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ON ORTHOGONAL POLYNOMIALS.(U)

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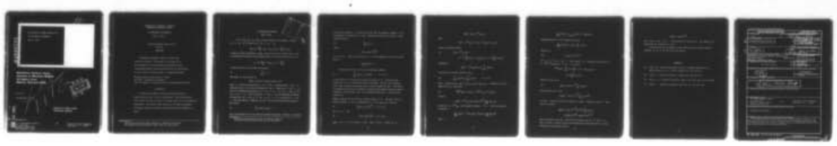

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ON ORTHOGONAL POLYNOMIALS

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ON ORTHOGONAL POLYNOMIALS

Paul G. Nevai

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ABSTRACT

Orthogonal polynomials satisfy a three term recurrence relation. The purpose of the paper is to give estimates for the orthogonal polynomials and the corresponding weight function provided that the coefficients in the recurrence formula behave in a prescribed manner.

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EXPLANATION

Orthogonal polynomials provide a convenient means to expand functions into series in polynomials. When investigating these series one has to be able to estimate the size of the orthogonal polynomials. The present paper shows how to estimate orthogonal polynomials when the recurrence relation for these polynomials is given.

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ON ORTHOGONAL POLYNOMIALS

Paul G. Nevai

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\gamma_n > 0\}_{n=0}^{\infty}$ be given sequences of real members. Putting $p_{-1} = 0$, $p_0 = \gamma_0$ and defining p_n for $n = 1,2,\ldots$ by

$$xp_{n-1}(x) = \frac{\gamma_{n-1}}{\gamma_n} p_n(x) + \alpha_{n-1} p_{n-1}(x) + \frac{\gamma_{n-2}}{\gamma_{n-1}} p_{n-2}(x)$$

we obtain a system of polynomials $\{p_n\}_{n=0}^{\infty}$ which by a result of J. Favard (see e.g. [2]) is orthonormal with respect to some positive measure $d\alpha$ acting on the real line. Let

$$c_n = \left|1-2\frac{\gamma_{n-1}}{\gamma_n}\right| + 2|\alpha_{n-1}| + \left|1 - 2\frac{\gamma_{n-2}}{\gamma_{n-1}}\right|$$

It has been shown in [3] that under the assumption

$$\sum_{n=0}^{\infty} c_n < \infty \tag{1}$$

the measure $d\alpha$ can be written as

$$d\alpha(x) = \alpha'(x)dx + \sum \{\text{jumps outside } (-1,1)\} ,$$

where α' is positive and continuous on $(-1,1)$ and α' vanishes outside $[-1,1]$. At the present time it is not clear that assuming (1) how α' behaves near -1 and 1. In case of the Tschebyshev polynomials ($\alpha_n = 0$ for $n = 0,1,\ldots$, $\gamma_0 = \gamma_1 = 1$ and $\gamma_n = 2^{n-1}$ for $n = 2,3,\ldots$) α' is not continuous at -1 and 1. For the Tschebyshev polynomials of second kind ($\alpha_n = 0$ and $\gamma_n = 2^n$ for $n = 0,1,\ldots$) α' is not positive at -1 and 1. Since the works of G. Szegö (see e.g. [4]) it has become known that those measures $d\alpha$ for which

$$\int_{-\pi}^{\pi} \log \alpha'(\cos\theta)d\theta > -\infty \tag{2}$$

play a very important role in the theory of orthogonal polynomials. Therefore it is natural

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to ask if (1) implies (2). It is easy to see that under the assumption $\operatorname{supp}(d\alpha) = [-1,1]$ the inequality (2) follows from (1) ([3]). Otherwise the question is still open. It was proved in [3] that

(3) $$\sum_{n=0}^{\infty} n\, c_n < \infty$$

implies

$$\alpha'(x) \geq \text{const} \sqrt{1-x^2}$$

for $-1 \leq x \leq 1$. Hence (2) follows from (3). K. M. Case suggested in [1] that (2) holds whenever

$$\limsup_{n \to \infty} n^2 c_n < \infty .$$

The purpose of this note is to show that the weaker condition

(4) $$\sum_{n=0}^{m} (n+1)c_n \leq A \log(m+1) \qquad (m = 1,2,\ldots)$$

not only implies (2) but also gives a pointwise estimate for α'. We will see that assuming (4) $\log \alpha'$ is very far from being nonintegrable. Our plan is the following. First, using an absolutely elementary method, we obtain estimate for $|p_n|$. This method is somewhat miraculous since we establish an inequality which improves itself when applied repeatedly. Having bound for $|p_n|$ the corresponding estimate for α' follows from a result in [3].

THEOREM. Suppose that (4) holds with a suitable constant $A > 0$. Then there exist two positive constants A_1 and A_2 depending only on A and $\inf_n \gamma_{n-1}/\gamma_n$ such that

(5) $$|p_n(x)| \leq A_1(1-x^2)^{-A_2} \qquad (-1 \leq x \leq 1)$$

for $n = 1,2,\ldots$ and

(6) $$\alpha'(x) \geq A_1^{-1}(1-x^2)^{A_2} \qquad (-1 \leq x \leq 1) .$$

<u>**Proof.**</u> Let $x \in [-1,1]$ and put $x = \cos\theta$ where $0 \leq \theta \leq \pi$. Define ϕ_n by

$$\phi_n(\theta) = p_n(x) - e^{i\theta} p_{n-1}(x) .$$

Then

$$\phi_n(\theta) - e^{-i\theta} \phi_{n-1}(\theta) = p_n(x) - 2xp_{n-1}(x) + p_{n-2}(x)$$

and by the recurrence formula

(7) $$\phi_n(\theta) - e^{-i\theta} \phi_{n-1}(\theta) =$$

$$= [1 - 2 \frac{\gamma_{n-1}}{\gamma_n}] p_n(x) - 2\alpha_{n-1}p_{n-1}(x) + [1 - 2 \frac{\gamma_{n-2}}{\gamma_{n-1}}] p_{n-2}(x) .$$

Consequently

$$|\phi_n(\theta) - e^{-i\theta} \phi_{n-1}(\theta)| \le c_n \sum_{k=n-2}^{n} |p_k(x)| .$$

Using again the recurrence formula we obtain

(8) $$\sum_{k=n-2}^{n} |p_k(x)| \le K \sum_{k=M-1}^{M} |p_k(x)| \qquad (M = n-1,n)$$

where K depends only on $\sup_n \alpha_n$, $\inf_n \gamma_{n-1}/\gamma_n$ and $\sup_n \gamma_{n-1}/\gamma_n$. Furthermore, from the definition of ϕ_n follows that

(9) $$\sqrt{1-x^2}\, |p_n(x)| \le |\phi_n(\theta)|, \quad \sqrt{1-x^2}\, |p_{n-1}(x)| \le |\phi_n(\theta)| .$$

Therefore

$$|\phi_n(\theta) - e^{-i\theta} \phi_{n-1}(\theta)| \le 2Kc_n(1-x^2)^{-\frac{1}{2}} \max_{|x|\le 1} |\phi_{n-1}(\theta)| .$$

Recall that $\phi_n - e^{-i\theta} \phi_{n-1}$ is a polynomial of degree n in x. Thus by a theorem of S. Bernstein

$$\max_{|x|\le 1} |\phi_n(\theta) - e^{-i\theta} \phi_{n-1}(\theta)| \le 2Kc_n(n+1) \max_{|x|\le 1} |\phi_{n-1}(\theta)| ,$$

that is

$$\max_{|x|\leq 1} |\phi_n(\theta)| \leq \max_{|x|\leq 1} |\phi_{n-1}(\theta)| \, [1 + 2Kc_n(n+1)] \; .$$

Repeated application of this inequality shows that

$$\max_{|x|\leq 1} |\phi_n(\theta)| \leq \gamma_0 \exp\{2K \sum_{j=1}^{n} (j+1)c_j\} \; .$$

Hence by (4)

$$|\phi_n(\theta)| \leq \gamma_0 (n+1)^{2KA} \tag{10}$$

for $-1 \leq x \leq 1$ and $n = 0,1,\ldots$. Now we return to (7). Multiplying both sides of (7) by $e^{in\theta}$ and summing for $n = 0,1,\ldots,m$ we obtain

$$e^{im\theta} \phi_m(\theta) = \sum_{n=0}^{m} \{[1-2 \frac{\gamma_{n-1}}{\gamma_n}] \, p_n(x) - 2\alpha_{n-1} \, p_{n-1}(x) +$$

$$+ [1-2 \frac{\gamma_{n-2}}{\gamma_{n-1}}] \, p_{n-2}(x)\} \; .$$

Therefore by (8) and (9)

$$|\phi_m(\theta)| \leq 2K(1-x^2)^{-\frac{1}{2}} \sum_{n=0}^{m} c_n |\phi_n(\theta)| \; . \tag{11}$$

Using inequality (10) we get

$$|\phi_m(\theta)| \leq 2K\gamma_0 (1-x^2)^{-\frac{1}{2}} \sum_{n=0}^{m} c_n (n+1)^{2KA} \; .$$

If $2KA < 1$ then by (4) and (9) the estimate (5) follows. Suppose that $2KA > 1$. Then using (4) we obtain

$$|\phi_m(\theta)| \leq 2K\gamma_0 (1-x^2)^{-\frac{1}{2}} (m+1)^{2KA-1} \sum_{n=0}^{m} c_n (n+1) \leq$$

$$\leq 2KA\gamma_0 (m+1)^{2KA-1} \log(m+1) (1-x^2)^{-\frac{1}{2}}$$

which is much better than (10). Now plug this inequality into (11). If $2KA - 1 < 1$ then (5) follows. Otherwise we get a new estimate which we again plug into (11). After finitely many similar steps we obtain

$$|\phi_m(\theta)| \leq B_1(1-x^2)^{-B_2}$$

for $-1 \leq x \leq 1$ and $n = 1,2,...$ which combined with (9) yields (5). The inequality (6) follows from (5) and Theorem 7.5 of [3].

Finally we note that the example of Jacobi polynomials shows that apart from the constants A_1 and A_2 our result cannot be improved.

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