	1	1		1				1
467	-				-		-	-	-		-	-	4	-	-	-	-	1	1	1	3	1	1		1				1
567	-				-		4	-	-		-	3	-	-	-	3	-	-	-	-	-	-	1		1				1

4. BIB Designs With $v = 7$, $k = 3$ and Support Sizes 30, 32, 33, 34.

In Section 3 we pointed out that the following designs do not exist.

Table 5

Basic Parameters					Support Size
v	b	5	k	λ	b^*
7	35	5	3	5	30
7	35	5	3	5	32
7	35	5	3	5	33
7	35	5	3	5	34

Suppose we hold the values of v, k and b^* as given in Table 5. Now a question of interest is, can we construct such designs if we allow the value of b (hence r and λ) increases, i.e., can we construct these designs if we are allowed to take more blocks. In case the answer is affirmative then what is the minimum value of b? As we pointed out before, any BIB design based on $v = 7$, $k = 3$ has $b \equiv 0 \pmod 7$ blocks. Therefore, we are interested to investigate the existence or nonexistence of BIB designs with $v = 7$, $b = 42$, $k = 3$ which are supported on exactly 30, 32, 33 and 34 distinct blocks. In case $b = 42$ is small we would like

Table 6

BIB designs With v = 7, k = 3, b = 42

and Support Sizes 30, 32, 33

b*	30	32	33
123	1	1	1
124	1	1	1
125	1	1	1
126	2	2	2
127	1	1	1
134	1	1	2
135	1	2	1
136	2	1	1
137	1	1	1
145	3	1	1
146	-	2	1
147	1	1	1
156	-	-	1
157	1	2	2
167	2	1	1
234	2	2	1
235	2	1	2
236	-	-	-
237	1	2	2
245	-	1	1
246	2	1	2
247	1	1	1
256	1	2	1
257	2	1	1
267	1	1	1
345	-	1	1
346	1	1	1
347	2	1	1
356	2	2	2
357	1	-	-
367	1	2	2
456	2	1	1
457	1	2	2
467	1	1	1
567	1	1	1

to search for such design with higher b's, namely 49, 56, etc. Fortunately, there are designs with support size 30, 32, and 33 needing only $b = 42$ blocks. Examples of such designs are displayed in Table 6. The three designs in Table 6 are constructed by the method of trade off. At the end of this section we shall explain the ways we have constructed these designs. Unfortunately, there is no design with $b = 42$ and $b^* = 43$ as the following theorem shows.

Theorem 4.1. The following two designs co-exist.

	v	b	r	k	λ	b^*
d_1:	7	28	4	3	4	27
d_2:	7	42	6	3	6	34

Proof: By taking two copies of the complete design and removing from it the blocks of d_1 we obtain a design with parameters of d_2. A similar argument can be used to show that $d_2 \Rightarrow d_1$.

Since by Theorem 3.1 no design with parameters of d_1 exists, thus no design with parameters of d_2 exists. We have not been able to show the existence or nonexistence of a design with support size 34 with 49 blocks. However, we now show that if we allow to take $b = 56$ blocks then we can have a design whose support size is 34.

Consider the following $B_1 \overset{2}{\approx} B_2$ and $B_3 \overset{2}{\approx} B_4$.

125		235		125		123
137		347		136	$\overset{2}{\approx}$	156
146	$\overset{2}{\approx}$	246		234		245
247		136	,	256		346
345		146				
236		127		B_3	.	B_4
B_1		B_2				

Thus the blocks of B_1 together with blocks of B_3 are equivalent for covering pairs as the blocks of B_2 together with blocks of B_4.

	# of copies			# of copies
125	2		235	1
136	1		347	1
137	1		246	1
146	1	$\overset{2}{\approx}$	136	1
234	1		146	1
247	1		127	1
345	1		123	1
236	1		156	1
456	1		245	1
B_1 and B_2			346	1
			B_2 and B_4	

Now add the following 21 blocks to the complete design (note that the blocks in each column is a BIB design).

125	136	456
137	234	124
146	145	135
247	127	167
345	256	347
236	357	236
567	467	257

From the resulting designs remove the blocks of B_1 and B_2 and add blocks of B_2 and B_4. In these processes we lose the block $\{1,2,5\}$ only and the total number of blocks will be $35 + 21 = 56$.

We shall now explain in detail the way the designs exhibited in Table 6 were obtained by the method of trade off.

<u>Design with</u> $b = 42$ <u>and</u> $b^* = 30$:.

<u>Step 1</u>. Add the following BIB design to the complete design.

1	2	3		3	5	6
1	4	5		2	4	6
1	6	7		3	4	7
2	5	7				

<u>Step 2</u>. Trade off the blocks of B_1 with blocks of B_2 in the resulting design in Step 1.

$$B_1 = \begin{matrix} 1 & 2 & 3 \\ 1 & 4 & 6 \\ 1 & 5 & 6 \\ 2 & 3 & 6 \\ 2 & 4 & 5 \\ 3 & 4 & 5 \end{matrix}, \quad B_2 = \begin{matrix} 1 & 2 & 6 \\ 1 & 3 & 6 \\ 1 & 4 & 5 \\ 2 & 3 & 4 \\ 2 & 3 & 5 \\ 4 & 5 & 6 \end{matrix}$$

Since $B_1 \overset{2}{\approx} B_2$ the net result is a design with $b = 42$ and $b^* = 30$.

<u>Design with</u> $b = 42$ <u>and</u> $b^* = 32$:

<u>Step 1</u>. Add the following BIB design to the complete design.

1	2	7		3	6	7
1	3	5		2	3	4
1	4	6		4	5	7
2	5	6				

Step 2. Trade off the blocks of B_1 with blocks of B_2 in the resulting design in Step 1.

$$B_1 = \begin{matrix} 1 & 2 & 7 \\ 1 & 5 & 6 \\ 2 & 3 & 6 \\ 3 & 5 & 6 \end{matrix}, \quad B_2 = \begin{matrix} 1 & 2 & 6 \\ 1 & 5 & 7 \\ 2 & 3 & 7 \\ 3 & 5 & 6 \end{matrix}$$

Since $B_1 \overset{2}{\approx} B_2$ the net result is a design with $b = 42$ and $b^* = 32$.

Design with $b = 42$ and $b^* = 33$:

Step 1. Add the following BIB design to the complete design.

$$\begin{matrix} 1 & 2 & 7 \\ 1 & 5 & 6 \\ 1 & 3 & 4 \\ 2 & 4 & 6 \end{matrix} \qquad \begin{matrix} 4 & 5 & 7 \\ 2 & 3 & 5 \\ 3 & 6 & 7 \end{matrix}$$

Step 2. Trade off the blocks of B_1 with blocks of B_2 in the resulting design in Step 1.

$$B_1 = \begin{matrix} 1 & 2 & 7 \\ 1 & 5 & 6 \\ 2 & 3 & 6 \\ 3 & 5 & 7 \end{matrix}, \quad B_2 = \begin{matrix} 1 & 2 & 6 \\ 1 & 5 & 7 \\ 2 & 3 & 7 \\ 3 & 6 & 5 \end{matrix}$$

Since $B_1 \overset{2}{\approx} B_2$ the net result is a design with $b = 42$ and $b^* = 33$.

We close this section with the following Theorem.

<u>Theorem 4.2</u>. <u>If there exists a</u> d_1, <u>a</u> BIB <u>design with</u>

$$v = 7,\ k = 3,\ b_1 = 28,\ b_1^*$$

<u>with no block repeated more than twice, then there exists a</u> d_2, <u>a</u> BIB <u>design with</u>

$$v = 7,\ k = 3,\ b_2 = 42,\ b_2^* = b_1^* + 7.$$

<u>Proof</u>: Take two copies of the complete design based on $v = 7$, $k = 3$. Delete from it the blocks of d_1. The resulting design has the parameters of d_2.

Thus the existence of a d_1 with $b_1^* = 23, 25, 26, 27$ whose blocks are repeated not more often than twice implies the existence of d_2 with $b_2 = 42$ and $b_2^* = 30, 32, 33, 34$.

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