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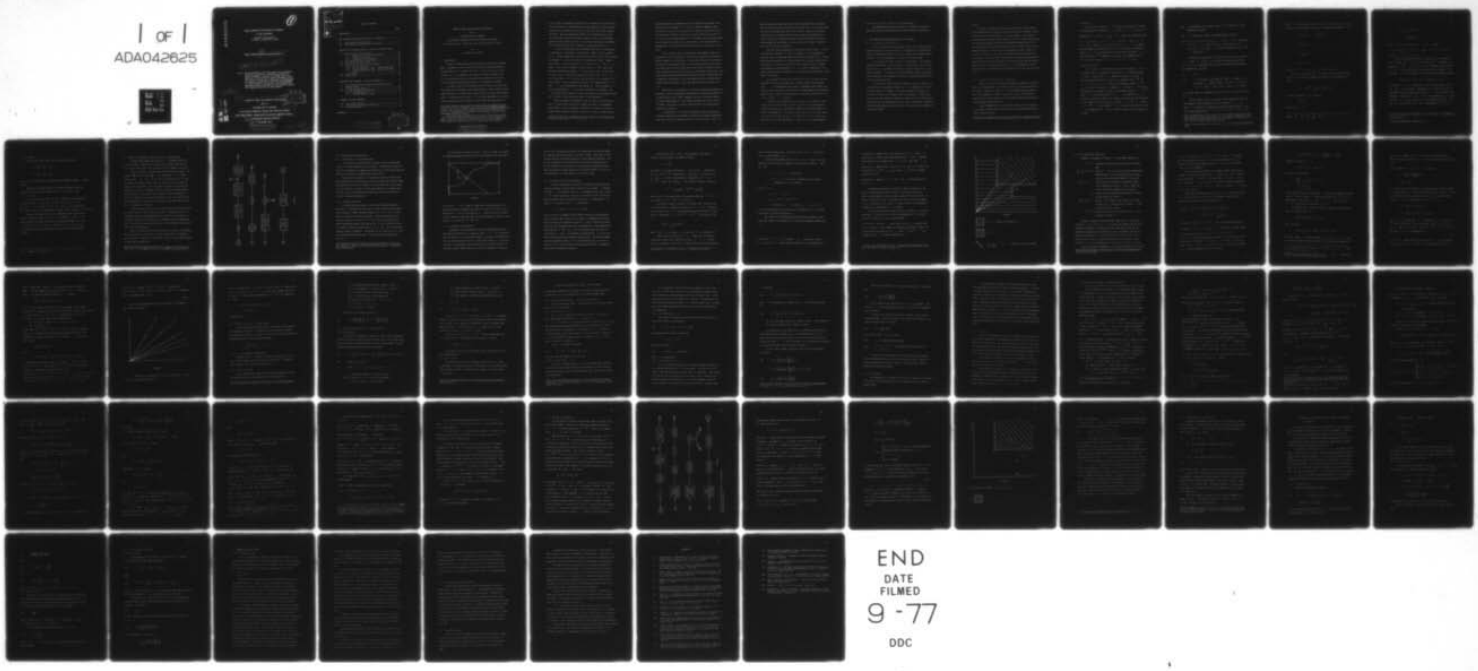

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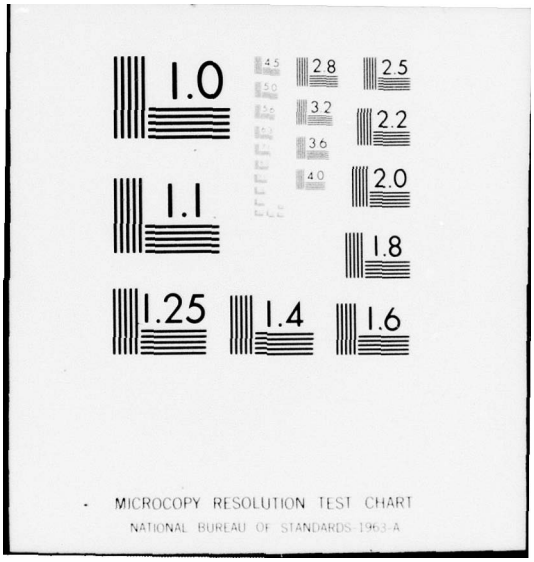
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A THEORY OF MONEY AND FINANCIAL INSTITUTIONS,

PART 36.

THE MONEY RATE OF INTEREST

(A MULTIPERIOD NONATOMIC TRADING AND PRODUCTION ECONOMY WITH OUTSIDE MONEY, INSIDE MONEY AND OPTIMAL BANKRUPTCY RULES),

Pradeep Dubey and Martin Shubik

April 27, 1977

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PART 36

THE MONEY RATE OF INTEREST

(A MULTIPERIOD NONATOMIC TRADING AND PRODUCTION ECONOMY WITH OUTSIDE MONEY, INSIDE MONEY AND OPTIMAL BANKRUPTCY RULES)*

by

P. Dubey and M. Shubik

1. INTRODUCTION

A model of trade and production through time using fiat and bank money is presented. Many of the detailed points concerning modelling of trade using fiat (= outside) and bank (= inside) money are covered elsewhere [1, 2, 3]; however in this paper an attempt is made to give both a self contained yet not overly discursive analysis of the roles of the two types of money and production in a multiperiod economy.

We suggest that an outside money serves to finance the float that comes about when individuals trade simultaneously in even a single period trading economy. An inside money, however serves to finance intertemporal trade. In several previous papers [2, 4] we have shown that for a single period economy which uses a money for trade, where the money has no in-

*This work relates to Department of the Navy Contract N00014-76-C-0085 issued by the Office of Naval Research under Contract Authority NR 047-006. However, the content does not necessarily reflect the position or the policy of the Department of the Navy or the Government, and no official endorsement should be inferred.

trinsic value, a bankruptcy penalty must be introduced to prevent individuals from trying to obtain arbitrarily large amounts of money, knowing that there is no penalty for failure to repay the source which supplied the money. When that bankruptcy penalty is set appropriately we have also shown that a nonatomic trading economy will have a noncooperative equilibrium point which coincides in the final distribution of resources and in relative prices with a competitive equilibrium point [5].

In the one period model we may imagine that fiat or outside money is supplied by a government or outside bank. Suppose that this amount is M. We may imagine that the traders all bid for a part of this supply by offering I.O.U. notes which must be redeemed after trade. Suppose each trader i offers an amount of I.O.U. notes of u^i; let $u = \int u^i$ then i will obtain an amount of outside money of $u^i M/u$ and he will owe the outside bank an amount u^i at the end of trade. We may regard the ratio u/M as defining an interest rate where $u/M = 1+\sigma$. If $\sigma > 0$ it is clear that $u > M$ and hence some individual will be unable to repay his debts. When the bankruptcy penalty is set appropriately* we have that $\sigma = 0$ at a noncooperative equilibrium [5]. When the penalty is not set appropriately $\sigma > 0$ and some individuals are bankrupt. The σ is not the type of interest rate we associated with intertemporal trade but is really a "loss reserve premium" to cover bankruptcy.

When we consider intertemporal trade, a supply of outside money which is fixed at the start may not supply the needs of intertemporal trade except if there is hoarding. Although, as we prove in Section 3.2 we may obtain a perfectly general solution to a trading economy using an

*Except for small set of bankruptcy penalties where C.E. allocations may be obtained even though $\sigma > 0$ (see [5]).

outside money where intertemporal prices are adjusted by hoarding, this solution is not reasonable. It is easy to construct examples, where, for instance, spot prices continuously fall and more and more money goes into hoard until finally at the end of economic activity--say after the k^{th} period in an economy defined for k periods--all of the money comes out of hoard to be repaid to the outside bank. Such a model might be mathematically correct but does not match either our experience or intuition.

A more "natural" way of incorporating intertemporal trade is to introduce an inside bank which is held by the traders and can create an arbitrary amount of inside money. In the latter sections of this paper we construct a multiperiod model of an economy with both outside and inside money and we assert and then prove that with the appropriate bankruptcy penalty there will be two types of noncooperative games whose equilibria have relative prices and distributions of goods coincidental with a competitive equilibrium. One of the games will have internal and external rates of interest of zero, the other will have them both positive.

When there is no production, even with the appropriate bankruptcy penalties, at the noncooperative equilibrium all traders go bankrupt for an amount equal to the "float" payment they incur by having to borrow to finance what amounts to a lag between expenditures and the receipt of income. This float is positive precisely when the inside rates of interest are positive.

The unrealistic or at least unreasonable feature of having all traders go bankrupt can be removed by explicitly introducing production that takes time and a "salvage value" for left over stocks. For a k

period economy with trade and production from capital goods, in general, there will be some capital goods left over at the end. If this were truly the end of the economy they would have no value. However if they had a "salvage value" or could be sold to others starting up elsewhere this credit could serve to balance the debt created by financing the gap between expenditure and income at a positive rate of interest. We prove that this is the case in Section 4.

The idea that an economy will end at some specific time k is not a satisfactory construct. In order to justify it, it is desirable to indicate, that at least under moderately general conditions the k period model can be interpreted as an approximation for a related infinite horizon problem. In Section 4 we do this, and are able to show a relationship between an equilibrium point to a $k+1$ period economy and a k-stationary infinite horizon economy (i.e. one in which the stationarity occurs every k^{th} period).

We conclude this section by setting up a series of finite horizon economies for which all traders have the same "natural time discount" rate, β, we show that as the horizon is lengthened that although our noncooperative game models differ from the related competitive equilibrium models at the infinite horizon stationary state,they coincide and a *money rate of interest* $\rho = 0$ emerges.

Although in an extremely special instance it is possible to link a money rate of interest with a natural discount in an obvious manner we must stress that our noncooperative games are defined extremely generally, and depend upon fine detail such as the size of the float caused by interfirm purchases and sales. The money rate of interest will vary with the float and the noncooperative model will give results somewhat

different to the usual competitive equilibrium model.

The remarks above are at best cryptic. Hopefully the specific models and the proofs which follow will provide further clarification.

2. THE COMPETITIVE EQUILIBRIUM AND RATE OF INTEREST

2.1. Some Modelling Considerations

Prior to constructing our nonatomic games to be solved for noncooperative equilibria an investigation of the Walrasian or Arrow-Debreu model of exchange modified for a role for a money may be helpful and instructive. This investigation is carried out in this section.

In this paper we limit our scope in several ways. In particular exogenous uncertainty is not considered. Our models are, for the most part, finite; i.e. problems involving the infinite horizon are not examined in general. In setting up the noncooperative game we are explicit in our description of information conditions, trading mechanisms and positions of disequilibrium.

In recent years there has been a growth of work on temporary general equilibria in sequential markets [6, 7]. An example of this type of work as applied to banking is given by Grandmont and Laroque [8]. Our work is close to but different from this approach. In particular we use a full game theoretic formulation which calls for a greater level of specification of sequencing of moves, trading mechanisms, information conditions and bankruptcy conditions than is required by the competitive equilibrium models.

A Caveat

We believe that both the competitive equilibrium* and noncooperative equilibrium are relatively unsatisfactory solution concepts on which to base the long term development of economic dynamics. They tend to be psychologically shallow. Most "dynamic stories" involving expectations are cut from the whole cloth and the adjustment mechanisms are *ad hoc*. In this paper we believe that a major value of our efforts comes in being absolutely explicit in describing the mechanisms of trade. Even if it is *ad hoc* at least it is as simple as possible given the conditions to be satisfied. We lack a story justifying the noncooperative equilibrium, beyond the usual attractive features of decentralization and the manifestation of a self fulfilling prediction, i.e. the noncooperative equilibrium path will be maintained if all predict that it will, and act accordingly.

2.2. *The Modified Walrasian Model of Exchange*

Implicit in the general equilibrium model of exchange is that payments are instantaneous, or the float is zero, or that the financing of the float is supplied at zero cost. In our extended models presented here this assumption is removed, although it is easy to include any interest rate into the context of an Arrow-Debreu model and thereby distinguish futures from spot prices.

We now commence to set up the notation needed for the remainder of this paper starting with a description of an Arrow-Debreu world with futures and spot prices.

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*Including its offspring the "temporary equilibria in sequential markets."

Some Notation

For a positive integer n, Ω^n denotes the non-negative orthant of Euclidean space of dimension n. For any $x \in \Omega^n$, x_j is the j^{th} component of x, and $\bar{x}$ is $\sum_{j=1}^{n} x_j$. $\| \ \|$ stands for the maximum norm, i.e., $\|x\| = max \{x_j : j = 1, \ldots, n\}$. For any x and y in Ω^n, $x > y$ [$\geq$, =] means $x_j > y_j$ [$\geq$, =] for $j = 1, \ldots, n$. 0 denotes both the number zero and the origin of Ω^n (this origin is sometimes written 0_n), but the meaning will be clear from the context. e_j is the vector whose j^{th} component is 1 and all other components are 0. T^n is the unit simplex in Ω^n, i.e., $T^n = \{x \in \Omega^n : \bar{x} = 1\}$.

The Nonatomic Economy

Let $\{I, \mathcal{C}, \mu\}$ be the measure space of traders where $I \equiv$ the set of traders, $\mathcal{C} \equiv$ the σ-algebra of coalitions, $\mu \equiv$ a nonatomic measure on $\{I, \mathcal{C}\}$. Trade occurs in m commodities through k periods. It will be convenient to label the axes of the commodity space Ω^{mk} by $\{th : 1 \leq t \leq k, \ 1 \leq h \leq m\}$. Thus for any commodity bundle $x \in \Omega^{mk}$, x_{th} represents the quantity of commodity h in period t. The initial endowments of the traders is given by a measurable function $a : I \to \Omega^{mk}$, where a^i is the initial bundle of $i \in I$. We assume that $\int a > 0$. Finally to complete the data of the market we must specify the preferences of the traders. This is given by a function $U : I \times \Omega^{mk} \to \Omega^1$, where $U(i, \cdot) : \Omega^{mk} \to \Omega^1$ (also denoted U^i) is the utility function of trader i. Our assumptions on U are as follows (compare with the assumptions in [6]):

(A1) U is measurable in the product space $I \times \Omega^{mk}$ (where Ω^{mk} is the standard Borel space).

(A2) U^i is continuous, concave, and nondecreasing for each i.

(A3)*** There is an integrable function $v : I \to \Omega^{mk}$, $v > a$, such that:

(i) if $x_{th} \geq v^i_{th}$, then $U^i(x + \Delta e_{th}) = U^i(x)$ for all $\Delta > 0$;

(ii) if $x_{th} < v^i_{th}$, then U^i is strictly increasing in the th^{th} variable at x.

A *competitive equilibrium in futures prices* (C.E.F.P.) is a pair (p^*, x), where $p^* \in \Omega^{mk} \setminus \{0\}$ is the vector of futures prices,* and $x : I \to \Omega^{mk}$ is a measurable function (which describes a reallocation of the commodities) such that:

(1) $\int x = \int a$

(2) x^i is optimal in the budget set $B^i_*(p^*)$ of trader i at the prices p^*, where $B^i_*(p^*) = \{y \in \Omega^{mk} : p^* \cdot y \leq p^* \cdot a^i\}$ i.e., $x^i \in B^i_*(p^*)$, and $U^i(x^i) = \max\{U^i(y) : y \in B^i_*(p^*)\}$

It is clear, under our assumptions, that if (p^*, x) is a C.E.F.P. then $p^* > 0$.

We now suppose as part of the data of the economy that there are intertemporal inside nonnegative rates of interest** $\rho_1, \ldots, \rho_k$, where ρ_t is the rate of interest that operates between period t and

---

*I.e. Arrow Debreu prices.

**If individuals were *forced* by the rules of the game to keep all of their money or credit resources in a bank, i.e., they are not permitted to hoard then negative rates of interest would also be feasible. It is the *strategic option* of hoarding that rules them out.

***(A3) is required only for Propositions (2)-(5), and may be dropped thereafter.

period $t+1$. Before we describe a *competitive equilibrium in spot prices* (C.E.S.P.) for the economy, it will help to build up some notation. Let*

$$\rho^* = \prod_{t=1}^{k} (1+\rho_t) \ , \quad \delta_t = \prod_{l=1}^{t-1} (1+\rho_l)$$

$$\xi_t = \prod_{l=t}^{k} (1+\rho_l)$$

$$\xi'_t = \prod_{l=t+1}^{k} (1+\rho_l)$$

$$\xi''_t = \xi_t - \xi'_t \ .$$

For any $x \in \Omega^{mk}$, $x_t = (x_{t1}, \ldots, x_{tm})$ for $1 \leq t \leq k$.

A C.E.S.P. is again a pair (p,x) which satisfies the conditions (1) and (2), but with the following modification of the budget set of trader i:

$$B^i(p) = \{y \in \Omega^{mk} : \sum_{t=1}^{k} \xi_t p_t \cdot (x_t^i - a_t^i) \leq 0\}$$

Consider the mapping $p \leftrightarrow p^*$ given by:

$$\delta_t p_t^* = p_t \ .$$

Then for any $y \in \Omega^{mk}$

---

*Define $\prod_{l=1}^{0}$, $\prod_{l=k}^{k-l}$, and $\prod_{l=k+1}^{k-l}$ to be 1.

$$p^* \cdot y = \sum_{t=1}^{k} (\delta_t)^{-1} p_t \cdot y_t$$

$$= \sum_{t=1}^{k} \frac{1}{\rho^*} \xi_t p_t \cdot y_t \ .$$

Thus $y \in B_*^i(p^*)$ if, and only if, $y \in B^i(p)$. Indeed,

$p^* \cdot (y - a^i) = \frac{1}{\rho^*} \sum_{t=1}^{k} \xi_t p_t \cdot (x_t^i - a_t^i)$. Hence (p^*, x) is a C.E.F.P. if, and only if, (p,x) is a C.E.S.P.*

Let us also recall the notion of "shadow prices of income"** at a C.E. Suppose (p,x) is a C.E. Then each trader i maximizes $U^i(y)$ subject to $y \in \Omega^{mk}$, $p \cdot y - p \cdot a^i \leq 0$, and the maximum is obtained at $x^i \in B^i(p)$. By the Kuhn-Tucker theorem there exists a number $\lambda^i \geq 0$ such that x^i is also a solution of the following unconstrained problem:

$$max[U^i(y) + \lambda^i(p \cdot a^i - p \cdot y)] \ , \quad \text{subject to} \quad y \in \Omega^{mk} \ .$$

The function $\lambda : I \to \Omega^1$ constitutes a choice of shadow prices at the C.E. (p,x) . We have shown in Appendix A of [5] that for any C.E. a measurable choice of shadow prices exists, i.e. λ can be picked to be measurable. Note that if (p,x) is a C.E. with shadow prices λ , then for any $K > 0$, (Kp,x) is also a C.E. with shadow prices $\frac{1}{K}\lambda$. For any $B > 0$ we will call a C.E. (p,x) B-normalized if $p \cdot \int a^i = B$.

---

*A C.E.F.P. and C.E.S.P. coincide if $\rho_t = 0$ for all t ; in this case we call both a C.E.

**Henceforth abbreviated "shadow prices."

A Simple Example

Consider two traders with utility functions as follows:

$$U^1 = .9x_1^1 + .9x_2^1 + x_3^1$$

$$U^2 = x_1^2 + x_2^2 + .9x_3^2$$

where x_t^i is the amount of the consumer good consumed by trader i during period t.

Assume that the good cannot be inventoried between periods and that the holdings of the two traders are* $a^1 = (20, 20, 0)$ and $a^2 = (0, 0, 40)$.

It is easy to observe that there is a competitive equilibrium set of futures prices (p_1^*, p_2^*, p_3^*) such that $p_1^* = p_2^* = p_3^* = 1$ and $x_1^1 = x_2^1 = 0$, $x_3^1 = 40$ and $x_1^2 = x_2^2 = 20$, $x_3^2 = 0$. At these prices the present values of the endowments of traders 1 and 2 are both $I^1 = I^2 = 40$.

Suppose that there were arbitrary nonnegative short term rates of interest between periods 1, 2 and 2, 3. Let them be ρ_1 and ρ_2. We may now define three spot prices for the goods being traded. $p_1 = p_1^* = 1$, $p_2 = p_2^*(1+\rho_1) = (1+\rho_1)$ and $p_3 = p_3^*(1+\rho_1)(1+\rho_2) = (1+\rho_1)(1+\rho_2)$. The futures prices are not changed by these rates of interest, neither is trade.

---

* a^1, stands for (a_1^1, a_2^1, a_3^1).

3. MODELLING AN EXCHANGE ECONOMY AS A GAME IN STRATEGIC FORM

Consider many traders each trading in m consumable goods for k time periods. Imagine that all goods are perishable and last only for a single time period. This condition will be relaxed in Section 4.

Figure 1 presents a flow diagram of how this model proceeds. The symbols $\rightarrow(i)$ connect with $(i)\rightarrow$. The instruction $t = t+1$ means "replace time t by $t+1$." At the start individuals bid I.O.U. notes in exchange for outside money, which they then use in the purchase of shares in an inside bank. They then borrow or deposit at the bank and proceed to the market. Depending upon the convention we adopt traders may be required to sell all of their holdings or may be permitted to send to the market only that which they wish to sell. The various trading conventions will cause different volumes of trade and size of float. This is illustrated in Figure 2.

After trade has taken place, the process goes into the next period and income from the sale of goods is received. Following this the inside bank credits all stockholders with the profits it has accrued. As the bank is permitted to "roll-over" or refinance loans no bankruptcies need to occur before the end thus profits can be calculated on an accrued rather than received basis.*

After the final period has been reached, the economy is liquidated and in the process of doing this, first the inside bank is liquidated and its capital is paid out to stockholders, then the outside money supply is called and the game ends.

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*This rules out the voluntary election by a traders to go bankrupt before period $k+1$, as the bank can "carry" him whether he likes it or not.

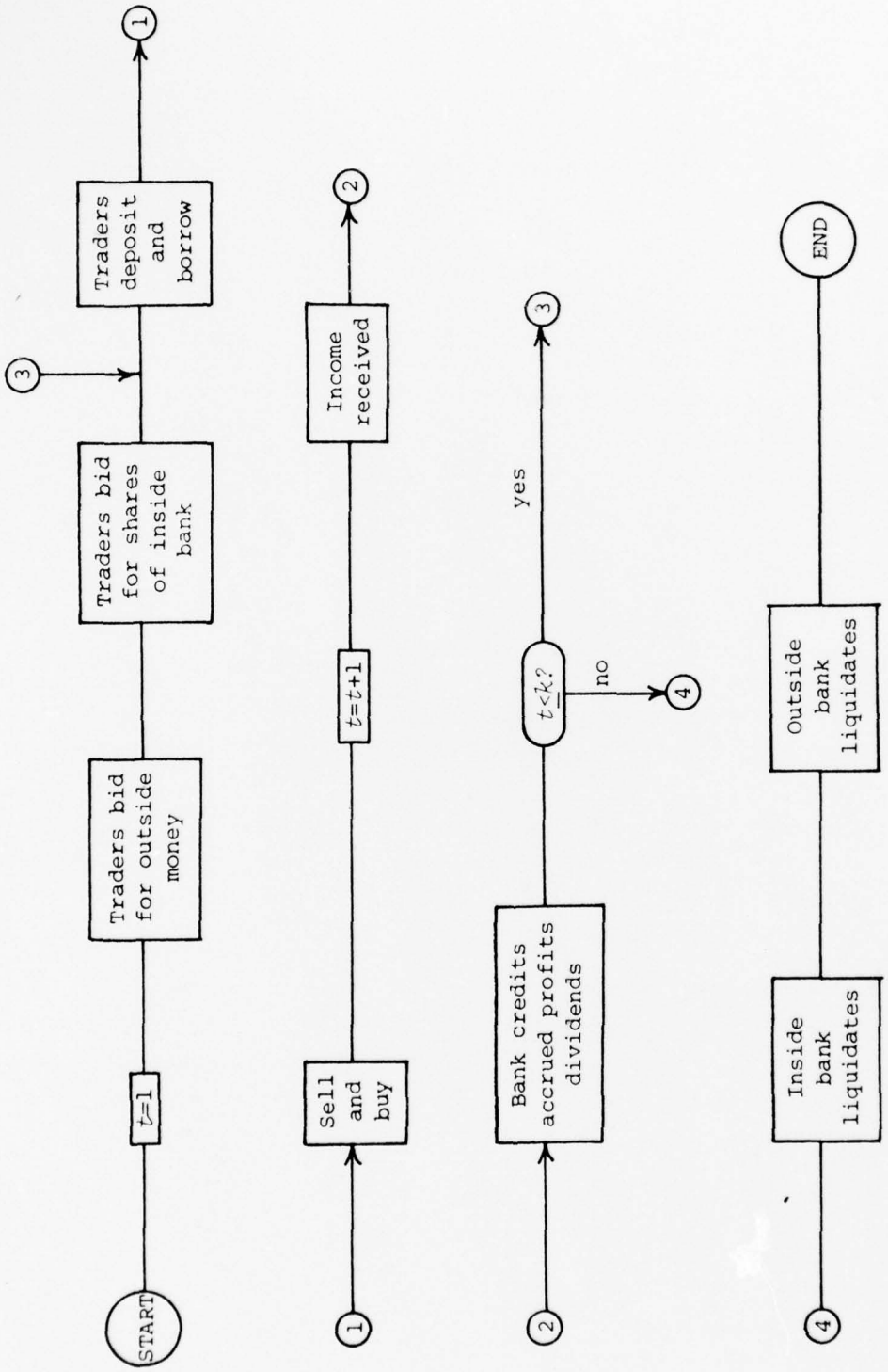
START
t=1
Traders bid for outside money
Traders bid for shares of inside bank
3
Traders deposit and borrow
1
1
Sell and buy
t=t+1
Income received
2
2
Bank credits accrued profits dividends
t<k?
yes
3
no
4
4
Inside bank liquidates
Outside bank liquidates
END

FIGURE 1

3.1. More Modelling Considerations

3.1.1. Competitive or Controlled Banking

In this model we assume that the interest rate for inside money is set by a central authority for each period. Alternatively we may let the rate be determined by competitive bidding only if the outside authority controls the quantity of loanable funds each period.

The question of whether it is possible to design a closed system which produces both the appropriate interest rates and the supply of loanable funds* by the forces of competition is not dealt with in this paper. It is not obvious to us that it can be done always, but a detailed specification of the difficulties and examination of the possibilities is left for a further investigation.

3.1.2. Trading Conventions

In several previous papers we have noted that when exchange is modelled as a noncooperative game there are many different ways by which we can describe a market clearing mechanism [4, 10, 11, 12]. In this paper we concentrate on the two simplest. They are respectively the "sell-all" and the "hold back" models. In the first all goods must be passed through the markets each time period. It is as though we forced the society to monetize national wealth [4, 10, 11]. In the second only national income is monetized. Individuals are not required to sell everything and buy back what they want. They can hold back anything they wish [12].

---

*Intuitively the supply of loanable funds and the rate of interest are dual variables. The design of a mechanism that enables us to determine both seems unlikely.

The distinction is made in Figure 2. This is a simple two trader, two commodity Edgeworth box with the initial holdings of the individuals

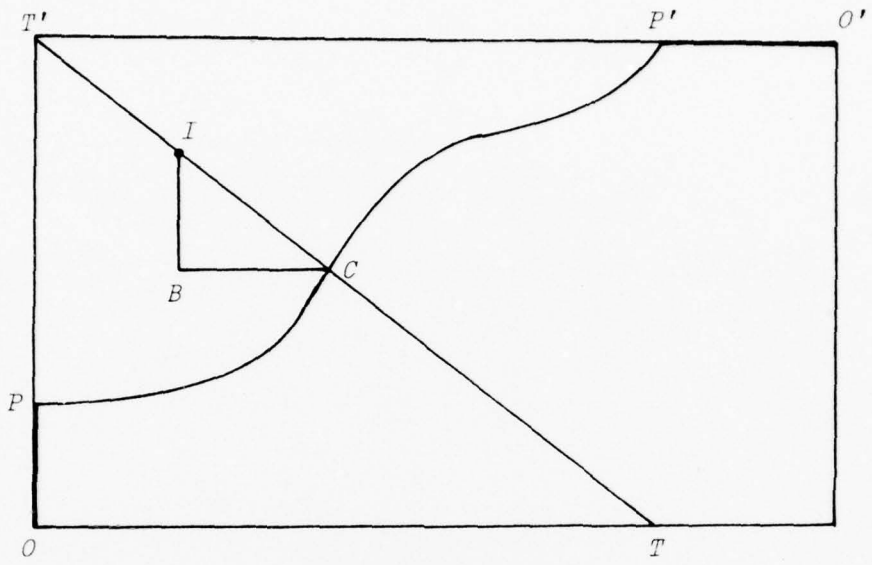

FIGURE 2

at the point I . $T'CT$ shows the competitive equilibrium price ray. Under the hold back assumption the amount of trade in the first good is measured by BI and in the second by BC . Under the sell all assumption the volume of trade (and hence the need for money or credit) is much larger. It is given by OT' and $O'T'$.

3.1.3. Strategies and Information

Elsewhere we have discussed in detail the problems incurred in well-defining strategies in a strategic game [1, 13]. It is our belief that the essence of the ideas which enable us to extend our analysis beyond the single move game and into a multiperiod model is a *low information state*. By assuming little information we can avoid having to let strategies become complex functions of previous knowledge. The way we

can do this is by having individuals not necessarily even know the amount of resources they have on hand at any point in time. They still can make feasible moves by allocating fractions of their expected resources. This may not necessarily be the most natural or the best convention but it is well defined for all positions of equilibrium or disequilibrium and when individuals are small relative to the overall market it is not even as "unrealistic" as it may appear to be at first glance.

3.2. *The Outside Bank Only and Hoarding*

To recast the market in the form of a strategic game we must describe the strategy sets and payoff functions of the traders. A strategy of trader i is to announce a bid u^i of promissory notes, and a vector $\tilde{g}^i \in T^{mk}$ which constitutes a decision as to how he will divide the fiat money he obtains from the bank into bids on the m trading-posts. Denoting i's strategy set by S^i, we then have

$$S^i = \{\{u^i, \tilde{g}^i\} : u^i \in \Omega^1, \tilde{g}^i \in T^{mk}\} .$$

Given a choice of strategies by the traders, how are the trading-posts and the outside bank cleared? We are beset by a fundamental difficulty when we consider the mechanism of Section 2. This is because the mechanism calls for aggregating the bids in the bank and in each trading-post. These would be $\int u$ and $\int b_{th}$. But the integrals make sense only if the functions u and b are assumed to be measurable. It is not clear how we would justify this assumption heuristically. Why should independent decision-makers behave in a jointly measurable way? We refer to Section 5 of [14], where a model of noncooperative behavior is suggested which leads to measurable strategies.

Assuming then that u and g^i are measurable, and letting M stand for the fiat money in the bank, we define

$$1 + \sigma = \frac{\int u}{M} \quad .$$

The amount of fiat money obtained by i is* $u^i/(1+\sigma)$, and thus his bid b^i_{th} on the th^{th} trading-post is $\tilde{a}^i_{th}u^i/(1+\sigma)$. The price p_{th} is $\int b_{th}/\int a_{th}$, and the final holding (of commodities) of i is: $x^i \in \Omega^{mk}$, where $x^i_{th} = b^i_{th}/p_{th}$. His credit at the bank is given by

$$\eta^i = -u^i + \sum_{t,h} p_{th}a^i_{th} = -(1+\sigma)\bar{b}^i + \sum_{t,h} p_{th}a^i_{th}$$

Hence, given a preassigned choice of bankruptcy penalties $\lambda : I \to \Omega^1$, the payoff to i is: $U^i(x^i) + \lambda^i \min[0, \eta^i]$.

We now have a game in strategic (or normal) form. With the rest of the data fixed, it depends on the choice of $\lambda : I \to \Omega^1$, and hence we will denote it by Γ_λ. A non-cooperative equilibrium (N.E.) of this game is a measurable $s_* : I \to \Omega^1 \times T^{mk}$, $s^i_* \in S^i$, such that, for all i,

$$\Pi^i(s_*) = \max_{s^i \in S^i} \Pi(s_*|s^i) \ ,$$

where $(s_*|s^i)$ is the same as s_* except that s^i_* is replaced by s^i.

There exists a trivial N.E. of Γ_λ, namely the collection of strategies in which each trader bids nothing, i.e., $u = 0$. We will focus our attention on *active* N.E.'s, namely those which produce positive

*Division by 0 is defined to yield 0 throughout this paper.

prices in each trading post. This implies that $1+\sigma > 0$, and in fact (as is easily checked) $\sigma \geq 0$.

We wish to investigate the N.E.'s of Γ_λ as λ varies. For this purpose it will be useful to demarcate certain regions in which λ may lie. Let

$$\Lambda = \{\lambda : T \to \Omega^1 : \lambda \text{ is measurable}\}$$

$$\Lambda_{\mathcal{C}} = \{\lambda \in \Lambda : \lambda \text{ is a choice of shadow prices at some } M\text{-normalized C.E. of the market}\}.$$

For any ${}^*\lambda \in \Lambda$,

$$\Lambda_{{}^*\lambda} = \{\lambda \in \Lambda \quad \lambda \geq {}^*\lambda\}$$

$$\Lambda_{(\lambda,\sigma)} = \{{}^*\lambda \in \Lambda : (1+\sigma){}^*\lambda = \lambda\}$$

$$\Lambda' = \Lambda \setminus [\Lambda_{\mathcal{C}} \cup \{\Lambda_{{}^*\lambda} : {}^*\lambda \in \Lambda_{\mathcal{C}}\} \cup \{\Lambda_{(\lambda,\sigma)} : \lambda \in \Lambda_{\mathcal{C}},\ \sigma > 0\}].$$

In the following when we say that an N.E. coincides with a C.E., we mean coincidence in prices and allocation.

The proofs of the following propositions follow by making slight notational changes in the proofs of the identically-numbered propositions in [5].

Proposition 2. If ${}^*\lambda \in \Lambda_\lambda$ for some $\lambda \in \Lambda_{\mathcal{C}}$, there exists an N.E. of $\Gamma_{{}^*\lambda}$ with $\sigma = 0$, which coincides with the C.E. associated with λ.

Proposition 3. Suppose there is an (active) N.E. of Γ_{*_λ} with $\sigma = 0$. Then this N.E. coincides with some *M*-normalized C.E. (p,x). Moreover, there exists a set of shadow prices λ for (p,x) such that ${}^*\lambda \in \Lambda_\lambda$.

Proposition 4. Suppose there is an N.E. of Γ_{*_λ} with $\sigma > 0$ which coincides with a C.E. Then ${}^*\lambda \in \Lambda_{(\lambda,\sigma)}$ where λ is a choice of shadow prices for the C.E.

Proposition 5. Suppose $\lambda \in \Lambda'$. Then $inf\{\sigma : \sigma$ occurs at an N.E. of $\Gamma_\lambda\} > 0$.

We summarize these result in Figure 3 which is drawn for a non-atomic economy with two types of traders. We consider only those λ which are type-symmetric, hence we may represent them by vectors in Ω^2. Figure 3 shows that λ-space. We assume that there are a finite number of C.E.'s, and for any C.E. there is a unique choice of (type-symmetric) shadow prices, represented by $\cdot$ in Figure 3.

In this model we have bounded credit by the amount M of outside bank money. If we were to let credit be unbounded then,* by the same proof as in Theorem 1 of [14], we would obtain:

Proposition 6. For any λ, every active N.E. of Γ_λ coincides with a C.E. Moreover, if the economy is "competitively bounded,"* every C.E. coincides with an active N.E. of $\Gamma_{\hat{\lambda}}$, where $\hat{\lambda}^i \geq sup\{\lambda^i : \lambda^i$ is a shadow price for i at some *B*-normalized C.E.$\}$.

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*For details of the model, see [14]. "Competitively bounded" means that the *sup* above is finite for almost all i.

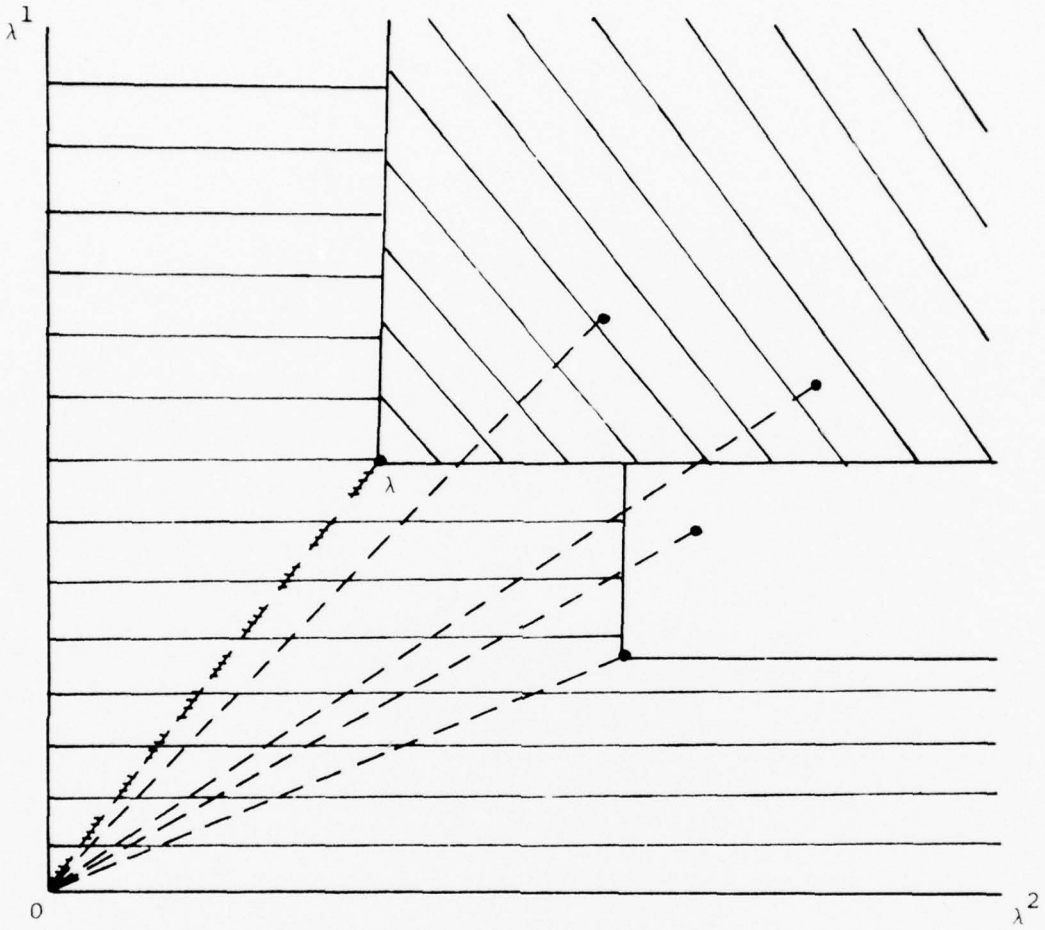
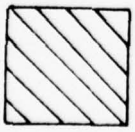


FIGURE 3

is Λ' , deleting the broken line



is Λ_{λ}

is $U\{\Lambda_{\lambda,\sigma} : \sigma > 0\}$. σ increases as we move towards the origin.

3.3. The Outside and Inside Bank

Formally, a strategy of a trader i in this model consists of:

$\omega^i \in \Omega^1$: the amount of promissory notes bid at the outside bank.

$(\omega_1^i, \omega_2^i, \omega_3^i) \in T^3$: $\omega_1^i(\omega_2^i, \omega_3^i) \equiv$ the fraction of the outside bank money obtained that is bid on inside bank shares (deposited in the inside bank, hoarded in the future).

$(d_{t1}^i, d_{t2}^i) \in T^2$: the fraction of the money at hand before trade at the beginning of period t that is deposited in the inside bank (hoarded into the future).

b_t^i : amount of inside bank money borrowed before trade at the beginning of period t.

$(\eta_{t1}^i, \eta_{t2}^i) \in T^2$: $\eta_{t1}^i(\eta_{t2}^i) \equiv$ the fraction of the money at hand that is bid on the m trading posts in period t (that is hoarded into the future).

$\underset{\sim}{\eta}_t^i \in T^m$: decision on how to split the bids on the m trading posts in period t.

However, a moment's reflection makes clear that i does best by setting $\omega_2^i = \omega_3^i = d_{t2}^i = \eta_{t2}^i = 0$. We describe the payoffs now for these simplified strategies. Our description will make quite clear how one would define payoffs for an arbitrary choice of strategies (and how cumbersome that would be!). *It will become obvious that every N.E. of the general game is an N.E. of the simplified game, and conversely. Thus nothing is lost in our simplification so far as the study of N.E.'s goes.*

With this excuse* a strategy of i will now be taken to be a vector

*We apologize in advance to Lloyd Shapley for not having completely specified the general game.

$s^i = (u^i, b^i)$ where $u^i \in \Omega^1$, and $b^i \in \Omega^{mk}$. Here u^i is the quantity of promissory notes bid by i at the outside bank, and b^i_{th} is the amount of inside bank money borrowed by i at the *start* of period t for bidding on commodity h.

Once each trader has picked his strategy the market functions as follows: Outside rate of interest is σ, where $1+\sigma = \int u/M$. Outside money obtained by i = bid by i for bank shares = $u^i/(1+\sigma)$. Price of h in period $t = p_{th} = \int b_{th}/\int a_{th}$. Amount of inside money deposited by i at *start* of period $t+1 = p_t \cdot a^i_t$. Amount of commodity th obtained by $i = x^i_{th} = b^i_{th}/p_{th}$.

Finally we must compute the net credit of i at the end of period k. For this purpose, first note that the amount of money held by i prior to the dissolution of the inside bank is:

$$\sum_{t=1}^{k} \xi_t \bar{b}^i_t - \sum_{t=1}^{k} \xi'_t p_t \cdot a^i_t = \sum_{t=1}^{k} \xi_t p_t \cdot (x^i_t - a^i_t) - \sum_{t=1}^{k} \xi''_t p_t \cdot a^i_t$$

$$= \rho^* p^* \cdot (x^i - a^i) - \sum_{t=1}^{k} \xi''_t p_t \cdot a^i_t .$$

Let us denote $\sum_{t=1}^{k} \xi''_t p_t \cdot a^i_t$ by $h^i(p)$. Thus the profit made by the inside bank is $\int h^i(p) = h(p) = \sum_{t=1}^{k} \xi''_t p_t \cdot \int a_t$. {Perhaps this is made clearer by noting that the profit $\int[\sigma^* p^* \cdot (x^i - a^i) + h^i(p)] = \int h^i(p)$ $\int x^i = \int a^i$.} When the bank dissolves, trader i gets his share of the profits which is [letting $v^i = u^i/(1+\sigma)$] $(v^i/\int v^i)h(p) = v^i[h(p)/M]$. Denote $h(p)/\rho^*$ by $H(p)$, $h^i(p)/\rho^*$ by $H^i(p)$. Then the net credit of i on which bankruptcy penalty is levied is:

$$\eta^i = -p^* \cdot (x^i - a^i) - H^i(p) + v^i\left[\frac{H(p)}{M} + 1 - (1+\sigma)\right]$$

Therefore the payoff to i is

$$\Pi^i(s) = U^i(x^i) + \lambda^i \min[0, \eta^i] .$$

At an N.E., we claim that

$$\frac{H(p)}{M} + 1 - (1+\sigma) = 0 , \quad \text{i.e.,}$$

(*) $$\boxed{H(p) = \sigma M .}$$

If the term were positive, each trader i could acquire arbitrarily large bundles x^i by increasing v^i which would violate $\int x^i = \int a^i$; if it were negative, each trader* i would set $v^i = 0$ contradicting that $\int v^i = M$. This proves our claim that $H(p)/M = \sigma$.

To study the N.E.'s it will be useful to rewrite the payoff functions in a simpler form. Let

$$\hat{U}^i_p(\beta) = \max\{U^i(y) : p^* \cdot y = \beta\}$$

Then, at an N.E.,

$$\Pi^i(\beta) = \hat{U}^i_p(\beta) + \lambda^i \min[0, -\beta + p^* \cdot a^i - H^i(p)]$$

At an N.E. trader i in effect maximizes $\Pi^i(\beta)$ for $\beta \geq 0$. β represents the (discounted) sum spent by him in the purchase of commodities.**

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*We are assuming that the inside bank is "active," i.e. $\int v^i > 0$. There is a trivial solution with $\int v^i = 0$.

**Note as we are running out of symbols we use β twice. β beyond this section is used for the natural discount rate.

Theorem (1a). Suppose (p^*, x) is a C.E.F.P. with shadow prices λ. Then there exists an active N.E. of Γ_λ which coincides with the C.E.S.P. (p,x).

Proof. Construct s^i, for $i \in I$, as follows:

$$u^i = \left[M\left(1 + \frac{H(p)}{M}\right)\right] \Big/ \mu(I)$$

$$y^i_{th} = p^*_{th} x^i_{th} .$$

It is clear that this collection of strategies produces the spot prices p, the allocation x, and an outside bank rate of interest $\sigma = H(p)/M$. To check that the collection constitutes an N.E. we must verify that, for all i, $\Pi^i(\beta)$ is maximized at $\beta^* = p^* \cdot a^i$. Let

$${}^1\Pi^i = \hat{U}^i_p(\beta) + \lambda^i[-\beta + p^* \cdot a^i]$$

$${}^2\Pi^i = \hat{U}^i_p(\beta) + \lambda^i[-\beta + p^* \cdot a^i - H^i(p)]$$

Since λ^i is a shadow price, ${}^1\Pi^i$ is maximized at β^*; hence so is ${}^2\Pi^i$ which differs from ${}^1\Pi^i$ by a constant. But $\Pi^i(\beta) \leq {}^2\Pi^i(\beta)$ for all β, and $\Pi^i(\beta^*) = {}^2\Pi^i(\beta^*)$ since $-\beta^* + p^* \cdot a^i - H^i(p) = -H^i(p) \leq 0$. Therefore $\Pi^i(\beta)$ is also maximized at β^*.

Q.E.D.

Theorem (1b). Suppose there is an active N.E. of Γ_λ which coincides with the C.E.S.P. (p,x). Then λ is a shadow price for (p^*, x).

Proof. Suppose not. Then let S be the non-null set of traders for whom λ is not a shadow price at the C.E.F.P. (p^*, x). In what follows, i is always an arbitrary element of S. Consider

$$ {}^1\Pi^i(\beta) = \hat{U}^i_p(\beta) + \lambda^i[-\beta + p^*\cdot a^i] $$

Let N^i be the closed convex set on which the maximum of the concave function ${}^1\Pi^i$ is attained. Then we first assert that $N^i \subset [0,\ p^*\cdot a^i - H^i(p)]$. (Recall that for all i, $H^i(p) > 0$.) Otherwise since $p^*\cdot a^i \notin N^i$, and since N^i is convex, there are two possibilities

(A) $N^i = [A,B]$, where $p^*\cdot a^i - H^i(p) < B < p^*\cdot a^i$.

(B) $N^i \subset (p^*\cdot a^i,\ \infty)$

Recall that $\Pi^i(\beta) = {}^1\Pi^i(\beta) - H^i(p)$ if $\beta \geq p^*\cdot a^i - H^i(p)$, and that since (p,x) are the prices and allocation produced at the N.E., the maximum of Π^i must be achieved at $p^*\cdot a^i$ for all i. But, in case (A), we have

$$ {}^1\Pi^i(\beta) > {}^1\Pi^i(p^*\cdot a^i) $$

hence

$$ \Pi^i(\beta) = {}^1\Pi^i(\beta) - H^i(p) > {}^1\Pi^i(p^*\cdot a^i) - H^i(p) = \Pi^i(p^*\cdot a^i), $$

a contradiction. Similarly, in case (B), taking c in M^i, we have ${}^1\Pi^i(c) > {}^1\Pi^i(p^*\cdot a^i)$, which implies $\Pi^i(c) > \Pi^i(p^*\cdot a^i)$, again a contradiction. This proves the assertion that for all $i \in S$, $N^i \subset [0,\ p^*\cdot a^i - H^i(p)]$. Pick an $\alpha^i \in N^i$, and an $\hat{\alpha}^i \in (p^*\cdot a^i - H^i(p),\ p^*\cdot a^i)$, and let $\hat{\alpha}^i = \delta^i\alpha^i + (1-\delta^i)p^*\cdot a^i$ for $0 < \delta^i < 1$. Since ${}^1\Pi^i$ is concave, ${}^1\Pi^i(\hat{\alpha}^i) \geq \delta^i\,{}^1\Pi^i(\alpha^i) + (1-\delta^i){}^1\Pi^i(p^*\cdot a^i)$. But $p^*\cdot a^i \notin N^i$,

and $\alpha^i \in N^i$, therefore ${}^1\Pi^i(\alpha^i) > {}^1\Pi^i(p^* \cdot a^i)$. This implies $\Pi^i(\alpha^i) = {}^1\Pi^i(\alpha^i) - H^i(p) > {}^1\Pi^i(p^* \cdot a^i) - H^i(p) = \Pi^i(p^* \cdot a^i)$, a contradiction. We conclude that $\mu(S) = 0$.

Q.E.D.

With the same labelling as was used for Figure 3, we summarize these results in Figure 4.

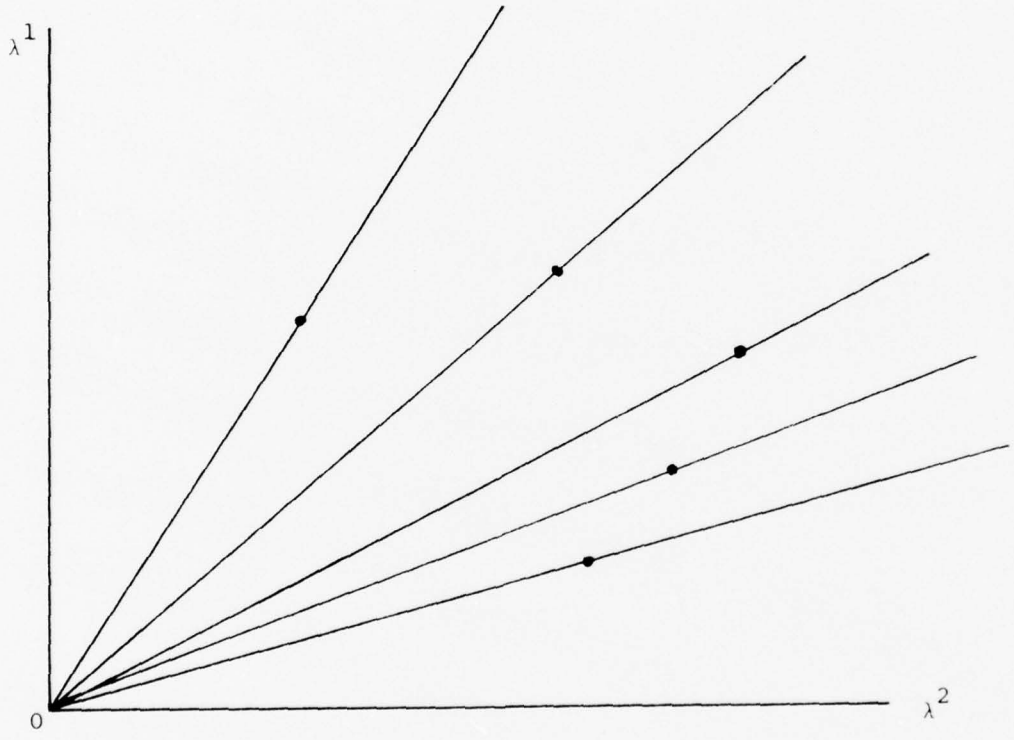

FIGURE 4

Γ_λ has an N.E. coincident with a C.E. if, and only if, λ lies on the ray corresponding to that C.E.

Remark: No matter what λ *we pick, some non-null subset of traders necessarily goes bankrupt at any N.E. of* Γ_λ . To see this, suppose there were an N.E. of Γ_λ with no one going bankrupt, i.e. $\eta^i = 0$ for almost all i . Then

$$0 = \int \eta^i = \int [-p^* \cdot (x^i - a^i) - H^i(p)]$$

$$= 0 - \int H^i(p) < 0$$

a contradiction.

3.4. Financing the Float: A Simple Example

In the remainder of this paper we use a relatively simple example to illustrate our results. In particular in this section we consider an economy which runs for k periods. There are two types of individuals distinguished only by their initial endowments.

$$(1) \qquad U^i = \sum_{t=1}^{k} \beta^{t-1} \log x_t^i , \quad i = 1, 2$$

$0 \leq \beta < 1$ is a "natural" time discount.

A trader of type i receives a "manna" each period $a_t^1 = 100$ units of the consumer good which cannot be inventoried.* A trader of type 2 obtains $a_t^2 = 1{,}000$ units each period.

3.4.1. The C.E. Problem

It is clear that the competitive equilibrium problem is trivial. No trade takes place as the distribution is already optimal.

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*Note: inventorying is really best considered as a production process.

λ^i = the Lagrangian multiplier for a trader of type i ;

x_t^i = the consumption of a trader of type i at time t ;

p_t^* = the (futures) price of the good at time t ;

p_1^* = 1 the first price is set to equal one;

x_t^1 = 100 and x_t^2 = 1,000 for = 1, ..., k ;

p_t^* = β^{t-1} for t = 1, ..., k ;

λ^1 = 1/100 and λ^2 = 1/1000 .

The gain to each trader is given by

$$(2) \qquad U^1 = 2\left\{\frac{1 - \beta^k}{1 - \beta}\right\} \quad \text{and} \quad U^2 = 3\left\{\frac{1 - \beta^k}{1 - \beta}\right\}$$

3.4.2. The Non-Cooperative Game: Outside Bank Only

(a) Hold Back Model

If we permit individuals to consume any goods in their possession without having to pass them through the market, then in this simple model there is no need for trade and hence no need to utilize money. The non-cooperative game payoff functions are given below:

$$(3) \qquad U^i = \sum_{t=1}^{k} \beta^{t-1} log\, x_t^i + \mu^i\, min[0, \{\sum_{t=1}^{k} p_t^*(a_t^i - x_t^i) + u^i - (1+\sigma)u^i\}]$$

$$(4) \qquad \text{subject to } p_t^* x_t^i \leq R_t^i$$

$$(5) \qquad \text{where } I_1^i = u^i \ , \quad I_t^i = p_{t-1}^* a_{t-1}^i \quad \text{for } t = 2, \ldots, k$$

μ^i = bankruptcy penalty on a trader of type i ;

$(1+\gamma)u^i$ = amount of I.O.U.'s bid for outside money;

σ = rate of interest on outside money;

I_t^i = income obtained by a trader of type i at time t ;

R_t^i = money held by a trader of type i at start of t ;

M = total amount of outside money supplied at the start.

(6) $$R_1^i = I_1^i$$

(7) $$R_t^i = R_{t-1}^i + p_{t-1}^* a_{t-1}^i - p_{t-1}^* x_{t-1}^i$$

If we set $\mu^i = \lambda^i$: $\mu^1 = 1/100$ and $\mu^2 = 1/1000$ it is straightforward to check from (3) that this noncooperative game has the same physical outcomes as the Competitive Equilibrium. There are however some minor differences. If we wish that $p_1^* = 1$ we must set $M = 1100$. Furthermore we have $u^1 = 100$ and $u^2 = 1000$.

The spot prices are $1, \beta, \ldots, \beta^{k-1}$. These coincide with the C.E. futures prices. The outside rate of interest is given by

$$1+\sigma = \frac{u}{M} = 1 \quad \text{or} \quad \sigma = 0 .$$

The traders bid for the outside money, hoard it and return at the final settlement date.*

(b) Sell All Model

The difference here is that a maximal volume of trade is forced on the model. All goods must be sold. Thus unlike in the hold back model trade in goods is active. In particular the volume of trade at period t is given by:

*There are also many other N.E.'s for this problem including having the traders spend all.

1,100 units of good and $1{,}100\beta^{t-1}$ value in money.

In this model, gradually more and more of the outside money may wind up in hoard until it is to be paid back at the final settlement date.

3.4.3. The Noncooperative Game: Outside and Inside Banks

We must consider two cases: (1) the inside rate of interest equals zero, or (2) it is positive.

(a)-1. Hold Back Model: $\rho = 0$

Let ρ_t = the inside rate of interest at time t (for ease in this simple example we may set $\rho_t = \rho$ for all t).

If $\rho = 0$ then it is easy to check that the solution as in 3.4.2(a) with the modification that the amount spent on the purchase of shares of the inside bank is indeterminate. For example: let v^i = the amount of outside money spent on buying shares of the inside bank. Further let S = the total number of shares issued. For simplicity set $S = 1100$. Price of shares is denoted by p_s .

There is a solution* as follows:

(9) $$v^1 = 100\ , \quad v^2 = 1000 \quad \text{and} \quad p_s = 1$$

with all other information as in 3.4.2.(a).

(a)-2. Sell All Model: $\rho = 0$

This has a solution similar to 3.4.3.(a)-1 above with the difference that the inside bank will lend the individuals at least enough to finance trade each period. They could borrow more, hoard it and return it at some point.

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*There is a definitional problem when no bids are made for the inside bank shares. Is the offer withdrawn or is the issue given away? If the latter, then how?

It is important to note the difference between 3.4.3.(a)-2 and 3.4.2.(b). In the first as there is no inside bank the strategic choice for the disposal of funds during any period is to hoard or buy. In the second case the choice is to hoard, buy or deposit. The distinction between hoarding and depositing when $\rho = 0$ may be trivial, but it is there and in any dynamic model with time lags or frictions; this distinction is of importance.

(b)-1. Hold Back Model: $\rho > 0$

If we adopt the convention that bank capital earns a rate of interest then we have a solution with:

$$(10) \qquad u^1 = v^1 = 100 \ , \quad u^2 = v^2 = 1{,}000$$

and although futures prices are given by

$$1, \ \beta, \ \beta^2, \ \ldots, \ \beta^{k-1}$$

spot prices are now

$$(11) \qquad 1, \ \beta(1+\rho), \ \ldots, \ [\beta(1+\rho)]^{k-1}$$

where ρ is indeterminate.

(b)-2. Sell All Model: $\rho > 0$

It is at this point that we see a serious divergence from the general equilibrium results for a k period market. The need to sell all creates a need for financing the trade that is to take place. The presence of a positive rate of interest on inside money in effect creates a gap between the buying and selling prices of the same commodity caused by a lag in payments. We may write the payoff function to a trader of type

i as follows:

$$(12) \qquad U^i = \sum_{t=1}^{k} \beta^{t-1} \log x_t^i + \mu^i \min[0, \eta^i]$$

where η^i = net balances of a trader of type i at the start of period $k+1$:

$$(13) \qquad \eta^i = \sum_{t=1}^{k} p_t(a_t^i - x_t^i) - F_k^i + \frac{s^i}{S}\{\bar{F}_k - \sigma S\}$$

F_k^i = the float debt incurred by a trader of type i up to period k ;
$\bar{F}_k$ = the float debt for the economy as a whole.

An examination of (13) shows that in a nonatomic market the first and third terms must be equal to zero in equilibrium hence if $F_k^i > 0$ (which will be the case for any model with active trade and a lag in the receipt of income) $\eta^i < 0$ and even though a C.E. distribution of resources is achieved all traders will go bankrupt; each by F_k^i .

For this simple example we may note that the following supplies a solution*

$$(14) \qquad F_k^i = a_t^i \left(\frac{1 - \beta^k}{1 - \beta}\right)\left(\frac{\rho}{1+\rho}\right) \quad \text{or}$$

$$F_k^1 = 100\left(\frac{1 - \beta^k}{1 - \beta}\right)\left(\frac{\rho}{1+\rho}\right) \quad \text{and} \quad F_k^2 = 10F_k^1 \ .$$

$$(15) \qquad \bar{F}_k = 1100\left(\frac{1 - \beta^k}{1 - \beta}\right)\left(\frac{\rho}{1+\rho}\right)$$

*We may wish to consider k borrowing periods including financing from 1 to 2 up to k to $k+1$ where trade ends in period k .

From the third term in (13) we obtain the equation in equilibrium that

$$\sigma = \left(\frac{1 - \beta^{k}}{1 - \beta}\right)\left(\frac{\rho}{1+\rho}\right) \tag{16}$$

This solution has the feature that $\rho > 0$ is a parameter. However once ρ is given the outside money rate of interest is determined (or vice-versa).

In Section 4 we justify being able to consider certain infinite horizon problems. Here we note that if this model can be solved for the infinite horizon then (16) becomes:

$$\sigma_{\infty} = \frac{1}{(1-\beta)}\,\frac{\rho}{(1+\rho)} \tag{17}$$

If we were to select $1+\rho = 1/\beta$ then

$$\sigma_{\infty} = 1 \quad \text{and spot prices become:} \tag{18}$$

$$\begin{matrix} 1, & 1, & 1, & 1, \ldots & \text{in contrast with futures prices of} \\ 1, & \beta, & \beta^2, & \beta^3, \ldots & \end{matrix} \tag{19}$$

We note that this selection of an inside money rate of interest yields stationary spot prices as well as stationary trade. However all traders go bankrupt at the end of time! Or more precisely traders go bankrupt for any finite horizon.

3.5. On Bankruptcy

The bankruptcy of all traders at period $k+1$ appears to be counter-intuitive but on closer inspection of the nonatomic game it is what we should expect.

In a model without exogenous uncertainty we would hope to show that bankruptcy should not occur under an optimal allocation. In Section 4 below we show that this is the case. But in order to establish the appropriate conditions it is necessary to introduce production and capital goods. When this is done we find that in the $k+1^{st}$ period we are left with an indebtedness which has come about through financing the float which can be offset by credits given for the salvage value of the capital goods left over at period $k+1$. This possibility for avoiding bankruptcy will not be unique but is tied in with attaching prices to the salvage values. In Section 4 we note the connection between determining the salvage values endogenously and searching for an infinite horizon stationary state.

3.6. Credit Only

There is an alternative model we can formulate which is more akin to the models employed by Postlethwaite and Schmeidler [21] and Dubey and Shapley [14] than those in 3.4. We dispense with the issue of fiat money and instead imagine that a bank offers all unlimited drawing rights each period subject only to penalty for default at a preannounced penalty level. It also stands willing to accept deposits at the same rate as loans.

If the shares of the bank are held by the government or the "referee" there must be a credit drain to the bank if $\rho > 0$ and it can be seen that the results of 3.4.3 hold.

If the shares of the bank were distributed in some fashion to the traders then all float payments would be internalized and the device of introducing a salvage value for capital stock is not necessary for balancing the books. It can be shown that there is a Pareto optimal solution without bankruptcy for every $\rho > 0$ for this model.

4. PRODUCTION AND EXCHANGE: THE FINITE HORIZON

4.1. The Float-Constrained Competitive Equilibrium with Salvage Values

We will establish the existence of a "float-constrained" competitive equilibrium (F.C.C.E.) for an economy with a finite number of consumers and firms. It seems to us that this can be shown in the nonatomic case as well, by a modification of Aumann's proof of the existence of a C.E. in [9], but we postpone a rigorous demonstration of this to a future paper. However note that our proof here for the finite economy also immediately implies the existence of a F.C.C.E. for a *finite-type* nonatomic economy.

Let $I = \{1, \ldots, n\}$ be the set of consumers, $J = \{n+1, \ldots, n+l\}$ the set of firms. The symbols i and j will be reserved exclusively to denote elements of I and J. The firms are completely consumer-owned, and $\theta^i \in \Omega^l$ is the share vector of i, where θ^i_j denotes the share of i in firm j. We require, then, that $\sum_{i \in I} \theta^i_j = 1$. There are m consumer goods and g producer goods, and the economy runs through $k+1$ periods. The utility function U^i of $i \in I$ maps Ω^{mk} to Ω^1. Firms are endowed with initial endowments $a^j \in \Omega^{(m+g)k}$, and with production sets $Y^j \subset R^{(m+g)(k+1)}$. Each $y^j \in Y^j$ denotes a possible production, with the negative (positive) components of y^j standing for inputs (outputs). Our assumptions on the Y^j are as follows:

(A) Y^j is closed, convex, and* $Y^j \cap \Omega = \{0\}$

(B) Inputs in period t yield outputs in period $t+1$

(C) Y^j permits a "constrained-inventorying"** i.e., there exists a $K > 0$ such that: for any $a \in \Omega^{(m+g)1}$ there is a

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* Ω is an abbreviation for $\Omega^{(m+g)(k+1)}$.

**Note that inventorying is a special form of production.

$$y^j_{k+1} \geq Ka \quad \text{with} \quad (-a,\ 0_{(k-1)m+g},\ y^j_{k+1}) \in Y^j$$

(D) $Y^j \supset -\Omega$, for each $j \in J$.

Besides, we will assume that $\theta^i > 0$ for each i , and that $\sum_{j\in J} a^j > 0$.

The state of the economy $\{(x^i)_{i\in I},\ (y^j)_{j\in J}\}$ is called *attainable* if:

(a) $x^i \in \Omega^{mk}$, $y^j \in a^j + Y^j$

(b) $\sum_{i\in I} x^i \times 0_{kg} \times 0_{m+g} - \sum_{j\in J} y^j = 0$ for all components except perhaps the last $m+g$.

The set of attainable states of the economy is a subset A of $R^{(n+1)(m+g)(k+1)}$, and by (2) in Section 5.4 of [20], A is compact. Let $G > max\{\|x^i\|,\ \|y^j\| : x^i$ or y^j occurs in an attainable state of the economy$\}$. The positive intertemporal rates of interest are given as before by $\sigma_1, \ldots, \sigma_k$. We will find it convenient to split the prices into two parts: the spot prices p in periods 1 through k , $p \in \Omega^{(m+g)k}$, and the salvage price π in period $k+1$, $\pi \in \Omega^{m+g}$. Introduce also the mapping $p \leftrightarrow \underset{\sim}{p}$ given by: $\underset{\sim}{p}_t = \xi_t p_t$ for $t = 1, \ldots, k$. Let P denote the simplex in Ω defined* by $P = \{(\underset{\sim}{p}, \pi) : \sum_{t,h} p_{th} + \sum_h \pi_h = 1\}$.

Let C^Q_q be the cube of size Q in R^q , for any $Q \geq 0$, and any integer q , i.e., $C^Q_q = \{x \in R^q : \|x\| < Q\}$. Define

$$\hat{X}^i = \Omega^{mk} \cap C^G_{mk}$$

$$\hat{Y}^j = (a^j + Y^j) \cap C^G_{(m+g)(k+1)}$$

The $\hat{X}^i$ and $\hat{Y}^j$ are clearly compact and convex, and

---

* $\sum_{t,h} \equiv \sum_{t=1}^{k} \sum_{h=1}^{m}$.

$$A \subset \left(\prod_{i=1}^{n} \hat{X}^i \times \{0_{kg+m+g}\} \right) \times \prod_{j=n+1}^{n+l} \hat{Y}^j$$

Given prices (p, π), the production y^j yields to firm j the following revenue:

$$R^j((p,\pi), y^j) = \sum_{t,h} \xi'_{th} p_{th} \max\{0, y^j_{th}\} + \sum_{t,h} \xi_{th} p_{th} \min\{0, y^j_{th}\} + \pi \cdot y^j_{k+1}$$

$$= \sum_{t,h} \xi_{th} p_{th} y^j_{th} - \sum_{t,h} \xi''_{th} p_{th} \max\{0, y^j_{th}\} + \pi \cdot y^j_{k+1}$$

Clearly $R^j((p,\pi), y^j)$ is a concave function of y^j for a fixed (p,π). Let* $\mathbf{q}^j(p,\pi) = \{{}^+y^j \in \hat{Y}^j : R^j((p,\pi), {}^+y^j) = \max_{y^j \in \hat{Y}^j} R^j((p,\pi), y^j)\}$ and denote the value of $R^j((p,\pi), {}^+y^j)$ for ${}^+y^j \in \mathbf{q}^j(p,\pi)$ by $R^j(p,\pi)$. Clearly $\mathbf{q}^j(p,\pi)$ is compact, convex and nonempty and is u.s.c. in $(p,\pi) \in P$. Also $R^j(p,\pi)$ is continuous in its variables. Next, define, for $(p,\pi) \in P$,

$$B^i(p,\pi) = \{x^i \in \hat{X}^i : p \cdot x^i \leq \sum_{j \in J} \theta^i_j R^j(p,\pi)$$

and

$${}_*\gamma^i(p,\pi) = \{{}^+x^i \in B^i(p,\pi) : U^i({}^+x^i) = \max_{x^i \in B^i(p,\pi)} U^i(x^i)\}$$

Since B^i is continuous** in its variables, and U^i is continuous on $\hat{X}^i$, ${}_*\gamma^i$ is u.s.c. in its variables. ${}_*\gamma^i$ is also clearly compact, convex, and nonempty.

---

*For the etymologists: since we have run out of Latin and Greek alphabets, we have had to take recourse to Hindi ($\mathbf{q}$ is pronounced "wo" as in "wonder").

**Note that, on account of (C) and the fact that $\theta^i > 0$ and $\sum_{j \in J} a^j > 0$, we have $B^i(p,\pi) = \hat{X}$ whenever $\|p\|$ is sufficiently small.

We digress from our development to make the

Definition. An F.C.C.E. of the economy is a price vector (p,π), a set of consumptions, $(x^i)_{i\in I}$, and productions $(y^j)_{j\in J}$ such that:

(i) $\left(\sum_{i\in I} x^i\right) \times 0_{kg+m+g} - \sum_{j\in J} y^j = 0$ except possibly for the last $m+g$ components.

(ii) $x^i \in B^i(p,\pi)$ and maximizes U^i on $B^i(p,\pi)$.

(iii) $y^j \in \alpha^j + Y^j$ and maximizes $R^j((p,\pi), y)$ for $y \in \alpha^j + Y^j$.

Our aim in this section is to demonstrate the existence of a F.C.C.E. To this end, we return to our development.

For any $(p,\pi) \in P$ and $y^j \in \hat{Y}^j$, $j \in J$, put

$$H^i((p,\pi), y^{n+1}, \ldots, y^{n+l})) = \sum_{j\in J} \theta^i_j(p,\pi)\cdot y^j - \sum_{j\in J} \beta^i_j R^j((p,\pi), y^j) .$$

Note that $H^i \geq 0$ and continuous in its variables. Take an $e \in \Omega^{m+g}$, $e > 0$, $\|e\| = 1$, and keep it fixed. For $\pi \neq 0$, put

$$K^i((p,\pi), y^{n+1}, \ldots, y^{n+l}) = \frac{H^i((p,\pi), y^{n+1}, \ldots, y^{n+l})}{\pi\cdot e} .$$

Define the set-valued mapping ζ^i by:

$$\zeta^i((p,\pi), y^{n+1}, \ldots, y^{n+l}) = \begin{cases} \{K^i((p,\pi), y^{n+1}, \ldots, y^{n+l})e\} & \text{if } \pi \neq 0 \text{ and } K^i \leq lG \\ \{lGe\} & \text{if } \pi \neq 0 \text{ and } K^i > lG \\ \{te : 0 \leq t \leq lG\} & \text{if } \pi = 0 \text{ and } H^i = 0 \\ \{lGe\} & \text{if } \pi = 0 \text{ and } H^i > 0 . \end{cases}$$

It can be checked that (i) ζ^i is u.s.c.; (ii) for any

$z \in \zeta^i((p,\pi), y^{n+1}, \ldots, y^{n+l})$, $\pi \cdot z \leq H^i((p,\pi), y^{n+1}, \ldots, y^{n+l})$; and (iii) ζ^i C^{lG}_{m+g} . Define $\gamma^i((p,\pi), y^1, \ldots, y^{n+l})$

$$_*\gamma^i(p,\pi) \times \{O_{kg}\} \times \{\zeta^i((p,\pi), y^1, \ldots, y^{n+l})\}$$

Note that for any (p,π) in P and y^j in $\hat{Y}^j$,

$$\gamma^i((p,\pi), y^1, \ldots, y^{n+l}) \subset \hat{X}^i \times \{O_{kg}\} \times C^{lG}_{m+g} ,$$

and that γ^i is convex, compact, nonempty and u.s.c. in its variables. Finally, for any $\hat{x}^i \in \hat{X}^i \times \{O_{kg}\} \times C^{lG}_{m+g}$ $(i \in I)$, and $y^j \in \hat{Y}^j$ $(j \in J)$, define

$$\theta(\hat{x}^1, \ldots, \hat{x}^n, y^{n+1}, \ldots, y^{n+l}) = \Big[(p',\pi') \in P :$$

$$(p',\pi') \cdot \Big[\sum_{i \in I} \hat{x}^i - \sum_{j \in J} y^j\Big]$$

$$= \max_{(p,\pi) \in P} \{(p,\pi) \cdot \Big[\sum_{i \in I} \hat{x}^i - \sum_{j \in J} y^j\Big]\}\Big]$$

Now for any $(p,\pi) \in P$, $\hat{x}^i \in \hat{X}^i \times \{O_{kg}\} \times C^{lG}_{m+g}$, $y^j \in \hat{Y}^j$ let

$$\Phi((p,\pi), \hat{x}^1, \ldots, \hat{x}^n, y^{n+1}, \ldots, y^{n+l})$$

$$= \theta(\hat{x}^1, \ldots, \hat{x}^n, y^{n+1}, \ldots, y^{n+l}) \times \Big[\prod_{i=1}^{n} \gamma^i((p,\pi), y^1, \ldots, y^j)\Big]$$

$$\times \Big[\prod_{j=n+1}^{n+l} \mathbf{d}^j(p,\pi)\Big] .$$

Φ is then a nonempty, compact, convex-valued, u.s.c. correspondence from

$$S = P \times [\prod_{i=1}^{n} (\hat{X}^i \times \{O_{kg}\} \times C_{m+g}^{lG})] \times [\prod_{j=n+1}^{n+l} \hat{Y}^j]$$

into itself.

Hence by Kakutani's theorem there is a point

$${}^*s = [({}^*\underset{\sim}{p}, {}^*\pi), {}^*\hat{x}^1, \ldots, {}^*\hat{x}^n, {}^*y^{n+1}, \ldots, {}^*y^{n+l}]$$

in S such that ${}^*s \in \phi({}^*s)$. Consider

$$z = \sum_{i \in I} {}^*\hat{x}^i - \sum_{j \in J} {}^*y^j$$

Now

$$({}^*\underset{\sim}{p}, {}^*\pi) \cdot {}^*\hat{x}^i \leq \sum_{j \in J} \theta_j^i ({}^*\underset{\sim}{p}, {}^*\pi) \cdot {}^*y^j$$

Summing over $i \in I$, we obtain

$$({}^*\underset{\sim}{p}, {}^*\pi) \cdot \sum_{i \in I} {}^*\hat{x}^i \leq ({}^*\underset{\sim}{p}, {}^*\pi) \cdot \sum_{j \in J} {}^*y^j$$

i.e.

$$({}^*\underset{\sim}{p}, {}^*\pi) \cdot z \leq 0 .$$

This shows that $z \leq 0$. For suppose some component, w.l.o.g. say component 1, of z is positive. Then take $(\underset{\sim}{p}', \pi')$ to be $(1, 0, \ldots, 0)$, to get $(\underset{\sim}{p}', \pi') \cdot z > 0$, which contradicts that $({}^*\underset{\sim}{p}, {}^*\pi) \in \theta({}^*\hat{x}^1, \ldots, {}^*\hat{x}^n, {}^*y^{n+1}, \ldots, {}^*y^{n+l})$.

Let ${}^*y$ denote $\sum_{j \in J} {}^*y^j$. Then ${}^*y \in Y$, and since z is non-positive, we have, by our conditions on Y [see (D)] that ${}^*y + z \in Y$. Hence there is, for each j in J , a ${}^*\hat{y}^j \in \alpha^j + Y^j$ such that

$$\sum_{j \in J} {}_{*}\hat{y}^{j} = {}^{*}y + z \ .$$

Then

$$\sum_{i \in I} {}^{*}x^{i} - \sum_{j \in J} {}_{*}y^{j} = 0 \ .$$

Thus* $\{({}^{*}x^{i}), ({}_{*}\hat{y}^{j})\} \in A$. Therefore each ${}^{*}x^{i}$ $({}_{*}\hat{y}^{j})$ is in the *interior* of the cube C^{G}_{mk} $(C^{G}_{(m+g)(k+1)})$. Next we claim that

$$({}^{*}\underset{\sim}{p},\ {}^{*}\pi)\cdot z = 0 \ .$$

To show this, we must establish that

$$\pi \cdot {}^{*}\underset{\sim}{x}^{i} = H^{i}(({}^{*}\underset{\sim}{p},\ {}^{*}\pi),\ {}^{*}y^{n+1},\ \ldots,\ {}^{*}y^{n+l})$$

for every $i \in I$. If this is not true then ${}^{*}\underset{\sim}{x}^{i} = lGe$ for some i , contradicting that $z \leq 0$ (since $\| \sum_{j \in J} {}^{*}y^{j} \| < lG$), which proves the claim. But $(\underset{\sim}{p}, \pi)\cdot z = 0$ and $z \leq 0$. Hence $z_{th} < 0$ implies that p_{th} (or π_{th} , if $t = k+1$) $= 0$. By our assumption (D) on the Y^{j} , we can find a ${}_{*}y^{j}$ in $\alpha^{j} + Y^{j}$ for each j in J such that: $\sum_{j \in J} {}_{*}y^{j} = \sum_{j \in J} {}^{*}y^{j} + z$, and ${}_{*}y^{j}_{th}$ differs from ${}^{*}y^{j}_{th}$ only if $z_{th} < 0$. Thus $(\underset{\sim}{p}, \pi)\cdot {}_{*}y^{j} = (\underset{\sim}{p}, \pi)\cdot {}^{*}y^{j}$, and $R^{j}((\underset{\sim}{p}, \pi), {}_{*}y^{j}) = R^{j}((\underset{\sim}{p}, \pi), {}^{*}y^{j})$ for each j . Also note that $\sum_{i \in I} {}^{*}\hat{x}^{i} - \sum_{j \in J} {}_{*}y^{j} = 0$, hence each ${}_{*}y^{j}$ is in the *interior* of $C^{G}_{(m+g)(k+1)}$.

---

* ${}^{*}x^{i}$ denotes the appropriate components of ${}^{*}\hat{x}^{i}$, i.e., ${}^{*}\hat{x}^{i} = {}^{*}x^{i} \times O_{gk} \times {}^{*}\underset{\sim}{x}^{i}$. Observe that ${}^{*}\underset{\sim}{x}^{i} \in \zeta^{i}(({}^{*}\underset{\sim}{p},\ {}^{*}\pi),\ {}^{*}y^{n+1},\ \ldots,\ {}^{*}y^{n+l})$; and thus ${}^{*}\pi \cdot {}^{*}\underset{\sim}{x}^{i} \leq H^{i}(({}^{*}\underset{\sim}{p},\ {}^{*}\pi),\ {}^{*}y^{n+1},\ \ldots,\ {}^{*}y^{n+l})$.

We now set out to establish that $\{({}^*x^i), ({}_*y^j), ({}^*\underset{\sim}{p}, {}^*\pi)\}$ is a F.C.C.E.

Properties of ${}^*x^i$. We know that ${}^*x^i$ maximizes U^i on the set $B^i({}^*\underset{\sim}{p}, {}^*\pi) = \{x^i \in \Omega^{mk} \cap C^G_{mk} : {}^*p \cdot x^i \leq \sum_{j \in J} \beta^i_j R^j({}^*\underset{\sim}{p}, {}^*\pi)\}$. We wish to show that in fact ${}^*x^i$ maximizes U^i on the set $B^i = \{x^i \in \Omega^{mk} : {}^*p \cdot x^i \leq \sum_{j \in J} \beta^i_j R^j({}^*\underset{\sim}{p}, {}^*\pi)\}$. Suppose not. Let $\underset{\sim}{x}^i$ be optimal in B^i. Define ${}_t x^i = (1-t)\underset{\sim}{x}^i + t\,{}^*x^i$ for $t \in (0,1)$. For each such t, $U^i({}_t x^i) > U^i({}^*x^i)$. But for t close enough to 0, ${}_t x^i$ would be in C^G_{mk} since ${}^*x^i$ is in the interior of C^G_{mk}, hence also in $B^i({}^*\underset{\sim}{p}, {}^*\pi)$, a contradiction.

Properties of ${}_*y^j$. Each ${}_*y^j$ maximizes profit on $\hat{Y}^j$. By an argument similar to the one used above for ${}^*x^i$, we can establish that in fact each ${}_*y^j$ maximizes profit on $\alpha^j + Y^j$ [recall that ${}_*y^j$ is in the interior of $C^G_{(m+g)(k+1)}$ for each j].

This shows that $[({}^*x^i)_{i \in I}, ({}_*y^j)_{j \in J}, ({}^*\underset{\sim}{p}, {}^*\pi)]$ is a F.C.C.E. We have proved*

Theorem 2. Under our assumptions, an F.C.C.E. always exists.

Remarks. (1) At an F.C.C.E.

$$(*) \qquad {}^*\pi \cdot \Big(\sum_{j \in J} {}_*y^j_{k+1}\Big) = \sum_{i \in I} H^i(({}^*\underset{\sim}{p}, {}^*\pi), {}_*y^{n+1}, \ldots, {}_*y^{n+l})$$

i.e. the salvage value of the left-over goods is equal to the total **float.**

*Our proof follows the outline of the proof of the existence of a C.E. in Chapter 5 of [20], but with enough variations to warrant having written it in detail. Ward Whitt is responsible for any errors. We are greatly indebted to him for a detailed midnight check of the proof.

(2) It is easily verified that there is no F.C.C.E. with $\rho > 0$ and ${}^{*}\pi = 0$. If $\rho = 0$, any F.C.C.E. has ${}^{*}\pi = 0$ and coincides with a C.E., and conversely.

(3) Note that the e in our proof was arbitrary. Thus an F.C.C.E. exists which has the left-over vector of $m+g$ goods in period $k+1$ in any proportions that we may specify.

To describe the nonatomic version of the production and exchange economy let $\{I \cup J, \mathcal{C}, \mu\}$ be the nonatomic measure space of agents, where $I \cap J = \phi$, $\mu(I) > 0$, $\mu(J) > 0$. We will use exactly the same notation as in the previous section to denote the characteristics of the agents, but with the added stipulation that they vary measurably. Thus the production sets of the firms is given by a measurable mapping Y from J to subsets of $R^{(m+g)(k+1)}$, the shares of consumers by a measurable function $\theta : I \times J \to \Omega^1$ where $\theta(i,j) \equiv \theta^i_j$, etc. The symbols $\beta^i(p,\pi)$, $R^j(p,\pi)$, $R^j((p,\pi), y)$ carry exactly the same meaning as before. An F.C.C.E is then defined as in 4.1 with " Σ " replaced by " $\int$ ".

At an F.C.C.E. $\{(x^i)_{i\in I}, (y^j)_{j\in J}, (p,\pi)\}$ there are shadow prices λ^i for each $i \in I$ such that

$$\max_{y\in\Omega^{mk}} \{U^i(y) + \lambda^i[-p\cdot y + \int_J \theta^i_j R^j(p,\pi)d\mu]\}$$

is attained for $y = x^i$. Moreover, as shown in the appendix of [5], the λ^i may be picked measurably.

4.2. The Game with Production

We now proceed to construct the nonatomic game in strategic form for the above model. The game will depend upon bankruptcy penalties $\lambda : I \to \Omega^1$, and the salvage prices $\pi \in \Omega^{m+g}$, hence we will denote it by $\Gamma_{\lambda,\pi}$. (M and ρ are fixed.)

The strategy set of $i \in I$ is, as in 3.3., the vector (u^i, b^i) with $u^i \in \Omega^1$, $b^i \in \Omega^{mk}$. The description of a strategy by firm j is rather more complicated. First, the firm must decide how much to borrow from the inside bank to bid on the $m+g$ trading posts in each period before the market functions. This we denote simply by a vector $b^j \in \Omega^{(m+g)k}$ (ruling out hoarding at N.E.'s as we did for the consumers). But in addition, a strategy of firm j, also includes a *plan* for production which tells him what to do for every possible situation that might arise in the course of the economy. We can best represent this by a set of functions $F^j = (f_1^j, \ldots, f_k^j)$ where

$$f_t^j : \Omega^{m+g} \to \Omega^{m+g} \times \Omega^{m+g}$$

and, letting $f_t^j(y) = (y', y'')$, where $y' \in (0_{(m+g)(t-1)}, y, 0_{(k-t)(m+g)}) + Y^j \cap \Omega^{m+g}$, $y'' \leq y'$. Here y' is the output in period $t+1$ given the inputs y in period t, and y'' is the portion of the output that is held back by j (the remainder $y' - y''$ being put up for sale). Thus *a* strategy of firm j is the tuple $(b^j, f_1^j, \ldots, f_k^j)$. Once all the firms and consumers announce their strategies, the market mechanism functions exactly as is described in Section 3.3, with the obvious changes. See Figure 5. The revenue earned by the firm is determined as on page 37. To describe the payoff to consumer i, however, we need to determine

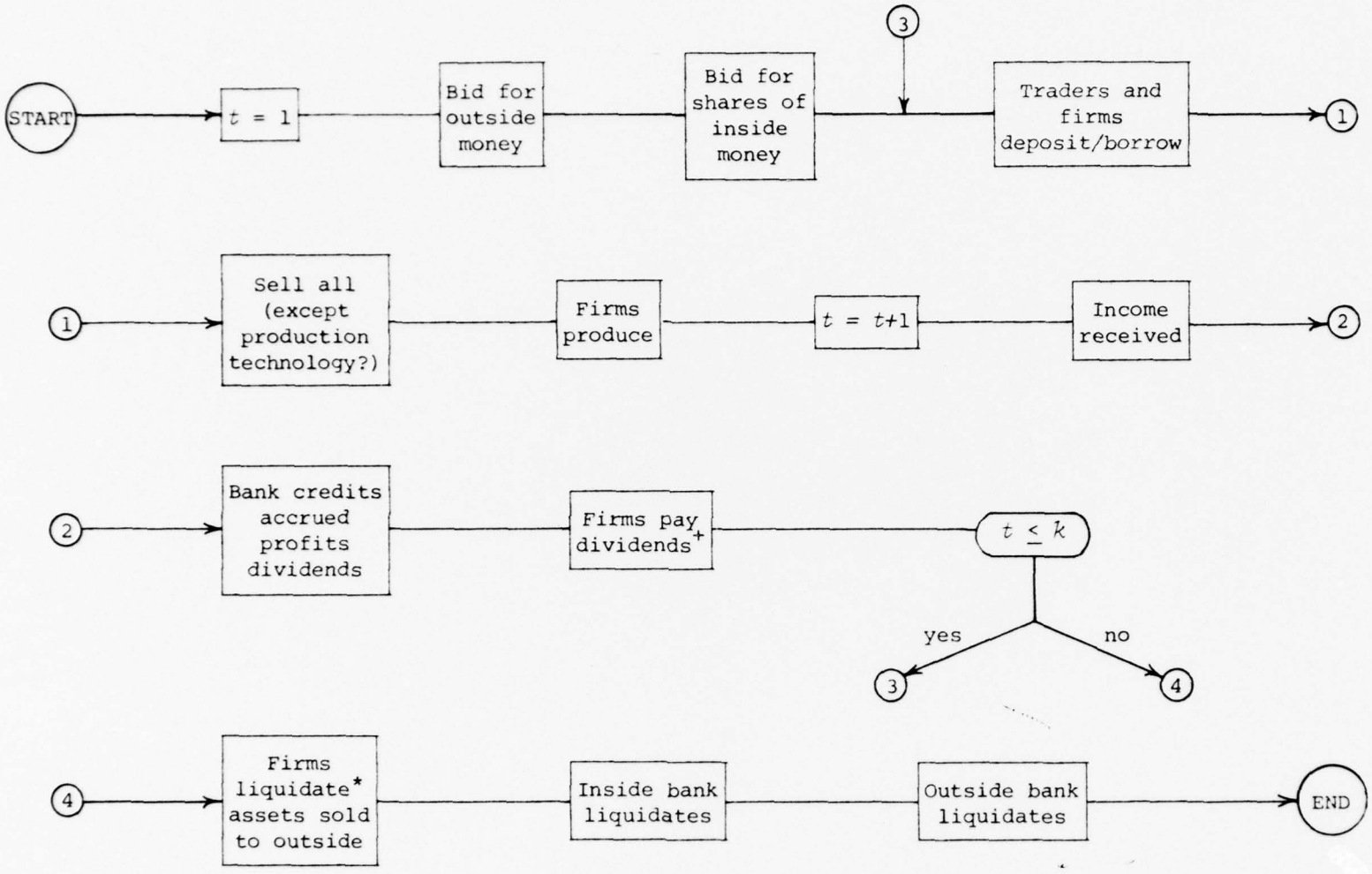


FIGURE 5

*Any debts are flowed through to stockholders.
+Easier to pay at end.

the bankruptcy penalty levied on him. For this purpose note that i's net credit at the end is

$$\eta^i = -\underset{\sim}{p}\cdot x^i + \int_J \theta^i_j R^j((\underset{\sim}{p},\pi), F^j)$$

(Note that $\underset{\sim}{p}$ depends upon the collection of all the consumers' and firms' strategies.) The payoff to i in the game is then defined to be $U^i(x^i) + \lambda^i \min[0, \eta^i]$. We wish to study the N.E.'s of $\Gamma_{\lambda,\pi}$ as (λ,π) varies. Define $\Lambda^F_C = \{\lambda \in \Lambda : \lambda$ is a choice of shadow prices at some F.C.C.E. of the market$\}$. For any $\lambda \in \Lambda^F_C$, let $\Pi_\lambda = \{\pi : \pi$ occurs as a salvage price at some F.C.C.E. with shadow prices $\lambda\}$. Our main result is

Theorem (3a). Suppose $\lambda \in \Lambda_{\lambda'}$, $\lambda' \in \Lambda^F_C$, and $\pi \in \overline{\Pi}_{\lambda'}$. Then for any F.C.C.E. associated with λ, there is an active N.E. of $\Gamma_{\lambda,\pi}$ which coincides with the F.C.C.E. *No consumer goes bankrupt at this N.E.*

Theorem (3b). Suppose there is an active N.E. of $\Gamma_{\lambda,\pi}$ at which no consumers go bankrupt. Then $\lambda \in \Lambda_{\lambda'}$ for some $\lambda' \in \Lambda_C$, $\pi \in \overline{\Pi}_{\lambda'}$, and the N.E. coincides with an F.C.C.E. associated with λ'.

The proof of (3b) is obvious, requiring only minor changes in the proof of Proposition 2 in [5].

Proof of Theorem (3a). Construct s^i and s^j as follows (where $\{(\underset{\sim}{p},\pi), (x^i)_{i\in I}, (y^j)_{j\in J}\}$ is the F.C.C.E.):

$$u^i = \left[\left(1 + \int_J \frac{\{p\cdot x^i - R^i(p,\pi)\}}{M}\right)\right]\cdot\frac{M}{\mu(I)}$$

$$b^i_{th} = p_{th}x^i_{th}$$

$$b^j_{th} = p_{th}\ min[0,\ y^j_{th}]$$

$$f^t(y) = \begin{cases} (y',\ y'') \text{ where } y' \text{ and } y'' \text{ are the produced and} \\ \text{held-back outputs obtained from } y^j\text{ , if} \\ y_h = min[0,\ y_{t-1,h}] \\ (0,0) \quad \text{otherwise} \end{cases}$$

It is clear that this choice of strategies produces the prices p, the consumptions $(x^i)_{i\in I}$, and the production $(y^j)_{j\in J}$. Also clearly $\eta^i = 0$ for all $i \in I$. That these strategies comprise an N.E. may be verified exactly as in the proof of Proposition 3 in [5].

Q.E.D.

For $K > 0$, $[(p,\pi),\ \lambda]$ is associated with an F.C.C.E. if, and only if, $[(Kp,\pi),\ \frac{1}{K}\lambda]$ is. Suppose we restrain π to satisfy $\bar{\pi} = 1$. Then this degree of freedom is lost, and in the λ-space, the set Λ^F_C will no longer be a union of 1-dimensional sets. We draw a picture again for an economy with two types of traders (compare with Figures 3 and 4).

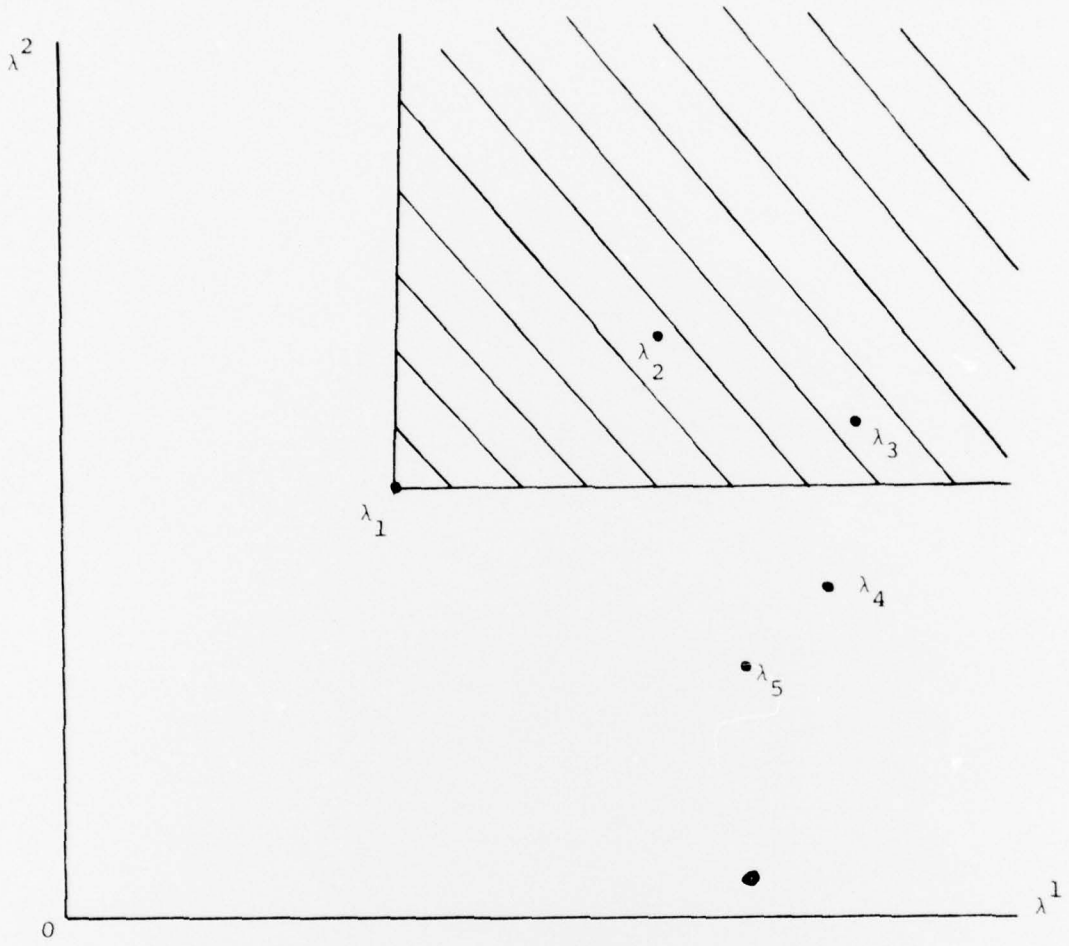
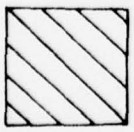


FIGURE 6

Λ_C^F consists of the points $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$

$\Lambda_{\lambda_1} =$ 

Remarks. In the game λ , π , ρ and M are specified exogenously [where ρ is the set $\{\rho_1, \ldots, \rho_k\}$]. For the game to have an N.E. without bankruptcy, there must be related by (*) and (**):

(#) $$\boxed{\sigma M = \Sigma\, H^i(p, \pi, \ldots) = \pi \cdot y \ .}$$

σ , p , π are linked also to λ as the proof shows even though this is not implicit in (#). We can think of λ , π , ρ and M as the control parameters that are specified by a government or outside agency. The choice of these parameters determines how production and exchange takes place, and in particular determines the vector of left-over goods. We may imagine that in period k+1 the outside agency turns over this vector to the successive generation of agents. Thus the agency can, by its choice* of λ , ρ , and M , guide the initial capital stock that is made available to each generation. The money $\pi \cdot y$ that it introduces at the end of each generational cycle as salvage value is precisely the float $\Sigma\, H^i$ which is recouped as the extra amount σM that goes into the outside bank.

This rivets our attention on the possibility of an invariant initial stock that is made available to each generation. We will show in [19] that, supposing that the generations have identical size, tastes and technology, there exists an initial stock which can remain invariant for each generation if λ , ρ and M are selected appropriately.

---

* π is indirectly determined by these according to (#).

*4.3. A Simple Example with Production**

We modify the examples presented in 3.3 by introducing a production technology. In particular we consider one type of firm which takes as input an amount y_j of the single consumer good and transmutes it into an output of z_{t+1} where:

(1) $$z_{t+1} = \alpha\sqrt{y_t} \qquad \alpha > 1 .$$

Let x_t^i be the amount consumed by a trader of type i

$$x_t = x_t^1 + x_t^2 .$$

Let y_t be the amount used by a typical firm. Then:

(2) $$z_t = x_t + y_t .$$

Let a trader of type i own θ^i% of the shares of the different firms.

As we are only looking closely at a position of equilibrium assuming symmetric treatment of identical traders or firms we dispense with using the extra identification needed to distinguish a specific trader of type i or a specific firm. Concerning holding of shares we assume that traders of the same type hold the same "market profile" of firms of the same type.

The easiest model to consider is where the only endowment of a trader of type i is his supply of shares; i holds $(0, 0, 0, 0, \theta^i)$ to start with where the vector measures:

---

*For this example no attempt is made to use the same notion as has been used for the general theorems. Furthermore the equation numbering refers only to this section.

(consumer good; outside money, bank shares, inside money, firm shares).

The firms are assumed to be run by fiduciorus for the stockholders who each attempt to maximize profits for their stockholders. This may also be interpreted as maximizing the present value of net worth.

Each firm owns its production function (which may be interpreted as comprising the special worth of untraded or unaccounted for inputs as well as unimputed organizational structure.

At the start each firm has $(a, 0, 0, 0, 0)$. We assume, for the sake of simplicity that beyond its production function each firm owns the supply of all resources available. It neither owns nor bids on outside money and on the shares of the inside bank or upon shares of its own or of other corporations. Thus we may keep track of the firm via two numbers (g_t, w_t) its current supply of consumer good and current wealth net of its inventories, where g_t is its inventories and w_t is wealth.

Let π^*_{k+1} = the futures salvage price per unit of the consumer-producer good remaining at period $k+1$ after all trade has ceased.

4.3.1. The Modified C.E.

A typical firm attempts to maximize:

$$(3) \qquad R = p^* \cdot a + \sum_{t=2}^{k} (p^*_t z_t - p^*_{t-1} y_{t-1}) - \hat{F}_k + \pi^*_{k+1} z_{k+1}$$

subject to $z_t = \alpha\sqrt{y_{t-1}}$;

and its cash flow constraints. Where $\hat{F}_k$ is the cost of the float to the firm and the p^* are futures prices.

A consumer of type i attempts to maximize

$$(4) \qquad \sum_{t=1}^{k} \beta^{t-1} \log x_t^i$$

subject to:

$$(5) \qquad (\theta^i R - \sum_{t=1}^{k} p_t^* x_t^i - F_k^i) = 0 .$$

In this simple example if the firm does not have to sell all of its output it will incur no float cost as it will finance its production from inventories. This is not true if we assumed "sell all" or if production involved firms buying from firms.

(a) The hold back model

In particular let us consider the case where $\beta = 1/2$, $\alpha = 76.594$ then there will be a stationary solution as follows:

$$M = 1{,}100 , \quad \theta^1 = 1/11 , \quad \theta^2 = 10/11 , \quad a = 1{,}100$$

$$z = 1{,}466\tfrac{2}{3} , \quad x = 1{,}100 , \quad y = 366\tfrac{2}{3}$$

$$\hat{F}_k = 0 , \quad p_t^* = \beta^{t-1} \quad \text{for} \quad t = 1, \ldots, k , \quad \pi_{k+1} = \frac{3\beta^k}{1-\beta}$$

$$(6) \qquad R = 1{,}100 \left(\frac{1 - \beta^k}{1 - \beta} \right) + \frac{1{,}100}{1-\beta} \beta^k$$

A consumer of type 1 obtains $R/11$ and of type 2 obtains $10R/11$.

Thus for a consumer of type 1 we have that his budget constraint gives:

(7) $$\frac{100\rho}{1+\rho}\left\{\frac{1-\beta^k}{1-\beta}\right\} = \frac{100}{1-\beta}\beta^k$$

and

(8) $$\lambda^1 = \frac{1}{100}\ , \quad \lambda^2 = \frac{1}{1000}$$

or

(9) $$\frac{\rho}{1+\rho} = \frac{\beta^k}{(1-\beta^k)} \quad \text{or} \quad \rho = \frac{\beta^k}{1-2\beta^k}$$

for $\beta = 1/2$ and $k = 2$, $\rho = 1/2$ and as $k \to \infty$, $\rho \to 0$.

(b) The sell all model

This model has a float for the firm and the need for covering the short term financing will change the production policy of the firm. In particular the stationary state maximization for the firm will be given by calculating the stationary values of y that maximizes

(10) $$\frac{\beta\alpha\sqrt{y}}{1+\rho} - y$$

hence, in this case: $y = 1100/3(1+\rho)^2$, $z = 4400/3(1+\rho)^2$, and $x = 1100/(1+\rho)^2$ and $\pi = 3\ /(1-\beta)(1+\rho)^2$.

In this simple model we observe that for k periods

(11) $$\rho = \frac{\beta^k}{1-2\beta^k}$$

and as $k \to \infty$, $\rho \to 0$ and the sell all and hold back models give the same outcomes.

4.3.2. The Noncooperative Game

The Sell All Model

The maximization for the firms is as before in 4.3.1. However the consumer faces a more complicated payoff

$$U^i = \sum_{t=1}^{k} \beta^{t-1} \log x_t^i + \lambda^i \min[0, \eta^i] \tag{12}$$

where

$$\eta^i = \theta^i R - \sum_{t=1}^{k} p_t^* x_t^i - F_k^i + \frac{s^i}{S}\{\bar{F}_k - \sigma S\} + \pi_{k+1}^* z_{k+1} \tag{13}$$

where $\bar{F}_k$ is the total float, S the value of the total number of shares in the inside bank and σ the outside money rate of interest.

This gives us the same solution as in 4.3.1 with the additional condition that we have on equation linking inside and outside rates of interest. In particular

$$\bar{F}_k = \sigma S \ . \tag{14}$$

In this instance all outside money will be spent on bank shares hence $S = M$, or

$$\frac{\rho}{1+\rho} \frac{1100}{3(1+\rho)^2} \left(\frac{1 - \beta^k}{1 - \beta}\right) = \sigma M \ . \tag{15}$$

In this example $M = 1100$ hence

$$\sigma = \frac{1}{3(1+\rho)^2} \left\{\frac{1 - \beta^k}{1 - \beta}\right\}\left\{\frac{\rho}{1+\rho}\right\} \ .$$

5. COMMENTS AND OPEN PROBLEMS

5.1. The Infinite Horizon

In 4.3 we illustrated an example in which as the horizon k becomes longer the money rate of interest approaches zero as does the future price of the capital goods left over for salvage. This is a result which will hold for any model with a "natural time discount" of the type

$$U^i = \sum_{t=1}^{k} \beta^{t-1} \phi(x_t^i) .$$

Although utility functions with a time discount has been used in some growth models [15], Ramsey in his seminal paper [16] chose not to consider this restriction. This avoidance of an easy way of bounding an infinite series of payoffs does not prevent us from considering models with an indefinite horizon which can nevertheless be analyzed. In particular when we consider stationary states we can consider economies in which the payoff per period is bounded rather than the total payoff.

Once we try to consider models of economies with infinite horizons several basic problems in the formulation of growth models must be dealt with. They concern conservation of matter and energy; the meaning of free disposal and unappropriated resources and the possibility of modelling successive generations of individuals in a model of this type following through the type of modelling suggested by Samuelson in his "Pure Consumption Loan" model [17, 18]. In a subsequent paper on "Growth in a Finite Universe" [19] we argue that it is reasonable to model both production and consumption as activities, to introduce a device for at least symbolically appropriating *all* resources and to dispose with the free disposal assumption. Under these circumstances we conjecture, but have not yet proved rigorously, that it is possible to sustain a stationary

state with a positive money rate of interest over the indefinite horizon.

The problem we face in sustaining a positive money rate of interest appears to be intimately related with the mechanism whereby one generation passes resources on to the next, i.e., the "bequest" or "inheritance" problem.

A monetary system can be designed to be "cash consuming" so that at the end of k periods all are indebted to the banking system for the financing of the float. This debt can be removed by the *pro forma* transaction of equating it to the worth of the remaining capital stock. We might then imagine that if the economy continues to function beyond k periods (say with a new generation); the referee hands out the capital stock in some manner to the next generation. This convention is not as arbitrary or far fetched as it may seem at first sight. In particular in a centralized state, or in a state without exogenous uncertainty which feels that growth policy is a public not an individual decision, the proposition that an individual should die with a net worth of zero is reasonable.

If however, we require specific individual motivation to lead to the survival and growth of future generations we need the present generation to have both the desire and the resources to provide the capital stock for the next generation.

The conditions on motivation appear to call for an explicit inclusion among the preferences of the present generation, the desires to leave an inheritance.

In order to ensure that the present generation owns the resources it wants to pass on we might wish to offset the cash consumption that appeared as a property of the models in 3.4 and the first model in 3.6.

There are at least two ways to achieve this, the first is indicated in the second model of 3.6, i.e. unrestricted inside credit with bank shares held by the traders. The second way involves having both an outside and inside bank, as in 3.4. , but the cash flow drain to the banking system can be offset by the introduction of interest paying bonds which are held by the traders. These models are considered further in a subsequent paper [2].

5.2. *Institutions, Tastes and Technology*

We have taken institutions, tastes and technology as given and fixed in all of the models discussed here. The mechanisms of change for these items are essentially exogenous to those models. Furthermore it should be noted that because implicit in this formulation is the proposition that tastes, technology and other aspects of organization can neither be created nor destroyed, then there is no need to attach accounting values to them; as they merely represent a constant added at the start and subtracted at the end. Thus the books always can be made to balance. Looked at this way the value of a firm as an entity over and above its mere sum of assets is accounted for by a "goodwill" entry that remains constant over time.

5.3. *Concluding Remarks*

There are a host of problems that we have not even begun to answer, yet which are directly relevant and related to this type of model. In particular exogenous uncertainty plays no role in our work (although endogenous uncertainty is implicit in the formulation of the model as a game).

Our bankruptcy conditions are levied at the end. A more reasonable condition is to have intermediate settlement dates. This point also ties in with the treatment of the length of life of the individual as contrasted with the species and with the role of bequests and inheritances.

The models we have presented do not have competitive banking or changes in the outside money supply. We conjecture that as the rate of interest and the quantity of money over the appropriate ranges must be dual variables that it is unlikely that a competitive system can be designed which produces both endogenously for an economy during a period of growth. We note that we have not modelled bank reserves and our central bank and private bank descriptions are rudimentary.

Details concerning the effects of the trading of corporate and bank shares appear to make no difference to the model without exogenous uncertainty. Furthermore neither does the difference in length between production and consumption processes.

It is not difficult to create many different patterns of float depending somewhat upon the depth of trade and production (i.e. the amount of intra-industry trade required for a unit of final sales). Furthermore, at least part of the float can be regarded as being generated by strategic variables. Thus "living off the float and checkkiting can be modelled."

Given the way our model has been set up the velocity of money appears to be fixed at one. This feature is fundamentally an artifact of the exchange technology (or mechanisms) and can be modelled otherwise.

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