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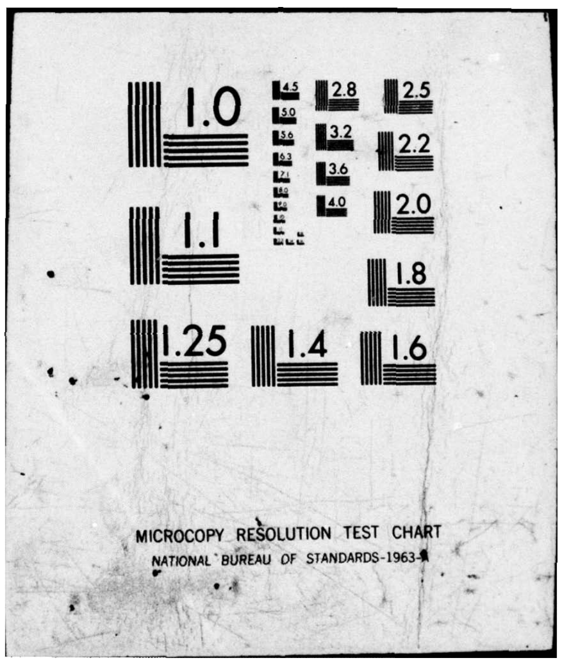
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A THEORETICAL ANALYSIS OF RESONATOR MODES IN THE PRESENCE OF HOMOGENEOUS MEDIA

DISSERTATION

DS/PH/77-2 GLENN R. DOUGHTY
MAJOR USAF

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A method of analyzing the modes of laser resonators containing homogeneous media is presented and established. This analysis is based on a derivation, which begins with Maxwell's equations and displays the required assumptions, of a pair of integral equations involving the tangential fields on the resonator mirrors. This pair of equations, which must be solved simultaneously, is specialized to apply to paraxial resonators with perfectly conducting mirrors. The result of the specialization is a pair of integral, eigenvalue equations for the current distributions induced on the resonator mirrors.

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After further specializing them to resonators for which the spatial dependence of the modes separates, these integral equations are solved using a straightforward technique based on a variational principle. This technique, which employs a novel method of obtaining modal expansion functions, reduces the analysis to a homogeneous matrix equation that is solved using well-known numerical methods.

The basic theory and method of solution presented not only produce comparable results in the classical cases considered by other authors in this field, but they also provide a framework for attacking general resonator problems.

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A THEORETICAL ANALYSIS OF RESONATOR MODES IN THE PRESENCE OF HOMOGENEOUS MEDIA

DISSERTATION

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology

Air University

in Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy

by

Glenn R. Doughty, M.S.
Major USAF

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Preface

In this effort to develop and establish a method for analyzing laser resonator modes, I have become indebted to many people. First, of course, is my thesis advisor, Dr. Donn Shankland, who must have wondered if I would ever answer any of his seemingly infinite list of questions. I am extremely grateful for his advice, which was especially helpful in the latter phases of this study, and for his computer routines, which reduced the effort required to obtain the vital numerical results. Special thanks is also extended to Dr. John Kenemuth who acted as my advisor here at the Weapons Laboratory. His encouragement and guidance during the difficult early phases were essential to the successful completion of this effort. I would also like to thank Dr. John Reichert, Capt. Ed Oliver, and Dr. Bob Shea for their continued assistance during the theoretical and computational phases of this study.

In the area of moral support, I thank my wife and family and many friends for their encouragement and patience. I am sure that they often felt that I would never finish this dissertation. Last, but also first, I thank my Lord and Savior, Jesus Christ, for his guidance and assistance throughout this effort.

Table of Contents

Table of Contents

List of Figures

Figure		Page

List of Tables

Notation

Symbol		Page
a	The radius of a circular mirror, the x-dimension of a rectangular mirror, or the inner radius of a toroidal mirror	7
$\hat{a}_x, \hat{a}_y, \hat{a}_z$	Unit vectors along the x, y, and z axes	33
A	A column matrix with n elements, A_q	51
$\overline{A}$	An electromagnetic field vector equal to $\overline{E}$ or $\overline{H}$	15
b	The y-dimension of a rectangular mirror or the outer radius of a toroidal mirror	7
c	The speed of light in vacuum	25
C_m	A contour enclosing a surface S_m'	28
$\cos(\hat{m},\hat{n})$	Cosine of the angle between the vectors $\hat{m}$ and $\hat{n}$	36
d	A parameter, $d^2 = 2\sqrt{g^2 - 1}$	72
$d\overline{s}_m$	Oriented differential arc length along a contour C_m	28
$\overline{D}$	Electric flux density	20
$D_\nu(z)$	A parabolic cylinder function of order ν	134
$\overline{E}$	Electric intensity	20
f_{mnq}	Oscillation frequency of the $(m,n,q)^{th}$ mode	166
$f(g,x)$	A function used in obtaining expansion functions for various resonators	60
F	A column matrix with n elements, F_q	183
$F_m^n(z)$	A generalized Laguerre polynomial	70
g	A mirror parameter, $g = 1 - \frac{L}{R}$	7
$g_j(r)$	The slowly varying portion of the current distribution for the j^{th} mode	114

Notation (Continued)

Symbol		Page
$\overline{\overline{G}}$	The free-space, outgoing wave, dyadic Green's function	24
$\overline{\overline{G}} \times \overleftarrow{\nabla}$	The vector operator ∇ operating to the left on the dyadic $\overline{\overline{G}}$	29
h	A parameter, $h^2 = 2\sqrt{1 - g^2} = \delta'^2$	137
H	The resonator hull	16
$\overline{H}$	Magnetic intensity	20
H_a	A limit of integration, $H_a = \sqrt{\frac{k'}{L}}\, a$. Similar notation applies for H_b, H_y, and H_x	66
i	$i = \sqrt{-1}$	15
$\overline{\overline{I}}$	The unit dyadic, $\overline{\overline{I}} \equiv \hat{a}_x\hat{a}_x + \hat{a}_y\hat{a}_y + \hat{a}_z\hat{a}_z$	24
Im z	The imaginary part of z	124
j	A subscript denoting the j^{th} mode	15
$\overline{J}$	Electric current density	20
$J_{x1}(x,y)$	The x-component of surface current on mirror #1 . . .	35
$J_\nu(z)$	Bessel function of the first kind of order ν	63
$\overline{k}$	Wave vector with magnitude, $\lvert\overline{k}\rvert = k$	35
k_j	Complex wave number for the j^{th} mode, $k_j = k_j' + ik_j''$ with k_j' and k_j'' real	22
k_ρ	Radial wave number	162
K	Square matrix of order n with elements K_{qm}	51
$K(\overline{r}_1 \vert \overline{r}_2)$	Kernel of an integral equation for a current distribution	45
$K(x_1 \vert x_2)$	Kernel of an integral equation for the x-component of a current distribution	48
$K_n(\rho_1 \vert \rho_2)$	Kernel of an integral equation for the radial coordinate of a current distribution with $e^{+in\theta}$ azimuthal variation	48

Notation (Continued)

Symbol		Page
$\hat{K}_{ij}$	An integral operator yielding the current on the i^{th} mirror due to a current on the j^{th} mirror	37
L'	Residue of an integrand	15
L	Mirror separation measured along the optic axis . . .	7
$\hat{L}$	A differential operator used in computing expansion functions for toroidal resonators	153
ln (z)	The natural logarithm of z	46
$\lim_{r \to r_{1+}} F$	The limit of a function F as r approaches r_1 in a manner so that $r > r_1$	35
M	A resonator parameter, $M \equiv \sqrt{8\pi N}$	63
$\overline{M}$	Magnetic current density	20
$\hat{M}$	A differential operator used in computing expansion functions	57
$M_{\kappa,\mu}(z)$	A Whittaker function of the first kind with parameters κ and μ	71
n	Index of refraction	21
$\hat{n}$	Unit normal vector	21
N	Mirror Fresnel number, $N \equiv \frac{a^2}{\lambda L}$	8
N_e	Equivalent Fresnel number. For a symmetric resonator, $N_e = \beta_q N$	77
$N(x \vert x')$	A function which is symmetric with respect to the spatial variables x and x'	56
$O\vert K^{-1}\vert$	Terms on the order of $\vert K^{-1}\vert$	190
$\overline{P}$	Polarization	20
q_m	Magnetic charge density	20
$\overline{r}$	Position vector of a field point	19
$\overline{r}'$	Position vector of a source point	35

Notation (Continued)

Symbol		Page
R	An upper right triangular matrix or the radius of curvature of a mirror	186
R_i	Radius of curvature of the i^{th} mirror	7
R_{12}	Distance between two points on mirrors #1 and #2 . . .	19
Re z	The real part of z	114
s	A parameter in an eigenvalue problem	62
S	The resonator shell, or a parameter, $S \equiv \frac{\lambda\beta_g}{2L}$	17
S_m'	The m^{th} open surface in a series of open surfaces	28
t	A variable of integration	165
T	A tridiagonal matrix	185
$u(x)$	The x-variation of a product solution for a current distribution $J_{x1}(x,y)$	47
$\hat{u}$	A unit vector	33
$u_j(\zeta)$	Current distribution, along the ζ coordinate, for the j^{th} mode	114
$v(y)$	The y-variation of a product solution for a current distribution $J_{x1}(x,y)$	47
$w_n(\xi)$	A function used in determining expansion functions for circular mirror resonators	69
$W_{\kappa,\mu}(z)$	A Whittaker function of the second kind with parameters κ and μ	70
x,y,z	Axes in a rectangular coordinate system	19
α	An argument of a confluent hypergeometric function	72
α'	A parameter, $\alpha' = \sqrt{1-g^2}$	69
α_{ij}	The angle between R_{ij} and the optic axis	40

Notation (Continued)

Symbol		Page
β	A constant, $\beta \approx 0.824$	63
β_g	A parameter $\beta_g = \sqrt{g^2-1}$	40
γ	An eigenvalue, or an argument of a confluent hypergeometric function	46
$\bar{\gamma}$	$\bar{\gamma} \equiv \frac{1}{\gamma}$	48
$\Gamma(z)$	The complex gamma function with argument z	135
δ	The variation of (or about) some function or parameter	48
$\delta(R)$	The Dirac delta function	24
δ'	A parameter, $\delta'^2 \equiv 2\sqrt{1-g^2} = h^2$	164
δ_{qm}	The Kronoker delta function	51
Δ	The projection, measured along the optic (or mirror axis for toroidal mirrors), between an arbitrary point on a mirror and the intersection of that mirror and the optic (mirror) axis	150
$\Delta\phi_1$	The change in the function ϕ_1	117
ε	Permittivity	19
ε_0	Permittivity of free space	20
$\tilde{\varepsilon}$	Complex permittivity	21
ζ	A spatial coordinate on a mirror surface ζ = x,y, or ρ depending on the mirror shape	59
η_j	Normalized y-coordinate; $\eta_j \equiv \sqrt{\frac{k}{L}}\, y_j$	131
Θ	The azimuthal coordinate of a circular cylindrical system of coordinates	63
ι	A complex constant	164
κ	A parameter characterizing Whittaker functions	70
λ	Wavelength	8

Notation (Continued)

Symbol		Page
Λ	A parameter, $\Lambda = g+i\sqrt{1-g^2}$	165
$\Lambda_m(x)$	An integral involved in the computation of matrix elements	52
μ	Permeability, or a parameter characteristic of Whittaker functions	19
ν	The order of various transcendental functions	69
ν_{nm}	The m^{th} root of the n^{th} order Bessel function of the first kind	63
ξ_i	Normalized x-coordinate; $\xi_i \equiv \sqrt{\frac{k'}{L}}\, x_i$. Also used as normalized radial coordinate	66
ξ_e	A normalized distance; $\xi_e = \sqrt{\frac{k'}{L}}\, \rho_e$	151
ρ	The radial coordinate in a circular cylindrical system of coordinates	63
ρ_e	A distance characteristic of toroidal mirrors	9
$\rho(\overline{r})$	Electric charge density	20
σ	Conductivity	19
τ	A parameter used with Whittaker functions in the analysis of circular mirror resonators	70
τ'	Volume of integration	25
ϕ	A function giving the spatial variation of the free-space Green's function; $\overline{\overline{G}} \equiv \overline{\overline{I}}\phi$	24
$\tilde{\phi}$	The Fourier transform of ϕ	99
$\phi_1(r)$	$\phi_1(r) \equiv kr$	116
$\phi_2(r)$	$\phi_2(r) \equiv \frac{k\beta_g}{2L} r^2$	116
ϕ_z	$\phi_z \equiv \frac{\partial\phi}{\partial z}$	33
$\Phi(\alpha,\gamma,z)$	A confluent hypergeometric function of the first kind	72

Notation (Continued)

Symbol		Page
χ	Electric susceptibility, $\chi = \chi' + i\chi''$ where χ' and χ'' are real	21
ψ_q	The q^{th} expansion function in a set of n expansion functions	49
Ψ	A square matrix of order n with elements Ψ_{qm}	51
$\Psi(\alpha,\gamma,z)$	A confluent hypergeometric function of the second kind	76
ω	Frequency (angular)	15
ω_j	Complex frequency of the j^{th} mode: $\omega_j = \omega_j' + i\omega_j''$ where ω_j' and ω_j'' are real	15
Ω	A parameter, $\Omega^2 \equiv \frac{1}{2}\sqrt{g^2-1}$	70
∇	$\nabla \equiv \hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}$	20

Abstract

A method of analyzing the modes of laser resonators containing homogeneous media is presented and established. This analysis is based on a derivation, which begins with Maxwell's equations and displays the required assumptions, of a pair of integral equations involving the tangential fields on the resonator mirrors. This pair of equations, which must be solved simultaneously, is specialized to apply to paraxial resonators with perfectly conducting mirrors. The result of the specialization is a pair of integral, eigenvalue equations for the current distributions induced on the resonator mirrors.

After further specializing them to resonators for which the spatial dependence of the modes separates, these integral equations are solved using a straightforward technique based on a variational principle. This technique, which employs a novel method of obtaining modal expansion functions, reduces the analysis to a homogeneous matrix equation that is solved using well-known numerical methods.

The basic theory and method of solution presented not only produce comparable results in the classical cases considered by other authors in this field, but they also provide a framework for attacking general resonator problems.

A THEORETICAL ANALYSIS OF RESONATOR MODES IN THE PRESENCE OF HOMOGENEOUS MEDIA

I. Introduction

Since the discovery of the laser in 1961, many workers have devoted considerable effort to analyzing the electromagnetic fields associated with laser resonators. The approaches employed in these efforts cover the spectrum with regard to derivation of basic equations, essential assumptions, and method of solution. However, despite this wide range of approaches and high level of effort, the vast majority of these approaches has not been derived in a completely general manner. In addition, all of these approaches leave much to be desired with regard to ease of application, depth of understanding and capability of prediction in resonator problems. This paper is directed not only towards establishing a sound, general theory but also towards improving the existing capabilities in these three aspects of the analysis of resonator fields.

The improvement in the ability to analyze resonator fields is important because such analyses have significant impact throughout the development and use of laser systems. For example, although the first step in the development of a new laser is usually the finding of a gain medium, the next step is devising an efficient means of extracting power from the medium so that the output beam has certain desirable characteristics. These desirable characteristics might include good beam quality (nearly uniform phase and amplitude of the field), low losses, and a large volume over which the field can interact with the active medium. For the effective design of resonators with these or other extraction characteristics, one needs to know not only the fields present within a given resonator but also how they are affected by changing the resonator parameters.

G. Fox and T. Li (ref. 1, pp. 453-488) were among the first to recognize the importance of analyzing the fields associated with lasers containing homogeneous media. They concentrated their efforts on resonators which (with regard to geometric optics) periodically refocus paraxial rays so that they always remain confined to the resonator volume (except for transmission through the mirrors). In addition to Fox and Li, many others (refs. 2-10) analyzed the fields associated with these so-called stable resonators by calculating their normal modes (which could be used to represent any resonator field). These normal modes were, by definition, the eigensolutions of a certain integral equation that was derived by applying the theory of diffraction to the resonator in question.

For several years, it was felt that only stable resonators could find practical application in the laser field. However, disadvantages such as small mode volume and poor mode discrimination prompted Siegman (ref. 11, p. 278) in 1965 to propose using resonators that do not confine paraxial rays to the resonator volume. These unstable resonators, which have since found widespread application, are characterized by large mode volume, diffraction output coupling, good transverse mode discrimination, and totally reflecting optics. This last characteristic is especially valuable for high power lasers for which the resonator mirrors need to be cooled.

Just as laser resonators have become more complex, the theory and analytical techniques used to study them have become more sophisticated. For example, the initial analyses (refs. 11-16) of unstable resonators were based on geometric optics. However, these approaches, which yield average values of the losses and rough estimates of the mode distributions, have been replaced by analyses that consider the diffractive effects introduced by the finite sizes of the mirrors. Although other approaches (refs. 17-19) also have merit, the most promising techniques can be grouped into two broad categories; the Waveguide Analogy (refs. 20-23) and the Integral Equation Method (refs. 6, 9, 10, 24-28). These categories, along with their specific deficiencies, are described in chapter II.

Despite the many positive aspects of the techniques in these two categories, no method exists which one can use to adequately determine and understand the characteristics and behavior of complex resonator modes. The objective of this work is to develop and establish such a method for resonators containing a homogeneous medium and two perfectly conducting mirrors. The purpose of this report is to present this analytical method and its supporting results.

As discussed later in this report, this analysis provides two significant contributions for determining and understanding laser resonator modes. The first is a derivation of integral equations for the tangential fields on the resonator mirrors; it begins with Maxwell's equations and explicitly displays all required assumptions. The second contribution is a straightforward technique for solving these equations for a wide variety of resonators with perfectly conducting mirrors. This technique includes a novel method for obtaining expansion functions to be used with the variational principle upon which the technique is based.

This report is organized in the following manner. First, following a presentation of important background material, the basic laser resonator problem is formulated. The formulation is followed by derivations of the integral equations for the current induced on the mirrors of open and closed laser resonators. Then, in chapter IV, the general problem of solving these integral equations is discussed, and the method to be used to approximately solve these equations for paraxial resonators is presented. In chapter V, the theory and method of solution of the previous two chapters are specialized to apply to resonators for which the spatial dependence of the modes can be separated to yield two independent governing equations. The results of the specialization are presented and compared to existing published work in the following chapter. The text ends with chapter VII, which presents specific conclusions and recommendations. The eight appendices contain detailed mathematical derivations and calculations in support of this work.

II. Background and Theoretical Preliminaries

This chapter presents and then formulates the basic resonator problem. To that end, the body of this chapter begins with a discussion of important background material and a definition of a laser resonator mode. This definition, as well as the insight gained from the discussion which follows, forms the basis for the formulation, which includes a discussion of the analytical approach used and a presentation of the basic equations applied in the following chapters.

Background

Despite the large number of configurations in use today, laser resonators can be grouped into the two rather broad categories of open and closed resonators. Since the analysis presented in this paper is performed by first considering closed resonators and then extending the results to open resonators, some care must be exercised in distinguishing between the two resonator types. As one might expect, the essential difference between the two types involves the nature of the resonator mirrors and the surfaces with which one can enclose them.

As a first step in determining whether a particular resonator is either open or closed, one uses one or more closed surfaces to enclose the resonator mirrors such that only the mirrors are included within the enclosed volume(s). Thus, for each case, the number and shape of the closed surfaces will be chosen to correspond to that series of closed surfaces which most nearly conforms to the mirror shapes. If only one closed surface is used, the resonator is an open resonator containing one mirror with one or more "holes" in the mirror surface. If two surfaces are used, the resonator is either a closed resonator or an open resonator with two mirrors. If the two closed surfaces are such that one surface completely encloses the second surface, the resonator is closed. However, if one surface does not include the other, the resonator is open. Finally, if three or more closed surfaces are used to enclose the resonator mirrors, the resonator is an open resonator.

To clarify these ideas concerning open and closed resonators, two examples are shown below. First, a planar view of a closed resonator is shown in figure 1. As one can see from the figure,

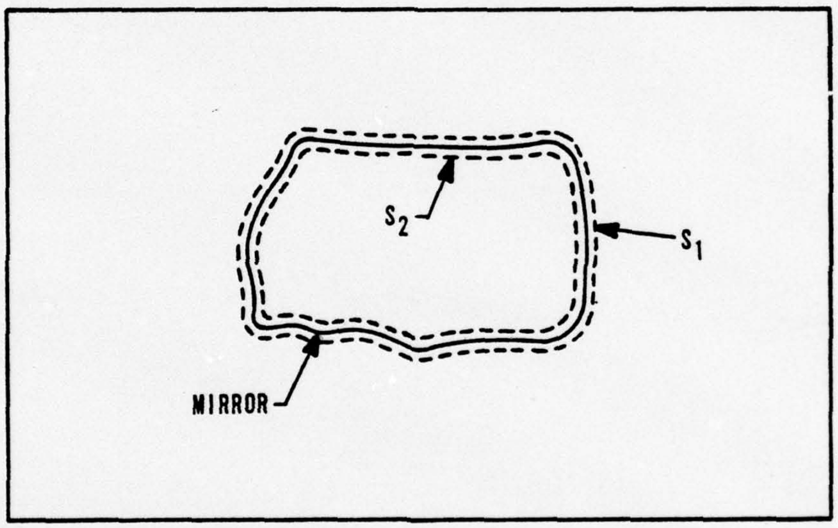

Figure 1. A Planar View of a Closed Resonator

the surface enclosing the unbroken material (mirror) boundary, which may include segments of several materials, consists of two closed surfaces S_1 and S_2. As indicated in the previous paragraph, one surface (S_1) completely encloses the second surface (S_2). In a similar fashion, a planar view of an open resonator containing two mirrors is shown in figure 2. From this figure, one can see that neither surface encloses the other.

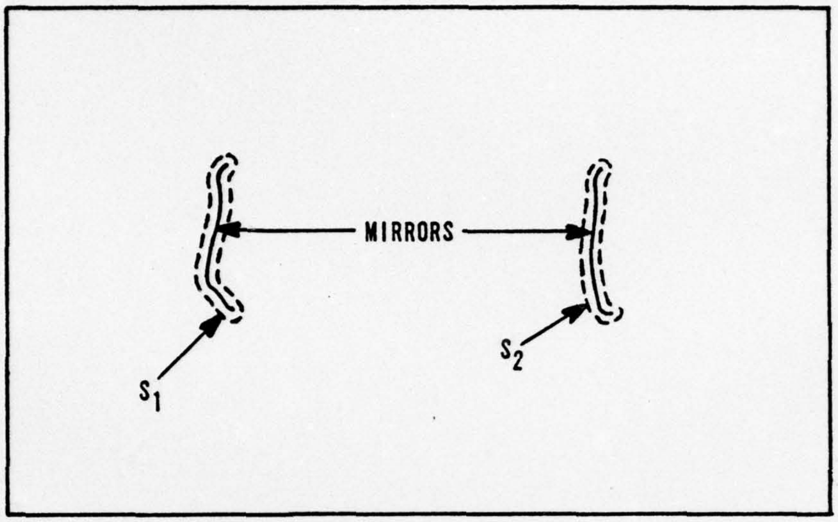

Figure 2. A Planar View of an Open Resonator

Open Resonators with Spherical Mirrors. A significant fraction of the open resonators in use today employs two spherical mirrors of either rectangular or circular projection. In addition to the material properties of the host and lasing media, these resonators are characterized by the following mirror parameters:

1. The transverse mirror dimensions (diameter or width) a and/or b;

2. The mirror radii of curvature, R; and

3. The mirror separation, L.

These last two parameters play an important role in the classification of laser resonators containing spherical mirrors. For that role, the mirror separation and radii of curvature are combined into so-called "g-parameters" as shown below,

$$g_i = 1 - \frac{L}{R_i} \qquad (1)$$

where the subscript denotes the i^{th} mirror. The ranges and/or values of these g-parameters for several resonators are shown below in table I.

Table I

Resonator g-parameters

RESONATOR TYPE	g-PARAMETERS
STABLE	$0 < g_1 g_2 < 1$
UNSTABLE	$g_1 g_2 < 0$ OR $g_1 g_2 > 1$
QUASI-STABLE	$g_1 g_2 = 0$ OR $g_1 g_2 = 1$
CONFOCAL*	$2g_1 g_2 = g_1 + g_2$
PLANE PARALLEL	$g_1 = g_2 = 1$

*A confocal resonator is a resonator for which the foci of the two resonator mirrors are colocated.

The resonator Fresnel number N is another parameter that is often used to characterize these spherical resonators. Actually, a Fresnel number, which is a quantity often reserved for mirrors of circular projection, is designated for each resonator mirror and is defined by

$$N = \frac{a^2}{\lambda L} \qquad (2)$$

where a is the mirror radius and λ is the wavelength of the radiation within the resonator. To adapt this quantity to mirrors of rectangular projection, one uses Eq. (2) to define two Fresnel numbers for each mirror. However, for this case, the quantity "a" denotes each of the transverse mirror dimensions (length and width) instead of the mirror radius.

Even though spherical mirror resonator configurations of the types shown in table I are widely used, they do have several shortcomings. Unfortunately, these shortcomings are not limited to the small mode volume and poor mode discrimination exhibited by stable resonators. For instance, for many of the applications involving unstable resonators with mirrors of circular projection, the output beam has an annular shape. This annular shape not only makes the beam difficult to use, but it can also prevent a nontrivial (several percent or more) portion of the energy from reaching the far-field central spot. Also, spherical unstable resonators do not efficiently accommodate new lasers having central obscurations (to the radiation within the resonator) caused by the use of a radial gas flow or a radial electron beam. These and other deficiencies have led to the study and development of some new and rather exotic resonator configurations.

Toroidal Resonators. Many of these new resonator configurations involve resonators which include at least one toroidal mirror. As the name implies, the mirrors falling into this category simply correspond to different portions or cross sections of a toroid; however, there are two kinds of toroidal mirrors. The first kind, which is rarely used, is simply characterized by two radii of curvature. The second kind is a mirror that is characterized by a particular surface of revolution; that is, a plane which passes through the resonator axis and intersects the mirror yields two arcs. These two arcs, which have radii of curvature R and which may or may not be connected, have centers of curvature which are displaced from the resonator axis by a distance ρ_e.

As with spherical mirror resonators, one may use a pair of g-parameters to characterize or classify toroidal resonators. However, for these resonators, one modifies Eq. (1) by replacing the radius of curvature of a spherical mirror with the radius of curvature of the individual arcs discussed above. With this modification, one can use table I to classify toroidal resonators with regard to stability, etc. Although this method of classifying toroidal resonators is somewhat artificial, it does provide a useful framework for the analysis.

Four resonators using this second kind of toroidal mirror are depicted in figure 3. Actually, each of the sketches in this figure shows the intersection of a toroidal resonator with a plane passing through the resonator axis (denoted AA' in the figure). To obtain a three dimensional view of each resonator, it is necessary to revolve each sketch through an angle of 180° about the resonator axis. In figure 3(d), the lines with the tic-marks designate the boundaries of a central obscuration to the radiation within the resonator. The lines and/or arcs in each of the other three sketches depict a wavefront as it passes through ard out of that resonator.

Past Work. Existing analytical techniques are deficient in that they cannot adequately treat many spherical or toroidal resonator configurations. This is especially true of the Waveguide Analogy (refs. 20-23) in which the resonator is treated as if it were a waveguide section (with the resonator mirrors as the guiding surface) which is coupled to the surrounding space. The guide is taken to be operating near cut-off, and the modes are expanded in terms of the fields which would be present if the waveguide section were infinite. The expansion coefficients are calculated by first determining how the

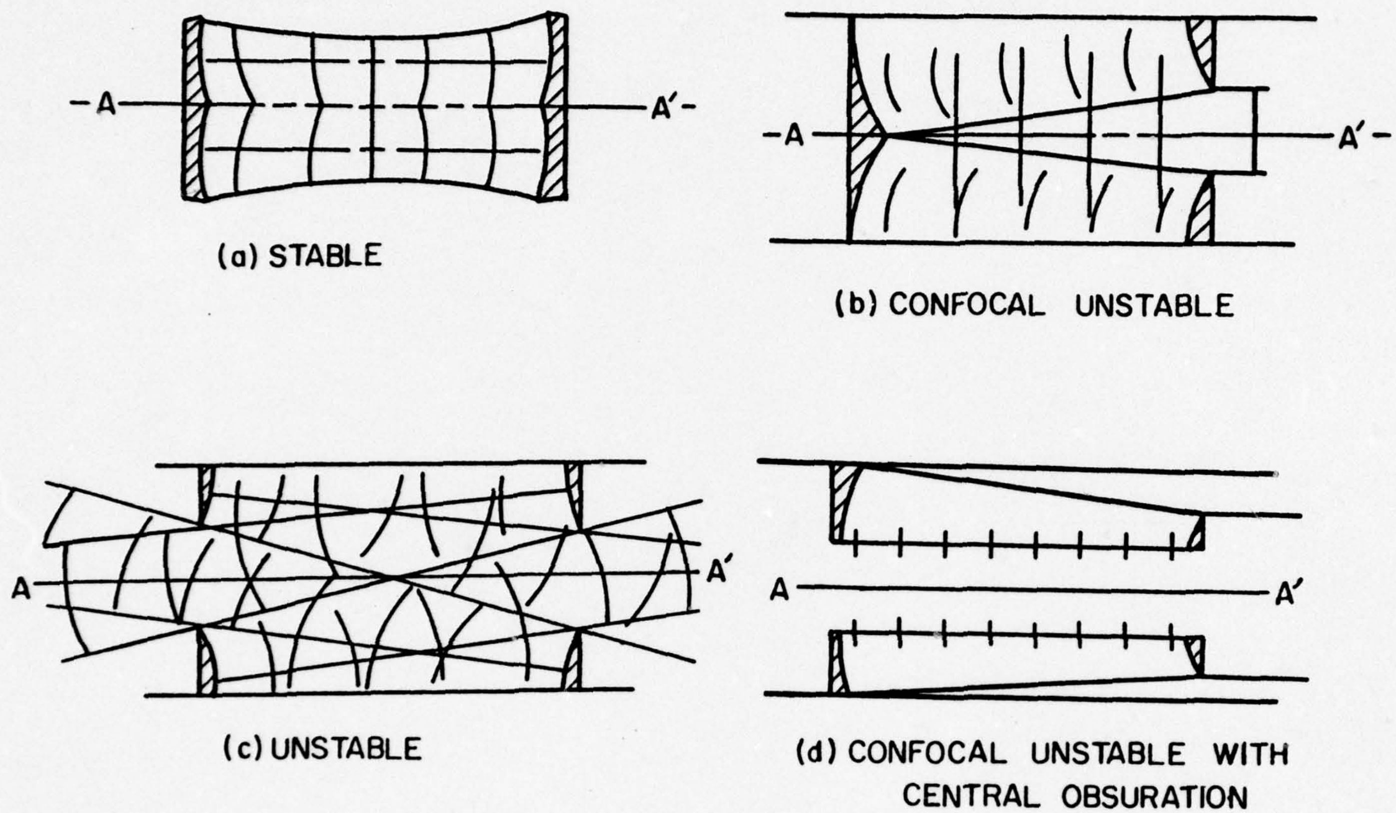

Figure 3. Toroidal Resonators

infinite guide modes are reflected by the interfaces present between the guide section and its surroundings, and then requiring that the resulting fields be self-reproducing. Of the two approaches mentioned earlier, it is this Waveguide Analogy which lends the greatest insight into the physical behavior of the modes. However, since its application not only requires solving rather involved transcendental equations in the complex plane, but also calculating reflection coefficients at all interfaces (between the guide and its surroundings), it would be very difficult to apply to resonators with output coupling apertures or toroidal mirrors.

Despite these limitations, the Waveguide Analogy has been used to excellent advantage by Vainshtein (ref. 21, p. 711) to obtain closed form expressions for the modes of finite, plane parallel resonators. As will be seen on page 63, these expressions play an important role in the techniques applied in this paper.

The second and more widely used analytical approach is based on solving certain integral equations for components of the electromagnetic field (precisely which components depends on the problem) on the resonator mirrors. The popularity of this approach is largely a result of the availability of mathematical methods for approximately solving integral equations; it is also due to the fact that once the basic forms have been derived, the equations can be easily modified to apply to a wide variety of complex resonators.

Several different authors (refs. 1, 29-31) have derived integral equations which are widely considered to be the basic forms used in this approach. However, each of these derivations and, in some cases, the equations themselves are deficient in one or more aspects of the problem. One major deficiency of many of the derivations is the tendency to make important assumptions at the beginning of the analysis concerning the nature of the electromagnetic field. This tendency not only limits the applicability of the entire analysis, but it can inhibit understanding the behavior of the field by obscuring the precise implications of the various assumptions. For example, many derivations are based on the implicit assumption that the mirror material properties do not affect the mode distributions.

This assumption, which may not always be valid, obscures the fact that in determining the mode distributions one must consider the effect of both the electric and magnetic fields on the mirror surfaces. For some cases, either the electric or magnetic field at the mirror boundary may actually have a negligible effect on the mode distributions. However, one should show that this is true in each case rather than assuming it is true in general. A second example of these obscuring assumptions is the assumption that the field (or current) on one resonator mirror can be expressed entirely in terms of the field (or current) on the second resonator mirror. This assumption inherently obscures the fact that, in some cases, one must include the effect of the current on both resonator mirrors.

A second deficiency applies to derivations that employ assumptions which are inconsistent with known mathematical theorems. One such assumption is that a scalar field, along with its normal derivative, vanishes identically on a finite surface element. Finally, many derivations are deficient in that they do not properly include the damping of the modes of open resonators and its effect on the resulting mode distributions. For some cases, the damping has a negligible effect on the actual distributions; however, in other cases, its effect can be significant. By including the effects of damping at the beginning of the analysis, one can show that it is not always possible to formulate the laser resonator problem in terms of a linear eigenvalue equation. Unfortunately, the bulk of the numerical work performed, especially for complex resonators, has been based on derivations containing this last deficiency.

As indicated in the introduction, the remainder of this paper is directed toward developing and establishing an analytical approach which will overcome many of the deficiencies of the existing theories.

Laser Resonator Modes

In a wide range of problems in electromagnetic theory, it is standard procedure to work in terms of fields (modes) which depend only on the characteristics of the material bodies and surrounding media. This useful procedure was adapted to this analysis by defining a resonator mode to be a member of that class of linearly independent, source free, electromagnetic fields which satisfy the boundary conditions imposed by the resonator.

To gain some insight into the nature of these modes, consider the following experiment. A radiation source is placed in the vicinity of a resonator. Prior to time t=0, the source is turned off, and only the null field is present. Then at t=0, the source is pulsed and immediately turned off. After sufficient time has passed for the wavefront to reach and interact with the resonator, the resulting field, which is composed entirely of the modes of the resonator, is expressed in terms of the Fourier decomposition shown in Eq. (3).

$$\overline{A}(\overline{r},t) = \int_{-\infty}^{\infty} \overline{A}(\overline{r},\omega)\, e^{+i\omega t}\, d\omega \qquad (3)$$

To analyze the temporal behavior of that field, the function $\overline{A}(\overline{r},\omega)$ is analytically continued into the complex ω-plane, and the resulting form is examined using contour integration in conjunction with the theory of residues. That examination reveals that since the field is nonvanishing after time t=0, the integrand must have singularities in the upper half of the complex ω-plane. The nature of these singularities, which occur for $\omega_j = \omega_j' + i\omega_j''$ $(\omega_j''>0)$, can be used to determine the behavior of the field through the relation

$$\overline{A}(\overline{r},t) = 2\pi i \sum_j \overline{L}_j'(\overline{r},\omega,t) \qquad (4)$$

where $\overline{L}_j'(\overline{r},\omega,t)$ is the residue of the integrand at ω_j. Although the exact form depends on the details of the source and the resonator, each residue can be written in the form

$$\overline{L}_j'(\overline{r},\omega,t) = \overline{P}_j(\overline{r},\omega_j)\, e^{+i\omega_j' t}\, e^{-\omega_j'' t} \qquad (5)$$

where $\overline{P}_j(\overline{r},\omega_j)$ is the residue of $\overline{A}(\overline{r},\omega)$ at $\omega = \omega_j$. The result obtained by substituting Eq. (5) into Eq. (4) expresses the field as a combination of the modes of the resonator, where each mode has a time dependence of the form $e^{+i\omega_j' t}\, e^{-\omega_j'' t}$. In this last expression, ω_j' is the frequency and ω_j'' is the decay constant of the j^{th} mode.

Another important aspect of the nature of these modes involves the type of energy flow associated with each mode. For this discussion, the two types of energy flow of importance are (1) energy flow from the resonator to the surroundings and (2) energy flow from the surroundings to the resonator. Of course, in the typical laser application, radiation emanates from within the resonator, and the net radiative energy flows from the resonator to the surroundings.

For closed resonators, such as the one shown below in figure 4, distinguishing between the two types of energy flow is relatively straightforward. For these resonators, which require two closed surfaces to enclose the unbroken material (mirror) boundary, one distinguishes between the two types of flow by considering energy flow across the unbroken material boundary.

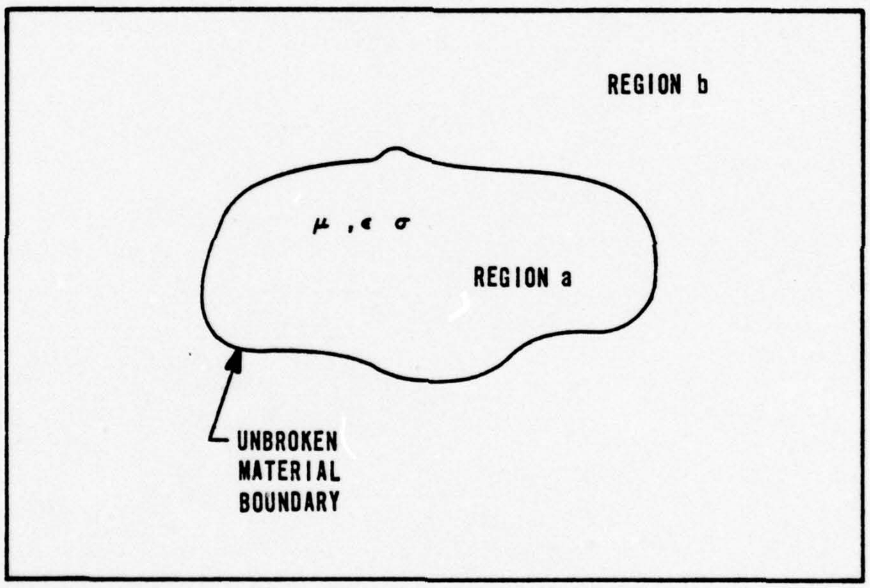

Figure 4. A Closed Resonator

However, for resonators not characterized by an unbroken material boundary, these ideas need to be made more precise. To do that, consider the two closed surfaces involving the open resonator shown in figure 5. The first surface, which I call the resonator hull, is designated H in the figure. This surface consists of the front surfaces of the two mirrors and the family of straight lines connecting the mirrors such that H encloses the maximum possible volume. The second surface, which I call the resonator shell, is designated S in the figure. The resonator shell consists of the front surfaces of the two mirrors and the family of straight lines enclosing the maximum volume such that each straight line begins and ends at a mirror edge.

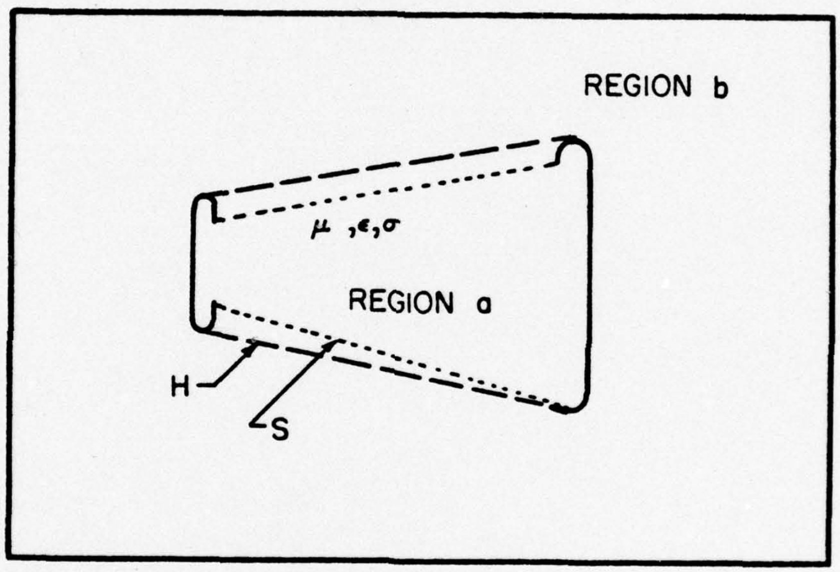

Figure 5. Surfaces Defining the Inside and Outside of an Open Resonator

If the resonator hull encloses a volume greater than that enclosed by the resonator shell (which might correspond to the presence of a lip on one of the mirrors), the ideas related to energy flow are not easily understood, and such a resonator will not be treated in this analysis. However, in the event that the resonator hull and shell coincide, the important energy flow ideas can be precisely formulated. To do this, the volume contained within and on the closed surface H, designated region a, is considered to be inside the resonator. Obviously then, the volume outside H, designated region b, is outside the resonator. For this analysis, only those fields for which the net energy flows from region a to region b will be designated as laser resonator modes.

Formulation of the Problem

General Approach. Based on the material covered in the previous section, the modes of a laser resonator can be found by determining the linearly independent members of that class of electromagnetic fields, with time dependence $e^{+i\omega' t} e^{-\omega'' t}$, which satisfy the boundary conditions imposed by the resonator to produce energy flowing from the resonator to the surroundings. Figure 6 depicts a planar view of this basic problem, which is analyzed in the remainder of this paper.

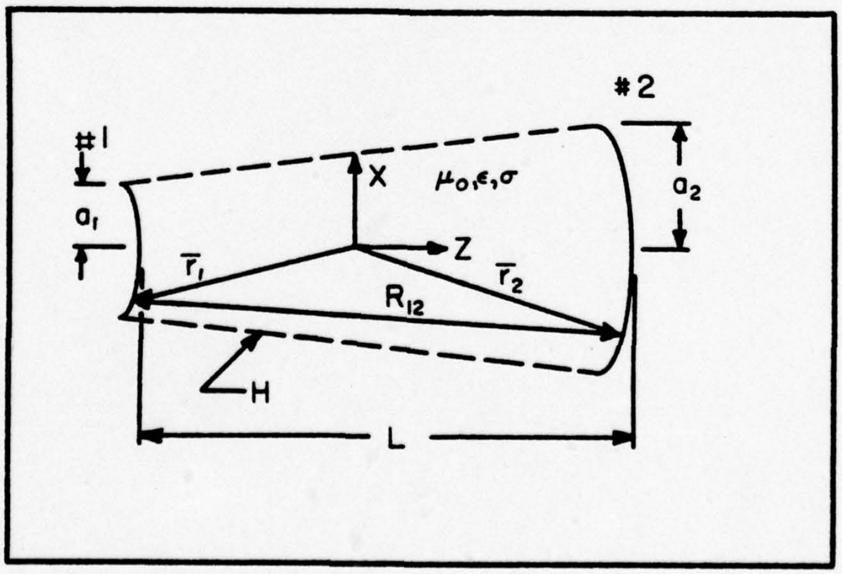

Figure 6. The Laser Resonator Problem

As indicated in the figure, a rectangular system of coordinates (only the x-z plane is shown in the figure) is established in the vicinity of the mirrors with the z-axis chosen as the optic axis of the resonator. In addition to the two mirrors, the resonator contains a homogeneous medium with parameters μ, ε, and σ. The position vector of a point on the j^{th} mirror is denoted $\overline{r}_j$, and the distance between two arbitrary points on the two mirrors is R_{12}. Finally, the mirror separation, measured along the optic axis, is denoted L. Figure 6 also applies to closed resonators, except in that case, there is one mirror rather than two separate ones.

The following approach is used to determine the fields and other characteristics associated with the laser resonator depicted in figure 6. First, the basic equations of electromagnetic theory are used to derive a pair of coupled integral equations for the electric and magnetic fields within the resonator. These equations express these fields within the resonator volume in terms of the electric and magnetic fields tangential to the mirror surfaces. This pair of equations is then specialized to apply to resonators with perfectly conducting mirrors. The result of this specialization is a pair of equations which relate the electric and magnetic fields within the resonator to the currents induced on the perfectly conducting mirrors. Then, using the boundary conditions for perfect conductors and letting the field point approach a point on the mirror surfaces, an integral equation is obtained for the current distributions on the resonator mirrors. Once this equation has been solved, these current distributions can be used in the equations that relate the electric and magnetic fields to the currents induced on the resonator mirrors. In this manner, one may specify the electromagnetic fields associated with each mode throughout the resonator volume. The fields obtained using this approach are unique (Harrington, ref. 32, p. 102).

<u>Basic Equations</u>. Since the modes are electromagnetic in nature, they satisfy Maxwell's equations, which are written below in mks units for uniform media (Stratton, ref. 33, p. 464).

$$- \nabla \times \overline{E}(\overline{r},t) = + \mu\dot{\overline{H}}(\overline{r},t) + \overline{M}(\overline{r},t) \tag{6}$$

$$\nabla \times \overline{H}(\overline{r},t) = \dot{\overline{D}}(\overline{r},t) + \overline{J}(\overline{r},t) \tag{7}$$

$$\nabla \cdot \overline{H}(\overline{r},t) = q_m/\mu \tag{8}$$

$$\nabla \cdot \overline{D}(\overline{r},t) = \rho \tag{9}$$

where $\overline{D} = \varepsilon_0\overline{E} + \overline{P}$. In these equations, $\overline{P}$ is the polarization of the medium, $\overline{M}$ is a fictitious magnetic current, and q_m is a fictitious magnetic charge. These equations can be manipulated to yield the following equations of continuity.

$$\nabla \cdot \overline{J}(\overline{r},t) = -\dot{\rho}(\overline{r},t) \tag{10}$$

$$\nabla \cdot \overline{M}(\overline{r},t) = -\dot{q}_m(\overline{r},t) \tag{11}$$

In addition to Eqs. (6) through (11), the following two equations govern the behavior of the fields in two different regions separated by a surface electric or magnetic current (Harrington, ref. 32, p. 34).

$$\hat{n} \times [\overline{H_1 - H_2}] = \overline{J}_s \tag{12}$$

$$[\overline{E_1 - E_2}] \times \hat{n} = \overline{M}_s \tag{13}$$

where $\hat{n}$, the unit normal to the surface, points into region #1.

To apply these equations to the resonator problem, the following conditions concerning the medium and fields within the resonator are assumed to hold:

1. The homogeneous medium is assumed to be linear and isotropic so that $\overline{D} = (n^2 + \chi)\varepsilon_0\overline{E}$ where n is the index of refraction of the host material, and χ is the susceptibility of the lasing material.
2. Within the medium, no magnetic sources or electric charge exists. Then $\rho = q_m = 0$ and $\overline{M} = 0$.
3. Within the medium, $\overline{J} = \sigma\overline{E}$.
4. For each mode, the fields have a time dependence of the form $e^{+i\omega_j t}$ where $\omega_j = \omega_j' + i\omega_j''$; $\omega_j'' \geq 0$.

With these assumptions, the field equations within the resonator become

$$\nabla \times \overline{E}_j(\overline{r}) = -\,i\omega_j\mu\overline{H}_j(\overline{r}) \tag{14}$$

$$\nabla \times \overline{H}_j(\overline{r}) = i\omega_j\tilde{\varepsilon}(\omega_j)\overline{E}_j(\overline{r}) \tag{15}$$

$$\nabla \cdot \overline{H}_j(\overline{r}) = 0 \tag{16}$$

$$\nabla \cdot \overline{E}_j(\overline{r}) = 0 \tag{17}$$

where $\tilde{\varepsilon}(\omega_j) = \varepsilon_0 \left[n^2 + \chi - \frac{i\sigma}{\omega_j \varepsilon_0} \right]$. By taking the curl of Eq. (14), substituting Eq. (15) into Eq. (14), and simplifying the result, the wave equation for $\overline{E}_j$ can be obtained as

$$\nabla^2 \overline{E}_j(\overline{r}) + k_j^2 \, \overline{E}_j(\overline{r}) = 0 \tag{18}$$

where $k_j^2 = \omega_j^2 \mu \tilde{\varepsilon}(\omega_j)$. A similar procedure yields an equation identical to Eq. (18) with $\overline{E}_j$ replaced by $\overline{H}_j$.

These wave equations form the basis for the derivation of the modal integral equations in the following chapter.

III. Derivation of the Integral Equations

The purpose of this chapter is to derive the basic integral equations which govern the behavior of the modal fields of laser resonators. The text begins with derivations of these integral equations for two classes of closed resonators. The derivations are then modified to apply to open resonators. Following these modifications, the integral equations for open resonators are specialized to apply to resonators with two perfectly conducting mirrors. The specialization is then manipulated to yield the integral equations tc be solved for the modal currents induced on the resonator mirrors.

Integral Equations for Closed Resonators

The two classes of closed resonators to be considered in this section are closed resonators with either homogeneous or segmented boundaries. As one might expect, a homogeneous boundary consists of a single homogeneous medium. On the other hand, a segmented boundary contains at least two materials with different electromagnetic properties.

For closed resonators containing either homogeneous or segmented boundaries, it is assumed that the boundary is smooth. This assumption, which also applies to open resonators, implies that (Taylor, ref. 30, pp. 360, 371)

1. The boundary (surface) does not intersect itself,
2. The boundary (surface) has a tangent plane at each point whose direction varies continuously as the point moves along the boundary.

It is further assumed that any discontinuities in the fields are such that, at the points of discontinuity, the changes in the fields are finite. These field discontinuities could occur at the intersection of the different materials in a closed resonator with a segmented boundary or at the edge of a mirror in an open resonator. It is also assumed that the resonator fields are differentiable at all points in space with the exception of these points of discontinuity along the resonator boundary.

Closed Resonators with Homogeneous Boundaries. As indicated at the end of the previous chapter, the wave equations for $\overline{E}_j$ and $\overline{H}_j$ form the basis for the integral equation derivations contained in this chapter. Those two wave equations are summarized by a single equation

$$\nabla'^2 \, \overline{A}_j(\overline{r'}) + k_j^2 \, \overline{A}_j(\overline{r'}) = 0 \tag{19}$$

where $\overline{A}_j = \overline{E}_j$ or $\overline{H}_j$ as desired and $k_j^2 = \omega_j^2 \mu \tilde{\varepsilon}(\omega_j)$. Equation (19) can be solved using Green's function techniques by first solving the equation

$$\nabla'^2 \overline{\overline{G}} + k_j^2 \, \overline{\overline{G}} = - \, \overline{\overline{I}} \; \delta(\overline{r-r'}) \tag{20}$$

where $\delta(\overline{r-r'})$ is the Dirac delta function (Collin, ref. 34, p. 565) and $\overline{\overline{I}} = \hat{a}_x\hat{a}_x + \hat{a}_y\hat{a}_y + \hat{a}_z\hat{a}_z$ is the unit dyadic in rectangular coordinates. One solution to this equation is a Green's function corresponding to outgoing waves (see appendix A) of the form $\overline{\overline{G}} = \overline{\overline{I}}\phi(R)$ where

$$\phi(R) = \frac{e^{-ik_jR}}{4\pi R}. \tag{21}$$

$$k_j = +\frac{\omega_j}{c}\sqrt{(n^2 + \chi_j) - \frac{i\sigma}{\omega_j\varepsilon_0}} \; . \tag{22}$$

with $R = |\overline{r-r'}|$.

To apply this Green's function to a closed resonator with a homogeneous boundary, the scalar products of $\bar{\bar{G}}$ with Eq. (19) and $\overline{A}_j(\overline{r})$ with Eq. (20) are taken, the two resulting equations are subtracted, and the difference is integrated over the volume inside the resonator to obtain Eq. (23).

$$\overline{A}_j(\overline{r}) = \int_{\tau'} \left\{ \nabla'^2\overline{A}_j(\overline{r}') \cdot \bar{\bar{G}} - \overline{A}_j(\overline{r}') \cdot \nabla'^2\,\bar{\bar{G}} \right\} d\tau' \tag{23}$$

Applying the identity (Collin, ref. 34, p. 60),

$$\nabla' \cdot \left[\overline{A}_j \times (\nabla' \times \bar{\bar{G}}) + (\nabla' \times \overline{A}_j) \times \bar{\bar{G}} + \overline{A}_j\,(\nabla' \cdot \bar{\bar{G}}) - (\nabla' \cdot \overline{A}_j)\,\bar{\bar{G}}\right]$$

$$= \overline{A}_j \cdot \nabla'^2\bar{\bar{G}} - \nabla'^2\overline{A}_j \cdot \bar{\bar{G}} \tag{24}$$

in conjunction with the divergence theorem applied to dyadics (Collin, ref. 34, p. 569), the volume integral is converted to the surface integral shown below

$$\overline{A}_j(\overline{r}) = \oint_{S'} \left\{ (\hat{n} \times \overline{A}_j) \cdot \nabla' \times \bar{\bar{G}} + (\hat{n} \times \nabla' \times \overline{A}_j) \cdot \bar{\bar{G}} + (\hat{n} \cdot \overline{A}_j)\nabla' \cdot \bar{\bar{G}} - (\hat{n} \cdot \bar{\bar{G}})\nabla \cdot \overline{A}_j \right\} dS' \tag{25}$$

where $\hat{n}$ points into the resonator volume, and S' is the surface just inside the resonator boundary. Then substituting $\overline{H}_j$ and $\overline{E}_j$ for $\overline{A}_j$ and using Eqs. (14) and (15) to eliminate $\nabla \times \overline{E}_j$ and $\nabla \times \overline{H}_j$, the following pair of equations is obtained.

$$\overline{E}_j(\overline{r}) = \oint_{S'} \Big\{ (\hat{n} \times \overline{E}_j) \cdot \nabla' \times \bar{\bar{G}} - i\omega_j\mu\, (\hat{n} \times \overline{H}_j) \cdot \bar{\bar{G}} + (\hat{n} \cdot \overline{E}_j)\nabla' \cdot \bar{\bar{G}} \Big\} dS' \tag{26}$$

$$\overline{H}_j(\overline{r}) = \oint_{S'} \Big\{ (\hat{n} \times \overline{H}_j) \cdot \nabla' \times \bar{\bar{G}} + i\omega_j\tilde{\varepsilon}\, (\hat{n} \times \overline{E}_j) \cdot \bar{\bar{G}} + (\hat{n} \cdot \overline{H}_j)\nabla' \cdot \bar{\bar{G}} \Big\} dS' \tag{27}$$

Equations (26) and (27) are the integral equations which govern the behavior of the modal fields within closed resonators with homogeneous boundaries. However, before applying these equations throughout the resonator volume, they must be specialized to apply to points on the surface S', and the specializations must be solved simultaneously for $(\hat{n} \times \overline{E}_j)$ and $(\hat{n} \times \overline{H}_j)$. When the surface fields have been determined, Eqs. (26) and (27) can be used to compute the fields within the resonator volume.

To compute the fields outside the resonator, one must specify how the fields behave as the surface S' is crossed. This specification, which simply amounts to describing the particular resonator boundary or interface, will yield known values for the field over the closed surface just outside the resonator boundary. This knowledge of the tangential fields over the closed surface uniquely specifies the fields throughout any bounded volume outside the resonator (Harrington, ref. 32, p. 102).

It is interesting to note that Eqs. (26) and (27) can be applied directly to a closed resonator with a perfectly conducting boundary ($\hat{n} \times \overline{E} = 0$). No modification of the equations at any point is required as the functions described by the closed surface integrals change discontinuously as the surface S' is crossed (from region a to region b in figure 5). These discontinuous changes are such that the fields outside S' vanish identically (Stratton, ref. 33, p. 468), thus corresponding to the fields of a closed conducting boundary.

Closed Resonators with Segmented Boundaries. In the above derivation, the divergence theorem was applied to a closed surface separating the resonator from its surroundings. That application is completely justified as the fields and their derivatives are continuous throughout the volume of a closed, homogeneous resonator with a smooth, homogeneous boundary. However, if the resonator boundary is segmented, discontinuities in the fields and their derivatives may occur at the intersection of the different materials in the boundary surface. As a result, any derivation applied to a resonator with a segmented boundary must allow for the presence of discontinuities in the fields at points along the resonator boundary.

As discussed in Stratton (ref. 33, p. 468), the presence of such discontinuities can be reconciled with the field equations only if one assumes the existence of charges or currents at the points of discontinuity. These sources produce fields which, when added to the fields in Eqs. (26) and (27), yield net fields that satisfy Maxwell's equations

(Baker and Copson, ref. 35, pp. 114-117; appendix B of this paper). These net fields are given by Eqs. (28) and (29) below,

$$\bar{E}_j(\bar{r}) = \sum_m \int_{S_m'} \Big\{ (\hat{n} \times \bar{E}_{jm}) \cdot \nabla' \times \bar{\bar{G}} + (\hat{n} \times \nabla' \times \bar{E}_{jm}) \cdot \bar{\bar{G}}$$

$$+ (\hat{n} \cdot \bar{E}_{jm})\nabla' \cdot \bar{\bar{G}} \Big\} dS_m'$$

$$- \frac{1}{i\omega_j \tilde{\varepsilon}} \sum_m \oint_{C_m} \nabla' \phi \bar{H}_{jm} \cdot d\bar{s}_m \tag{28}$$

$$\bar{H}_j(\bar{r}) = \sum_m \int_{S_m'} \Big\{ (\hat{n} \times \bar{H}_{jm}) \cdot \nabla' \times \bar{\bar{G}} + (\hat{n} \times \nabla' \times \bar{H}_{jm}) \cdot \bar{\bar{G}}$$

$$+ (\hat{n} \cdot \bar{H}_{jm}) \nabla' \cdot \bar{\bar{G}} \Big\} dS_m'$$

$$+ \frac{1}{i\omega_j \mu} \sum_m \oint_{C_m} \nabla' \phi \bar{E}_m \cdot d\bar{s}_m \tag{29}$$

where S_m' is the surface of the m^{th} continuous segment of the boundary, C_m is the contour enclosing S_m', and $\bar{E}_m$ and $\bar{H}_m$ are the fields just inside C_m.

Thus, for resonators with segmented boundaries, it is Eqs. (28) and (29) which must be specialized to apply to the tangential fields on the surface S'. Once these specializations are obtained, the procedure for computing the fields at other points in space is the same as the one described above for closed resonators with homogeneous boundaries.

However, for either case, the final equations may be simplified somewhat. This simplification results from taking the curl of the expressions for $\overline{E}_j$ and $\overline{H}_j$, applying the fact that $\nabla' \cdot \bar{\bar{G}} = \nabla'\phi = -\nabla\phi$, and using Eqs. (14) and (15). The final result is the pair of equations shown below,

$$\overline{H}_j(\overline{r}) = \int_{S'} \left\{ \frac{i}{\omega_j \mu} (\hat{n} \times \overline{E}_j) \cdot \nabla' \times \bar{\bar{G}} \times \overleftarrow{\nabla} + \nabla \times \bar{\bar{G}} \cdot (\hat{n} \times \overline{H}_j) \right\} dS' \qquad (30)$$

$$\overline{E}_j(\overline{r}) = \int_{S'} \left\{ \frac{-i}{\omega_j \tilde{\varepsilon}} (\hat{n} \times \overline{H}_j) \cdot \nabla' \times \bar{\bar{G}} \times \overleftarrow{\nabla} + \nabla \times \bar{\bar{G}} \cdot (\hat{n} \times \overline{E}_j) \right\} dS' \qquad (31)$$

where the surface integrals are evaluated over the closed boundary with any points of discontinuity removed. (See appendix B.)

Integral Equations for Open Resonators

For open resonators, these integral equations can be simplified further by eliminating that portion of the closed surface integral which corresponds to the open surface of the resonator hull. To do that, the open resonator is considered to be a special case of a closed resonator with a segmented boundary. Then, Eqs. (28) and (29) are applied to

1. The resonator hull, and
2. The closed surface consisting of the backs of the mirrors and the open portion of the hull.

The equations for $\overline{E}$ and $\overline{H}$ in these two cases are then subtracted to yield the following pair of equations for the fields,

$$\bar{E}_j(\bar{r}) = \sum_{m=1}^{4} \int_{S'_m} \left\{ (\hat{n} \times \bar{E}_{jm}) \cdot \nabla' \times \bar{\bar{G}} + (\hat{n} \times \nabla' \times \bar{E}_{jm}) \cdot \bar{\bar{G}} \right.$$

$$\left. + (\hat{n} \cdot \bar{E}_{jm}) \nabla' \cdot \bar{\bar{G}} \right\} dS'_m - \frac{1}{i\omega_j \tilde{\varepsilon}} \sum_{m=1}^{4} \oint_{C_m} \nabla' \phi \bar{H}_m \cdot d\bar{s}_m \qquad (32)$$

$$\bar{H}_j(\bar{r}) = \sum_{m=1}^{4} \int_{S'_m} \left\{ (\hat{n} \times \bar{H}_{jm}) \cdot \nabla' \times \bar{\bar{G}} + (\hat{n} \times \nabla' \times \bar{H}_{jm}) \cdot \bar{\bar{G}} \right.$$

$$\left. + (\hat{n} \cdot \bar{H}_{jm}) \nabla' \cdot \bar{\bar{G}} \right\} dS'_m + \frac{1}{i\omega_j \mu} \sum_{m=1}^{4} \int_{C_m} \nabla' \phi \bar{E}_m \cdot d\bar{s}_m \qquad (33)$$

where a plane view of the surfaces and contours is shown in Figure 7.

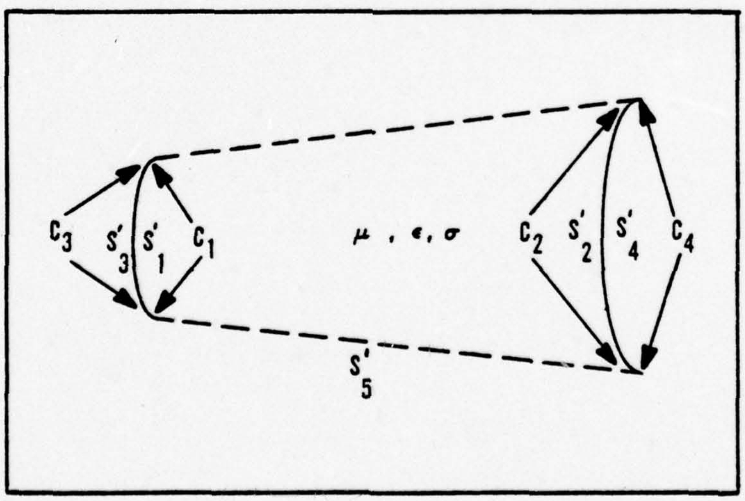

Figure 7. A Plane View of the Surfaces of an Open Resonator

The next step is to specialize these equations to apply to laser resonator modes at points inside the resonator hull. For these fields, the effect of the fields on the backs of the mirrors (S_3' and S_4') can be dropped. This is justified because S_3' and S_4' are outside the resonator, and laser resonator modes are fields for which the vast majority of the energy flows from the resonator to the surroundings.

It should be apparent that this procedure (of neglecting S_3' and S_4') will yield accurate results only to the degree that laser resonators actually produce fields such that the energy flow from the surroundings is negligible. For some or perhaps all resonators, there may be some modes for which no energy flows into the resonator. However, for a resonator operating at a low frequency, it is easy to imagine that the fields on the backs of the mirrors could affect the field inside the hull. Conversely, for resonators containing mirrors much larger than the wavelength of operation (which corresponds to the conditions for the physical optics approximation), the effect of the fields on S_3' and S_4' should be negligible. As a result, they will be neglected for the remainder of this analysis, and the sums in Eqs. (32) and (33) will be evaluated from m=1 to m=2.

As with the closed resonator case, one takes the curl of the two equations, applies $\nabla \times \nabla\phi = 0$, and uses Eqs. (14) and (15) to eliminate $\nabla \times \overline{E}_j$ and $\nabla \times \overline{H}_j$. The resulting equations, which are the chief result of this section, are shown below.

$$\overline{H}_j(\overline{r}) = \sum_{m=1}^{2} \int_{S_m'} \left\{ \frac{i}{\omega_j \mu} (\hat{n}\times\overline{E}_{jm})\cdot\nabla' \times \overline{\overline{G}} \times \overleftarrow{\nabla} + \nabla \times \overline{\overline{G}}\cdot(\hat{n}\times\overline{H}_{jm}) \right\} dS_m' \qquad (34)$$

$$\overline{E}_j(\overline{r}) = \sum_{m=1}^{2} \int_{S'_m} \left\{ \frac{-i}{\omega_j \tilde{\varepsilon}} (\hat{n} \times \overline{H}_{jm}) \cdot \nabla' \times \overline{\overline{G}} \times \overleftarrow{\nabla} + \nabla \times \overline{\overline{G}} \cdot (\hat{n} \times \overline{E}_{jm}) \right\} dS'_m \tag{35}$$

Open Resonators with Perfectly Conducting Mirrors

For resonators with perfectly conducting mirrors, the fields on the mirror surfaces satisfy the boundary conditions

$$\hat{n} \times \overline{E}_j(\overline{r}') = 0 \tag{36}$$

$$\hat{n} \times \overline{H}_j(\overline{r}') = \overline{J}_j(\overline{r}') \tag{37}$$

Substituting these two equations into Eqs. (34) and (35), the basic equations for this case take the form shown below.

$$\overline{H}_j(\overline{r}) = \sum_{m=1}^{2} \int_{S'_m} \left[\nabla \times \overline{\overline{G}} \cdot \overline{J}_{jm} \right] dS'_m \tag{38}$$

$$\overline{E}_j(\overline{r}) = - \frac{i}{\omega_j \tilde{\varepsilon}} \sum_{m=1}^{2} \int_{S'_m} \left[\overline{J}_{jm} \cdot \nabla' \times \overline{\overline{G}} \times \overleftarrow{\nabla} \right] dS'_m \tag{39}$$

Now if Eqs. (37) and (38) can be manipulated to yield forms which can be solved for the $\overline{J}_{jm}$, then Eqs. (38) and (39) can be used to compute the fields throughout the resonator volume. The procedure for obtaining equations for the $\overline{J}_{jm}$ is the subject of the next several pages, while the methods used for solving these equations are discussed in Chapter IV.

The Derivation. To begin the procedure, the cross product between Eq. (38) and an arbitrary unit vector $\hat{u}$ is computed. Then the surface current $\overline{J}_{jm}$ is expressed in terms of its rectangular components, and the resulting equation is projected onto the x, y, and z axes to obtain the following three equations relating the rectangular components of $\overline{H}_j$ and $\overline{J}_j$,

$$\hat{a}_x \cdot [\hat{u} \times \overline{H}(\overline{r})] = \sum_{m=1}^{2} \hat{u} \cdot \int_{S'_m} \Big[\hat{a}_y (J_{ym}\phi_x - J_{xm}\phi_y) - \hat{a}_z (J_{xm}\phi_z - J_{zm}\phi_x) \Big] \, dS'_m \qquad (40)$$

$$\hat{a}_y \cdot [\hat{u} \times \overline{H}(\overline{r})] = \sum_{m=1}^{2} \hat{u} \cdot \int_{S'_m} \Big[\hat{a}_z (J_{zm}\phi_y - J_{ym}\phi_z) - \hat{a}_x (J_{ym}\phi_x - J_{xm}\phi_y) \Big] \, dS'_m \qquad (41)$$

$$\hat{a}_z \cdot [\hat{u} \times \overline{H}(\overline{r})] = \sum_{m=1}^{2} \hat{u} \cdot \int_{S'_m} \Big[\hat{a}_x (J_{xm}\phi_z - J_{zm}\phi_x) - \hat{a}_y (J_{zm}\phi_y - J_{ym}\phi_z) \Big] \, dS'_m \qquad (42)$$

where the subscript j has been dropped and $\phi_x \equiv \frac{\partial \phi}{\partial x}$, etc.

From these equations, which apply to resonators containing mirrors of arbitrary curvature, it is evident that the rectangular components of the current induced on the mirrors and the magnetic field within the resonator are related in a very complicated manner. However, of the open resonators which find practical application, a large fraction employs mirrors for which

$$\frac{a}{|R|} << 1 \tag{43}$$

where a is the radius and R is the radius of curvature of the mirror. For these mirrors, the surface normal is nearly parallel to the optic axis, and the component of current parallel to that axis is very small. Since the effect of these deviations from currents which are transverse and mirror normals which are parallel to the optic axis is likely to be small, their effect will be neglected for the remainder of the analysis.

It is worth noting that for plane parallel resonators there is no deviation from transverse currents or longitudinal mirror normals. Although this fact does not prove anything for curved mirrors for which $\frac{a}{|R|} << 1$, it does support the idea that, for these mirrors, the longitudinal component of the current and the transverse components of the mirror normals are negligible. Then setting $J_z = 0$ and letting $\hat{u} = \pm\hat{a}_z$ in Eqs. (40) through (42), the following pair of equations is obtained.

$$\hat{a}_x \cdot [\pm\hat{a}_z \times \overline{H}(\overline{r})] = \mp \sum_{m=1}^{2} \int_{S_m'} J_{xm}\phi_z \; dS_m' \tag{44}$$

$$\hat{a}_y \cdot [\pm\hat{a}_z \times \overline{H}(\overline{r})] = \mp \sum_{m=1}^{2} \int_{S_m'} J_{ym}\phi_z \; dS_m' \tag{45}$$

These equations are identical in form and the x and y components of the current are not coupled. Then without loss of generality, only the x-component of the current will be treated for the remainder of this paper.

To convert Eq. (44) into integral equations in which the only unknowns are the currents on the mirrors, the field point, with position vector $\bar{r}$, will be allowed to approach an arbitrary point on each mirror surface in the limit as the distance between the field point and the mirror goes to zero. In each case, the approach will be made from within the resonator. As the field point approaches first one mirror and then the other, the appropriate sign will be chosen so that $\pm\hat{a}_z$ corresponds to the unit normal at the point of interest on the surface.

Then letting the field point approach a point with position vector $\bar{r}_1$ on mirror #1, and choosing the upper sign in Eq. (44), Eq. (46) is obtained,

$$J_{x1}(\bar{r}_1) = - \lim_{\bar{r}\to\bar{r}_{1+}} \frac{\partial}{\partial\hat{n}_1} \int_{S_1'} J_{x1}(\bar{r}_1')\phi dS_1' - \int_{S_2'} J_{x2}(\bar{r}_2')\phi_z\Big|_{\bar{r}=\bar{r}_1} dS_2' \tag{46}$$

where Eq. (37) and the relation $\phi_z = \hat{a}_z \cdot \nabla\phi$ with $\hat{n}_1 = \hat{a}_z$ have been employed. Using the expression for ϕ given in Eq. (21), the normal derivative of ϕ is written in the form shown below.

$$\frac{\partial\phi}{\partial\hat{n}_1} = e^{-ikR_{12}} (1 + ikR_{12})\hat{n}_1 \cdot \nabla\left(\frac{1}{4\pi R_{12}}\right) \tag{47}$$

This expression is used in Eq. (46) to rewrite the limit in the form

$$\lim_{\overline{r}\to\overline{r}_{1+}} \frac{\partial}{\partial \hat{n}_1} \int_{S_1'} J_{x1}(\overline{r}_1')\phi dS_1'$$

$$= \lim_{\overline{r}\to\overline{r}_{1+}} \int_{S_1'} u(\overline{r}_1')\ \hat{n}_1 \cdot \nabla\left(\frac{1}{4\pi R_{12}}\right) dS_1' \tag{48}$$

where

$$u(\overline{r}_1') = J_{x1}(\overline{r}_1')\ e^{-ikR_{12}}\ (1 + ikR_{12}) \tag{49}$$

Using Eq. (50) to evaluate this limit (Stakgold, ref. 36, vol. 2, p. 119),

$$\lim_{\overline{r}\to\overline{r}_{1+}} \int_{S_1'} u(\overline{r}_1')\hat{n}_1 \cdot \nabla\left(\frac{1}{4\pi R_{12}}\right) dS_1'$$

$$= -\frac{1}{2}\, u(\overline{r}_1) + \int_{S_1'} u(\overline{r}_1')\ \frac{\cos\ (\overline{r_1 - r_1'}, \hat{n}_1)}{4\pi|\overline{r_1 - r_1'}|^2}\ dS_1' \tag{50}$$

and substituting the result into Eq. (46), the following equation relating the currents on the mirrors is obtained.

$$J_{x1}(\overline{r}_1) = -\,2\int_{S_1'} J_{x1}(\overline{r}_1') \left[\frac{1 + ik|\overline{r_1 - r_1'}|}{|\overline{r_1 - r_1'}|}\right]$$

$$\phi(\overline{r}_1|\overline{r}_1')\ \cos(\overline{r_1 - r_1'}, \hat{n}_1)\ dS_1' - 2\int_{S_2'} J_{x2}(\overline{r}_2')\ \phi_z\Big|_{\overline{r}=\overline{r}_1} dS_2' \tag{51}$$

A similar procedure yields the following expression for the current on mirror #2.

$$J_{x2}(\overline{r}_2) = 2\int_{S_1'} J_{x1}(\overline{r}_1')\phi_z\Big|_{\overline{r}=\overline{r}_2} dS_1'$$

$$+ 2\int_{S_2'} J_{x2}(\overline{r}_2')\left[\frac{1 + ik|\overline{r_2 - r_2'}|}{|r_2 - r_2'|}\right]\phi(\overline{r}_2|\overline{r}_2')\cos(\overline{r_2 - r_2'}, \hat{n}_2)dS_2' \qquad (52)$$

Classification of the Modes. Based on the analysis presented on the past few pages, the problem of determining the modes for laser resonators with perfectly conducting mirrors has been reduced to solving the following pair of integral equations:

$$J_{x1} = \hat{K}_{11}J_{x1} + \hat{K}_{12}J_{x2} \qquad (53)$$

$$J_{x2} = \hat{K}_{21}J_{x1} + \hat{K}_{22}J_{x2} \qquad (54)$$

where

$$\hat{K}_{q\ell}J_{x\ell} = 2(-1)^q\int_{S_\ell'} J_{x\ell}(\overline{r}_\ell')\phi_z(\overline{r}|\overline{r}_\ell')\Big|_{\overline{r}=\overline{r}_\ell} dS_\ell' \qquad (55)$$

$$\hat{K}_{qq}J_{xq} = 2(-1)^q\int_{S_q'} J_{xq}(\overline{r}_q')\left\{\frac{1 + ik|\overline{r - r_q'}|}{|\overline{r - r_q'}|}\right\}\phi(\overline{r}_q'|\overline{r}_q)$$

$$\cos(\overline{r_q - r_q'}, \hat{n}_q)dS_q' \qquad (56)$$

The solutions of this pair of equations, which are laser resonator modes, can be divided into the three classes defined below.

1. A Class I mode is defined to be a mode for which the self-induction terms in Eqs. (53) and (54) are negligible in comparison to the mutual-induction terms; that is $|\hat{K}_{qq}J_{xq}| << |J_{xq}|$.
2. A Class II mode is defined to be a mode for which the self-induction terms are proportional to the mutual-induction terms; that is, $\hat{K}_{qq}J_{xq} = \gamma_q \hat{K}_{q\ell}J_{x\ell}$.
3. A Class III mode is any mode not falling into Classes I or II. Since the modes falling into this last class are beyond the scope of this paper, they will not be considered further.

For the modes falling into Class II, Eqs. (53) and (54) take the form shown below.

$$J_{x1} = (1 + \gamma_1)\ \hat{K}_{12}J_{x2} \tag{57}$$

$$J_{x2} = (1 + \gamma_2)\ \hat{K}_{21}J_{x1} \tag{58}$$

Substituting Eq. (58) into Eq. (57) and vice versa, one obtains the familiar eigenvalue problems of the form,

$$J_{x1} = (1 + \gamma_1)(1 + \gamma_2)\ \hat{K}_{12}\hat{K}_{21}J_{x1} \tag{59}$$

$$J_{x2} = (1 + \gamma_2)(1 + \gamma_1)\ \hat{K}_{21}\hat{K}_{12}J_{x2} \tag{60}$$

which apply for all values of k such that $k'>0$.

It should be apparent that not all of the solutions to Eqs. (59) and (60) are solutions to the more general set, Eqs. (53) and (54). Thus, not all solutions of Eqs. (59) and (60) are laser resonator modes. For such solutions to be Class II modes, they must also satisfy the consistency conditions,

$$\hat{K}_{11}J_{x1} = \gamma_1\hat{K}_{12}J_{x2} \tag{61}$$

$$\hat{K}_{22}J_{x2} = \gamma_2\hat{K}_{21}J_{x1} \tag{62}$$

for all points on mirrors #1 and #2. However, due to the difficulty involved in demonstrating these consistency conditions, this potentially important class of modes will not be considered further in this paper.

For Class I modes, Eqs. (53) and (54) take the form,

$$J_{x1} = \hat{K}_{12}J_{x2} \tag{63}$$

$$J_{x2} = \hat{K}_{21}J_{x1} \tag{64}$$

As with Class II modes, one can obtain an eigenvalue formulation by substituting for J_{x2} and vice versa. The result is the following pair of equations.

$$J_{x1} = \hat{K}_{12}\hat{K}_{21}J_{x1} \tag{65}$$

$$J_{x2} = \hat{K}_{21}\hat{K}_{12}J_{x2} \tag{66}$$

However, not all solutions of Eqs. (65) and (66) are solutions to Eqs. (53) and (54). Thus, for solutions of Eqs. (65) and (66) to be laser resonator modes, they must satisfy the consistency conditions

$$|\hat{K}_{11}J_{x1}| << |J_{x1}| \tag{67}$$

$$|\hat{K}_{22}J_{x2}| << |J_{x2}| \tag{68}$$

at all points on the resonator mirrors.

In appendix C, calculations are performed to estimate the conditions under which the inequalities in Eqs. (67) and (68) will apply.

For mirrors such that $a << |R|$ and $a < \frac{L}{\beta_g}\left\{1 - \sqrt{\frac{\lambda\beta_g}{2L}}\right\}$, this estimate (which was obtained by specializing Eqs. (67) and (68) to apply to azimuthally symmetric modes at the center of circular mirrors) corresponds to the conditions shown below,

$$\lambda << \frac{2L}{\beta_g} \tag{69}$$

$$\frac{1}{2|R|} e^{+k_j''\left(a + \frac{\beta_g}{2L} a^2\right)} \left[\frac{\lambda}{4}\sqrt{1 + \left(\frac{\pi}{4}\right)^2} + \frac{\pi}{2} a\right] << 1 \tag{70}$$

where $\beta_g = \sqrt{g^2 - 1}$. Other conditions, which correspond to cases where $a > \frac{L}{\beta_g}\left\{1 - \sqrt{\frac{\lambda\beta_g}{2L}}\right\}$, are also given in appendix C.

Not surprisingly, these estimates of the consistency conditions yield rather complex results that depend on λ, a, $|R|$, L, and the values of k'' for each candidate mode. For ranges of these parameters where these conditions are marginal, the rigorous conditions of Eqs. (67) and (68) should be applied.

Then assuming that the inequalities in Eqs. (69) and (70) hold, the equations for Class I modes take the form

$$J_{x2}(\bar{r}_2) = -\frac{ik}{2\pi}\int_{S_1'} J_{x1}(\bar{r}_1')\frac{e^{-ikR_{21}}}{R_{21}}\cos\alpha_{21}\, dS_1' \tag{71}$$

$$J_{x1}(\bar{r}_1) = -\frac{ik}{2\pi}\int_{S_2'} J_{x2}(\bar{r}_2')\frac{e^{-ikR_{12}}}{R_{12}}\cos\alpha_{12}\, dS_2' \tag{72}$$

where it has been assumed that $|k| >> 1/R_{ij}$ and α_{ij} is the angle between R_{ij} and the optic axis. Finally, substituting Eq. (72) into Eq. (71) and vice versa, one obtains the following pair of symmetric integral equations for the currents on the mirrors.

$$J_{x1}(\bar{r}_1) = \left(\frac{ik}{2\pi}\right)^2 \int_{S_1'} J_{x1}(\bar{r}_1') \int_{S_2'} \frac{e^{-ik(R_{12} + R_{21})}}{R_{12}\, R_{21}} \cos\alpha_{12} \cos\alpha_{21}\, dS_2'\, dS_1' \tag{73}$$

$$J_{x2}(\bar{r}_2) = \left(\frac{ik}{2\pi}\right)^2 \int_{S_2'} J_{x2}(\bar{r}_2') \int_{S_1'} \frac{e^{-ik(R_{21} + R_{12})}}{R_{21}\, R_{12}} \cos\alpha_{21} \cos\alpha_{12}\, dS_1'\, dS_2' \tag{74}$$

This last pair of equations bears considerable resemblance to the integral equations that are normally used (Fox and Li, ref. 1, p. 454) to analyze the modes of laser resonators. However, three important differences are listed below.

1. Eqs. (73) and (74) are not in the form of a linear eigenvalue problem. In this paper, the eigenvalue k is not only a multiplier, but it is also included in the integrand.
2. The obliquity factor, $\cos\alpha_{12}$, used in this paper is different from the factor, $1/2\ (1 + \cos\alpha_{12})$, which is normally used (Fox and Li, ref. 1, p. 454).

3. Eqs. (73) and (74) apply to the current distribution induced on perfectly conducting mirrors, while the equations normally used are widely considered to apply to the field (electric or magnetic) on resonator mirrors of any material.

These first two differences, which correspond to differences in the form of the equations, can have a significant effect on the mode distributions and losses. However, for some of the modes of paraxial resonators, the effects of these two differences in form are negligible. Although the third difference will have no effect on the forms of the solutions obtained, it does affect the physical interpretation of the terms in the equations as well as the conditions under which the equations can be applied. Finally, it is important to note that, (1) not all solutions of Eqs. (73) and (74) are resonator modes; and (2) these modes are not the only modes of the resonator. Other possible modes are Class II and III laser resonator modes and the entire set of incoming wave modes (which have not been addressed in this paper).

In any case, to determine the Class I modes of a laser resonator with perfectly conducting mirrors, a procedure must be developed for solving Eqs. (73) and (74). That procedure, which must specify simultaneously both the real and imaginary parts of k as well as the current distributions on the two mirrors, is the subject of the next chapter.

IV. Solution of the Integral Equation

The purpose of this chapter is to discuss the general problem of solving the integral equations derived in the previous chapter and to present the method, based on a variational principle, to be used to obtain approximate solutions to these equations for open resonators satisfying the paraxial approximation. To accomplish these objectives, this chapter begins with a discussion of the general form of the integral equations and the relationships between the solutions to these equations and the fundamental resonator parameters. The paraxial approximation is then discussed and applied to the general equations to obtain an eigenvalue formulation for paraxial resonators. Following the formulation, the variational method for obtaining approximate solutions to these equations is presented and discussed.

The Basic Problem

As indicated at the end of the preceding chapter, the first step in determining the modal properties of open resonators is solving Eqs. (73) and (74). To solve these equations, the quantities J_{x1}, J_{x2}, and k must be specified simultaneously for each mode. Once k is known, the resonator mode and material parameters may be determined using Eq. (22), which is repeated without subscripts below.

$$k = + \frac{\omega}{c} \sqrt{(n^2 + \chi) - \frac{i\sigma}{\omega\varepsilon_0}} \qquad (22)$$

Writing the square root in series form, assuming that $\frac{1}{n^2}\left|\chi - \frac{i\sigma}{\omega\varepsilon_0}\right| << 1$, and writing k, ω, and χ in terms of their real (') and imaginary ('') parts, the following relationships are obtained.

$$k' = \frac{\omega' n}{c}\left(1 + \frac{\chi'}{2n^2}\right) - \frac{\omega''}{2nc}\chi'' \tag{75}$$

$$k'' = \frac{\omega'' n}{c}\left(1 + \frac{\chi'}{2n^2}\right) + \frac{\omega'\chi''}{2nc} - \frac{\sigma}{2nc\varepsilon_0} \tag{76}$$

Using these relations with the computed values of k' and k" (which have been obtained as part of the solution of the integral equations) and the given values of four of the parameters ω', ω'', χ', χ'', n, and σ, the remaining two quantitites can be determined.

The main difficulty with the general approach just described lies in obtaining the solutions to Eqs. (73) and (74). This is especially true when information concerning higher loss modes is desired. One possible approach to obtaining such solutions begins with an iteration approach similar to that used by other workers (refs. 1, 26) in the field. By applying that procedure to Eqs. (73) and (74), one can determine the current distributions and the value of k for the lowest loss mode. To extend that procedure to the next lowest loss mode, one must insure that each successive approximation for that mode is orthogonal to the solution for the lowest loss mode. Similarly, the approximation for each higher loss mode must be made orthogonal to all of the lower loss modes that have been obtained.

This iteration/orthogonalization procedure is, in general, very difficult and time consuming to apply. However, as shown in the next section, a considerably simpler procedure is available for open resonators.

Formulation of the Eigenvalue Problem for Paraxial Resonators

For many resonators containing mirrors for which $a/|R| \ll 1$, the mirror separation L is much greater than the mirror radius a. Such resonators satisfy the paraxial approximation. Some of the modes of these paraxial resonators satisfy the more stringent condition that, for the j^{th} mode, the quantity $e^{+k_j''R_{12}}$ is accurately approximated by $e^{+k_j''L}$ for any two points on the mirror surfaces. For these paraxial modes, the coupled integral equations take the form below.

$$J_{x1}(\bar{r}_1) = e^{+2k''L} \int_{S_1'} J_{x1}(\bar{r}_1') \left\{ \left(\frac{ik}{2\pi L}\right)^2 \int_{S_2'} e^{-ik'(R_{12} + R_{21})} dS_2' \right\} dS_1' \tag{77}$$

$$J_{x2}(\bar{r}_2) = e^{+2k''L} \int_{S_2'} J_{x2}(\bar{r}_2') \left\{ \left(\frac{ik}{2\pi L}\right)^2 \int_{S_1'} e^{-ik'(R_{21} + R_{12})} dS_1' \right\} dS_2' \tag{78}$$

For resonators operating at or near optical wavelengths, the relations $k' >> k''$ and $\omega' >> \omega''$ hold for all but the lossiest modes. Using these relations, Eq. (77) is rewritten in the form

$$J_{x1}(\bar{r}_1) = e^{+2k''L} \int_{S_1'} J_{x1}(\bar{r}_1') K_{k'}(\bar{r}_1|\bar{r}_1') dS_1' \tag{79}$$

where

$$K_{k'}(\bar{r}_1|\bar{r}_1') = \left(\frac{i}{\lambda L}\right)^2 \int_{S_2'} e^{-ik'(R_{12} + R_{21})} dS_2' \tag{80}$$

Using this symmetric kernel, which is not a function of k'', Eq. (79) is now cast in the form of a linear eigenvalue problem with an external parameter k'.

$$J_{x1}(\bar{r}_1) = \gamma \int_{S_1'} J_{x1}(\bar{r}_1') K_{k'}(\bar{r}_1|\bar{r}_1') dS_1' \qquad (81)$$

Then subject to the constraint of finding the proper value of k' such that $k'' = \frac{1}{2L} \ln \gamma$, where γ is real and positive, the problem of determining the modes has been reduced to a linear eigenvalue problem.

When the precise frequency spectrum is not required, this constraint on γ can be relaxed without significantly affecting the current distributions or the damping rates for the modes of interest in this analysis (relatively low loss modes at optical or infrared wavelengths). This relaxation is a result of the relatively high mode density which is characteristic of even lossy resonators at the wavelengths of interest. This high mode density limits the difference between an arbitrarily chosen value of k' and one which will produce a real, positive value of γ. For example, the axial mode spacing of a plane parallel resonator in free space with mirror separation L is

$$\Delta f_a = c/2L \qquad (82)$$

As the maximum shift off a resonance is one-half of this spacing, the maximum shift or error in the value of k' (at threshold and line center) is $\pi/2L$. This maximum shift corresponds to a relative change in the resonator Fresnel number, $N = a^2/\lambda L$, of

$$\Delta N = a^2/4L^2 \qquad (83)$$

where a is the mirror radius.

For paraxial resonators of all but the largest magnification, this small change in the Fresnel number will have negligible effect on the modal parameters. As a result, the requirement that γ must be real and positive will be dropped. This reduces the analysis to solving an eigenvalue problem (with complex eigenvalue) for any value of k' corresponding to optical or infrared wavelengths. As complex values of γ will now be allowed, the expression for k" must now be written

$$k'' = \frac{1}{2L} \ln |\gamma| \tag{84}$$

For certain cases, the problem can be simplified further. Those cases correspond to problems for which the spatial dependence of the kernel, $K_{k'}(\bar{r}_1|\bar{r}_1')$, can be separated or the dependence on one coordinate can be solved in closed form. In the first case, which corresponds to resonators with rectangular mirrors, the separation of variables reduces to solving two eigenvalue problems as shown below.

$$u(x_1) = \gamma_1 \int u(x_1')U_{k'}(x_1|x_1')dx_1' \tag{85}$$

$$v(y_1) = \gamma_2 \int v(y_1')V_{k'}(y_1|y_1')dy_1' \tag{86}$$

with $e^{+2k''L} = |\gamma_1\gamma_2|$, $J_{x1}(\bar{r}_1) = u(x_1)v(y_1)$, and $K_{k'}(\bar{r}_1|\bar{r}_1') = U_{k'}(x_1|x_1')V_{k'}(y_1|y_1')$.

In the second case, which corresponds to mirrors of circular cross section (where the azimuthal dependence can be solved exactly), a single eigenvalue equation involving only the radial coordinate results. The corresponding equation has the form

$$J_{x1}(\rho_1) = \gamma \int J_{x1}(\rho_1')K_{k'}(\rho_1|\rho_1')d\rho_1' \qquad (87)$$

with $e^{+2k''L} = |\gamma|$ as before.

The Variational Method of Solution

To see how a variational approach can be applied to these eigenvalue problems, it is useful to determine the variational principle (or problem) satisfied by the eigensolutions of the type under consideration. To determine that principle, one first considers eigensolutions $\bar{\gamma}$ and $u(x)$ which satisfy

$$\bar{\gamma}u(x) = \int_{x'} u(x')K(x|x')dx' \qquad (88)$$

where $K(x|x')$ is symmetric with respect to x and x', and $\bar{\gamma} = \gamma^{-1}$. One then multiplies Eq. (88) by an arbitrary function $w(x)$ and integrates the result over x to obtain Eq. (89).

$$\bar{\gamma} \int_x u(x)w(x)dx = \int_x w(x) \int_{x'} u(x')K(x|x')dx'dx \qquad (89)$$

Next, one varies the quantities $\bar{\gamma}$, $w(x)$, and $u(x)$, uses the fact that $K(x|x')$ is symmetric, and recombines terms to obtain the result shown in Eq. (90).

$$\int_x \delta w(x) \left\{ \bar{\gamma}u(x) - \int_{x'} K(x|x')u(x')dx' \right\} dx + \int_x \delta u(x) \left\{ \bar{\gamma}w(x) - \int w(x')K(x|x')dx' \right\} dx = -\delta\bar{\gamma} \int_x u(x)w(x)dx \qquad (90)$$

Letting $w(x) = u(x)$ in this equation, it can be seen that if $\int u^2(x)dx \neq 0$, then the eigensolutions $\overline{\gamma}$ and $u(x)$ satisfy the condition

$$\delta\overline{\gamma} = 0 \tag{91}$$

where

$$\overline{\gamma} = \frac{\int_x u(x) \int_{x'} K(x|x')u(x')dx'dx}{\int_x u^2(x)dx} \tag{92}$$

Thus the eigenfunctions $u(x)$ are those functions which make $\overline{\gamma}$ stationary for all variations about the functions $u(x)$. This implies that if one were to substitute all possible trial functions for $u(x)$ in Eq. (92), by writing $u(x)$ in terms of an infinite number of parameters $\{A_n\}$, $\overline{\gamma}$ would have stationary values for those values of the $\{A_n\}$ which would yield the eigenfunctions $u(x)$. One cannot in practice generate <u>all</u> possible trial functions, but one can generate all possible variations within a given class of functions and then require that $\delta\overline{\gamma} = 0$. To the degree that the chosen class has the capability to represent the modes, one can obtain accurate approximations to these modes. One well-known technique for obtaining such stationary approximations is the Rayleigh-Ritz procedure (Morse and Feshback, ref. 37, vol. 2, p. 1118) described in the following paragraphs.

<u>The Rayleigh-Ritz Procedure</u>. To apply the Rayleigh-Ritz procedure to the eigenvalue problem shown in Eq. (88), one begins with the variational principle just described. One then expands $u(x)$ in terms of a set of known functions $\{\psi_i(x)\}$,

$$u(x) = \sum_i A_i \psi_i(x) \tag{93}$$

and substitutes the result into Eq. (92) to obtain the following expression.

$$\bar{\gamma} \sum_{i,m} A_i A_m \int_x \psi_i(x) \psi_m(x) dx$$

$$= \sum_{i,m} A_i A_m \int_x \psi_i(x) \int_{x'} K(x|x') \psi_m(x') dx' dx \tag{94}$$

The next step in the procedure is to find those values of the independent set $\{A_i\}$ which make $\bar{\gamma}$ stationary. It can be shown that those values satisfy (Morse and Feshback, ref. 37, vol. 2, p. 1119)

$$\frac{\partial \bar{\gamma}}{\partial A_i} = 0 \tag{95}$$

for each value of i. Applying that condition, the following set of equations is obtained for i = 1,2,--,n,

$$\bar{\gamma} \sum_m A_m \Psi_{mi} = \sum_m A_m K_{mi} \tag{96}$$

where

$$\Psi_{mi} = \int_x \psi_m(x) \psi_i(x) dx \tag{97}$$

and

$$K_{mi} = \int_x \psi_m(x) \int_{x'} K(x|x') \psi_i(x') dx' dx \tag{98}$$

This set of homogeneous equations corresponds to the matrix problem

$$\overline{\gamma}\Psi A = KA \tag{99}$$

where Ψ and K are square matrices with elements Ψ_{mi} and K_{mi} as defined above and A is a column matrix with elements A_i.

In selecting the variational method to solve the linear eigenvalue problem, consideration was given to the fact that the eigenfunctions can be normalized so that they obey the orthogonality condition (Siegman and Miller, ref. 28, p. 2730),

$$\int_x u_n(x)u_m(x)dx = \delta_{nm} \tag{100}$$

Then using the Rayleigh-Ritz procedure, *stationary approximations* to the eigenvalues $\overline{\gamma}$, which yield approximations to the orthogonal eigenfunctions, can be found if the matrix elements Ψ_{mi} and K_{mi} can be evaluated. These approximations can also be made to obey Eq. (100) by using the procedure discussed in appendix G to solve the matrix problem.

The General Form of the Matrix Elements. To evaluate the matrix elements, it will be necessary to know the specific form for the $\psi_i(x_1)$ and the kernel $K(x_1|x_1')$. Although these precise forms are covered in the following chapter, the basic equations are summarized below.

Referring to Eq. (80), the kernel $K(x_1|x_1')$ can be written in the general form

$$K(x_1|x_1') = \int_{x_2} K_{12}(x_1|x_2)K_{21}(x_1'|x_2)dx_2 \tag{101}$$

where the integration over x_2 is taken over the space corresponding to the second mirror. Substituting Eq. (101) into Eq. (98) yields

$$K_{mi} = \int_{x_2} dx_2 \left\{ \int_{x_1'} dx_1' \; K_{12}(x_1'|x_2) \; \psi_m(x_1') \right\} \left\{ \int_{x_1} dx_1 \; K_{21}(x_1|x_2) \psi_i(x_1) \right\} \quad (102)$$

To simplify the notation, the function $\Lambda_m(x_2)$ is defined according to Eq. (103) below.

$$\Lambda_m(x_2) = \int_{x_1} K_{12}(x_1|x_2) \; \psi_m(x_1) dx_1 \quad (103)$$

Using this definition, the matrix elements can be compactly summarized as shown below.

$$K_{mi} = \int_{x_2} dx_2 \; \Lambda_m(x_2) \; \Lambda_i(x_2) \quad (104)$$

$$\Psi_{mi} = \int_{x_1} dx_1 \; \psi_m(x_1) \; \psi_i(x_1) \quad (105)$$

As previously indicated, the next step in the procedure for obtaining the modes of open resonators is the application of these general expressions to the particular resonators of interest. This specialization is the subject of the next chapter.

However, before proceeding to that chapter, it may be helpful to summarize the types of resonators and modes for which the basic theory and method of solution just outlined are applicable. First, the basic equations themselves (Eqs. (79) and (80) are applicable only for resonators with perfectly conducting mirrors where the effect of any gain is essentially constant for all points on the resonator mirrors. In addition, these equations are applicable only for resonators for which the oscillation wavelength is small in comparison to all resonator dimensions and where the mirror diameters are small in comparison to the mirror separation and radii of curvature. Finally, these equations are applicable only for cases in which the self-induction integrals in Eqs. (53) and (54) are negligible, and the method of solution should be employed only for cases that do not require knowledge of the precise frequency spectrum.

V. Application of the Variational Method

The purpose of this chapter is to specialize the basic theory and method of solution to open resonators of rectangular or circular projection for which the spatial dependence separates. These specializations include obtaining the specific form of the integral equation as well as the expansion functions for each geometry.

Since it is to be applied to several cases, the general procedure for obtaining these specializations is the first subject of discussion in this chapter. This procedure is then applied to resonators with spherical mirrors of circular projection; and to illustrate its various aspects, the calculations are covered in some detail. Finally, the chapter ends with a similar section for rectangular mirror resonators; however, the details for this case are covered in appendix D.

The General Procedure

Specific Forms of the Integral Equations. The procedure for obtaining the integral equation is usually relatively simple. To apply it, one begins with Eqs. (71) and (72) as the basic forms. One then makes the following assumptions:

1. The mirror radii are much smaller than the mirror radii of curvature,
2. The paraxial approximation holds,
3. $e^{+k''R_{12}} \approx e^{+k''L}$ across the mirror surfaces,
4. The precise frequency spectrum is not of interest,
5. $k' >> k''$.

Using these assumptions and substituting Eq. (72) into Eq. (71) and vice versa, one obtains Eqs. (80) and (81) from chapter IV. Then, the distance between two arbitrary points on the mirrors, which is expressed in terms of the resonator constants and transverse mirror coordinates, is written in series form using a binomial expansion. In this expansion, a sufficient number of terms is kept to accurately approximate the exponential $e^{-ik'R}$ across the mirror surfaces. Finally, where possible, the spatial variation of the equation is separated to yield one or more equations of the type shown in Eqs. (85) through (87).

Selection of Expansion Functions.

Desirable Characteristics--Although the precise forms are somewhat arbitrary, the expansion functions used to calculate the resonator modes should have two desirable characteristics. First, one should be able to establish the precise form of the functions from the resonator parameters with relative ease. Second, the functions should exhibit (to whatever degree is practical) the behavior expected from the actual modes so that

1. Trends can be identified with respect to changes in parameters,
2. Physical insight can be gained into the behavior of the modes,
3. The modes can be adequately represented using a relatively small number of expansion functions.

Excluding the actual modes themselves, three categories that exhibit some of these desired characteristics are listed below:

1. Approximate solutions of the derived resonator integral equation,
2. Exact solutions of the derived integral equation for a resonator similar to the one being considered,

3. Approximate solutions of the derived integral equation for a resonator similar to the one being considered.

Unfortunately, to obtain the functions corresponding to the first two cases, one must solve a problem which is at least as difficult as the original. As a result, the expansion functions will be chosen to be approximate solutions of an integral equation for a resonator similar to the one being considered. The "similar" resonator chosen for each case is a symmetric resonator, which consists of two mirrors such that

1. The mirrors are identical to the one on which the current is being analyzed, and
2. The mirror separation is the same as for the original problem.

Thus, to determine the expansion functions, approximate solutions of integral equations of the following form must be obtained,

$$J(\overline{x}) = \gamma \int_{\overline{x}'} J(\overline{x}')\ K(\overline{x}|\overline{x}')\ d\overline{x}' \tag{106}$$

$$K(\overline{x}|\overline{x}') = \frac{ik'}{2\pi L}\, e^{-\frac{ik'}{2L} N(\overline{x}|\overline{x}')} \tag{107}$$

where the subscript x denoting the x-component of the current has been dropped, and $N(\overline{x}|\overline{x}')$ is symmetric with respect to the spatial coordinates $\overline{x}$ and $\overline{x}'$. As indicated before, when the spatial dependence separates, Eq. (106) reduces to one-dimensional equations with symmetric kernels.

Possible Approaches for Obtaining the Approximate Solutions-- To discuss the procedure for approximately solving these symmetric integral equations, Eq. (106) is shown below in operator notation.

$$J(\overline{x}) = \gamma\hat{K}\ J(\overline{x}) \tag{108}$$

For the first approach, one assumes that an operator $\hat{M}$ can be found such that $\hat{M}$ commutes with $\hat{K}$. Then, by definition, Eq. (109) holds,

$$\hat{M}\,\hat{K}\,J(\overline{x}) = \hat{K}\,\hat{M}\,J(\overline{x}) \tag{109}$$

and the operators $\hat{K}$ and $\hat{M}$ have simultaneous eigenfunctions. This implies that if the eigenfunctions of the operator $\hat{M}$ can be found, the eigenfunctions of $\hat{K}$ can be expressed as a linear combination of them. Thus, if functions $v_s(\overline{x})$ can be found such that

$$\hat{M}\,v_s(\overline{x}) = -s^2 v_s(\overline{x}) \tag{110}$$

then the eigenfunctions $J(\overline{x})$ of $\hat{K}$ can be expressed in the form

$$J_\gamma(\overline{x}) = \sum_s A_{\gamma s}\; v_s(\overline{x}) \tag{111}$$

The particular linear combination to be used in each case can be determined using the variational procedure discussed in chapter IV to specify the $A_{\gamma s}$.

Unfortunately, for all but the simplest cases, finding an operator $\hat{M}$, and its eigenfunctions, so that $\hat{M}$ commutes with $\hat{K}$ is very difficult. However, in some cases it may be possible to find an operator $\hat{M}$ such that the commutator between $\hat{M}$ and $\hat{K}$ is relatively small, especially for low loss modes. For such cases, Eq. (109) must be modified to include this relatively small difference, which is represented by $\hat{R}J(\overline{x})$ in each of the following equations,

$$\hat{M}\,\hat{K}\,J(\overline{x}) = \hat{K}\,\hat{M}\,J(\overline{x}) - \hat{R}\,J(\overline{x}) \tag{112}$$

$$\hat{M}\,J(\overline{x}) = \gamma\hat{K}\,\hat{M}\,J(\overline{x}) - \gamma\hat{R}\,J(\overline{x}) \tag{113}$$

where Eq. (113) was obtained by combining Eq. (108) with Eq. (112). For this case, $\hat{M}$ and $\hat{K}$ do not have simultaneous eigenfunctions, but if $|\hat{R}\, J(\overline{x})|$ is indeed small in comparison to $|\hat{K}\, \hat{M}\, J(\overline{x})|$, the eigenfunctions of $\hat{M}$ may still form an excellent expansion set for the eigenfunctions of $\hat{K}$.

Unfortunately, it may be quite difficult to actually show that Eq. (114) holds.

$$|\hat{R}\, J(\overline{x})| << |\hat{K}\, \hat{M}\, J(\overline{x})| \tag{114}$$

However, even if one cannot find operators $\hat{M}$ and $\hat{R}$ such that Eq. (114) can be shown to hold, it may still be possible to find forms for $\hat{M}$ and $\hat{R}$ such that the eigenfunctions of $\hat{M}$ form an excellent expansion set for the eigenfunctions of $\hat{K}$. With that in mind, the following rather simple procedure is used to obtain the approximate solutions to Eq. (108).

The Chosen Procedure--First, using the operations discussed in the following paragraphs, Eq. (108) will be manipulated to yield an equation of the form shown in Eq. (113). The particular manipulations used will be selected to yield an operator $\hat{M}$ such that one can solve the eigenvalue problem shown in Eq. (110). In addition, the manipulations will be chosen so $\hat{R}\, J(\overline{x})$ contains only terms which either

1. Depend on $J(\overline{x})$ and its derivatives only at the mirror edges, or
2. Involve products of the coordinates of both resonator mirrors.

When Eq. (110) has been solved, the functions $v_s(x)$ will be chosen as the approximate solutions of the symmetric integral equation shown in Eq. (108). For mirrors which are conic sections, this selection will yield expansion functions that are obtained by neglecting the finite sizes of the resonator mirrors.

The operations used to obtain Eq. (113) from Eq. (108) are similar to those used by Bergstein (ref. 2, p. 497) and others (refs. 22, 23). These manipulations are outlined below for resonators such that

1. The mirrors are conic sections, and
2. The integral equation separates.

The approach used for extending this procedure to toroidal mirrors is covered in appendix D.

One begins by separating Eq. (106) to yield one or more of the equations of the form of Eqs. (85) through (87), which are represented by Eq. (115),

$$u(\zeta) = \gamma \int_{\zeta'} u(\zeta')\, K(\zeta|\zeta')\, d\zeta' \tag{115}$$

where $\zeta = x, y,$ or ρ. One then computes the second derivative of Eq. (115) to obtain Eq. (116).

$$\frac{d^2u(\zeta)}{d\zeta^2} = \gamma \int_{\zeta'} u(\zeta')\, \frac{d^2K(\zeta|\zeta')}{d\zeta^2}\, d\zeta' \tag{116}$$

Next, one integrates the term $K(\zeta|\zeta')\,\frac{d^2u(\zeta')}{d\zeta'^2}$ by parts twice to yield

$$\gamma \int_{\zeta'} K(\zeta|\zeta')\, \frac{d^2u(\zeta')}{d\zeta'^2}\, d\zeta' = \gamma \int_{\zeta'} u(\zeta')\, \frac{d^2K(\zeta|\zeta')}{d\zeta'^2}\, d\zeta' + \gamma \hat{R}' u(\zeta') \tag{117}$$

where the expression

$$\hat{R}'u(\zeta') = \left\{K(\zeta|\zeta') \frac{du(\zeta')}{d\zeta'} - u(\zeta') \frac{dK(\zeta|\zeta')}{d\zeta'}\right\} \qquad (118)$$

is evaluated at the corresponding limits of integration. One then subtracts Eq. (117) from Eq. (116) to obtain the following rather complicated relation.

$$\frac{d^2u(\zeta)}{d\zeta^2} = \gamma \int_{\zeta'} K(\zeta|\zeta') \frac{d^2u(\zeta')}{d\zeta'^2} d\zeta' - \gamma\hat{R}'u(\zeta')$$

$$+ \gamma \int_{\zeta'} u(\zeta') \left\{\frac{d^2}{d\zeta^2} - \frac{d^2}{d\zeta'^2}\right\} K(\zeta|\zeta')\, d\zeta' \qquad (119)$$

For symmetric, separable kernels of the type shown in Eq. (107), the derivatives of $K(\zeta|\zeta')$ in Eq. (119) can be written as the sum of three terms as shown below,

$$\left\{\frac{d^2}{d\zeta^2} - \frac{d^2}{d\zeta'^2}\right\} K(\zeta|\zeta') = \left\{- f(g,\zeta) + f(g,\zeta') + h(g,\zeta\zeta')\right\} K(\zeta|\zeta') \qquad (120)$$

where

1. The function $h(g,\zeta\zeta')$ contains all terms involving products of ζ and ζ' (the coordinates of both resonator mirrors),*

2. The function $f(g,\zeta)$ contains all terms involving ζ but not ζ', and

3. The function $f(g,\zeta')$ contains all terms involving ζ' but not ζ.

*For cases where the mirrors are conic sections, $h(g,\zeta\zeta')$ vanishes (Bergstein, ref. 2, p. 497).

Then, substituting Eq. (120) into Eq. (119), and using Eq. (115) to show that

$$\gamma \int_{\zeta'} f(g,\zeta)\ K(\zeta|\zeta')\ \ u(\zeta')d\zeta' = f(g,\zeta)\ u(\zeta) \tag{121}$$

Eq. (119) can be rewritten in the form shown below.

$$\left\{\frac{d^2}{d\zeta^2} + f(g,\zeta)\right\} u(\zeta) = \gamma \int_{\zeta'} K(\zeta|\zeta') \left\{\frac{d^2}{d\zeta'^2} + f(g,\zeta')\right\} u(\zeta')d\zeta'$$

$$+ \gamma \int_{\zeta'} K(\zeta|\zeta')\ h(g,\zeta\zeta')u(\zeta')d\zeta' - \gamma\hat{R}'u(\zeta'). \tag{122}$$

Examining the last two terms in Eq. (122), one can see that

1. $\hat{R}'u(\zeta')$ depends on $u(\zeta')$ and its derivatives only at the limits of integration, and
2. The integral involves products of both resonator coordinates.

Then, subject to the condition that $u(\zeta')$ must be selected so that $\hat{R}'u(\zeta')$ depends on $u(\zeta')$ and its derivatives only at the mirror edges, one identifies $\hat{R}u(\zeta')$ by comparing Eq. (122) with Eq. (113). The result is

$$\hat{R}u(\zeta') = \hat{R}'u(\zeta') - \int_{\zeta'} K(\zeta|\zeta')\ h(g,\zeta\zeta')u(\zeta')d\zeta' \tag{123}$$

Substituting this form into Eq. (122), one obtains the following result.

$$\left\{\frac{d^2}{d\zeta^2} + f(g,\zeta)\right\} u(\zeta) = -\gamma\hat{R}u(\zeta')$$

$$+ \int_{\zeta'} K(\zeta|\zeta') \left\{\frac{d^2}{d\zeta'^2} + f(g,\zeta')\right\} u(\zeta')d\zeta' \qquad (124)$$

Specializing this equation to resonators containing mirrors which are conic sections, and comparing the result with Eq. (113), the operator $\hat{M}$ is identified,

$$\hat{M} = \frac{d^2}{d\zeta^2} + f(g,\zeta) \qquad (125)$$

and the eigenvalue problem to be solved has the form shown below.

$$\frac{d^2u(\zeta)}{d\zeta^2} + f(g,\zeta)\, u(\zeta) = -s^2u(\zeta) \qquad (126)$$

The eigenfunctions obtained by solving Eq. (126) will be too general to be applied to the resonator problem without restrictions. Three appropriate restrictions are listed below.

1. For rectangular mirror resonators, the modes (and the expansion functions) are either even or odd functions.
2. The expansion functions must be finite at all points on the mirrors.
3. The expansion functions must be consistent with other known behavior for the modes.

For unstable resonators, the last condition corresponds to the requirement that as $g \to 1$, the eigenfunctions of $\hat{M}$ must reduce to the known forms for plane parallel resonators. These known forms, which were obtained by Vainshtein (ref. 21, p. 711) using the Waveguide

Analogy and which are to be used as expansion functions for plane parallel resonators, are listed below.

For rectangular mirrors of width 2a and separation 2L, the expansion functions for modes of odd symmetry are

$$f_n(x) = \sin \frac{n\pi x}{2a\,\{1 + (1 + i)\frac{\beta}{M}\}} ; n = 2,4,6 \text{ ---} \quad (127)$$

and the expansion functions for modes of even symmetry are

$$f_n(x) = \cos \frac{n\pi x}{2a\,\{1 + (1 + i)\frac{\beta}{M}\}} ; n = 1,3,5 \text{ ---} \quad (128)$$

In these expressions, $\beta = -\frac{\zeta(\frac{1}{2})}{\sqrt{\pi}} \approx 0.824$, $M = \sqrt{8\pi N}$, and ζ is the Riemann zeta function (Erdélyi, ref. 41, vol. 2, p. 32, Eq. 2).

For circular mirrors of radius a and separation 2L, the expansion functions are given by

$$f_{nm}(\rho,\theta) = J_n\left(\frac{\nu_{nm}\frac{\rho}{a}}{1 + (1 + i)\frac{\beta}{M}}\right)e^{+in\theta} \quad (129)$$

where ν_{nm} is the m^{th} root of the n^{th} order Bessel function of the first kind, $J_n(x)$, and β and M are given above.

<u>Resonators with Spherical Mirrors of Circular Projection</u>

To apply this procedure to a circular mirror resonator with mirrors of radii a_1 and a_2, one begins with Eqs. (71) and (72). The appropriate forms of these equations, which express the current on mirror #1 in terms of the current on mirror #2 and vice versa, are shown below,

$$J_{x2}(\rho_2,\theta_2) = -\frac{ik}{2\pi}\int_0^{2\pi}\int_0^{a_1} J_{x1}(\rho_1,\theta_1)\,\frac{e^{-ikR_{21}}}{R_{21}}\cos\alpha_{21}\rho_1 d\rho_1 d\theta_1 \tag{130}$$

$$J_{x1}(\rho_1,\theta_1) = -\frac{ik}{2\pi}\int_0^{2\pi}\int_0^{a_2} J_{x2}(\rho_2,\theta_2)\,\frac{e^{-ikR_{12}}}{R_{12}}\cos\alpha_{12}\rho_2 d\rho_2 d\theta_2 \tag{131}$$

where R_{12} is given by Eq. (132).

$$R_{12}^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\theta_2 - \theta_1) + (z_2 - z_1)^2 \tag{132}$$

To obtain the expansion functions for the current on mirror #1, one specializes Eq. (131) to apply to a symmetric resonator containing two mirrors identical to mirror #1 with the same mirror separation present in the original problem. Since the resonator being considered (to obtain the expansion functions) is symmetric, $a_1 = a_2$, $R_1 = R_2$, and

$$J_{x2}(\rho_2,\theta_2) = e^{+i\Omega'} J_{x1}(\rho_2,\theta_2) \tag{133}$$

where $\Omega' = \pi q$, and q is an integer.* Thus, the equation for the current on mirror #1 of this symmetric resonator can be written in the following form.

$$J_{x1}(\rho_1,\theta_1) = -\frac{ik}{2\pi}e^{+i\Omega'}\int_0^{2\pi} d\theta_2 \int_0^{a_1} J_{x1}(\rho_2,\theta_2)\frac{e^{-ikR_{12}}}{R_{12}}\cos\alpha_{12}\rho_2 d\rho_2 \tag{134}$$

*One can see this by choosing the origin midway between the mirrors and realizing that the fields must be even or odd with respect to z.

To simplify the procedure for obtaining these expansion functions, the following approximations are made.*

$$\frac{a_1}{|R_1|} << 1 \tag{135}$$

$$\frac{a_1^{\,2}}{\lambda L} << \left(\frac{L}{a_1}\right)^2 \tag{136}$$

$$k'' << k' \tag{137}$$

$$a_1 << L \tag{138}$$

Under these conditions, $\cos\alpha_{12} \approx 1$ and

$$e^{-ik'R_{12}} \approx e^{-ik'L}\, e^{-\frac{igk'}{2L}(\rho_1^{\,2} + \rho_2^{\,2})}\, e^{+\frac{ik'}{L}\rho_1\rho_2 \cos(\theta_2 - \theta_1)} \tag{139}$$

where $g = 1 - \frac{L}{R_1}$ and the first two terms of the binomial expansion have been used to approximate R_{12}. Finally, it is assumed that

$$e^{+k''R_{12}} \approx e^{+k''L} \tag{140}$$

across the entire mirror surface. With these approximations, Eq. (134) can be manipulated to yield

*For resonators where Eqs. (135) through (138) are not valid, using these approximations will simply require the use of a relatively large number of expansion functions to represent the modes.

$$J_{x1}(\rho_1,\theta_1) = \frac{ik'}{2\pi L}\gamma \int_0^{2\pi}\int_0^{a_1} J_{x1}(\rho_2,\theta_2)\, e^{-\frac{ik'g}{2L}(\rho_1^2 + \rho_2^2)}$$

$$e^{+\frac{ik'}{L}\rho_1\rho_2\cos(\theta_2 - \theta_1)}\rho_2 d\rho_2 d\theta_2 \qquad (141)$$

where $\gamma = -e^{+i\Omega'-ikL}$.

Next, one assumes solutions of the form,

$$J_{x1}(\rho,\theta) = u_n(\rho)\, e^{+in\theta} \qquad (142)$$

and applies the identity (Erdélyi, ref. 41, vol. 2, p. 7),

$$i^n\, 2\pi J_n(z) = \int_0^{2\pi} e^{+iz\cos\phi+in\phi}\, d\phi \qquad (143)$$

where $J_n(z)$ is a Bessel function of the first kind. The result is the following integral equation for the radial mode function,

$$u_n(\xi_1) = i^{n+1}\gamma \int_0^{H_a} u_n(\xi_1')e^{-\frac{ig}{2}(\xi_1^2 + \xi_1'^2)} J_n(\xi_1\xi_1')\xi_1' d\xi_1' , \qquad (144)$$

where $\xi_1 = \sqrt{\frac{k'}{L}}\rho_1$, $\xi_1' = \sqrt{\frac{k'}{L}}\rho_2$, and $H_a = \sqrt{\frac{k'}{L}}\, a_1$.

To complete this step in the procedure, one makes the substitutions $u_n(\xi_1) = \nu_n(\xi_1)/\sqrt{\xi_1}$ and $\gamma_n = \gamma i^{n+1}$, to obtain the following integral equation,

$$\nu_n(\xi_1) = \gamma_n \int_0^{H_a} \nu_n(\xi_1')K_n(\xi_1|\xi_1')\, d\xi_1' \tag{145}$$

where the kernel, $K_n(\xi_1|\xi_1')$, which is given by

$$K_n(\xi_1|\xi_1') = \sqrt{\xi_1\xi_1'}\, e^{-\frac{ig}{2}(\xi_1{}^2 + \xi_1'^2)} J_n(\xi_1\xi_1') \tag{146}$$

is symmetric with respect to ξ_1 and ξ_1'.

The next step in the procedure is to obtain the operator $\hat{M}$, using the manipulations described in the section entitled "The Chosen Procedure." These manipulations involve the following pair of equations,

$$\frac{d^2\nu_n(\xi_1)}{d\xi_1{}^2} = \gamma_n \int_0^{H_a} \nu_n(\xi_1') \frac{d^2K_n(\xi_1|\xi_1')}{d\xi_1'^2}\, d\xi_1' \tag{147}$$

$$\gamma_n \int_0^{H_a} K_n(\xi_1|\xi_1') \frac{d^2\nu_n(\xi_1')}{d\xi_1'^2}\, d\xi_1'$$

$$= \gamma_n \int_0^{H_a} \nu_n(\xi_1') \frac{d^2K_n(\xi_1|\xi_1')}{d\xi_1'^2}\, d\xi_1' + \gamma_n \hat{R}\, \nu_n(\xi_1') \tag{148}$$

where

$$\hat{R}\nu_n(\xi_1') = \left\{ K_n(\xi_1|\xi_1') \frac{d\nu_n(\xi_1')}{d\xi_1'} - \nu_n(\xi_1') \frac{dK_n(\xi_1'|\xi_1)}{d\xi_1'} \right\}_0^{H_a} \tag{149}$$

Equation (147) was obtained by operating on Eq. (145) with $\frac{d^2}{d\xi_1{}^2}$ while Eq. (148) was obtained by integrating the term $K_n(\xi_1|\xi_1')$ $\frac{d^2\nu_n(\xi_1')}{d\xi_1'^2}$ by parts twice.

Continuing to follow the section entitled "The Chosen Procedure," one subtracts Eq. (148) from Eq. (147), computes the derivatives of $K_n(\xi_1|\xi_1')$, and uses Eq. (145) to obtain the result shown below.

$$\left\{\frac{d^2}{d\xi_1{}^2} + (g^2 - 1)\xi_1{}^2 + \frac{1}{\xi_1{}^2}\left(\frac{1}{4} - n^2\right)\right\} \nu_n(\xi_1)$$

$$= - \gamma_n\hat{R}\ \nu_n(\xi_1') + \gamma_n \int_0^{H_a} K_n(\xi_1|\xi_1')$$

$$\left\{\frac{d^2}{d\xi_1'^2} + (g^2 - 1)\xi_1'^2 + \frac{1}{\xi_1'^2}\left(\frac{1}{4} - n^2\right)\right\} \nu_n(\xi_1')d\xi_1' \qquad (150)$$

Referring to Eq. (124), the following conditions may be shown to apply.

1. $f(g,\xi) = (g^2 - 1)\ \xi^2 + \frac{1}{\xi^2}\left(\frac{1}{4} - n^2\right)$
2. $h(g,\xi_1\xi_1') = 0$, and
3. Unless the term in braces in Eq. (149) vanishes at $\xi_1' = 0$, $\hat{R}\nu_n(\xi_1')$ will depend on $\nu_n(\xi_1)$ and its derivative at the origin as well as the mirror edge.

Thus, to meet all the conditions described on page 58, the $\nu_n(\xi)$ must be selected so that at $\xi_1' = 0$,

$$K_n(\xi_1|\xi_1') \frac{d\nu_n(\xi_1')}{d\xi_1'} - \nu_n(\xi_1') \frac{dK_n(\xi_1|\xi_1')}{d\xi_1'} = 0 \qquad (151)$$

For the actual demonstration that Eq. (151) holds for the selected forms of the $\nu_n(\xi_1')$, the reader is referred to a more detailed version of this discussion in appendix D. For this presentation, it is simply assumed that Eq. (151) holds.

Therefore, subject to this assumption and in accordance with Eq. (125), the operator $\hat{M}$ is chosen to be

$$\hat{M} = \frac{d^2}{d\xi^2} + (g^2 - 1)\, \xi^2 + \frac{1}{\xi^2}\left(\frac{1}{4} - n^2\right) \qquad (152)$$

Thus, the eigenvalue problem of Eq. (110) has been reduced to solving the following differential equation and applying the restrictions presented on page 62.

$$\frac{d^2\nu_n(\xi)}{d\xi^2} + \left\{(g^2 - 1)\, \xi^2 + \frac{\frac{1}{4} - n^2}{\xi^2} + s^2\right\} \nu_n(\xi) = 0 \qquad (153)$$

For stable resonators, the most convenient form of this equation is

$$\frac{d^2w_n(z)}{dz^2} - \frac{1}{4}\left\{1 + \frac{(2\nu + 1)}{z} + \frac{(n^2 - 1)}{z^2}\right\} w_n(z) = 0 \qquad (154)$$

This equation, which was obtained using the substitutions $w_n(\xi) = \nu_n(\xi)\sqrt{\xi}$, $z = \alpha'\xi^2$, $s^2 = -(2\nu + 1)\alpha'$, and $\alpha' = \sqrt{1 - g^2}$, is Whittaker's differential equation (Whittaker and Watson, ref. 39, p. 337). It has solutions of the form,

$$w_{\kappa,p}(z) = A\, W_{\kappa,p}(z) + B\, W_{-\kappa,p}(-z) \tag{155}$$

where the $W_{\kappa,p}(z)$ are Whittaker functions of the second kind, $\kappa = -\frac{1}{4}(2\nu + 1)$, and $p = \frac{n}{2}$. With z given above, these functions are characterized by the following:

1. Singularities at the origin, except for the special case where $\kappa = -m - (\frac{n+1}{2})$ with m = 0,1,2, ----, and
2. Exponential decay (for $W_{\kappa,p}(z)$) or exponential growth (for $W_{-\kappa,p}(-z)$) with increasing values of the radial coordinate ρ.

Thus, to make the solutions given by Eq. (155) consistent with the known behavior for the modes of stable resonators (i.e., the fields do not increase exponentially with increasing mirror radius), one sets B = 0 and $\kappa = -m - (\frac{n+1}{2})$ with m = 0,1,2,---. Using these conditions and well known relationships involving Whittaker functions (see appendix H), one can show that the expansion functions for stable resonators have the form

$$u_{nm}(\xi) = \xi^n e^{-\frac{\alpha'\xi^2}{2}} F_m^n(\alpha'\xi^2) \tag{156}$$

where $F_m^n(\alpha'\xi^2)$ is a generalized Laguerre Polynomial (Erdélyi, ref. 41, vol. 1, p. 268).

For unstable resonators, it is advantageous not only to convert Eq. (153) back to an equation for $u_n(\xi)$, but also to make the substitutions $z = \Omega\xi, \tau = -(s/2\Omega)^2$, and $\Omega^2 = \frac{1}{2}\sqrt{\sigma^2 - 1}$ to obtain the following equation.

$$\frac{d^2u_n(z)}{dz^2} + \frac{1}{z}\frac{du_n(z)}{dz} + \left\{4z^2 - \frac{n^2}{z^2} - 4\tau\right\} u_n(z) = 0 \qquad (157)$$

This equation has solutions of the form

$$u_{n,\tau}(z) = \frac{1}{z}\left\{AM_{i\tau,\frac{n}{2}}(iz^2) + BW_{i\tau,\frac{n}{2}}(iz^2)\right\} \qquad (158)$$

where $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ are Whittaker functions of the first and second kind, respectively. As indicated in the discussion for stable resonators, the function $W_{\kappa,\mu}(z)$ has a singularity at the origin unless $i\tau = -(m + \frac{n+1}{2})$ with m = 0,1,2,---. However, as solutions with these values of τ do not reduce to the functions shown in Eq. (129), they will not be considered further. Then, in line with the fact that $i\tau \neq -(m + \frac{n+1}{2})$, the coefficient B in Eq. (158) must be set equal to zero. The result is

$$u_{n,\tau}(z) = A\,\frac{M_{i\tau,\frac{n}{2}}(iz^2)}{z} \qquad (159)$$

In appendix D, it is shown that,

$$u_{n,\tau}(z) \underset{|\Omega^2|\to 0}{\to} C\,J_n(s\xi) \qquad (160)$$

where $J_n(s\xi)$ is a Bessel function of the first kind and C is a constant. Thus, if s is chosen so that

$$s = \frac{2\nu_{nm}}{M + (1 + i)\beta} \qquad (161)$$

then the functions $u_{n,\tau}(z)$ defined in Eq. (160) will reduce to the functions specified in Eq. (129). Accordingly, the expansion functions for unstable resonators with circular mirrors are given by Eq. (159) with

$$\tau = - \left\{ \frac{\frac{\nu_{nm}}{\Omega}}{M + (1 + i)\beta} \right\}^2 \tag{162}$$

where $\beta = 0.824$, $M = \sqrt{8\pi N}$, and N is the resonator Fresnel number.

For resonators containing mirrors with centered coupling apertures, there are no difficulties involving singularities at the origin. As a result, the functions $W_{i\tau,\frac{n}{2}}(iz^2)$, with τ given by Eq. (162), must be included in the expansion set.

Finally, to determine the expansion functions to be used for resonators with toroidal mirrors, the reader is referred to appendix D.

Resonators with Spherical Mirrors of Rectangular Projection. If the same procedure is applied to resonators with mirrors of rectangular projection, the following expansion functions are obtained,

$$u(\xi) = \begin{cases} e^{-\frac{d^2\xi^2}{4}} \Phi\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{d^2\xi^2}{2}\right) \text{ for even functions} \\ e^{-\frac{d^2\xi^2}{4}} \xi\Phi\left(\frac{1-\nu}{2}, \frac{3}{2}, \frac{d^2\xi^2}{2}\right) \text{for odd functions} \end{cases} \tag{163}$$

where $d^2 = 2\sqrt{g^2 - 1}$, $\xi = \sqrt{\frac{k'}{L}}\, x$, and $\Phi(\alpha,\gamma,x)$ is a confluent hypergeometric function of the first kind (see appendix H). For stable resonators, the parameter ν must be zero or a positive integer. For unstable resonators,

$$\nu = -\frac{1}{2} - i\left(\frac{s}{d}\right)^2 \qquad (164)$$

with $s = \frac{m\pi}{M + (1 + i)\beta}$. For even modes, m = 1,3,5,---; and for odd modes, m = 2,4,6,---.

To support the basic theory and method of solution presented in this paper, these expansion functions have been used in conjunction with the Rayleigh-Ritz procedure to calculate the modes of several laser resonators. The results of these calculations are presented and discussed in the following chapter.

VI. Computational Procedure and Results

This chapter has two important objectives. The first is to present the results of the numerical computations performed by applying the basic theory and method of solution discussed in the previous chapters. The second is to support the validity of that theory and method by comparing the results with existing published work.

To accomplish these objectives, the chapter is organized in the following manner. First, the computational procedure used to obtain these results is outlined and discussed. Then, the results obtained for circular resonators are presented in some detail, followed by a summary of the results for rectangular mirror resonators, which are detailed in appendix F. This chapter ends with an evaluation of these calculations and the associated procedure. However, before discussing the computational procedure, the manner in which the term "mode" is used throughout this chapter is briefly discussed.

The reader may recall from chapter III that laser resonator modes were divided into three classes. The modes in Classes II and III were considered beyond the scope of this paper and were eliminated from further consideration. In addition, the following procedure was established for determining the Class I modes of a laser resonator.

1. Assume the self-induction terms in the coupled integral equations (Eqs. (53) and (54)) are negligible.
2. Solve the resulting integral equations (Eqs. (65) and (66)).
3. Show that the solutions are consistent with the assumption that the self-induction terms are negligible.

Since the calculations in this chapter were performed only to check the basic theory and method of solution presented earlier, the third step of this procedure has not been accomplished. Despite that fact, to remain consistent with commonly used terminology, the term "mode" is used to designate all solutions to Eqs. (65) and (66) (or specialized forms of these equations) regardless of whether these solutions produce self-induction terms that are negligible.

Computational Procedure

This section is divided into three parts. The first discusses an ideal procedure for using the Rayleigh-Ritz technique to determine laser resonator modes. The second part covers the computational difficulties encountered as a result of the limitations of the computer programs used by the author for these calculations. This ideal procedure and these limitations combine to yield the actual procedure, which is discussed in part three.

Ideal Procedure. The steps of an ideal procedure for applying the Rayleigh-Ritz technique to laser resonator problems are listed below.

1. Select a number of expansion functions for the first stage of the calculation.
2. Using that set of expansion functions, compute the eigenvalues and corresponding mode distributions (magnitude and phase).
3. Increase the number of expansion functions and repeat steps 1 and 2.
4. Compare the results obtained using the two sets of functions to determine if the desired accuracy has been achieved.
5. If the desired accuracy has been achieved, the procedure is terminated. If not, steps 3 and 4 are repeated.

The convergence criterion used in step 4 depends on the data desired from the calculation. If only eigenvalues are required, the details of the mode distributions need not be considered since the eigenvalues are relatively insensitive to small perturbations in the mode distributions. Thus, convergence with respect to eigenvalue data may be achieved with a few expansion functions. On the other hand, many expansion functions may be required (at least 10) to determine the details of the modes of large magnification resonators.

Limitations of Existing Programs. Experience obtained from many applications of this procedure indicates that, for resonators departing significantly from $g = 1.0$ ($g \geq 1.25$), accurate calculations with at least 10 expansion functions are required to determine

1. The eigenvalues of several modes, or
2. Any mode distributions.

As shown in chapter V, the expansion functions selected for all resonators considered in this paper are directly related to the confluent hypergeometric functions of the first ($\Phi(\alpha,\gamma,z)$) and second ($\Psi(\alpha,\gamma,z)$) kind. In this analysis, the parameter α has a strong dependence on the number of functions used (see Eqs. (162) and (163) in chapter V). That is to say, as the number of expansion functions increases, the magnitude of α for the last function(s) becomes large. This large magnitude makes it difficult to compute these functions accurately for certain ranges of the spatial variable z. These ranges correspond to intermediate values of z, where z is too small to apply asymptotic series expansions and where round-off error difficulties are encountered with the usual Taylor series expansions.

As a result of these limitations, which were inherent in the computer routines readily available to the author, it was not always possible to perform accurate computations with the 10 expansion functions needed. Due to these inaccuracies, the ideal procedure was modified as discussed in the following paragraphs.

Actual Procedure. The procedure actually employed in this paper depended on the particular resonator being studied. For resonators with small equivalent Fresnel number, N_e, the ideal procedure was used. Typically, however, only the first two steps of that procedure were performed since the results obtained for the lowest loss modes at that stage of the calculation were in close agreement with those obtained by other authors. For these cases, only the results for the first three or four modes are reported. The eigenvalues for higher loss modes, which are not reported, are considered as not having converged.

For resonators corresponding to the regions of difficulty mentioned on the previous page, this procedure was modified further. For these cases, a series of eigenvalue computations was performed using a relatively small number (usually six or seven) of expansion functions for each case. For these resonators only, the series reported corresponds to the one yielding the best agreement in the eigenvalue of the lowest order mode. For example, the data reported later for g = 2.6 fall into this category.

The rationale behind selecting the series yielding the best agreement points up the main computational difficulty encountered in this paper. To understand the difficulty, it is helpful to recall that, with any expansion procedure, one expects the accuracy of calculations to increase as the number of expansion functions is increased. However, due to the difficulties with the higher order expansion functions (those with large $|\alpha|$ discussed on the previous page), that accuracy could actually decrease as the number of functions is increased. Thus, with the tools available, there was an optimum number of expansion functions for each resonator in this category. It was assumed that the optimum number corresponded to the one yielding the best agreement (with existing published work) for the lowest loss mode.

The results obtained using this modified procedure are presented in the following two sections.

Presentation of Circular Mirror Results

Two types of results are presented in this section. First, data related to the eigenvalues in the integral equation are presented for a wide range of symmetric resonators with circular mirrors. Except when mentioned, these data are presented in the same format used by other authors. Second, plots are presented of the relative magnitude and phase of the current induced on one mirror of a plane parallel resonator with N = 10. For brevity, only plots for the first two azimuthally symmetric modes are included in this section. For other mode distributions involving this resonator, see the first section of appendix F.

Eigenvalue Data. As indicated earlier, the eigenvalue data presented in this section cover a broad range of laser resonators. The results for the first case, a plane parallel resonator with N = 10, are summarized in tables II and III. The data used for comparison were taken from Fox and Li (ref. 25, p. 465, table I), and the indicated percentage power loss corresponds to a single pass through the resonator.

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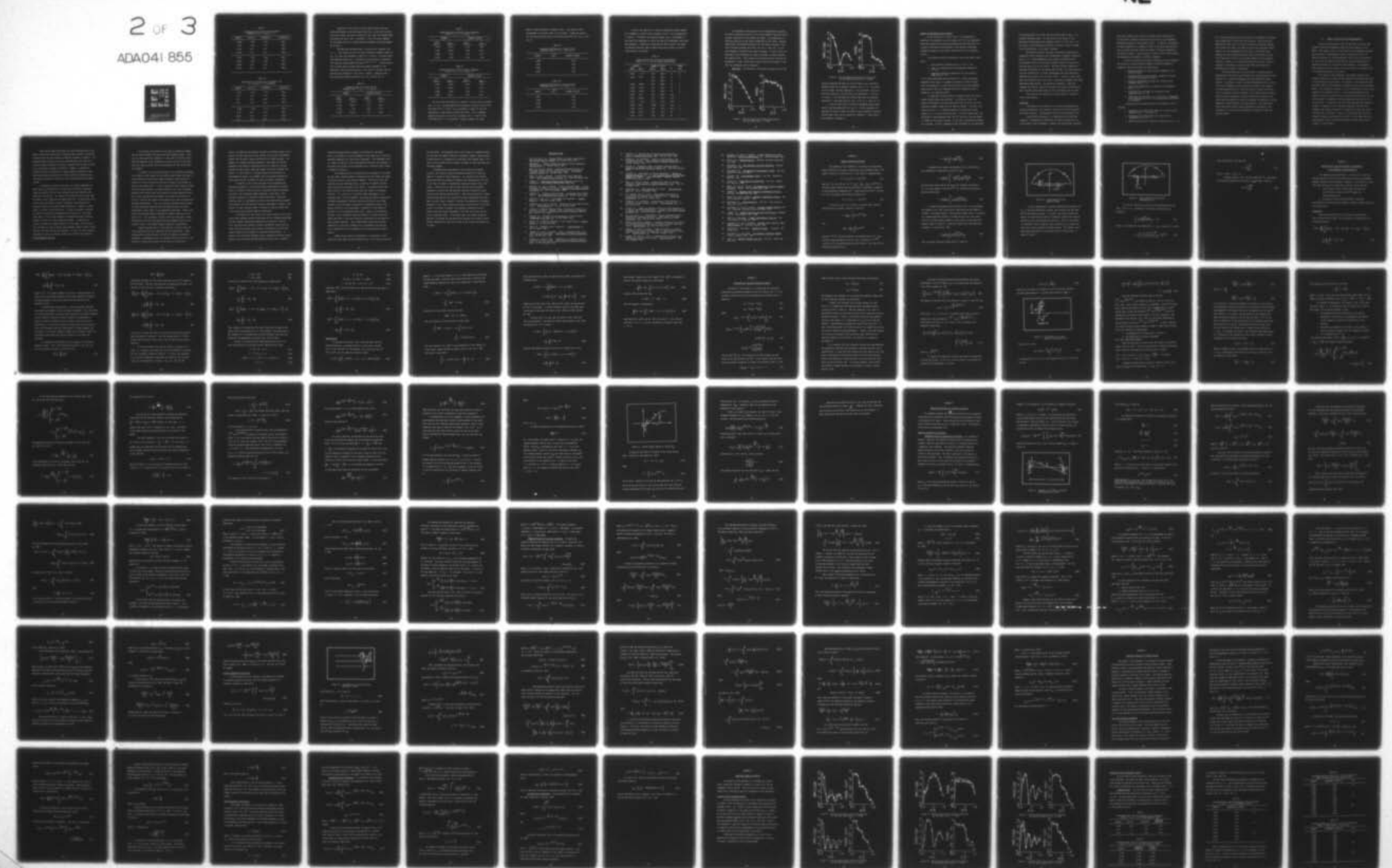

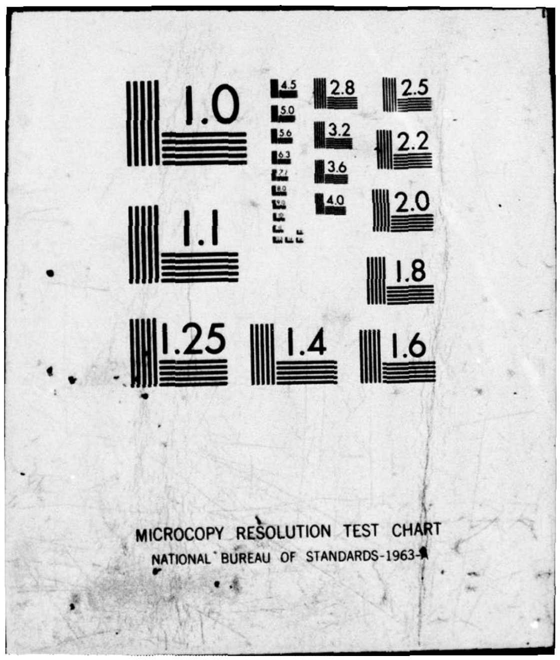
1.0
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Table II

Percentage Power Loss for Circular Mirror Resonator with N = 10 and g = 1.0

DOUGHTY	FOX & LI	VAINSHTEIN	DESIGNATION
0.848	0.84	0.82	TEM_{00}
4.447	4.45	4.31	TEM_{10}
11.198	10.6	10.6	TEM_{20}
19.579	18.7	19.7	TEM_{30}
2.019	2.02	2.08	TEM_{01}
6.857	6.66	6.96	TEM_{11}
14.452	14.4	14.6	TEM_{21}
24.737	23.3	25.1	TEM_{31}

Table III

Phase Shift for Circular Mirror Resonator with N = 10 and g = 1.0 (in degrees)

DOUGHTY	FOX & LI	VAINSHTEIN	DESIGNATION
2.36	2.36	2.75	TEM_{00}
12.5	12.4	11.9	TEM_{10}
30.7	30.7	29.2	TEM_{20}
57.2	57.0	54.2	TEM_{30}
6.025	6.03	5.72	TEM_{01}
20.15	20.1	19.2	TEM_{11}
42.35	42.2	40.3	TEM_{21}
73.0	73.0	69.2	TEM_{31}

Comparison of the results in these tables reveals that the maximum difference in the percentage power loss is less than 6 percent; and in most cases, the results obtained in this paper fall between those obtained by Fox and Li and L. Vainshtein. Also, the values computed for the phase shift are in almost perfect agreement with those computed by Fox and Li.

The next case considered was a circular mirror resonator with g = 1.1. The results for the first four azimuthally symmetric modes for N_e = 1,2, and 3 are presented in tables IV, V, and VI. For those cases, the eigenvalue phase, arg γ, corresponds to the round trip eigenvalue into which has been absorbed the terms $e^{-2ik'L}\, i^{n+1}$. A brief examination of these results reveals that the maximum difference in the magnitude of the one-way eigenvalue is less than 3.5 percent, and the maximum phase difference is less than 11 degrees. Comparison data were taken from Siegman and Miller (ref. 28, p. 2734, fig. 5).

Table IV

Eigenvalue Data for Circular Mirror
Resonator with n = 0, g = 1.1, and N_e = 1

DOUGHTY		SIEGMAN & MILLER	
$\sqrt{\lvert\gamma\rvert}$	ARG γ	$\sqrt{\lvert\gamma\rvert}$	ARG γ
0.816	143.8°	0.82	140.°
0.778	-134.2°	0.78	-140.°
0.544	+ 2.9°	0.54	+ 10.°
0.308	-136.9°	0.30	-140.°

Table V

Eigenvalue Data for Circular Mirror Resonator
with $n = 0$, $g = 1.1$, and $N_e = 2$

DOUGHTY		SIEGMAN & MILLER	
$\sqrt{\lvert\gamma\rvert}$	ARG γ	$\sqrt{\lvert\gamma\rvert}$	ARG γ
0.778	145.7°	0.76	140.°
0.761	-147.3°	0.78	-140.°
0.692	19.5°	0.67	+ 30.°
0.594	- 41.8°	0.60	- 40.°

Table VI

Eigenvalue Data for Circular Mirror Resonator
with $n = 0$, $g = 1.1$, and $N_e = 3$

DOUGHTY		SIEGMAN & MILLER	
$\sqrt{\lvert\gamma\rvert}$	ARG γ	$\sqrt{\lvert\gamma\rvert}$	ARG γ
0.767	148.4°	0.76	140.°
0.740	-150.5°	0.74	-140.°
0.680	45.8°	0.68	+ 40.°
0.624	-104.4°	0.62	-100.°

The third case considered was a symmetric, circular mirror resonator with $g = 1.25$. The magnitudes of the eigenvalues (one-way) for the first four modes for $N = 4$ and $N = 8$ are presented in tables VII and VIII. Examination of these results reveals two things. First, agreement between the results for the first two modes with $N = 4$ and for the first mode with $N = 8$ is excellent. Second, agreement for other modes of these resonators is generally poor. The reason for this disagreement is discussed later in this chapter. Comparison results for this case were also taken from Siegman and Miller (ref. 28, p. 2733, fig. 3).

Table VII

Eigenvalue Magnitudes for Circular Mirror
Resonator with n = 0, g = 1.25, N = 4

DOUGHTY	$\sqrt{\lvert\gamma\rvert}$	SIEGMAN & MILLER
0.596		0.59
0.560		0.56
0.559		0.46
0.429		0.42

Table VIII

Eigenvalue Magnitudes for Circular Mirror
Resonator with n = 0, g = 1.25, N = 8

DOUGHTY	$\sqrt{\lvert\gamma\rvert}$	SIEGMAN & MILLER
0.631		0.63
0.468		0.58
0.425		0.46
0.420		0.45

Finally, the results for a range of equivalent Fresnel numbers for a symmetric, circular mirror resonator with g = 2.6 are summarized in table IX. As before, the eigenvalue phase, arg γ, corresponds to the round trip eigenvalue for which the spatially independent terms have been absorbed. In addition to reporting the basic results, the number of expansion functions used to obtain these data is shown in the far right column of this table.

Table IX

Eigenvalue Data for Circular Mirror Resonator with n = 0, g = 2.6 for Several Values of N_e

DOUGHTY		SIEGMAN & MILLER			Number
$\sqrt{\lvert\gamma\rvert}$	ARG γ	$\sqrt{\lvert\gamma\rvert}$	ARG γ	N_e	Used
0.235	- 4.4°	0.24	-100.°	6	6
0.202	-179.°	0.24	+100.°	6	6
		0.20	-180.°	6	
0.255	108.6°	0.22	+100.°	12	6
0.225	- 94.8°	0.23	- 80.°	12	6
0.125	127.4°	0.10	+100.°	12	6
0.333	107.°	0.30	+160.°	12.5	7
0.212	-135.6°	0.15	+ 20.°	12.5	7
0.173	- 3.4°	0.15	- 10.°	12.5	7
0.278	-159.°	0.285	-176.°	20.5	6
0.149	-103.°	0.23	+ 20.°	20.5	6
0.284	-115.1°	0.26	-110.°	24	7
0.129	+112°	0.26	140°	24	7

As indicated in the discussion of the computational procedure, the results obtained using only six or seven expansion functions may be marginal, especially for the higher loss modes. That expectation is certainly borne out by the results summarized in this table. However, despite that, the maximum difference for the lowest loss mode is less than 16 percent (Siegman and Miller, ref. 28, p. 2732, fig. 2), and the difference is typically less than 10 percent. As can be seen, the magnitude of the difference for higher loss modes is rather sporadic, but generally high. These results are discussed further following the presentation of the circular mirror mode plots and the summary of the data for rectangular mirror resonators.

Mode Plots. This portion of the chapter contains plots of the

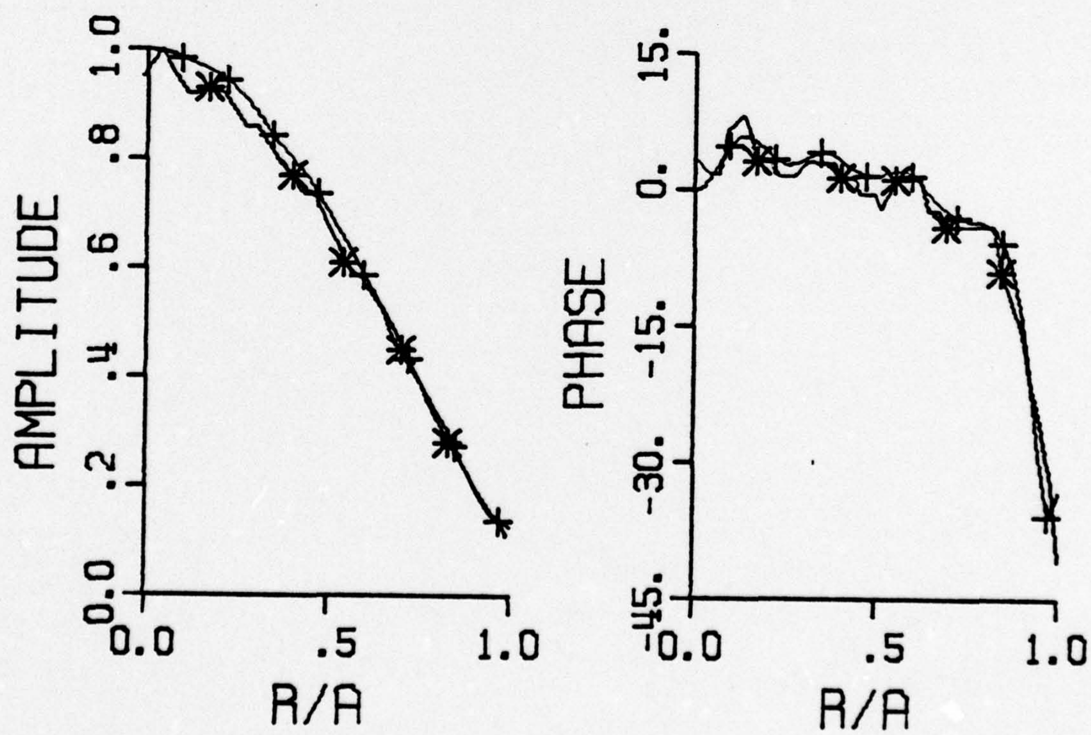

Figure 8. Relative Magnitude and Phase Distributions for the Lowest Loss, n = 0 Mode

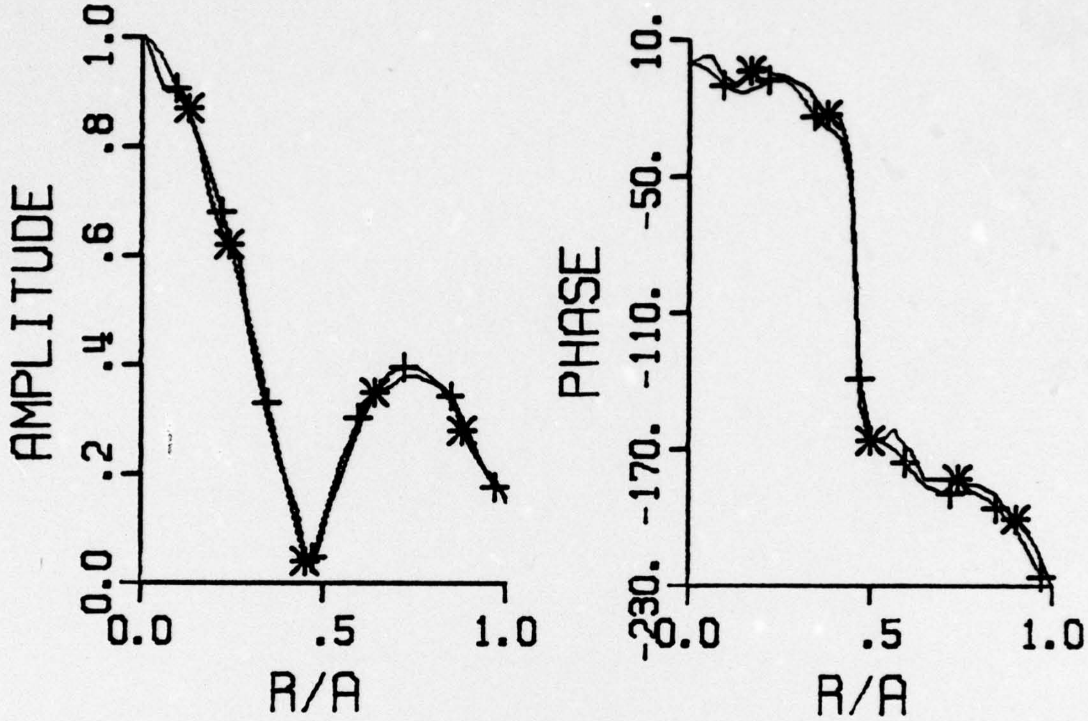

Figure 9. Relative Magnitude and Phase Distributions for the Second Lowest Loss, n = 0 Mode

relative magnitude and phase for the two lowest loss, azimuthally symmetric modes for a symmetric resonator with N = 10. The results for the lowest loss mode, denoted by +, are presented in figure 8, while the results for the next mode, also denoted by +, are given in figure 9. The distributions to which these data are compared, denoted by *, were taken from Fox and Li (ref. 25, p. 464, fig. 2).

As can be seen, there is excellent agreement between these results and those obtained by Fox and Li. Similar agreement was obtained for several higher loss modes of this resonator (including several modes which are not azimuthally symmetric). These results are presented in appendix F.

Summary of Rectangular Mirror Results

As was the case for circular mirrors, the computations performed for rectangular mirror resonators fall into two categories. These two categories are eigenvalue computations and mode plots consisting of the relative distribution of the intensity (magnitude squared) and phase for the current induced on one of the resonator mirrors.

The eigenvalue results correspond to one of the cases listed below.

1. Plane parallel resonators with $N = 10$ or $N = 8/\pi$.
2. Power loss computations for odd symmetric modes with $g = 1.2$.
3. Eigenvalue magnitude computations for even symmetric modes with $g = 1.8$.

All of these computations were performed for a single rectangular component (strip case) of a symmetric resonator so that the eigenvalue corresponded to a single pass (one-way) through the resonator. These results, which are presented along with comparative data in appendix F, are summarized below.

As expected, the plane parallél resonator results for the two Fresnel numbers shown were excellent. For these two cases, the relative difference in the percentage power loss for any of the first three even symmetric modes was less than 2.5 percent. Similarly, the results for the $g = 1.2$ unstable resonator for equivalent Fresnel numbers of 1, 1.5, and 2 were quite good. In each of these cases, the difference in the percentage power loss for the first two even symmetric modes was less than 2 percent. In any case, the maximum difference was 7 percent. Finally, computations were performed for the magnitudes of the eigenvalues for at least the first three modes of the $g = 1.8$ resonator mentioned above. For that series, nine values for the equivalent Fresnel number were selected so that $2.5 \leq N_e \leq 20/\pi$. For this series, the difference was typically 4 percent or less, although in one isolated case, it did reach 7 percent.

The relative intensity and phase distributions determined for rectangular mirror resonators also corresponded to symmetric resonators with $g = 1.8$. These computations were limited to determining the distributions for the first two even symmetric modes for values of N_e such that $2\pi N_e = 18, 34, 36$, and 40. With the possible exception of the phase distribution of the second mode for $2\pi N_e = 34$, these distributions are certainly not in strong disagreement with those obtained by Sanderson and Streifer (ref. 42, figs. 13-21). In fact, in most cases, these distributions can be said to be in qualitative agreement with those obtained by these authors. By qualitative agreement, I mean that although the relative values for the peaks and troughs may differ somewhat, the basic nature and trends for the distributions are the same.

The reason for any deviations in these results is covered in the following section.

Discussion

The purpose of this section is to evaluate the computational procedure employed for this analysis, as well as the results obtained using that procedure. The procedure itself will be considered first.

From earlier discussions, it is apparent that the existing procedure is inadequate for determining the modal characteristics of a wide range of laser resonators. However, the shortcomings associated with that procedure result from the limited range of applicability of the author's computer routines and are not due to any fundamental deficiency inherent in the procedure itself. In addition, the purpose of these computations is somewhat limited in that they were performed only to support the basic theory and method of solution presented in the earlier chapters. When evaluated in terms of this limited purpose, the computational procedure is certainly adequate.

To aid in evaluating the results obtained using this procedure, the results for which excellent agreement (difference generally 5 percent or less) was achieved are listed below.

1. Mode distributions for plane parallel resonators with circular mirrors.
2. Eigenvalue data for plane parallel resonators with both circular and rectangular mirrors.
3. Eigenvalue magnitudes and phases for circular mirror resonators with $g = 1.1$.
4. Lowest loss eigenvalues for circular mirror resonators with $g = 1.2$.
5. Eigenvalues for odd modes for rectangular mirror resonators with $g = 1.2$.
6. Eigenvalues for even modes for a wide range of rectangular mirror resonators with $g = 1.8$ (one point had a difference of 7 percent).

Listed below are the results which are considered as fair or marginal.

1. Eigenvalues for higher order modes for a circular mirror resonator with $g = 1.2$ and $N = 8$.
2. Eigenvalues for several modes for circular mirror resonators with $g = 2.6$.
3. Mode distributions for rectangular resonators with $g = 1.8$.

All of the cases falling into this second list correspond to instances where computational difficulties were expected. Difficulties were expected for cases 1 and 2 as they correspond to eigenvalue computations (mostly for higher loss modes) for which the equivalent Fresnel number is in the intermediate to high range. Difficulties were expected for case 3 because the actual mode distributions are especially sensitive to errors in the values computed for the expansion functions. However, it is worth mentioning that the basic nature or character of the modes is correctly described, even though some differences are present in these plots.

From this discussion, one can see that in regions where no computational difficulties were expected, the results obtained in this analysis are in excellent agreement with those obtained by other workers. In addition, in every case where agreement was fair or poor, that lack of agreement can be correctly attributed to the computer routines employed and not to the basic theory or method of solution. Thus, the numerical results discussed in this chapter and the analytic results covered in appendix E strongly support the validity of the basic theory and method of solution for symmetric resonators of various geometries, Fresnel numbers, and magnifications. More importantly, these same results should inspire considerable confidence in the applicability of this theory and method of solution for other more complex systems.

VII. Summary, Conclusions, and Recommendations

The basic theory needed to analyze the modes of complex laser resonators containing homogeneous media has been presented in this report. That theory, which is firmly based on the fundamental equations and principles of electromagnetic theory, culminates in a pair of integral equations for the electromagnetic field within the resonator. It is this pair of equations which must be solved simultaneously to determine the modes of an arbitrary laser resonator. Unfortunately, these equations are very complex, and solving them for the general case is far beyond the scope of this effort.

As a result of this complexity, the integral equations were gradually simplified by making various assumptions concerning the nature of the resonators being analyzed. For example, it was assumed that the resonators consist of two perfectly conducting mirrors and a homogeneous medium. This assumption resulted in a pair of equations for the currents induced on the resonator mirrors. It was further assumed that the effect of any gain or amplification (resulting from polarization of the lasing medium) is essentially constant between any two points on different mirrors. In addition, the oscillation wavelength was assumed to be small in comparison to all resonator dimensions, and the mirror diameters were assumed to be small in comparison to the mirror separation and radii of curvature. Finally, it was assumed that the self-induction integrals are negligible and that it is the mode losses and distributions, and not the precise frequency spectrum, which are of interest in this analysis.

After making these assumptions, the resulting equations for the modal currents were solved using a technique that combines a variational principle with the novel expansion functions discussed in chapter V. By comparing these solutions to existing published work, the validity of the basic theory and method of solution presented here has been verified for a wide range of symmetric resonators. However, in addition to directly verifying the analysis for these symmetric resonators, these same calculations and comparisons strongly support the assertion that the theory and method of solution are also applicable for more complex systems.

In addition to providing the basis for actually computing the mode losses and distributions, the derivation of the basic equations sheds considerable light on laser resonators and their physical characteristics and fields. For example, the derivation clearly shows that the modes of open resonators are damped with time. For many cases, such as the lower loss modes of paraxial resonators, this damping has a negligible effect on the actual mode distributions. However, in other cases, its effect can be significant. In fact, by properly including the effect of damping at the beginning of the analysis, it has been shown that it is not always possible to formulate the laser resonator problem in terms of a linear eigenvalue problem. In addition, by properly including the temporal behavior of the modes, this derivation has eliminated the need to define laser resonator modes in terms of some mythical round trip through the resonator. In fact, from this analysis, the modes are seen to be fields for which the relative distributions <u>do not change at any time</u>.

This analysis also points out the need to distinguish between the two types of energy flow associated with resonant systems. Moreover, by using continuity arguments, it shows that for energy flowing from the resonator to the surroundings, the open portion of the resonator hull does not enter into the calculations for the fields within the resonator.

In addition to the points mentioned in the preceding paragraphs, this analysis clearly shows that one cannot always analyze the modes of laser resonators by using only integral equations which express the current (or field) on one mirror entirely in terms of the current (or field) on the other mirror. In fact, since both mirrors normally affect the current distributions, one should generally use a pair of coupled equations which involve both self-induction and mutual-induction terms. As a result, analyses which do express the current on one mirror entirely in terms of the current on the second mirror have two potentially important deficiencies. First, such analyses completely neglect modes for which the mutual-induction and self-induction terms are either proportional to each other or have comparable magnitudes. Second, some of the distributions normally regarded as resonator modes may not be modes at all. This statement is based on the fact that some of these distributions may not be consistent with the assumption that the self-induction terms in the coupled integral equations are negligible.

Another important point in this analysis is the fact that the fields associated with open resonators may be discontinuous. These discontinuities, which may occur at the mirror edges or at material interfaces in segmented mirrors, are treated simply by allowing for the presence of charges and currents at the points of discontinuity.

Finally, by comparing the equations obtained at different stages of this analysis to those obtained by other authors, one can gain considerable insight into the actual range of applicability of other analyses. For example, the integral equations obtained in this paper for paraxial resonators with perfectly conducting mirrors are identical to those obtained by other authors under what appear to be less stringent conditions. As this analysis is strongly based on the principles of electromagnetic theory, this suggests that the equations obtained by those authors are not as general as they are normally considered.

In addition to the insight obtained from the equations and derivations themselves, one can gain further understanding of the behavior of resonator fields by actually calculating mode distributions and losses. The variational method presented in this paper is one excellent technique for performing these calculations. Using this technique, which is implemented by applying the Rayleigh-Ritz procedure in conjunction with the expansion functions developed in chapter V, the analysis of laser resonator modes is reduced to a matrix problem that can be solved using standard numerical techniques.

This reduction of the analysis to a matrix problem, which can be solved using standard numerical techniques, has two important advantages over the standard iteration/orthogonalization methods of solution. First, the variational method is usually considerably faster than the normal iteration approach, which actually corresponds to solving the integral equation by the method of successive approximations. In addition, the variational approach yields information concerning several modes at once, while the iteration approach requires that a complete iteration/orthogonalization sequence be performed for each mode. Finally, the method of solution presented in this paper has at least one important advantage over other matrix approaches. This advantage, which is a result of the way in which the expansion functions are obtained, is that modal calculations can be performed using a relatively small number of expansion functions.

As indicated earlier, the numerical work presented in this paper has the rather limited purpose of supporting the basic theory and method of solution covered in the previous chapters. With that in mind, the numerical work was performed using a collection of computer routines which were easily accessible to this author. No extensive effort was made to write or find routines which were very general or extremely efficient. As a result, this collection needs improvement in two important areas. First and foremost, the subroutine used to compute the confluent hypergeometric functions should be modified so that these functions can be accurately evaluated for intermediate values of the spatial variable z. This improvement is essential if this method of solution is to find widespread application. Second, the integration routines in this collection should be modified in a manner which will reduce the effect of the rapid variation associated with the kernels of these integral equations. Such a modification would allow accurate computation of the matrix elements with a considerable time savings. This is especially true for resonators with large equivalent Fresnel numbers.

Another aspect requiring improvement is the analysis of the conditions under which the self-induction terms in the integral equations are negligible. The estimates given in this paper will probably suffice for the lower loss modes of paraxial resonators; however, they may begin to break down as k" increases or as the mirror radii become large. As a result, more accurate and more general estimates of these conditions are definitely needed.

Following these improvements in the analysis and computer routines, this analysis should be extended and applied to study the modes of a variety of laser resonators. For instance, this analysis should definitely be extended to determine the characteristics of Class II and Class III modes of conventional stable and unstable laser resonators. It should also be used to study toroidal and hole-coupled resonators to verify that the expansion functions obtained for these cases have the desired characteristics. When the applicability of these functions has been verified, and when the above improvements have been made, the basic theory, method of solution, and computational procedure presented in this report should be extremely valuable tools in the modal analysis of paraxial resonators with perfectly conducting mirrors. However, even with these improvements, this work will not be the last word in resonator theory. On the other hand, the integral equations derived in this paper can and should form the basis for studies of more complex resonators for which one or more of the limiting assumptions do not apply.

References Cited

1. Fox, A.G. and T. Li. "Resonant Modes in a Maser Interferometer." Bell System Technical Journal, 40: 453-488 (March 1961).

2. Bergstein, L. "Modes of Stable and Unstable Optical Resonators." Applied Optics, 7: 495-504 (March 1968).

3. Boyd, G.D. and J.P. Gordon. "Confocal Multimode Resonator for Millimeter Through Optical Wavelength Masers." Bell System Technical Journal, 40: 489-508 (March 1961).

4. Boyd, G.D. and H. Kogelnik. "Generalized Confocal Resonator Theory." Bell System Technical Journal, 41: 1347-1369 (March 1967).

5. Gleason, T.J. Analysis of Complex Laser Cavities, HDL-TM-71-5. Washington: Harry Diamond Laboratories, July, 1971.

6. Heurtley, J.C. and W. Streifer. "Optical Resonator Modes - Circular Reflectors of Spherical Curvature." Journal of the Optical Society of America, 55: 1472-1479 (November 1963).

7. Kogelnik, H. "Imaging of Optical Modes -- Resonators with Internal Lenses." Bell System Technical Journal, 44: 455-494 (March 1965).

8. Kogelnik, H. and T. Li. "Laser Beams and Resonators." Applied Optics, 5: 1550-1567 (October 1966).

9. Sanderson, R.L. and W. Streifer. "Comparison of Laser Mode Calculations." Applied Optics, 8: 131-136 (January 1969).

10. Streifer, W. "Optical Resonator Modes - Rectangular Reflectors of Spherical Curvature." Journal of the Optical Society of America, 55: 868-877 (July 1965).

11. Siegman, A.E. "Unstable Optical Resonators for Applications." Proceedings of the IEEE, 53: 277-287 (March 1965).

12. Barone, S.R. "Optical Resonators in the Unstable Region." Applied Optics, 6: 861-863 (May 1967).

13. Kahn, W.K. "Unstable Optical Resonators." Applied Optics, 5: 407-413 (March 1966).

14. Siegman, A.E. and R. Arrathoon. "Modes in Unstable Optical Resonators and Lens Waveguides." IEEE Journal of Quantum Electronics, QE-7: 156-163 (April 1967).

15. Krupke, W.F. and W.R. Sooy. "Properties of an Unstable Confocal Resonator CO_2 Laser System." IEEE Journal of Quantum Electronics, QE-5: 575-586 (December 1969).

16. Streifer, W. "Unstable Optical Resonators and Waveguides." Journal of Quantum Electronics, QE-4: 229-230 (April 1968).

17. Bel'dyugin, I.M. and others. "Theory of Open Resonators with Cylindrical Mirrors." Soviet Journal of Quantum Electronics, 4: 485-490 (October 1974).

18. Horwitz, P. "Asymptotic Theory of Unstable Resonator Modes." Journal of the Optical Society of America, 63: 1528-1543 (December 1973).

19. Louisell, W. "From Maxwell to Optical Resonators." Physics of Quantum Electronics, Vol. III, Laser Induced Fusion and X-Ray Laser Studies, Ed. Stephen F. Jacobs and others. Reading: Addison-Wesley Publishing Co., 1976.

20. Chen, L.W. and L.B. Felsen. "Coupled Mode Theory of Unstable Resonators." IEEE Journal of Quantum Electronics, QE-9: 1102-1113 (November 1973).

21. Vainshtein, L.A. "Open Resonators for Lasers." Soviet Physics-JETP, 17: 709-719 (September 1963).

22. V.V. Lyubimov and I.B. Orlova. "Approximate Calculations of Oscillations in Resonators with Concave Mirrors." Optics and Spectroscopy, 29: 310-313 (August 1970).

23. Vinokurov, G.N. and others. "Investigation of the Selective Properties of Open Unstable Cavities." Opt. Spectrosk, 34: 741-751 (April 1973).

24. Carter, W.H. "Stable and Unstable Cylindrical-Laser-Resonators in an Angular Spectrum Representation." Journal of the Optical Society of America, 64: 1100-1106 (August 1974).

25. Fox, A.G. and T. Li. "Computation of Optical Resonator Modes by the Method of Resonance Excitation." IEEE Journal of Quantum Electronics, QE-4: 460-465 (July 1968).

26. Rensch, D.B. and A.N. Chester. "Iterative Diffraction Calculations of Transverse Mode Distributions in Confocal Unstable Laser Resonators." Applied Optics, 12: 997-1010 (May 1973).

27. Siegman, A.E. and E.A. Sziklas. "Mode Calculations in Unstable Resonators with Flowing Saturable Gain 1: Hermite-Gaussian Expansion." Applied Optics, 13: 2775-2792 (December 1974).

28. Siegman, A.E. and H.Y. Miller. "Unstable Optical Resonator Loss Calculations Using the Prony Method." Applied Optics, 9: 2729-2736 (December 1970).

29. Buldyrev, V.S. and E.F. Fradkin. "Integral Equations for Open Resonators." Optics and Spectroscopy, 17: 314-320 (October 1964).

30. Taylor, A.E. Advanced Calculus. Waltham: Blaisdell Publishing Co., 1955.

31. Vainshtein, L.A. Open Resonators and Open Waveguides. Boulder: Golem Press, 1969.

32. Harrington, R.F. Time Harmonic Electromagnetic Fields. New York: McGraw-Hill Book Co., 1961.

33. Stratton, J.A. Electromagnetic Theory. New York: McGraw-Hill Book Co., 1941.

34. Collin, R.E. Field Theory of Guided Waves. New York: McGraw-Hill Book Co., 1960.

35. Baker, B.B. and E.T. Copson. The Mathematical Theory of Huygen's Principle. Oxford: The Clarendon Press, 1939.

36. Stakgold, I. Boundary Value Problems of Mathematical Physics. New York: The MacMillian Co., 1967.

37. Morse, P.M. and H. Feshback. Methods of Theoretical Physics. New York: McGraw-Hill Book Co., 1953.

38. Mertzbacher, E. Quantum Mechanics. New York: John Wiley and Sons, Inc., 1970.

39. Whittaker, E.T. and G.N. Watson. A Course in Modern Analysis (4th Ed.). Cambridge: Cambridge University Press, 1950.

40. Lebedev, N.N. Special Functions and Their Applications, Englewood Cliffs: Prentice-Hall, Inc., 1965.

41. Erdélyi, A. and others. Higher Transcendental Functions. New York: McGraw Hill Book Co., 1953.

42. Sanderson, R.L. and W. Streifer. "Unstable Laser Resonator Modes." Applied Optics, 8: 2129-2136 (October 1969).

43. Pressley, R.J. and others. Handbook of Lasers. Cleveland: CRC Press, 1971.

44. Wilkinson, J.H. and others. The Algebraic Eigenvalue Problem. Oxford: Clarendon Press, 1965.

45. Acton, F.S. Numerical Methods that Work. New York: Harper and Row, 1970.

APPENDIX A

Green's Function Derivation

The objective of this appendix is to derive an outgoing wave Green's function to be used in calculating laser resonator modes. This Green's function is a solution of Eq. (20), which is repeated below

$$\nabla'^2 \bar{\bar{G}} + k_j^2 \bar{\bar{G}} = - \bar{\bar{I}}\, \delta(\overline{r - r'}) \tag{A1}$$

where $k_j = k_j' + k_j''$ with $k_j' > 0$, $\bar{\bar{I}} = \hat{a}_x\hat{a}_x + \hat{a}_y\hat{a}_y + \hat{a}_z\hat{a}_z$, $\delta(\overline{r - r'})$ is the Dirac delta function, and the Laplacian is understood to operate on the rectangular components of $\bar{\bar{G}}$. Substituting the form $\bar{\bar{G}} = \bar{\bar{I}}\phi$ into Eq. (A1), one obtains the following equation for ϕ.

$$\nabla'^2 \phi + k_j^2 \phi = - \delta(\overline{r - r'}) \tag{A2}$$

To solve Eq. (A2), it is helpful to express both ϕ and the delta function as Fourier integrals of the form

$$\phi = \frac{1}{(2\pi)^3} \int_{\bar{k}} d\bar{k}\; e^{-i\bar{k}\cdot\bar{R}}\; \tilde{\phi}(\bar{k}) \tag{A3}$$

and

$$\delta(\bar{R}) = \frac{1}{(2\pi)^3} \int_{\bar{k}} d\bar{k}\; e^{-i\bar{k}\cdot\bar{R}} \tag{A4}$$

where $\bar{R} = \overline{r - r'}$, and the integrals are evaluated over all $\bar{k}$. Substituting these expressions into Eq. (A2), evaluating $\nabla'^2 e^{-i\bar{k}\cdot\bar{R}}$, solving for $\tilde{\phi}$, and substituting the result into Eq. (A3) leads to the following expression for ϕ.

$$\phi = \frac{1}{(2\pi)^3} \int_{\bar{k}} d\bar{k} \frac{e^{-i\bar{k}\cdot\bar{R}}}{|\bar{k}|^2 - k_j^2} \tag{A5}$$

Integrating over angles and denoting $|\bar{k}|$ by k and $|\bar{R}|$ by R, this expression is simplified to yield Eq. (A6).

$$\phi = \frac{1}{(2\pi)^2 iR} \int_0^{+\infty} \frac{k\,[e^{+ikR} - e^{-ikR}]\,dk}{(k + k_j)\,(k - k_j)} \tag{A6}$$

By writing the right side as the sum of two integrals and letting $k \to -k$ in the integral involving e^{-ikR}, Eq. (A6) may be rewritten in the form shown below.

$$\phi = \frac{1}{(2\pi)^2 iR} \int_{-\infty}^{+\infty} \frac{k\,e^{+ikR}\,dk}{(k + k_j)\,(k - k_j)} \tag{A7}$$

To obtain the outgoing wave solution for $k_j'' < 0$, one evaluates the integral in Eq. (A7) using contour integration and the theory of residues. The closed contour in the complex k-plane, which is traversed in a counterclockwise direction, includes the real axis and a semicircle of infinite radius (centered at the origin) in the upper half-plane as shown in figure 10. Since the integral over the semicircle vanishes, ϕ is given by Eq. (A8),

$$\phi = \frac{1}{(2\pi)^2 iR} \int_C \frac{k\,e^{+ikR}\,dk}{(k + k_j)\,(k - k_j)} \tag{A8}$$

with the contour and pole diagram shown in Figure 10.

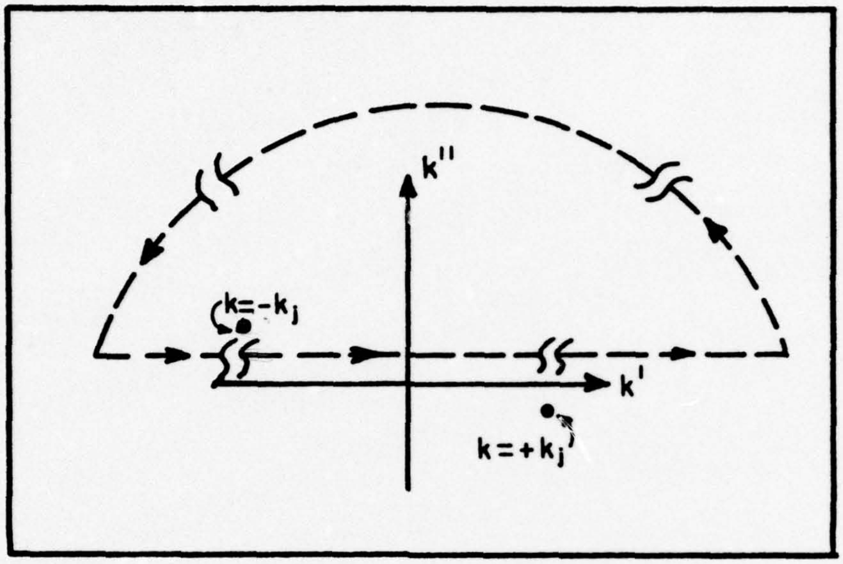

Figure 10. Closed Contour in the Complex k-Plane

To avoid systematically excluding outgoing wave solutions for $k_j'' \geq 0$, one must analytically continue this solution into the upper half of the complex k_j-plane. To do this, one simply distorts the contour to insure that the path of integration includes the outgoing wave pole as this pole crosses the real axis. That is, the contour is distorted so that it does not cross any poles as $-k_j$ crosses the real axis so that $-k_j''$ takes on negative values. This contour, which analytically continues ϕ into the upper half of the k_j-plane, is shown in figure 11.

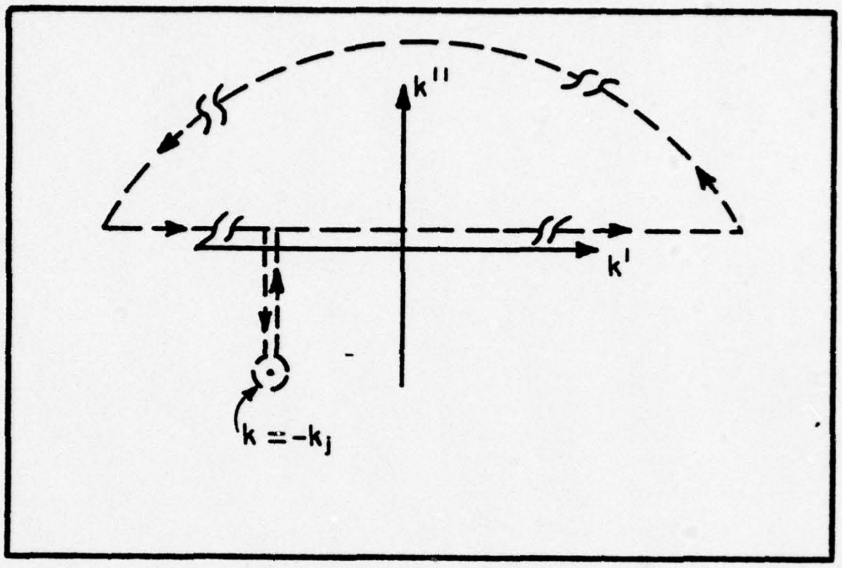

Figure 11. Contour Defining the Analytic Continuation of ϕ

Thus for all values of k_j'', the only pole enclosed by the contours is at $k = -k_j$. Accordingly, the contour integrals can be written as

$$\int_C \frac{k\, e^{+ikR}\, dk}{(k + k_j)\,(k - k_j)} = 2\pi i L' \tag{A9}$$

where L', the residue of the integrand at $k = -k_j$, is given by Eq. (A10).

$$L' = \lim_{k \to -k_j} \frac{(k + k_j)\, k\, e^{+ikR}}{(k + k_j)\,(k - k_j)} = \frac{1}{2} e^{-ik_j R} \tag{A10}$$

Thus, the function ϕ has the form

$$\phi = \frac{e^{-ik_j R}}{4\pi R} \qquad \text{(A11)}$$

for $k_j' > 0$ and $-\infty < k_j'' < +\infty$.

Finally, using Eq. (A11) and the relation $\bar{\bar{G}} = \bar{\bar{I}}\phi$, one obtains the following expression for the outgoing wave Green's function.

$$\bar{\bar{G}} = \bar{\bar{I}} \frac{e^{-ik_j R}}{4\pi R} \qquad \text{(A12)}$$

APPENDIX B

Application of the Field Equations to Resonators with Segmented or Open Boundaries

This appendix has two objectives. The first is to show that the expressions for the fields for both open resonators and segmented boundary resonators satisfy Maxwell's equations. The second is to show that these fields can be divided into m partial fields such that

1. The m^{th} partial field involves only the field over the m^{th} mirror surface or the m^{th} segment of a closed boundary; and
2. Each partial field satisfies Maxwell's equations.

The formulation and manipulations needed to accomplish these objectives are covered below.

Formulation

The field expressions for the two cases described above (Eqs. (28) and (29), and Eqs. (32) and (33))can be summarized by the following equations,

$$\overline{E}(\overline{r}) = \sum_m \int_{S'_m} \left\{ (\hat{n} x \overline{E}_m) \cdot \nabla' x \bar{\bar{G}} + (\hat{n} x \nabla' x \overline{E}_m) \cdot \bar{\bar{G}} + (\hat{n} \cdot \overline{E}_m) \nabla' \cdot \bar{\bar{G}} \right\} dS'_m$$

$$- \frac{1}{i\omega\tilde{\varepsilon}} \sum_m \oint_{C_m} \nabla' \phi \, \overline{H}_m \cdot d\overline{s}_m \qquad \text{(B1)}$$

$$\overline{H}(\overline{r}) = \sum_m \int_{S'_m} \left\{ (\hat{n}x\overline{H}_m) \cdot \nabla' x\bar{\bar{G}} + (\hat{n}x\nabla' x\overline{H}_m) \cdot \bar{\bar{G}} + (\hat{n}\cdot\overline{H}_m)\nabla' \cdot \bar{\bar{G}} \right\} dS'_m$$

$$+ \frac{1}{i\omega\mu} \sum_m \oint_{C_m} \nabla' \phi \, \overline{E}_m \cdot d\overline{s}_m \qquad \text{(B2)}$$

where $\bar{\bar{G}} = \bar{\bar{I}}\phi$. For closed resonators, the sums in these equations are taken over all continuous segments of the closed interface or boundary. For open resonators, the sums are taken over the surfaces of the resonator mirrors.

To obtain these two equations, it has been assumed that both the electric and magnetic fields are square integrable. Thus, the points of finite discontinuity have simply been removed from the above integrals, and the remaining surface and contour integrals are considered to extend to within an arbitrarily small distance δ from the points of discontinuity. As they do not actually include the points of discontinuity, the integrands (and their derivatives) of these integrals are continuous. As a result, integral theorems such as Stokes' theorem can be applied to the surface and contour integrals in the above two equations.

To accomplish the objectives of this appendix, the following approach is used. First, the fields $\overline{E}(\overline{r})$ and $\overline{H}(\overline{r})$ in Eqs. (B1) and (B2) are written as the sum of m partial fields,

$$\overline{E}(\overline{r}) = \sum_m \overline{E}_m(\overline{r}) \qquad \text{(B3)}$$

$$\overline{H}(\overline{r}) = \sum_m \overline{H}_m(\overline{r}) \tag{B4}$$

where $\overline{E}_m(\overline{r})$ and $\overline{H}_m(\overline{r})$ are the fields resulting from the m^{th} segment or mirror surface. Then Eqs. (B3) and (B4) are substituted into Eqs. (B1) and (B2) to yield the pair of equations shown below.

$$\sum_m \overline{E}_m(\overline{r}) = \sum_m \int_{S'_m} \left\{ (\hat{n}x\overline{E}_m) \cdot \nabla' x\bar{\bar{G}} + (\hat{n}x\nabla' x\overline{E}_m) \cdot \bar{\bar{G}} + (\hat{n}\cdot\overline{E}_m)\nabla' \cdot \bar{\bar{G}} \right\} dS'_m$$

$$- \frac{1}{i\omega\tilde{\varepsilon}} \sum_m \oint_{C_m} \nabla' \phi \, \overline{H}_m \cdot d\overline{s}_m \tag{B5}$$

$$\sum_m \overline{H}_m(\overline{r}) = \sum_m \int_{S'_m} \left\{ (\hat{n}x\overline{H}_m) \cdot \nabla' x\bar{\bar{G}} + (\hat{n}x\nabla' x\overline{H}_m) \cdot \bar{\bar{G}} + (\hat{n}\cdot\overline{H}_m)\nabla' \cdot \bar{\bar{G}} \right\} dS'_m$$

$$+ \frac{1}{i\omega\mu} \sum_m \oint_{C_m} \nabla' \phi \, \overline{H}_m \cdot d\overline{s}_m \tag{B6}$$

Then it will be shown that each of the partial fields $\overline{E}_m(\overline{r})$ and $\overline{H}_m(\overline{r})$ satisfies Maxwell's equations. From this demonstration, it follows directly that the total fields in Eqs. (B1) and (B2) satisfy Maxwell's equations.

To show that these fields satisfy Maxwell's equations in all space, it is necessary to assume that they satisfy Maxwell's equations over the indicated surfaces and contours. Of course, any assumption to the contrary immediately invalidates the assertion that the modal fields are electromagnetic in nature. Thus, Eqs. (B7) and (B8)

$$\nabla \times \overline{E} = -i\omega\mu\overline{H} \tag{B7}$$

$$\nabla \times \overline{H} = +i\omega\tilde{\varepsilon}\overline{E} \tag{B8}$$

can be used to simplify the field expressions as shown below.

$$\overline{E}_m(\overline{r}) = \int_{S_m'} \left\{ (\hat{n}\times\overline{E}_m) \cdot \nabla' \times \bar{\bar{G}} - i\omega\mu\,(\hat{n} \times \overline{H}_m) \cdot \bar{\bar{G}} + (\hat{n}\cdot\overline{E}_m)\nabla' \cdot \bar{\bar{G}} \right\} dS_m'$$

$$- \frac{1}{i\omega\tilde{\varepsilon}} \oint_{C_m} \nabla' \phi\, \overline{H}_m \cdot d\overline{s}_m \tag{B9}$$

$$\overline{H}_m(\overline{r}) = \int_{S_m'} \left\{ (\hat{n}\times\overline{H}_m) \cdot \nabla' \times \bar{\bar{G}} + i\omega\tilde{\varepsilon}\,(\hat{n}\times\overline{E}_m) \cdot \bar{\bar{G}} + (\hat{n}\cdot\overline{H}_m)\nabla' \cdot \bar{\bar{G}} \right\} dS_m'$$

$$+ \frac{1}{i\omega\mu} \oint_{C_m} \nabla' \phi\, \overline{E}_m \cdot d\overline{s}_m \tag{B10}$$

Then, subject to the condition that these fields must be square integrable (which was assumed earlier in this appendix), the objectives of this appendix will be accomplished if $\overline{E}_m(\overline{r})$ and $\overline{H}_m(\overline{r})$ given above are solenoidal (divergenceless) and satisfy Eqs. (B7) and (B8).

Finally, to further simplify these equations and to aid in later manipulations, the following vector identities are listed.

$$(\nabla \times \bar{\bar{G}}) \cdot \overline{A} = \nabla\phi \times \overline{A} \tag{B11}$$

$$\overline{A} \cdot (\nabla \times \bar{\bar{G}}) = \overline{A} \times \nabla\phi \tag{B12}$$

$$\nabla \times [\overline{A}(\overline{r}') \times \nabla' \phi] = \overline{A}(\overline{r}')\, \nabla'\nabla'\phi - \overline{A}(\overline{r}')\nabla'^2\phi \tag{B13}$$

$$\nabla \cdot \bar{\bar{G}} = \nabla\phi \tag{B14}$$

$$\overline{A} \cdot \overline{\overline{G}} = \overline{A}\phi \tag{B15}$$

$$\overline{A} \times \nabla\nabla\phi = (\nabla \times \overline{A})\nabla\phi - \nabla \times (\overline{A}\nabla\phi) \tag{B16}$$

$$\nabla \cdot (\overline{A} \times \overline{B}) = \overline{B} \cdot \nabla \times \overline{A} - \overline{A} \cdot \nabla \times \overline{B} \tag{B17}$$

Using Eqs. (B12), (B14) and (B15), Eqs. (B9) and (B10) are rewritten as shown below.

$$\overline{E}_m(\overline{r}) = \int_{S_m'} \left\{ (\hat{n}x\overline{E}_m) \times \nabla'\phi - i\omega\mu\ (\hat{n}x\overline{H}_m)\ \phi + \nabla'\phi\ (\hat{n}\cdot\overline{E}_m) \right\} dS_m'$$

$$- \frac{1}{i\omega\tilde{\varepsilon}} \oint_{C_m} \nabla'\ \phi\ \overline{H}_m \cdot d\overline{s}_m \tag{B18}$$

$$\overline{H}_m(\overline{r}) = \int_{S_m'} \left\{ (\hat{n}x\overline{H}_m) \times \nabla'\ \phi + i\omega\tilde{\varepsilon}\ (\hat{n}x\overline{E}_m)\ \phi + \nabla'\phi\ (\hat{n}\cdot\overline{H}_m) \right\} dS_m'$$

$$+ \frac{1}{i\omega\mu} \oint_{C_m} \nabla'\ \phi\ \overline{E}_m \cdot d\overline{s}_m \tag{B19}$$

Manipulations

To show that the fields in Eqs. (B18) and (B19) have the desired properties, a procedure similar to that used by Stratton (ref. 33, pp. 469-470) will be employed. One first takes the curl of Eq. (B18) with the subscript dropped to obtain

$$\nabla \times \overline{E}(\overline{r}) = \int_{S'} \left\{ \nabla \times [\hat{n}x\overline{E}(\overline{r}') \times \nabla'\phi] - i\omega\mu\nabla \times [\hat{n}x\overline{H}(\overline{r}')]\phi \right\} dS' \tag{B20}$$

where $\nabla\phi = -\nabla'\phi$ and the identity $\nabla \times \nabla\phi = 0$ have been used to eliminate the last two terms. Using Eq. (B13) and the fact that ϕ satisfies the scalar Helmholtz equation (Eq. (A2)), this expression is simplified to yield

$$\nabla \times \overline{E}(\overline{r}) = -i\omega\mu \int_{S'} \left\{ (\hat{n}x\overline{H}) \times \nabla'\phi + i\omega\tilde{\varepsilon}\, (\hat{n}x\overline{E})\, \phi \right\} dS'$$

$$+ \int_{S'} (\hat{n}x\overline{E}) \cdot \nabla'\nabla'\phi dS' \qquad \text{(B21)}$$

Using Eqs. (B7) and (B16), and the fact that

$$(\hat{n}x\overline{E}) \cdot \nabla\nabla\phi = \hat{n} \cdot \{\overline{E}x\nabla\nabla\phi\} \qquad \text{(B22)}$$

this last integral can be simplified as shown below.

$$\int_{S'} (\hat{n}x\overline{E}) \cdot \nabla'\nabla'\phi dS' = -i\omega\mu \int_{S'} (\hat{n}\cdot\overline{H})\, \nabla'\phi\, dS'$$

$$- \int_{S'} \hat{n} \cdot [\nabla' x(\overline{E}\nabla'\phi)]\, dS' \qquad \text{(B23)}$$

The last integral in Eq. (B23) is now converted to a line integral by using Stokes' theorem applied to dyadics (Collin, ref. 34, p. 569). The result is shown below.

$$\int_{S'} \hat{n} \cdot [\nabla' x(\overline{E}\nabla'\phi)]\, dS' = \oint_C \nabla'\phi\overline{E} \cdot d\overline{s} \qquad \text{(B24)}$$

Then substituting Eqs. (B23) and (B24) into Eq. (B21), one obtains the following result.

$$\nabla \times \overline{E}(\overline{r}) = - i\omega\mu \left[\int_{S'} \left\{ (\hat{n}\times\overline{H}) \times \nabla'\phi + i\omega\tilde{\varepsilon}\ (\hat{n}\times\overline{E})\phi \right.\right.$$

$$\left.\left. + (\hat{n}\cdot\overline{H})\nabla'\phi \right\} dS' + \frac{1}{i\omega\mu} \oint_C \nabla'\phi\overline{E} \cdot d\overline{s} \right] \tag{B25}$$

Comparing the right side of Eq. (B25) with Eq. (B19), one can see that Eq. (B7) is satisfied. A similar procedure, which is not shown here, can be used to show that the fields in Eqs. (B18) and (B19) satisfy Eq. (B8).

The next step is to show that the fields in Eqs. (B18) and (B19) are solenoidal. One begins by taking the divergence of Eq. (B19) and applying Eq. (ʋ17) to obtain

$$\nabla \cdot \overline{H}(\overline{r}) = \int_{S'} \left\{ i\omega\tilde{\varepsilon}\nabla \cdot [(\hat{n}\times\overline{E}(\overline{r}')\phi] - \nabla'^2\phi(\hat{n}\cdot\overline{H}) \right\} dS'$$

$$- \frac{1}{i\omega\mu} \oint_C \nabla'^2\phi\overline{E} \cdot d\overline{s} \tag{B26}$$

Applying the Helmholtz equation, this form is simplified to yield

$$\nabla \cdot \overline{H}(\overline{r}) = \int_{S'} \left\{ - i\omega\tilde{\varepsilon}\ (\hat{n}\times\overline{E}) \cdot \nabla'\phi + k^2\ (\hat{n}\cdot\overline{H}) \right\} dS'$$

$$+ \frac{k^2}{i\omega\mu} \oint_C \phi\overline{E} \cdot d\overline{s} \tag{B27}$$

Using Stokes' theorem, the line integral in Eq. (B27) is expressed in terms of two surface integrals as shown below.

$$\oint_C \phi\overline{E} \cdot d\overline{s} = \int_{S'} \left\{ \nabla'\phi \times \overline{E} + \phi(\nabla' \times \overline{E}) \right\} \cdot \hat{n} dS' \qquad \text{(B28)}$$

Using Eq. (B7) and the fact that

$$(\nabla'\phi \times \overline{E}) \cdot \hat{n} = -(\hat{n} \times \overline{E}) \cdot \nabla'\phi \qquad \text{(B29)}$$

the line integral is expressed as

$$\oint_C \phi\overline{E} \cdot d\overline{s} = \int_{S'} \left\{ -(\hat{n} \times \overline{E}) \cdot \nabla'\phi - i\omega\mu \, (\hat{n} \cdot \overline{H}) \, \phi \right\} dS' \qquad \text{(B30)}$$

Substituting Eq. (B30) into Eq. (B27) and using $k^2 = \omega^2\mu\tilde{\varepsilon}$, one can show that $\nabla \cdot \overline{H} = 0$. A similar procedure can be used to show that $\nabla \cdot \overline{E} = 0$.

APPENDIX C

Elimination of the Self-Induction Integral

On page 37 in this report, it is shown that for resonators containing two perfectly conducting mirrors for which $a/|R| << 1$, the currents on mirrors #1 and #2 are governed by the following pair of equations,

$$J_{x1} = \hat{K}_{11} J_{x1} + \hat{K}_{12} J_{x2} \tag{C1}$$

$$J_{x2} = \hat{K}_{21} J_{x1} + \hat{K}_{22} J_{x2} \tag{C2}$$

where

$$\hat{K}_{q\ell} J_{x\ell} = 2(-1)^q \int_{S'_\ell} J_{x\ell}(\overline{r}'_\ell) \; \phi_z(\overline{r}|\overline{r}'_\ell) \Big|_{\overline{r}=\overline{r}_\ell} dS'_\ell \tag{C3}$$

$$\hat{K}_{qq} J_{xq} = 2(-1)^q \int_{S'_q} J_{xq}(\overline{r}'_q) \left\{ \frac{1 + ik|\overline{r - r'_q}|}{|\overline{r - r'_q}|} \right\} \phi(\overline{r}'_q|\overline{r}_q)$$

$$\cos(\overline{r_q - r'_q}, \hat{n}_q) \, dS'_q \tag{C4}$$

$$\phi(\overline{r}_q|\overline{r}'_q) = \frac{e^{-ik|\overline{r_q - r'_q}|}}{4\pi|\overline{r_q - r'_q}|} \tag{C5}$$

and $\cos(\overline{r_q - r'_q}, \hat{n}_q)$ is the cosine of the angle between the unit normal at $\overline{r}_q$ and the vector $\overline{r_q - r'_q}$. It was further shown that when the self-induction integrals (or terms) are negligible, that is, when

$$|\hat{K}_{qq} J_{xq}| << |J_{xq}|, \text{ for } q = 1, 2 \tag{C6}$$

then a certain class of laser resonator modes obeys the equations

$$J_{x1} = \hat{K}_{12}\hat{K}_{21}J_{x1} \tag{C7}$$

$$J_{x2} = \hat{K}_{21}\hat{K}_{12}J_{x2} \tag{C8}$$

The purpose of this appendix is to estimate the conditions under which the self-induction integrals are negligible.

To obtain this estimate, the following approach is used. First, it is assumed that the resonators being considered contain circular mirrors of radius a. Then the inequality in Eq. (C6) is specialized to apply to the center of mirror #2 for modes which have no azimuthal variation. Finally, a simplified version of the expansion functions (see chapter V) is substituted for J_{x2}, and the resulting integral is approximately evaluated for mirrors such that $a/|R| \ll 1$. Thus, the estimate itself will take the form of an inequality involving the various resonator and mode parameters. However, since this inequality is merely an estimate of the conditions under which the self-induction integrals are negligible, the inequality is denoted by the symbol $\overset{?}{\ll}$.

It is apparent that this procedure involves many approximations and assumptions concerning the resonators considered and their associated fields. To show that this procedure at least leads to the right order of magnitude of the conditions under which the self-induction terms can be neglected, a second approach is used to check the conditions for one particular case. This second approach, which employs the method of steepest descents to approximate an integral, begins with Eq. (C33).

Using the first approach described and denoting the current distribution for the j^{th} mode by $u_j(r)$, one specializes the inequality in Eq. (C6) to obtain Eq. (C9).

$$\left| \int_0^a u_j(r_2') \, e^{-ik_j r_2'} \left(\frac{1 + ik_j r_2'}{r_2'^2} \right) \cos(\hat{a}_z, \bar{r}_2') \, r_2' dr_2' \right| \overset{?}{<<} \left| u_j(o) \right| \qquad \text{(C9)}$$

The expansion functions which were obtained in chapter V, have the form

$$u_j(r) = g_j(r) \, e^{\pm \frac{ik_j}{2L} \sqrt{g^2 - 1} \; r^2} \qquad \text{(C10)}$$

where $Re(k_j) = k_j' > o$, and $g_j(r)$ is assumed to be slowly varying in comparison to the exponentials $e^{-ik_j' r}$ and $e^{\pm \frac{ik_j'}{2L} \sqrt{g^2 - 1} \; r^2}$. Substituting this form for $u_j(r)$ into Eq. (C9), one obtains the inequality shown below,

$$\left| g_j(o) \right| \overset{?}{>>} \left| \int_0^a g_j(r_2') \exp \left\{ ik_j \left[-r_2' \pm \frac{\beta_g}{2L} r_2'^2 \right] \right\} \cos(\hat{a}_z, \bar{r}_2') \left(\frac{1 + ik_j r_2'}{r_2'} \right) dr_2' \right| \qquad \text{(C11)}$$

where $\beta_g = \sqrt{g^2 - 1}$

To simplify this expression further, one needs to approximate the term $\cos(\hat{a}_z, \bar{r}_2')$. To do this, one uses figure 12 to see that the vector $\bar{r}_2'$ can be expressed in the form

$$\overline{r}_2' \approx r_2' \, \hat{a}_\rho + \frac{r_2'^2}{2R} \, \hat{a}_z \qquad \text{(C12)}$$

where R is the radius of curvature of mirror #2 and the distance z has been approximated by $z = r_2'^2/2R$, which applies if $\frac{a}{|R|} << 1$.

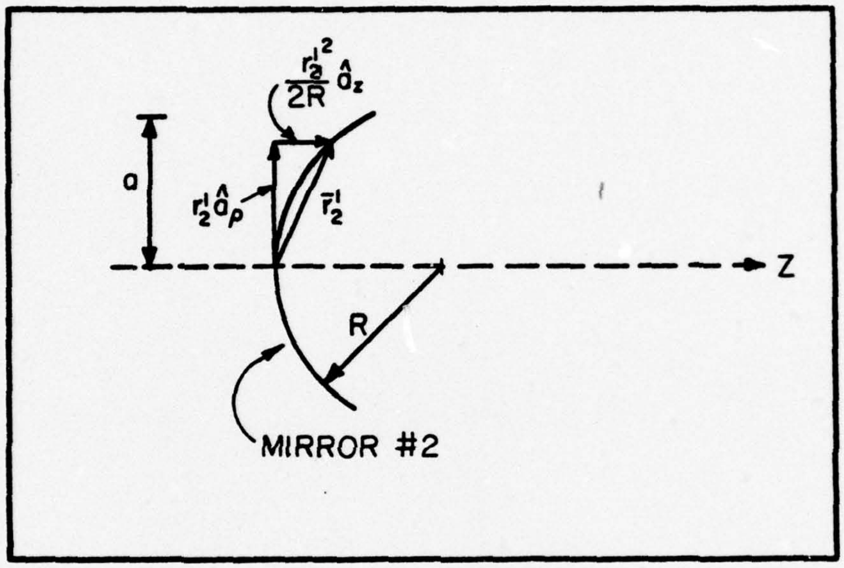

Figure 12. The Geometry of a Large Radius of Curvature Mirror

Then using Eq. (C13),

$$\cos(\hat{a}_z, \overline{r}_2') = \frac{\hat{a}_z \cdot \overline{r}_2'}{|\overline{r}_2'|} \approx \frac{\hat{a}_z \cdot \overline{r}_2'}{r_2'} \qquad \text{(C13)}$$

in conjunction with Eq. (C12), the estimated condition is simplified to yield

$$\left|g_j(o)\right| \overset{?}{>>} \frac{1}{2|R|}\left|\int_0^a g_j(r_2') \exp\left\{+ik_j\left[-r_2' \pm \frac{\beta_g}{2L} r_2'^2\right]\right\} (1 + ik_j r_2') dr_2'\right| \quad \text{(C14)}$$

Since the integrand contains terms of the form $e^{+k_j''r}$ and $e^{\mp k_j''\beta_g r^2/2L}$, the probability that Eq. (C14) will hold (for both + and - signs) should increase as k_j'' decreases. With this in mind, and to simplify the analysis as well, it is assumed that k_j'' is sufficiently small so that the functions $e^{+k_j''r}$ and $e^{\mp k_j''\beta_g r^2/2L}$ are slowly varying across the entire mirror surface. With this assumption, the magnitude of the integral in Eq. (C14) will be largely determined by the oscillatory functions $e^{-ik_j'r}$ and $e^{\pm ik_j'\beta_g r^2/2L}$. In fact, these two oscillatory functions should combine to produce a significant contribution to the integral in regions where

1. The exponentials oscillate nearly in phase, or
2. The exponentials are nearly conjugates

over some significant distance.

There are two types of regions over which these two conditions appear likely to occur. The first type consists of regions surrounding points where the two phase terms are equal; that is, where

$\phi_1(r) = \phi_2(r)$ with $\phi_1(r) \equiv k_j'r$ and $\phi_2(r) \equiv \frac{k_j'\beta_g}{2L} r^2$. Of course, $\phi_1(r) = \phi_2(r)$ at $r = o$ and at $r = 2L/\beta_g$.

Although the phase terms are equal at these two points, their rates of change are not equal there. In fact, at $r = o$,

$$\frac{d\phi_1(r)}{dr} = k_j' \text{ and } \frac{d\phi_2(r)}{dr} = o \qquad \text{(C15)}$$

while at $r = \frac{2L}{\beta_g}$,

$$\frac{d\phi_1(r)}{dr} = k_j' \text{ and } \frac{d\phi_2(r)}{dr} = 2k_j' \qquad \text{(C16)}$$

At the optical or infrared wavelengths of interest for this analysis, the difference in these two rates will be sufficiently large to prevent the exponentials from being in phase (or conjugates) over any significant distance. As a result, the contribution to the integral from these two regions (especially the one near the point $r = o$) will be small in comparison to the contribution from a stationary phase type region where $\phi_1(r)$ and $\phi_2(r)$ are changing at nearly the same rate. The only region of this second type is located in the vicinity of $r = L/\beta_g$. To estimate the width of this stationary phase type region, it is assumed that the exponentials remain nearly in phase or nearly conjugates over the distance for which,

$$\left| \Delta\phi_2(r) - \Delta\phi_1(r) \right| = \frac{\pi}{2} \qquad \text{(C17)}$$

where the changes in the ϕ_i are computed relative to their values at $r = L/\beta_g$. Substituting the forms for $\phi_1(r)$ and $\phi_2(r)$ into Eq. (C17), one obtains Eq. (C18) below.

$$\frac{\beta_g}{2L} r^2 - r + \frac{L}{2\beta_g} = \pm \frac{\lambda}{4} \qquad \text{(C18)}$$

This equation has real solutions of the form

$$r = \frac{L}{\beta_g}\left[1 \pm \sqrt{S}\right] \tag{C19}$$

where $S = \lambda\beta_g/2L$. Thus, the so-called stationary phase region will be taken as the region extending from $r = \frac{L}{\beta_g}(1 - \sqrt{S})$ to $r = \frac{L}{\beta_g}(1 + \sqrt{S})$.

To estimate the magnitude of the integral (in Eq. (C14)) when the mirror radius a is sufficiently large to include all or part of this region, the following procedure is used.

1. The assumed slowly varying functions, $g_j(r)$ and the exponentials involving k_j'', are evaluated at the midpoint of the integration interval and factored out of the integral.
2. The remaining exponentials are taken as being exactly in phase or exactly conjugates over the integration interval.
3. It is assumed that $k_j' >> k_j''$.

Then applying these steps, taking $\left| g_j\left(\frac{L}{\beta_g}\right) \right| \approx \left| g_j(o) \right|$, and assuming $a > \frac{L}{\beta_g}(1 + \sqrt{S})$, the proposed inequality becomes

$$1 \overset{?}{>>} \frac{e^{\pm\frac{k_j''L}{2\beta_g}} e^{+\frac{k_j''L}{\beta_g}}}{2|R|} \left| \int_{\frac{L}{\beta_g}(1-\sqrt{S})}^{\frac{L}{\beta_g}(1+\sqrt{S})} e^{-ik_j'r_2'} e^{\mp\frac{ik_j'}{2L}\beta_g r_2'^2} (1 + ik_j'r_2')\, dr_2' \right| \tag{C20}$$

For the case where the exponentials are in phase (upper sign), Eq. (C20) takes the form shown below.

$$1 \overset{?}{>>} \frac{e^{\frac{3k_j''L}{2\beta_g}}}{2|R|} \left| \int_{\frac{L}{\beta_g}(1-\sqrt{S})}^{\frac{L}{\beta_g}(1+\sqrt{S})} e^{-2ik_j'r_2'} dr_2' \right.$$

$$\left. + ik_j' \int_{\frac{L}{\beta_g}(1-\sqrt{S})}^{\frac{L}{\beta_g}(1+\sqrt{S})} e^{\frac{-ik_j'}{L} \beta_g r_2'^2} r_2' dr_2' \right| \tag{C21}$$

By determining the bounds on these two integrals, one can show that Eq. (C21) will hold if

$$1 \overset{?}{>>} \frac{1}{2|R|} e^{\frac{3k_j''L}{2\beta_g}} \left[\frac{\lambda}{2\pi} + \frac{L}{\beta_g} \right] \tag{C22}$$

If the exponentials are taken as conjugates (lower sign) over the stationary phase region, Eq. (C20) takes the form

$$1 \overset{?}{>>} \frac{1}{2|R|} e^{+\frac{k_j''L}{2\beta_g}} \left| \int_{\frac{L}{\beta_g}(1-\sqrt{S})}^{\frac{L}{\beta_g}(1+\sqrt{S})} (1 + ik_j'r_2') dr_2' \right| \tag{C23}$$

This inequality will hold if

$$1 \overset{?}{>>} \frac{\sqrt{S}}{|R|} e^{+\frac{k_j'' L}{2\beta_g}} \left[1 + \frac{2\pi L}{\lambda \beta_g} \right] \frac{L}{\beta_g} \tag{C24}$$

One can use this same procedure to estimate the conditions under which the self-induction integrals can be neglected for $\frac{L}{\beta_g}(1 - \sqrt{S}) \leq a \leq \frac{L}{\beta_g}(1 + \sqrt{S})$; however, for this case, a is taken as the upper limit of integration in Eq. (C20). The result is two rather complicated conditions which are not given in this report.

This same procedure is also used to estimate the integral in Eq. (C14) for the case where $a < \frac{L}{\beta_g}(1 - \sqrt{S})$. For this case, it is assumed that the integration region extends over the comparatively small distance (see the discussion following Eq. (C16)) corresponding to the condition,

$$|\Delta\phi_2(r) - \Delta\phi_1(r)| = \pm \frac{\pi}{2} \tag{C25}$$

where the changes in $\phi_1(r)$ and $\phi_2(r)$ are computed relative to their values at r = o. Application of this condition leads to the equation

$$k_j' \left(\frac{\beta_g r^2}{2L} - r \right) = \pm \frac{\pi}{2} \tag{C26}$$

which has solutions of the form

$$r = \frac{L}{\beta_g}\left\{1 \pm \sqrt{1 + S}\right\} \qquad \text{(C27)}$$

Since $r < \frac{L}{\beta_g}(1 - \sqrt{S})$, one chooses both minus signs; thus, the region of integration has a width w given by Eq. (C28).

$$w = \left(1 - \sqrt{1 - S}\right)\frac{L}{\beta_g} \qquad \text{(C28)}$$

For the case when $S << 1$, $w \approx \lambda/4$.

To at least partially include the fact that the exponentials $e^{+k_j''r}$ and $e^{+k_j''r^2\beta_g/2L}$ increase with increasing radius, this region of width w will be placed at the outer edge of the mirror of radius a. Thus, for this case, the integral in Eq. (C14) will be approximated using the procedure described earlier and limits of integration of $r = a-w$ and $r = a$. Then evaluating the exponentials involving k_j'' at $r = a - \frac{w}{2}$ and factoring them and $g_j(r)$ out of the integral, the proposed inequality takes the form

$$1 \overset{?}{>>} \frac{1}{2|R|} e^{k_j''(r \pm \frac{\beta_g}{2L} r^2)} \left| \int_{a-w}^{a} e^{-ik_j'r_2' \mp \frac{ik_j'}{2L}\beta_g r_2'^2} (1 + ik_j' r_2')dr_2' \right| \qquad \text{(C29)}$$

The inequality in Eq. (C29) will be satisfied if

$$\frac{w}{2|R|} e^{+k_j''(a + \frac{\beta_g}{2L} a^2)} \left| 1 + \frac{i\pi}{\lambda} (2a - w) \right| \overset{?}{<<} 1 \qquad \text{(C30)}$$

For the case where S << 1, Eq. (C30) reduces to Eq. (C31),

$$\frac{\lambda}{8|R|} e^{+k_j''(a + \frac{\beta_g}{2L} a^2)} \left| (1 - \frac{i\pi}{4}) + \frac{i2\pi a}{\lambda} \right| \overset{?}{<<} 1 \qquad \text{(C31)}$$

which will be satisfied if

$$\frac{1}{2|R|} e^{+k_j''(a + \frac{\beta_g}{2L} a^2)} \left[\frac{\lambda}{4} \sqrt{1 + (\frac{\pi}{4})^2} + \frac{\pi}{2} a \right] \overset{?}{<<} 1 \qquad \text{(C32)}$$

For future reference, the estimates of the conditions under which the self-induction integrals may be neglected are summarized below. First, for the case where $\frac{a}{|R|} << 1$, $a < \frac{L}{\beta_g}(1 - \sqrt{S})$, and $\lambda << \frac{2L}{\beta_g}$, this estimate corresponds to the condition shown in Eq. (C32). For the resonators considered in this paper (those for which the wavelength is small in comparison to all resonator dimensions), this condition will hold for all but the lossiest modes. For the case where $\frac{a}{|R|} << 1$ and $\frac{L}{\beta_g}(1 + \sqrt{S}) < a$, this estimate corresponds to the pair of conditions shown below and referenced for your convenience.

$$\frac{1}{2|R|} e^{+\frac{3k_j''L}{2\beta_g}} \left[\frac{\lambda}{2\pi} + \frac{L}{\beta_g} \right] \overset{?}{<<} 1 \qquad \text{(C22)}$$

$$\frac{L}{\beta_g} \frac{\sqrt{S}}{|R|} e^{+\frac{k_j'' L}{2\beta_g}} \left[1 + \frac{2\pi L}{\lambda \beta_g}\right] \overset{?}{<<} 1 \qquad \text{(C24)}$$

These conditions will hold only for cases where the mirror radius of curvature is very large in comparison to the mirror separation.

As indicated earlier in this appendix, various assumptions and approximations have been made to obtain these estimates of the conditions where the self-induction terms may be neglected. Many of these assumptions were made to simplify the integral in Eq. (C14). To at least show that this simplification process has led to the right order of magnitude for these estimates (Eqs. C22, C24, and C32), the integral

$$I_o = \int_0^a g_j(r_2') \, e^{-ik_j r_2' + \frac{ik_j}{2L} \beta_g r_2'^2} (1 + ik_j r_2') dr_2' \qquad \text{(C33)}$$

will be approximated for the case where $\frac{L}{\beta_g} < a$ using the method of steepest descents (Erdélyi, ref. 41, vol. 2, pp. 24-27). To remain consistent with the calculations performed earlier in this appendix, it is assumed that $k_j' >> k_j''$. With this assumption, I_o may be written in the form characteristic of the method of steepest descents; that is,

$$I_o = \int_0^a h(z) \, e^{k_j' f(z)} dz \qquad \text{(C34)}$$

where

$$h(z) = g_j(z)\,(1 + ik_j'z)\,e^{k_j''\left(z - \frac{\beta_g}{2L} z^2\right)} \qquad \text{(C35)}$$

$$f(z) = i\left(\frac{\beta_g}{2L} z^2 - z\right) \qquad \text{(C36)}$$

and $z = x + iy$.

To locate the saddle point, one determines the point at which

$$\frac{df(z)}{dz} = 0 \qquad \text{(C37)}$$

For this problem, the saddle point is located at $z = L/\beta_g$ and, the steepest descents contour, which is specified by the equation $\mathrm{Im}\, f(z) = \mathrm{Im}\, f(L/\beta_g)$, corresponds to the line $y = x - L/\beta_g$ in the complex z-plane. As part of the overall procedure to estimate I_o, this steepest descents contour (C_{Sd}) has been chosen as one segment of the closed contour shown below in figure 13, where C_1 and C_2 are lines parallel to the imaginary axis. C_1 begins at the point $z = a$ and ends at $z = a(1 + i)$ while C_2 begins at $z = -i\,L/\beta_g$ and ends at $z = 0$. The segment C_o extends along the real axis from $x = o$ to $x = +a$.

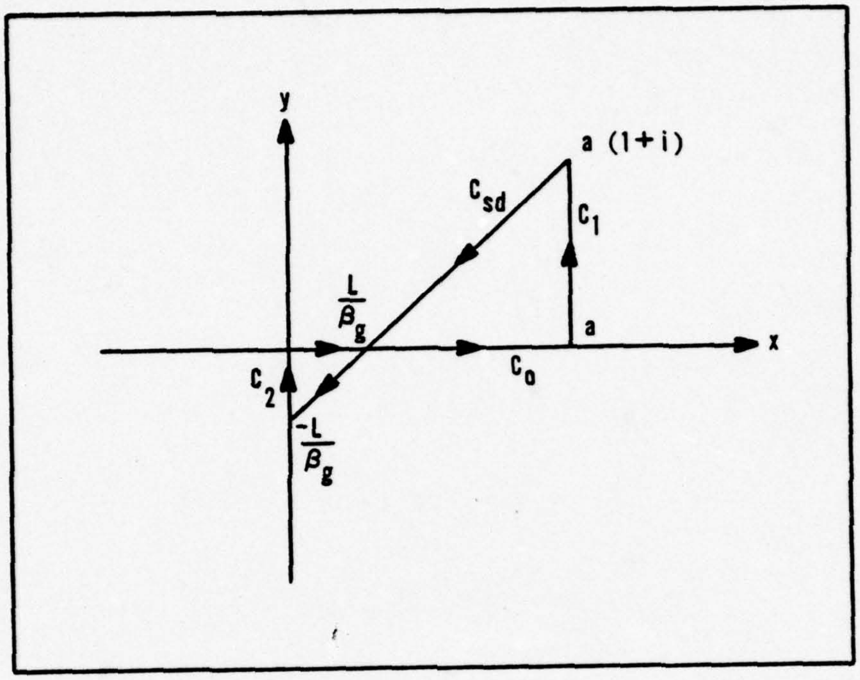

Figure 13. Closed Contour Chosen to Include C_{Sd}

By applying the theory of residues to the closed contour shown in figure 13, one obtains Eq. (C38)

$$I_o = -(I_1 + I_2 + I_{Sd}) \tag{C38}$$

where

$$I_1 = \int_{C_1} h(z)\, e^{+k_j' f(z)}\, dz \tag{C39}$$

and so forth. However, by writing out the expressions for I_1 and I_2 and using the fact that $k_j' >> k_j''$, one can show that these integrals involve exponentials which decay very rapidly with increasing distance from the real axis. As a result, $|I_1|$ and $|I_2|$ should be small in comparison to $|I_{Sd}|$. Therefore, they will be neglected for the remainder of this analysis.

Finally, to simplify the analysis, only the first term in the asymptotic expansion of I_{Sd} (Erdélyi, ref. 41, vol. 2, p. 26) will be used. One thus obtains the following expression.

$$I_{Sd} \approx \sqrt{\frac{2\pi L}{k_j' \beta_g}}\, g_j\left(\frac{L}{\beta_g}\right) e^{+\frac{1}{2} L K_j''/\beta_g} \left(1 + ik_j' \frac{L}{\beta_g}\right) e^{-\frac{ik_j' L}{\beta_g}} \tag{C40}$$

Taking the bound of the right side of Eq. (C40), the following condition is obtained.

$$\left|I_{Sd}\right| \leq \sqrt{\frac{\lambda L}{\beta_g}} \left|g_j\left(\frac{L}{\beta_g}\right)\right| e^{+\frac{k_j'' L}{2\beta_g}} \left[1 + 2\pi \frac{L}{\lambda \beta_g}\right] \tag{C41}$$

Substituting Eq. (C41) into Eq. (C14) and taking

$$\frac{\left|g_j(o)\right|}{\left|g_j\left(\frac{L}{\beta_g}\right)\right|} \approx 1 \tag{C42}$$

the proposed inequality for the case where $L/\beta_g < a$ takes the form

$$\frac{1}{\sqrt{2}} \frac{L}{\beta_g |R|} \sqrt{S}\; e^{+\frac{k_j'' L}{2\beta_g}} \left[1 + 2\pi \frac{L}{\lambda \beta_g}\right] \overset{?}{<<} 1 \tag{C43}$$

Comparing this expression with Eq. (C24), one can see that the two conditions differ by a factor $\frac{1}{\sqrt{2}}$. Although this small difference does not prove the validity of the inequalities in this appendix, it does indicate that they have the right order of magnitude.

APPENDIX D

Detailed Calculation of Expansion Functions

This appendix presents the detailed calculations of the expansion functions used in conjunction with the Rayleigh-Ritz procedure discussed in the body of this report. These calculations are performed for resonators containing toroidal as well as spherical mirrors. The spherical mirror cases are considered first.

Spherical Resonator Calculations

Spherical Mirrors of Rectangular Projection. As indicated in chapter V (page 56), the expansion functions used in computing laser resonator modes are approximate solutions to an integral equation for a symmetric laser resonator. The particular symmetric resonator chosen consists of two mirrors identical to the one on which the current is being analyzed. The mirror separation is the same as in the original problem. Thus, to determine the expansion functions for the current on mirror #2 with radius of curvature R and transverse dimensions 2a and 2b, one begins with Eq. (71) which is repeated below,

$$J_{x2}(\bar{r}_2) = -\frac{ik}{2\pi} \int_{S_1'} J_{x1}(\bar{r}_1') \frac{e^{-ikR_{21}}}{R_{21}} \cos\alpha_{21} dS_1' \qquad \text{(D1)}$$

where R_{21} is the distance between two points on mirrors #1 and #2, α_{21} is the angle between R_{21} and the optic axis, and S_1' is the surface of mirror #1.

However, as this equation is to be applied to a symmetric resonator,

$$J_{x1}(\bar{r}_2') = e^{i\Omega'} J_{x2}(\bar{r}_2') \qquad \text{(D2)}$$

where $\Omega' = \pi q$ and q is an integer. One can see this by choosing the origin midway between the two mirrors and realizing that the fields must be even or odd with respect to z. Then, writing Eq. (D1) in terms of rectangular coordinates, using Eq. (D2) to eliminate $J_{x1}(\bar{r}_2)$, and dropping the prime, one obtains the following result,

$$J_{x2}(x_2,y_2) = -\frac{ik}{2\pi} e^{+i\Omega'} \int_{-a}^{a} \int_{-b}^{b} J_{x2}(x_1,y_1) \frac{e^{-ikR_{21}}}{R_{21}} \cos\alpha_{21} dx_1 dy_1 \qquad \text{(D3)}$$

where the projection of R_{21} in the x-z plane, denoted R_{21}', is shown in figure 14.

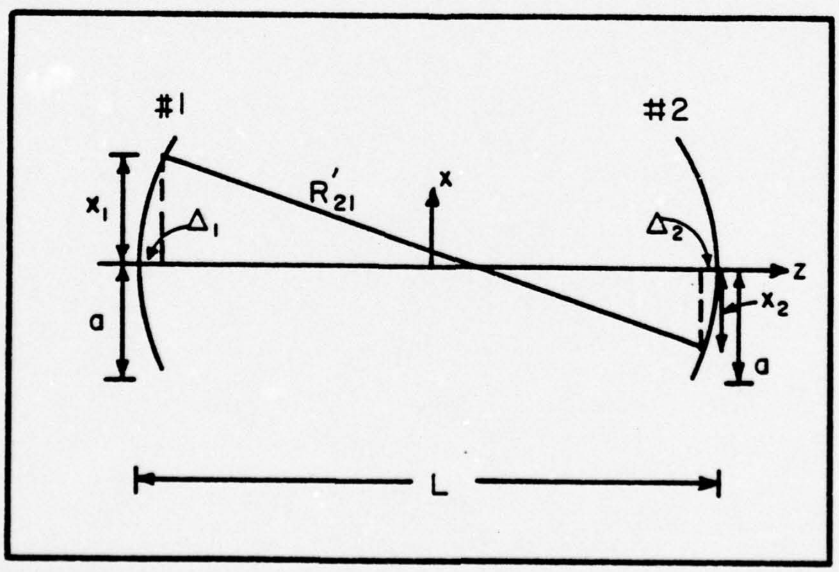

Figure 14. Geometry of a Spherical Resonator with Rectangular Mirrors

The distance R_{21} is given by

$$R_{21}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \qquad \text{(D4)}$$

To simplify the procedure for obtaining the expansion functions, it is assumed that*

$$\frac{d_m}{|R|} << 1 \qquad \text{(D5)}$$

$$\frac{d_m^2}{\lambda L} << \left(\frac{L}{d_m}\right)^2 \qquad \text{(D6)}$$

$$k'' << k' \qquad \text{(D7)}$$

$$d_m << L \qquad \text{(D8)}$$

where $d_m^2 = a^2 + b^2$.† With these assumptions, $\cos\alpha_{21} \approx 1$, and

$$e^{-ik'R_{21}} \approx e^{-ik'L} \, e^{-\frac{igk'}{2L}(x_1^2 + x_2^2 + y_1^2 + y_2^2)} \, e^{+\frac{ik'}{L}(x_1x_2 + y_1y_2)} \qquad \text{(D9)}$$

where $g = 1 - L/R$ and the first two terms of the binomial expansion have been used to approximate R_{21}. Finally, it is assumed that

$$e^{+k''R_{21}} \approx e^{+k''L} \qquad \text{(D10)}$$

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*For resonators for which Eqs. (D5) through (D8) are not valid, the effect of these approximations will be simply to require the use of a relatively large number of expansion functions to represent the modes.

†In general, $d_m^2 = (x_1^2 + y_1^2)_{max}$.

across the entire mirror surface. Using these approximations, Eq. (D3) can be manipulated to yield

$$u(\xi_2) = \gamma_x \sqrt{\frac{i}{2\pi}} \int_{-H_x}^{H_x} u(\xi_1)\, e^{-\frac{ig}{2}(\xi_1^2 + \xi_2^2) + i\xi_1\xi_2}\, d\xi_1 \qquad \text{(D11)}$$

$$v(\eta_2) = \gamma_y \sqrt{\frac{i}{2\pi}} \int_{-H_y}^{H_y} v(\eta_1)\, e^{-\frac{ig}{2}(\eta_1^2 + \eta_2^2) + i\eta_1\eta_2}\, d\eta_1 \qquad \text{(D12)}$$

$$\gamma_x\gamma_y = e^{+i\Omega' - ikL} \qquad \text{(D13)}$$

where $\xi_i = \sqrt{\frac{k'}{L}}\, x_i$, $\eta_i = \sqrt{\frac{k'}{L}}\, y_i$, $H_x = \sqrt{\frac{k'}{L}}\, a$, $H_y = \sqrt{\frac{k'}{L}}\, b$, and $J_{x1}(x,y) = u(x)v(y)$.

Since Eqs. (D11) and (D12) are uncoupled equations of identical form, the remaining calculations are performed considering only the x-variation. To simplify these calculations, one lets

$$K(\xi_2|\xi_1) = \sqrt{\frac{i}{2\pi}}\, e^{-\frac{ig}{2}(\xi_1^2 + \xi_2^2) + i\xi_1\xi_2} \qquad \text{(D14)}$$

so that Eq. (D11) can be written in the form shown below.

$$u(\xi_2) = \gamma_x \int_{-H_x}^{H_x} u(\xi_1)\, K(\xi_2|\xi_1)\, d\xi_1 \qquad \text{(D15)}$$

In the next step, one determines the operator $\hat{M}$ (see pages 57 - 62) associated with the eigenvalue problem for the expansion functions. To do this, one begins with Eqs. (D16) and (D17) below,

$$\frac{d^2u(\xi_2)}{d\xi_2^2} = \gamma_x \int_{-H_x}^{H_x} u(\xi_1)\,\frac{d^2K(\xi_2|\xi_1)}{d\xi_2^2}\,d\xi_1 \tag{D16}$$

$$\gamma_x \int_{-H_x}^{H_x} d\xi_1 K(\xi_2|\xi_1)\,\frac{d^2u(\xi_1)}{d\xi_1^2} = \gamma_x \int_{-H_x}^{H_x} u(\xi_1)\,\frac{d^2K(\xi_2|\xi_1)}{d\xi_1^2}\,d\xi_1 + \gamma_x \hat{R}u(\xi_1) \tag{D17}$$

where Eq. (D16) was obtained by computing $d^2/d\xi_2^2$ of Eq. (D15), and Eq. (D17) was obtained by integrating the term $K(\xi_2|\xi_1)\ d^2u(\xi_1)/d\xi_1^2$ by parts twice. The function $\hat{R}u(\xi_1)$ in Eq. (D17) is defined by Eq. (D18) below.

$$\hat{R}u(\xi_1) = \left\{ u(\xi_1)\,\frac{dK(\xi_2|\xi_1)}{d\xi_1} - K(\xi_2|\xi_1)\,\frac{du(\xi_1)}{d\xi_1} \right\}_{-H_x}^{H_x} \tag{D18}$$

Then, evaluating the derivatives, substituting the expression for $d^2K(\xi_2|\xi_1)/d\xi_2^2$ into Eq. (D16), and using Eq. (D15) to simplify terms of the form $\int_{-H_x}^{H_x} \xi_2^2\ u(\xi_1)\ K(\xi_2|\xi_1)\ d\xi_1$, one can obtain the integrodifferential equation shown below.

$$\left\{ \frac{d^2}{d\xi_2^2} + g^2\xi_2^2 + ig \right\} u(\xi_2) = - \gamma_x \int_{-H_x}^{H_x} u(\xi_1)\ K(\xi_2|\xi_1)\ \xi_1^2 d\xi_1$$

$$+ 2g\xi_2\ \gamma_x \int_{-H_x}^{H_x} \xi_1 u(\xi_1)\ K(\xi_2|\xi_1)\ d\xi_1 \qquad \text{(D19)}$$

A similar procedure involving $d^2K(\xi_2|\xi_1)/d\xi_1^2$ and Eq. (D17) yields Eq. (D20).

$$-(\xi_2^2 + ig)\ u(\xi_2) = \gamma_x \int_{-H_x}^{H_x} K(\xi_2|\xi_1) \left\{ \frac{d^2}{d\xi_1^2} + g^2\xi_1^2 \right\} u(\xi_1)\ d\xi_1$$

$$- 2g\xi_2\gamma_x \int_{-H_x}^{H_x} \xi_1 u(\xi_1)\ K(\xi_2|\xi_1)\ d\xi_1 + \gamma_x \hat{R}u(\xi_1) \qquad \text{(D20)}$$

One then adds Eq. (D19) to Eq. (D20) to obtain

$$\hat{M}u(\xi_2) = \gamma_x \int_{-H_x}^{H_x} K(\xi_2|\xi_1)\hat{M}u(\xi_1)d\xi_1 - \gamma_x \hat{R}u(\xi_1) \qquad \text{(D21)}$$

where

$$\hat{M} = \frac{d^2}{d\xi^2} + (g^2 - 1)\ \xi^2 \qquad \text{(D22)}$$

is the desired operator. Using this operator, the eigenvalue problem is formulated as the following differential equation.

$$\frac{d^2u(\xi)}{d\xi^2} + \left\{(g^2 - 1)\xi^2 + s^2\right\} u(\xi) = 0 \qquad \text{(D23)}$$

To solve this equation, one first considers the case where $g^2 > 1$ and makes the substitutions $d^2 = 2\sqrt{(g^2 - 1)}$ and $z = \sqrt{i}\,\xi d$ to obtain Eq. (D24),

$$\frac{d^2u(z)}{dz^2} + \left\{- \frac{z^2}{4} + (\nu + \frac{1}{2})\right\} u(z) = 0 \qquad \text{(D24)}$$

where $\nu + 1/2 = - is^2/d^2$. This equation is Weber's differential equation (Whittaker and Watson, ref. 39, p. 347), which, if ν is not an integer, has a general solution of the form

$$u(z) = AD_\nu(z) + BD_\nu(-z) \qquad \text{(D25)}$$

where the $D_\nu(z)$ are parabolic cylinder functions (Lebedev, ref. 40, chapter 10).

By letting $B = \pm A$, these solutions are specialized to apply to either odd or even modes. Making this substitution and writing the $D_\nu(z)$ in terms of the confluent hypergeometric functions of the first kind (Erdélyi, ref. 41, vol. 2, p. 123), one obtains the following expressions.

$$u(z) = \begin{cases} Ae^{-z^2/4}\ \Phi(-\frac{\nu}{2}, \frac{1}{2}, \frac{z^2}{2}) & \text{for even modes} \\ \\ Ae^{-z^2/4}\ z\Phi\ (\frac{1-\nu}{2}, \frac{3}{2}, \frac{z^2}{2}) & \text{for odd modes} \end{cases} \qquad \text{(D26)}$$

At this point, the only remaining step is to specify the parameter ν by following the procedure outlined on page 62. That procedure corresponds to the requirement that as $g \to 1$, the expansion functions must reduce to the plane parallel solutions of Vainshtein shown below,

$$u(\xi) = \begin{cases} \cos s\xi \text{ for even modes} \\ \sin s\xi \text{ for odd modes} \end{cases} \tag{D27}$$

where $s = m\pi/(M + (1 + i)\beta)$, $\beta = -\zeta(1/2)/\sqrt{\pi} \approx 0.824$, $M = \sqrt{8\pi N}$, and N is the resonator Fresnel number. For even modes, m = 1,3,5--- and for odd modes, m = 2,4,6---.

This procedure for specifying the parameter ν can be simplified considerably by realizing that as $g \to 1$, $d^2 \to 0$, and $|\nu| \to \infty$. Letting $\alpha = -\nu/2$ or $(1-\nu)/2$ and $y = z^2/2$, this procedure also corresponds to investigating the behavior of the $\Phi(\alpha,\gamma,z)$ as $|\alpha| \to \infty$. Several investigations of this type nave been performed.

To obtain the particular form used for one such investigation, one sets $\kappa = \gamma/2 - \alpha$, and requires αy to be bounded in absolute value. Then, since $\gamma = 1/2$ or $3/2$, κy is also bounded in absolute value. With these restrictions, it has been shown that (Erdélyi, ref. 41, vol. 1, p. 280)

$$\Phi(\alpha, \gamma, y) \underset{|\kappa| \to \infty}{\to} \Gamma(\gamma)\, (\kappa y)^{\frac{1-\gamma}{2}} e^{y/2} J_{\gamma-1}(2\sqrt{\kappa y}) \tag{D28}$$

One then uses the fact that since $\kappa = (\nu/2 + 1/4) = -is^2/2d^2$, $\kappa y = s^2\xi^2/4$. This expression for κy is then substituted into Eq. (D28) to obtain Eq. (D29).

$$\Phi(\alpha, \gamma, y) \underset{|\kappa| \to \infty}{\to} \Gamma(\gamma) \left(\frac{s\xi}{2}\right)^{1-\gamma} e^{\frac{id^2\xi^2}{4}} J_{\gamma-1}(s\xi) \tag{D29}$$

Then, for the even mode solutions of Eq. (D26), one lets $\gamma = 1/2$,

$$u(\xi) \underset{g \to 1}{\to} A\,\Gamma(\tfrac{1}{2}) \sqrt{\frac{s\xi}{2}}\; J_{-\frac{1}{2}}(s\xi) \tag{D30}$$

and for odd modes, $\gamma = 3/2$

$$u(\xi) \underset{g \to 1}{\to} Ad\xi\Gamma \left(\frac{3}{2}\right) \sqrt{\frac{2i}{s\xi}}\; J_{\frac{1}{2}}(s\xi) \tag{D31}$$

Substituting the well-known forms (Whittaker and Watson, ref. 39, p. 364)

$$J_{\frac{1}{2}}(s\xi) = \sqrt{\frac{2}{\pi s\xi}} \sin s\xi \tag{D32}$$

$$J_{-\frac{1}{2}}(s\xi) = \sqrt{\frac{2}{\pi s\xi}} \cos s\xi \tag{D33}$$

into Eqs. (D30) and (D31), one finds that for even modes,

$$u(\xi) \underset{g \to 1}{\to} A \cos s\xi \tag{D34}$$

and for odd modes,

$$u(\xi) \underset{g \to 1}{\to} \frac{A}{\sqrt{\kappa}} \sin s\xi \tag{D35}$$

Thus, for the desired reduction to occur, it must be true that $s = m\pi/(M + (1 + i)\beta)$. Therefore, ν has the value given below.

$$\nu = -\frac{1}{2} \left\{ 1 + 2i \left[\frac{m\pi/d}{M + (1 + i)\beta}\right]^2 \right\} \tag{D36}$$

To complete the procedure for obtaining the expansion functions, one returns to the differential equation, considers the case $g^2 < 1$, and makes the substitutions $h^2 = 2\sqrt{1 - g^2}$ and $y = h\xi$. The result is Weber's equation as shown below.

$$\frac{d^2u(y)}{dy^2} + \left\{(\nu + \frac{1}{2}) - \frac{y^2}{4}\right\} u(y) = 0 \tag{D37}$$

For all values of ν, the general solution of this equation may be written in the form (Whittaker and Watson, ref. 39, p. 348),

$$u(y) = AD_\nu(y) + BD_{-\nu\ -1}(iy) \tag{D38}$$

However, for real values of y, $D_{-\nu\ -1}$ (iy) increases exponentially as $|y|$ increases. Since this directly contradicts the known behavior of the modes of stable resonators, one requires that $B = 0$. Finally, the requirement that the modes be either even or odd yields $\nu = 0,2,4,6,$--- for even modes and $\nu = 1,3,5,$--- for odd modes. Thus, for $g^2 < 1$, the expansion functions are given by Eq. (D39).

$$u(y) = \begin{cases} e^{-\frac{y^2}{4}} \Phi\left(-\frac{n}{2}, \frac{1}{2}, \frac{y^2}{2}\right) & \text{for even modes, } n = 0,2,4,\text{---} \\ e^{-\frac{y^2}{4}} y\Phi\left(\frac{1-n}{2}, \frac{3}{2}, \frac{y^2}{2}\right) & \text{for odd modes, } n = 1,3,5,\text{---} \end{cases} \tag{D39}$$

Then combining the forms in Eqs. (D26) and (D39), the expansion functions for one transverse dimension are given by,

$$u(\xi) = \begin{cases} e^{-\frac{h^2\xi^2}{4}} \Phi\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{h^2\xi^2}{2}\right) & \text{for even modes} \\ \xi e^{-\frac{h^2\xi^2}{4}} \Phi\left(\frac{1-\nu}{2}, \frac{3}{2}, \frac{h^2\xi^2}{2}\right) & \text{for odd modes} \end{cases} \tag{D40}$$

where $h^2 = 2\sqrt{1 - g^2}$ and $\xi = \sqrt{k'/L}\, x$. For stable resonators, $\nu = 0,2,4,\text{---}$ (even modes) or $\nu = 1,3,5,\text{---}$ (odd modes). For unstable resonators, ν is given by Eq. (D36) where $m = 1,3,5,\text{---}$ (even modes) or $m = 2,4,6,\text{---}$ (odd modes).

Spherical Mirrors of Circular Projection. To obtain the expansion functions for spherical mirrors of circular projection, one begins with the integral equation for a symmetric resonator in circular cylindrical coordinates as shown below,

$$J_{x1}(\rho_1, \theta_1) = -\frac{ik}{2\pi} e^{+i\Omega'} \int_0^{2\pi} d\theta_2 \int_0^{a} J_{x1}(\rho_2, \theta_2) \frac{e^{-ikR_{12}}}{R_{12}}$$

$$\cos\alpha_{12}\rho_2 d\rho_2 \qquad \text{(D41)}$$

where $\Omega' = \pi q$ as before. Then, in addition to assuming that Eqs. (D5) through (D10) hold, one assumes solutions of the form

$$J_{x2}(\rho, \theta) = u_n(\rho)\, e^{+in\theta} \qquad \text{(D42)}$$

and applies the identity (Erdélyi, ref. 41, vol. 2, p. 7),

$$i^n\, 2\pi J_n(z) = \int_0^{2\pi} e^{iz\cos\phi + in\phi}\, d\phi \qquad \text{(D43)}$$

where $J_n(z)$ is a Bessel function of the first kind. The result is the following integral equation for the radial mode function $u_n(\xi)$,

$$u_n(\xi_1) = \gamma_n \int_0^{H_a} u_n(\xi_2)\, e^{-\frac{ig}{2}(\xi_1^2 + \xi_2^2)} J_n(\xi_1\xi_2)\xi_2 d\xi_2 \qquad \text{(D44)}$$

where $\gamma_n = e^{-ikL+i\Omega'} i^{n+1}$, $\xi_i = \sqrt{k'/L}\,\rho_i$, and $g = 1 - L/R$. Finally, one converts this equation to an integral equation with a symmetric kernel by making the substitution $u_n(\xi) = \nu_n(\xi)/\sqrt{\xi}$. The result is shown below in Eq. (D45),

$$\nu_n(\xi_1) = \gamma_n \int_0^{H_a} \nu_n(\xi_2)\, K_n(\xi_1|\xi_2)\, d\xi_2 \qquad \text{(D45)}$$

where

$$K_n(\xi_1|\xi_2) = \sqrt{\xi_1\xi_2}\, e^{-\frac{ig}{2}(\xi_1^2 + \xi_2^2)} J_n(\xi_1\xi_2) \qquad \text{(D46)}$$

To obtain the eigenvalue problem for the expansion functions, one begins with the following pair of equations,

$$\frac{d^2\nu_n(\xi_1)}{d\xi_1^2} = \gamma_n \int_0^{H_a} \nu_n(\xi_2)\, \frac{d^2K_n(\xi_2|\xi_1)}{d\xi_1^2}\, d\xi_2 \qquad \text{(D47)}$$

$$\gamma_n \int_0^{H_a} K_n(\xi_2|\xi_1)\, \frac{d^2\nu_n(\xi_2)}{d\xi_2^2}\, d\xi_2 = \gamma_n \int_0^{H_a} \nu_n(\xi_2)\, \frac{d^2K_n(\xi_2|\xi_1)}{d\xi_2^2}\, d\xi_2 + \gamma_n \hat{R}\nu_n(\xi_2) \qquad \text{(D48)}$$

where

$$\hat{R}\nu_n(\xi_2) = \left\{ K_n(\xi_2|\xi_1)\, \frac{d\nu_n(\xi_2)}{d\xi_2} - \nu_n(\xi_2)\, \frac{dK_n(\xi_2|\xi_1)}{d\xi_2} \right\}_0^{H_a} \qquad \text{(D49)}$$

The indicated derivatives of $K_n(\xi_2|\xi_1)$ are then evaluated, and a procedure identical to the one used for rectangular mirrors is followed to obtain Eqs. (D50) and (D51) shown below,

$$\left\{\frac{d^2}{d\xi_1^2} + g^2\xi_1^2 + 2ig + \frac{\left(\frac{1}{4} - n^2\right)}{\xi_1^2}\right\}\nu_n(\xi_1)$$

$$= - \gamma_n \int_0^{H_a} \nu_n(\xi_2)K_n(\xi_2|\xi_1)\xi_2^2 d\xi_2$$

$$- 2ig\ \gamma_n \int_0^{H_a} (\xi_1\xi_2)^{\frac{3}{2}}\ J_n'(\xi_1\xi_2)\ E(\xi_2|\xi_1)\ d\xi_2 \qquad \text{(D50)}$$

$$\left\{2ig + \xi_1^2\right\}\nu_n(\xi_1)$$

$$= \gamma_n \int_0^{H_a} K_n(\xi_2|\xi_1)\left\{\frac{d^2}{d\xi_2^2} + g^2\xi_2^2 + \frac{\left(\frac{1}{4} - n^2\right)}{\xi_2^2}\right\}\nu_n(\xi_2)d\xi_2$$

$$+ 2ig\ \gamma_n \int_0^{H_a} (\xi_1\xi_2)^{\frac{3}{2}}\ J_n'(\xi_1\xi_2)\ E(\xi_2|\xi_1)\ d\xi_2 - \gamma_n\hat{R}\nu_n(\xi_2) \qquad \text{(D51)}$$

where

$$E(\xi_2|\xi_1) = e^{-\frac{ig}{2}(\xi_2^2 + \xi_1^2)} \qquad \text{(D52)}$$

and $J_n'(x) = \dfrac{dJ_n(x)}{dx}$.

Finally, one adds Eqs. (D50) and (D51) to obtain Eq. (D53).

$$\left\{\frac{d^2}{d\xi_1^2} + (g^2 - 1)\ \xi_1^2 + \frac{\left(\frac{1}{4} - n^2\right)}{\xi_1^2}\right\} \nu_n(\xi_1) = -\ \gamma_n \hat{R} \nu_n(\xi_2)$$

$$+ \int_0^{H_a} K_n(\xi_2|\xi_1) \left\{\frac{d^2}{d\xi_2^2} + (g^2 - 1)\ \xi_2^2 + \frac{\left(\frac{1}{4} - n^2\right)}{\xi_2^2}\right\} \nu_n(\xi_2) d\xi_2 \qquad \text{(D53)}$$

One can see that this equation has the same form as Eq. (113) in chapter V. However, Eq. (D53) will not have the characteristic described in condition #1 following Eq. (122) unless the term in braces in Eq. (D49) vanishes at $\xi_2 = 0$. Since the functions $\nu_n(\xi_2)$ have not yet been selected, it is initially assumed that this term vanishes at the origin. The validity of this assumption is demonstrated later in this appendix (beginning with Eq. (D77)).

Then, subject to the above assumption and in accordance with Eq. (125), the operator $\hat{M}$ is chosen as shown below.

$$\hat{M} = \frac{d^2}{d\xi^2} + (g^2 - 1)\ \xi^2 + \frac{\left(\frac{1}{4} - n^2\right)}{\xi^2} \qquad \text{(D54)}$$

Thus, the eigenvalue problem for the expansion functions is expressed as the following differential equation.

$$\frac{d^2\nu_n(\xi)}{d\xi^2} + \left\{(g^2 - 1)\ \xi^2 + \frac{\left(\frac{1}{4} - n^2\right)}{\xi^2} + s^2\right\} \nu_n(\xi) = 0 \qquad \text{(D55)}$$

To solve this equation, one first considers stable resonators ($g^2 < 1$) and makes the substitutions,

$$w_n(\xi) = \nu_n(\xi)\ \sqrt{\xi} \tag{D56}$$

$$z = \alpha'\xi^2 \tag{D57}$$

where $\alpha' = \sqrt{1 - g^2}$. Then, letting $s^2 = -(2\nu + 1)\alpha'$, one obtains Eq. (D58).

$$\frac{d^2w_n(z)}{dz^2} - \frac{1}{4}\left\{1 + \frac{(2\nu + 1)}{z} + \frac{(n^2 - 1)}{z^2}\right\} w_n(z) = 0 \tag{D58}$$

Equation (D58) is Whittaker's equation (Whittaker and Watson, ref. 39, p. 337), which has a general solution of the form

$$w_{\kappa,p}\ (z) = A\ W_{\kappa,\ p}(z) + B\ W_{-\kappa,\ p}\ (-z) \tag{D59}$$

where $\kappa = -1/4\ (2\nu + 1)$, $p = n/2$, and $W_{\kappa,\ p}\ (z)$ is a Whittaker function of the second kind. Now, the Whittaker functions are related to the confluent hypergeometric functions of the second kind, $\Psi(\alpha,\gamma,z)$, by the following equation (Lebedev, ref. 40, p. 274),

$$W_{\kappa,\ \frac{n}{2}}\ (z) = e^{-\frac{z}{2}}\ z^{\frac{\gamma}{2}}\ \Psi(\alpha,\ \gamma,\ z) \tag{D60}$$

where $\alpha = (n + 1)/2 - \kappa$ and $\gamma = n + 1$. When n is zero or a positive integer, which it is for this problem, $\Psi(\alpha,\ n + 1,\ z)$ is represented by the series (Lebedev, ref. 40, p. 264),

$$\Psi(\alpha,n+1,z) = \frac{(-1)^n}{\Gamma(\alpha-n)} \sum_{k=0}^{\infty} \frac{(\alpha)_k z^k}{(n+k)!k!} \left\{ \psi(\alpha+k)-\psi(1+k)-\psi(n+k+1)+\ln(z) \right\}$$

$$+ \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \frac{(-1)^k(n-k-1)!(\alpha-n)_k}{k!} z^{k-n} \qquad \text{(D61)}$$

where $(\alpha)_k = \Gamma(\alpha+k)/\Gamma(\alpha)$, and $\psi(x)$ is the logarithmic derivative of the gamma function (Erdélyi, ref. 41, vol. 1, p. 15).

From this series, one can see that, unless $\alpha = -m$ where m is zero or a positive integer, $\Psi(\alpha, n + 1, z)$ has a singularity at $z = 0$. However, if $\alpha = -m$, the series is indeterminant, and $\Psi(\alpha, n + 1, z)$ must be evaluated using a limiting process. For this case, one finds that (Erdélyi, ref. 41, vol. 1, p. 268)

$$\Psi(-m, n + 1, z) = (-1)^m m! \, F_m{}^n(z) \qquad \text{(D62)}$$

where $F_m{}^n(z)$ is a generalized Laguerre polynomial. Thus, if the origin ($z = 0$) is included, the solution to Eq. (D58) becomes

$$w_{nm}(\xi) = A' e^{-\frac{\alpha'\xi^2}{2}} \xi^{n+1} F_m{}^n(\alpha'\xi^2)$$

$$+ B' e^{+\frac{\alpha'\xi^2}{2}} \xi^{n+1} F_m{}^n(-\alpha'\xi^2) \qquad \text{(D63)}$$

However, since stable resonators do not produce current distributions for which the current grows exponentially with increasing mirror radius (Bergstein, ref. 2, p. 500), B' must be set equal to zero. Thus, the expansion functions are given by Eq. (D64).

$$w_{nm}(\xi) = A'e^{-\frac{\alpha'\xi^2}{2}} \xi^{n+1} F_m^{\,n} (\alpha'\xi^2) \qquad \text{(D64)}$$

For unstable resonators ($g^2 > 1$), it is advantageous to convert the differential equation back to an equation for $u_n(\xi)$. To do this, one substitutes $u_n(\xi) = \nu_n(\xi)/\sqrt{\xi}$ in Eq. (D55) to obtain,

$$\frac{d^2u_n(z)}{dz^2} + \frac{1}{z}\frac{du_n(z)}{dz} + \left\{4z^2 - \frac{n^2}{z^2} - 4\tau\right\} u_n(z) = 0 \qquad \text{(D65)}$$

where $z = \Omega\xi$, $\tau = -(s/2\Omega)^2$, and $\Omega^2 = 1/2\sqrt{g^2 - 1}$. This equation has a general solution of the form (Erdélyi, ref. 41, vol. 2, p. 126),

$$u_{n,\tau}(z) = \frac{1}{z}\left\{AM_{i\tau,\frac{n}{2}}(iz^2) + B\,W_{i\tau,\frac{n}{2}}(+iz^2)\right\} \qquad \text{(D66)}$$

where the $M_{\kappa,\mu}$ and $W_{\kappa,\mu}$ are Whittaker functions of the first and second kind.

As with rectangular mirror resonators, one must now insure that these solutions,

1. Have no singularities, and
2. Reduce to the proper forms as $g \to 1$ ($\Omega \to 0$).

Dealing with the singularities first, one uses the fact that $W_{i\tau,\, n/2}(x)$ has a singularity at the origin unless $i\tau = -(m + (n+1)/2)$, where $m = 0,1,2,---$. For this case (Lebedev, ref. 40, p. 274),

$$M_{i\tau,\frac{n}{2}}(iz^2) = e^{-\frac{iz^2}{2}} (iz^2)^{\frac{n+1}{2}} \Phi(m + n + 1; n + 1, +iz^2) \qquad \text{(D67)}$$

$$W_{i\tau,\frac{n}{2}}(+iz^2) = e^{-\frac{iz^2}{2}}(+iz^2)^{\frac{n+1}{2}}\Psi(m + n + 1, n + 1, +iz^2) \tag{D68}$$

and

$$u_{n,\tau}(z) = z^n \left\{ A'e^{-\frac{iz^2}{2}}\Phi(\overline{\beta}, \overline{\delta}, iz^2) + B'e^{-\frac{iz^2}{2}}\Psi(\overline{\beta}, \overline{\delta}, +iz^2) \right\} \tag{D69}$$

where $\overline{\beta} = m + n + 1$ and $\overline{\delta} = n + 1$. However, as $\Omega \to 0$, $z \to 0$ and $\Phi(\overline{\beta}, \overline{\delta}, iz^2) \to 1$ and $\Psi(\overline{\beta}, \overline{\delta}, +iz^2) \to 1$ (see appendix H). Thus, as $\Omega \to 0$, the solution $u_n(z)$ approaches a constant value, which does not correspond to one of the solutions (for the plane parallel case) shown below,

$$u_{nm}(\rho) = J_n\left(\frac{\nu_{nm}\rho/a}{1 + (1 + i)\beta/M}\right) \tag{D70}$$

where ν_{nm} is the m^{th} root of the n^{th} order Bessel function of the first kind, $\beta = 0.824$, $M = \sqrt{8\pi N}$, and N is the resonator Fresnel number. As a result, the case where $i\tau = -(m + (n+1)/2)$ will not be considered further. Therefore, to avoid the singularity at the origin, one must set B = 0 in Eq. (D66) to obtain

$$u_{n,\tau}(z) = AM_{i\tau,\frac{n}{2}}(iz^2)\frac{1}{z} \tag{D71}$$

where the yet to be determined values of τ correspond to those for which the $u_{n,\tau}(z)$ reduce to the functions in Eq. (D70) as $\Omega \to 0$.

From the relation $\tau = -(s/2\Omega)^2$, one can see that as $\Omega \to 0$, $|i\tau| \to \infty$. Thus, as with the rectangular mirror case, it is necessary to investigate the behavior of the solution as a parameter becomes large. To do this, one first writes $u_{n,\tau}(z)$ in terms of $\Phi(\alpha, \gamma, iz^2)$ as shown below (see appendix H).

$$(-i)^{\frac{n+1}{2}} \frac{A}{z} M_{i\tau, \frac{n}{2}}(iz^2) = Ae^{-\frac{iz^2}{2}} z^n \Phi\left(\frac{n+1}{2} - i\tau, n+1, iz^2\right) \quad \text{(D72)}$$

Then, using Eqs. (D71) and (D72) in conjunction with Eq. (D28), where $\alpha = (n+1)/2 + i(s/2\Omega)^2$, $\gamma = n + 1$, $\kappa = \gamma/2 - \alpha$, and $\kappa y = \kappa i z^2 = (s\xi/2)^2$, one can show that if $|\kappa y|$ is bounded, then

$$u_{n,\tau}(\xi) \underset{g \to 1}{\to} A \left\{ i \, (i\tau)^{-\frac{n}{2}} n! \right\} J_n(s\xi) \quad \text{(D73)}$$

Thus, the $u_{n,\tau}(\xi)$ reduce to the desired form if

$$\sqrt{\frac{k'}{L}}\, s = \frac{\nu_{nm}}{a\left\{1 + \frac{(1+i)\beta}{M}\right\}} \quad \text{(D74)}$$

where β and M were defined in the discussion following Eq. (D27). Eq. (D74) corresponds to a value of τ such that

$$\tau = -\left\{\frac{\nu_{nm}}{M + (1+i)\beta}\right\}^2 \frac{1}{\Omega^2} \quad \text{(D75)}$$

For resonators containing mirrors with central coupling apertures, this expansion set must be altered somewhat. The alteration, which results from the fact that the singularity at the origin is no longer present, simply corresponds to retaining the terms

$$w_{n,\tau}(z) = \frac{B}{z} W_{i\tau,\, \frac{n}{2}}(iz^2) \tag{D76}$$

in Eq. (D66) with τ given by Eq. (D75).

In the discussion just following Eq. (D53), it was assumed that

$$\left[K_n(\xi_2|\xi_1) \frac{d\nu_n(\xi_2)}{d\xi_2} - \nu_n(\xi_2) \frac{dK_n(\xi_2|\xi_1)}{d\xi_2} \right]_{\xi_2 = 0} = 0 \tag{D77}$$

where $K_n(\xi_2|\xi_1)$ is given by Eq. (D46) and the $\nu_n(\xi_2)$ are the expansion functions for resonators with spherical mirrors of circular projection. Using Eqs. (D64) and (D71), one can show that for stable resonators,

$$\nu_{n,m}(\xi_2) = A' e^{-\frac{\alpha'\xi^2}{2}} \xi_2^{\,n + \frac{1}{2}} F_m^{\,n}(\alpha'\xi_2^2) \tag{D78}$$

and for unstable resonators,

$$\nu_{n,\tau}(\xi_2) = B'\xi_2^{-\frac{1}{2}} M_{i\tau,\, \frac{n}{2}}(i\Omega^2\xi_2^2) \tag{D79}$$

Further, by using a relation (see appendix H) between $M_{i\tau,\, n/2}(z)$ and $\Phi(\alpha, \gamma, z)$, Eq. (D79) can be rewritten in the form shown below.

$$\nu_{n,\tau}(\xi_2) = B'(i\Omega^2)^{\frac{n+1}{2}} \xi_2^{\frac{n+1}{2}} e^{-\frac{i\Omega^2\xi_2^2}{2}} \Phi\left(\frac{n+1}{2} - i\tau,\, n + 1,\, i\Omega^2\xi_2^2\right) \tag{D80}$$

Since the functions $F_m^{\,n}(\alpha'\xi_2^2)$ and $\Phi((n+1)/2 - i\tau, n+1, i\Omega^2\xi_2^2)$ are entire functions of ξ_2, the expansion functions of both stable and unstable resonators can be summarized by the form

$$\nu_n(\xi_2) = \xi_2^{\frac{1}{2}} F(\xi_2) \quad \text{(D81)}$$

where $F(\xi_2)$ is an entire function of ξ_2. The kernel, $K_n(\xi_2|\xi_1)$, can be written in a similar form as shown below,

$$K_n(\xi_2|\xi_1) = \xi_2^{\frac{1}{2}} G_n(\xi_2|\xi_1) \quad \text{(D82)}$$

where

$$G_n(\xi_2|\xi_1) = \xi_1^{\frac{1}{2}} e^{-\frac{ig}{2}(\xi_1^2 + \xi_2^2)} J_n(\xi_1\xi_2) \quad \text{(D83)}$$

is an entire function of ξ_2.

To show that Eq. (D77) holds, one differentiates $\nu_n(\xi_2)$ and $K_n(\xi_2|\xi_1)$ using the forms in Eqs. (D81) and (D82) to obtain the following pair of equations.

$$\frac{d\nu_n(\xi_2)}{d\xi_2} = \frac{1}{2}\xi_2^{-\frac{1}{2}} F(\xi_2) + \xi_2^{\frac{1}{2}} \frac{dF(\xi_2)}{d\xi_2} \quad \text{(D84)}$$

$$\frac{dK_n(\xi_2|\xi_1)}{d\xi_2} = \frac{1}{2}\xi_2^{-\frac{1}{2}} G_n(\xi_2|\xi_1) + \xi_2^{\frac{1}{2}} \frac{dG_n(\xi_2|\xi_1)}{d\xi_2} \quad \text{(D85)}$$

Substituting Eqs. (D84) and (D85) into the term in brackets in Eq. (D77), one obtains the following result.

$$K_n(\xi_2|\xi_1)\,\frac{d\nu_n(\xi_2)}{d\xi_2} - \nu_n(\xi_2)\,\frac{dK_n(\xi_2|\xi_1)}{d\xi_2}$$

$$= \xi_2 \left\{ G_n(\xi_2|\xi_1)\,\frac{dF(\xi_2)}{d\xi_2} - F(\xi_2)\,\frac{dG_n(\xi_2|\xi_1)}{d\xi_2} \right\} \qquad \text{(D86)}$$

Since the functions $F(\xi_2)$ and $G_n(\xi_2|\xi_1)$ are entire functions of ξ_2, the term in braces in Eq. (D86) is finite at $\xi_2 = 0$. Thus, Eq. (D77) holds as assumed.

Toroidal Resonator Calculations

As with spherical mirror resonators, one begins this procedure for obtaining expansion functions with the integral equation for a symmetric resonator as shown below,

$$J_{x1}(\rho_1, \theta_1) = -\frac{ik}{2\pi} e^{+i\Omega'} \int_a^b \int_0^{2\pi} J_{x1}(\rho_2, \theta_2)\,\frac{e^{-ikR_{12}}}{R_{12}}$$

$$\cos\alpha_{12}\rho_2 d\rho_2 d\theta_2 \qquad \text{(D87)}$$

where R_{12} is given by

$$R_{12}^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\theta_2 - \theta_1) + (z_2 - z_1)^2 \qquad \text{(D88)}$$

and a and b are the inner and outer mirror radii as shown in figure 15.

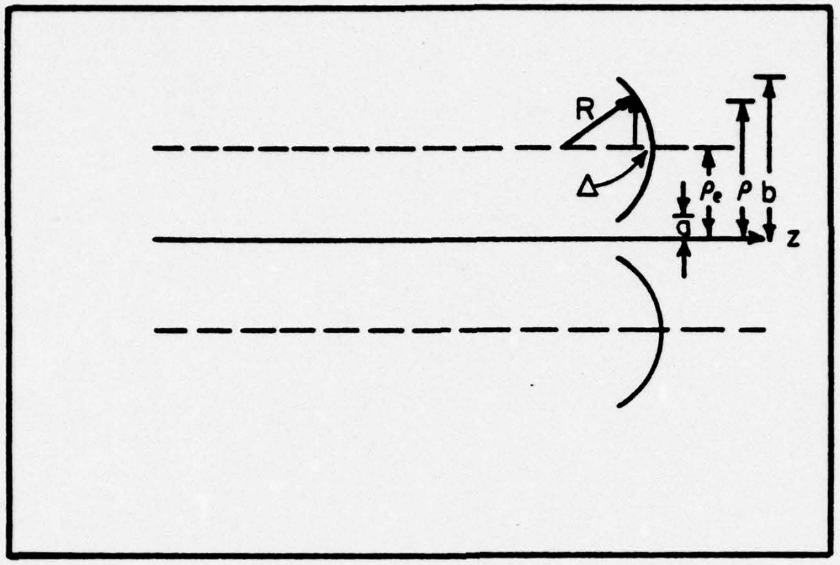

Figure 15. Intersection of a Toroidal Mirror With the x-z Plane

The distance $(z_2 - z_1)$ is equal to

$$(z_2 - z_1) = L - \Delta_1 - \Delta_2 \qquad \text{(D89)}$$

where the distance Δ_i, which is depicted by Δ in figure 15, is given by

$$\Delta_i = \frac{(\rho_i - \rho_e)^2}{2R} \qquad \text{(D90)}$$

where R is the radius of curvature of the individual arcs shown in figure 15 and ρ_e is the distance the axis of each of these arcs is displaced from the optic axis. Substituting Eqs. (D89) and (D90) into Eq. (D88) and neglecting terms of second order in Δ_i, one obtains the following expression for R_{12}.

$$R_{12} \approx L \left\{ 1 - \frac{\left[(\rho_1 - \rho_e)^2 + (\rho_2 - \rho_e)^2\right]}{LR} + \frac{\rho_1{}^2 + \rho_2{}^2}{L^2} - \frac{2\rho_1\rho_2}{L^2} \cos(\theta_2 - \theta_1) \right\}^{\frac{1}{2}} \tag{D91}$$

Then, in addition to assuming that Eqs. (D5) through (D10) hold, one assumes solutions of the form

$$J_{x1}(\rho,\theta) = u_n(\rho)\, e^{+in\theta} \tag{D92}$$

and applies Eq. (D43) to obtain the following equation for $u_n(\rho)$,

$$u_n(\rho_1) = \gamma_n' \frac{k'}{L} \int_a^b u_n(\rho_2)\, J_n\!\left(\frac{k'}{L}\rho_1\rho_2\right) e^{+\frac{ik'}{2R}\left\{(\rho_1-\rho_e)^2 + (\rho_2-\rho_e)^2\right\}} e^{-\frac{ik'}{2L}\left(\rho_1^2 + \rho_2^2\right)} \rho_2 d\rho_2 \tag{D93}$$

where $\gamma_n' = i^{n+1}\, e^{-ikL+i\Omega'}$.

Equation (D93) is simplified considerably by making the substitution $\xi_i = \sqrt{k'/L}\,\rho_i$. The result is shown in Eq. (D94),

$$u_n(\xi_1) = \gamma_n \int_{H_a}^{H_b} u_n(\xi_2)\, J_n(\xi_1\xi_2)\, e^{-\frac{ig}{2}(\xi_1^2 + \xi_2^2)} e^{+i(g-1)\xi_e(\xi_1+\xi_2)} \xi_2 d\xi_2 \tag{D94}$$

where $H_a = \sqrt{k'/L}\ a$, $H_b = \sqrt{k'/L}\ b$, $\gamma_n = \gamma_n' \, e^{+i(g-1)\xi_e^2}$, and $g = 1 - L/R$. Finally, to simplify the forthcoming computations, the following definitions

$$K_n(\xi_2|\xi_1) = J_n(\xi_2\xi_1)\ E(\xi_2|\xi_1)\xi_2 \qquad \text{(D95)}$$

$$E(\xi_2|\xi_1) = e^{-\frac{ig}{2}(\xi_1^2 + \xi_2^2)}\ e^{+i(g-1)\xi_e(\xi_1 + \xi_2)} \qquad \text{(D96)}$$

are employed to yield Eq. (D97).

$$u_n(\xi_1) = \gamma_n \int_{H_a}^{H_b} u_n(\xi_2)\ K_n(\xi_2|\xi_1)\ d\xi_2 \qquad \text{(D97)}$$

The manipulations used to obtain the expansion functions for these toroidal resonators are somewhat more complex than the manipulations already performed for spherical mirror resonators. For this case, one begins with the following two equations,

$$\frac{d^2u_n(\xi_1)}{d\xi_1^2} + \frac{1}{\xi_1}\frac{du_n(\xi_1)}{d\xi_1} = \gamma_n \int_{H_a}^{H_b} u_n(\xi_2) \left\{\frac{d^2}{d\xi_1^2} + \frac{1}{\xi_1}\frac{d}{d\xi_1}\right\} K_n(\xi_2|\xi_1)\ d\xi_2 \qquad \text{(D98)}$$

$$\gamma_n \int_{H_a}^{H_b} K_n(\xi_2|\xi_1) \left\{\frac{d^2}{d\xi_2^2} + \frac{1}{\xi_2}\ \frac{d}{d\xi_2}\right\} u_n(\xi_2)d\xi_2 = \gamma \int_{H_a}^{H_b} u_n(\xi_2) \left\{\frac{d^2}{d\xi_2^2} - \frac{1}{\xi_2}\ \frac{d}{d\xi_2} + \frac{1}{\xi_2^2}\right\} K_n(\xi_2|\xi_1)d\xi_2 + \gamma_n \hat{S} u_n(\xi_2) \qquad \text{(D99)}$$

where Eq. (D98) was obtained by operating on Eq. (D97) with $d^2/d\xi_1{}^2 + 1/\xi_1 \, d/d\xi_1$, and Eq. (D99) was obtained by integrating the integrand of the left side of Eq. (D99) by parts twice. The function $\hat{S}u_n(\xi_2)$ in Eq. (D99) is defined below in Eq. (D100).

$$\hat{S}u_n(\xi_2) = \left\{ K_n(\xi_2|\xi_1) \left[\frac{u_n}{\xi_2} + \frac{du_n}{d\xi_2}\right] - u_n \frac{dK_n(\xi_2|\xi_1)}{d\xi_2} \right\}_{H_a}^{H_b} \tag{D100}$$

One then calculates the indicated derivatives, substitutes the results into Eqs. (D98) and (D99), and uses Eq. (D97) for simplification purposes. The two simplified equations are then added to yield the rather complex equation shown below.

$$\hat{L}\, u_n(\xi_1) = \int_{H_a}^{H_b} K_n(\xi_2|\xi_1)\, \hat{L}\, u_n(\xi_2)\, d\xi_2 - \gamma_n \hat{S}u_n(\xi_2)$$

$$+ 2i\gamma_n \xi_e (1 - g) \int_{H_a}^{H_b} (\xi_1 - \xi_2)\, J_n'(\xi_1 \xi_2) \xi_2 E(\xi_2|\xi_1)\, d\xi_2 \tag{D101}$$

where

$$\hat{L} = \frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} + (g^2-1)\xi^2 - 2g(g-1)\xi_e \xi - \frac{i(g-1)}{\xi} - \frac{n^2}{\xi^2} \tag{D102}$$

To simplify the forthcoming eigenvalue problem for the expansion functions, it is desirable to eliminate the integral involving $J_n'(\xi_2 \xi_1)$. To do this, one applies the same sequence of operations (the sequence described between Eqs. (D100) and (D101)) to the pair of equations below,

$$\frac{d}{d\xi_1} u_n(\xi_1) = \gamma_n \int_{H_a}^{H_b} u_n(\xi_2) \frac{d}{d\xi_1} K_n(\xi_2|\xi_1)\, d\xi_2 \tag{D103}$$

$$\gamma_n \int_{H_a}^{H_b} K_n(\xi_2|\xi_1) \frac{du_n(\xi_2)}{d\xi_2}\, d\xi_2$$

$$= -\gamma_n \int_{H_a}^{H_b} u_n(\xi_2) \frac{dK_n(\xi_2|\xi_1)}{d\xi_2}\, d\xi_2 + \gamma_n \hat{T}u_n(\xi_2) \tag{D104}$$

where

$$\hat{T}u_n(\xi_2) = \left\{ u_n(\xi_1)\; K_n(\xi_2|\xi_1) \right\}_{H_a}^{H_b} \tag{D105}$$

The result is Eq. (D106).

$$\left\{ \frac{d}{d\xi_1} + ig\xi_1 \right\} u_n(\xi_1) = \gamma_n \int_{H_a}^{H_b} K_n(\xi_2|\xi_1)$$

$$\left\{ \frac{-d}{d\xi_2} + ig\xi_2 - \frac{1}{\xi_2} \right\} u_n(\xi_2)\, d\xi_2$$

$$- \gamma_n \int_{H_a}^{H_b} u_n(\xi_2)\, J_n'(\xi_2|\xi_1)\xi_2\, E(\xi_2|\xi_1)\, (\xi_1 - \xi_2)\, d\xi_2$$

$$+ \gamma_n \hat{T}u_n(\xi_2) \tag{D106}$$

One then multiplies Eq. (D106) by $2i\xi_e(1-g)$ and adds the result to Eq. (D101) to obtain

$$\hat{M}u_n(\xi_1) = \gamma_n \int_{H_a}^{H_b} K_n(\xi_2|\xi_1)\ \hat{M}u_n(\xi_2)\ d\xi_2 - \gamma_n\ \hat{R}u(\xi_2)$$

$$- 2i\xi_e\gamma_n(1 - g) \int_{H_a}^{H_b} K_n(\xi_2|\xi_1) \left\{2\frac{d}{d\xi_2} + \frac{1}{\xi_2}\right\} u_n(\xi_2)\ d\xi_2 \quad \text{(D107)}$$

where

$$\hat{M} = \frac{d^2}{d\xi^2} + \left\{\frac{1}{\xi} + 2i\xi_e(1 - g)\right\}\frac{d}{d\xi} + (g^2 - 1)\xi^2 + \frac{i\xi_e(1 - g)}{\xi} - \frac{n^2}{\xi^2} \quad \text{(D108)}$$

and

$$\hat{R}u_n(\xi_2) = \hat{S}u_n(\xi_2) - 2i\xi_e(1 - g)\ \hat{T}u_n(\xi_2) \quad \text{(D109)}$$

Then, using the operator $\hat{M}$ in the manner discussed in chapter V (pages 57-62), the eigenvalue problem for the expansion functions corresponds to the following differential equation.

$$\frac{d^2u_n(\xi)}{d\xi^2} + \left\{\frac{1}{\xi} + 2i\xi_e(1 - g)\right\} \frac{du_n(\xi)}{d\xi}$$

$$+\left\{(g^2 - 1)\xi^2 + \frac{i(1 - g)\xi_e}{\xi} - \frac{n^2}{\xi^2} + s^2\right\} u_n(\xi) = 0 \quad \text{(D110)}$$

As a first step in solving this equation, one lets $u(\xi) = w(\xi)\ e^{-i\xi_e(1 - g)\xi}$ and substitutes this form into Eq. (D110). This substitution leads to the following equation for $w(\xi)$.

$$\frac{d^2w(\xi)}{d\xi^2} + \frac{1}{\xi}\frac{dw(\xi)}{d\xi} + \left\{(g^2-1)\xi^2 - \frac{n^2}{\xi^2} + s^2 + (g-1)^2\xi_e^2\right\} w(\xi) = 0 \qquad \text{(D111)}$$

Then taking $g^2 > 1$, and letting $z = \Omega\xi$, $\Omega^2 = 1/2\sqrt{g^2 - 1}$, and

$\tau = -\frac{1}{4}\left\{\frac{s^2+(g-1)^2\xi_e^2}{\Omega^2}\right\}$, one obtains Eq. (D112).

$$\frac{d^2w(z)}{dz^2} + \frac{1}{z}\frac{dw(z)}{dz} + \left\{4z^2 - \frac{n^2}{z^2} - 4\tau\right\} w(z) = 0 \qquad \text{(D112)}$$

This equation, which is identical to Eq. (D65), has a general solution of the form

$$w_{n,\tau}(z) = \frac{1}{z}\left\{AM_{i\tau,\frac{n}{2}}(iz^2) + BW_{i\tau,\frac{n}{2}}(+iz^2)\right\} \qquad \text{(D113)}$$

For these toroidal resonators, the values of τ are chosen so that the solutions reduce to the spherical mirror expansion functions as $\xi_e \to 0$. This choice, which automatically incorporates the proper reduction to the plane parallel solutions, simply amounts to the requirement that (see Eq. (D75))

$$\tau = -\left\{\frac{\nu_{nm}}{M + (1 + i)\beta}\right\}^2 \frac{1}{\Omega^2} \qquad \text{(D114)}$$

Thus, for unstable resonators, the expansion set consists of functions $u_n(\xi)$ given by

$$u_{n,\tau}(\xi) = \begin{cases} A'e^{+i\xi_e(g-1)\xi}\,\frac{1}{\xi}M_{i\tau,\frac{n}{2}}(i\Omega^2\xi^2) \\ B'e^{+i\xi_e(g-1)\xi}\,\frac{1}{\xi}W_{i\tau,\frac{n}{2}}(+i\Omega^2\xi^2) \end{cases} \qquad \text{(D115)}$$

where τ is given by Eq. (D114).

For $g^2 < 1$, one returns to Eq. (D111) and makes the substitutions $w(\xi) = v(\xi)/\xi$ and $z = \alpha'\xi^2$, to obtain Eq. (D116),

$$\frac{d^2v(z)}{dz^2} - \frac{1}{4}\left\{1 + \frac{(n^2 - 1)}{z^2} + \frac{2\nu + 1}{z}\right\} v(z) = 0 \qquad \text{(D116)}$$

where $\alpha' = \sqrt{1 - g^2}$ and $s^2 + (g - 1)^2\ \xi_e^2 = -\alpha'(2\nu + 1)$. As previously indicated (see Eq. (D58)), the general solution of this equation is

$$v_{\kappa,p}(z) = AW_{\kappa,p}(z) + BW_{-\kappa,p}(-z) \qquad \text{(D117)}$$

where $\kappa = -\nu/2 - 1/4$ and $p = n/2$. By requiring these solutions to reduce to those for the spherical case as $\xi_e \to 0$, one obtains the following expression

$$w_{n,m}(\xi) = e^{+i\xi_e(g - 1)\xi}\ e^{-\frac{\alpha'\xi^2}{2}}\ \xi^{n + 1}\ F_m^n(\alpha'\xi^2) \qquad \text{(D118)}$$

for the expansion functions when $g^2 < 1$.

APPENDIX E

Analytical Results for Simple Systems

The purpose of this appendix is to present the results obtained from the analysis of three relatively simple resonant systems. These analytical results, which supplement the numerical results presented in chapter VI, were obtained to support the basic theory and method of solution discussed in the body of this report. The three systems considered are all symmetric resonators with perfectly conducting mirrors. They are an infinite radial waveguide, a stable resonator with rectangular mirrors, and a stable resonator with circular mirrors.

For each of these resonators, the analysis is performed in the following manner. First, the applicable integral equation is converted to an integral equation involving a surface integral over a single resonator mirror. Then using the expansion functions presented in chapter V, and replacing the finite limits of integration with infinite limits, the integral is evaluated to yield an oscillation condition for the resonator being considered. This oscillation condition is then compared to similar conditions obtained by other authors.

The Infinite Radial Waveguide

The system considered in this section consists of two plane mirrors that are infinite in extent. The region between the two mirrors, which are separated by a distance L, contains a homogeneous medium characterized by parameters μ, $\tilde{\varepsilon}$, and σ, where $\sigma > 0$. Since the mirrors in this system are infinite in extent, finite fields will be present only if any gain resulting from the polarization of the medium is less than the loss resulting from the conductivity σ. Thus, the propagation constant k is written in the form $k = k' + ik''$, with $k'' < 0$.

To take advantage of the symmetry, a system of coordinates is established so that the origin is located midway between the two mirrors. With this system, a point on the i^{th} mirror has position vector $\overline{r}_i = \overline{\rho}_i \pm L/2\ \hat{a}_z$, where $\overline{\rho}_i = \rho_i \hat{a}_\rho$, $\hat{a}_\rho$ is a unit vector in the radial direction, and $\hat{a}_z$ is a unit vector parallel to the optic axis. Then, using this system of coordinates and Eqs. (53) and (54) as the basic forms, one substitutes Eq. (54) into Eq. (53) and applies the fact that the self-induction integrals vanish identically ($\cos(\hat{n}, \overline{r-r'}) \equiv 0$). The result is the following expression for the current, $u(\overline{r}_1)$, on mirror #1,

$$u(\overline{r}_1) = \left(\frac{ik}{2\pi}\right)^2 \int_{S_1'} u(\overline{r}_1') \int_{S_2} \frac{e^{-ik(R_{12} + R_{21})}}{R_{12}R_{21}} \cos\alpha_{12} \cos\alpha_{21} dS_2 ds_1' \qquad \text{(E1)}$$

where $R_{12} = |\overline{r_2 - r_1}|$, $R_{21} = |\overline{r_1' - r_2}|$, and α_{ij} is the angle between R_{ij} and the optic axis of the guide.

The next step in the analysis is to assume that the current at a point with coordinate $\overline{\rho}_1$ on mirror #1 is primarily a result of the current over a small region surrounding the point with coordinate $\overline{\rho}_1$ on mirror #2. This effective region is assumed to be sufficiently small so that the paraxial approximation and the condition shown in Eq. (E2) are satisfied over the entire region.

$$e^{-ik|\overline{r_1 - r_2}|} \approx e^{-ikL} e^{-\frac{ik}{2L}|\overline{\rho_1 - \rho_2}|^2} \tag{E2}$$

As will be seen later, these assumptions, which certainly apply for lossy systems, correspond to paraxial modes within the radial guide.

Applying the paraxial approximation, the distances R_{ij} are approximated by $R_{ij} \approx L$ in the amplitude terms to yield

$$u(\overline{r}_1) = \left(\frac{ik}{2\pi L}\right)^2 \int_{S_1'} u(\overline{r}_1') \; K(\overline{r}_1|\overline{r}_1') \; dS_1' \tag{E3}$$

where

$$K(\overline{r}_1|\overline{r}_1') = \int_{S_2} e^{-ik\left\{|\overline{r_1 - r_2}| + |\overline{r_1' - r_2}|\right\}} dS_2 \tag{E4}$$

Substitution of Eq. (E2) into Eq. (E4) yields the following expression for the kernel $K(\overline{r}_1|\overline{r}_1')$.

$$K(\overline{r}_1|\overline{r}_1') = e^{-2ikL} \int_{S_2} e^{-\frac{ik}{2L}\left\{|\overline{\rho_1 - \rho_2}|^2 + |\overline{\rho_1' - \rho_2}|^2\right\}} dS_2 \tag{E5}$$

To evaluate this integral, one makes the substitution

$$\overline{q} = \overline{v} - \overline{\rho}_2 \tag{E6}$$

where $\overline{v} = 1/2\,(\overline{\rho_1 + \rho_1'})$ and $\overline{\rho} = (\overline{\rho_1 - \rho_1'})$ to obtain Eq. (E7).

$$K(\overline{v}|\overline{\rho}) = e^{-2ikL} \int_{\overline{q}} e^{-\frac{ik}{2L}\left\{|\overline{q} + \frac{1}{2}\overline{\rho}|^2 + |\overline{q} - \frac{1}{2}\overline{\rho}|^2\right\}} d\overline{q} \tag{E7}$$

Writing out the terms in the exponent and integrating over angles yields

$$K(\overline{v}|\overline{\rho}) = 2\pi e^{-2ikL} e^{-\frac{ik}{4L}\rho^2} \int_0^\infty e^{-\frac{ik}{L}q^2} q\, dq \tag{E8}$$

Recalling that k = k' + ik" with k" < 0, the integral in Eq. (E8) is easily evaluated by making a change of variables. Substituting the result into Eq. (E3), one obtains an integral equation in terms of an integral over a single mirror.

$$u(\overline{\rho}_1) = \left(\frac{ik}{4\pi L}\right) e^{-2ikL} \int_{S_1'} u(\overline{\rho}_1') e^{-\frac{ik}{4L}|\overline{\rho_1 - \rho_1'}|^2} dS_1' \tag{E9}$$

To reduce this equation to one involving only the radial coordinates, one assumes that the modes have the form

$$u(\overline{\rho}_i) = u_n(\rho_i) e^{+in\theta_i} \tag{E10}$$

and performs the azimuthal integration. The result is shown below.

$$u_n(\rho_1) = e^{-2ikL} \frac{k}{2L} i^{n+1} e^{-\frac{ik}{4L}\rho_1^2} \int_0^\infty u_n(\rho_1') e^{-\frac{ik}{4L}\rho_1'^2} J_n\left(\frac{k\rho_1\rho_1'}{2L}\right)\rho_1' d\rho_1' \tag{E11}$$

Finally, substituting the form for the plane parallel resonator expansion functions given in Eq. (129) ($J_n(k_\rho \rho)$, where k_ρ, the radial component of the wave vector, is chosen to be real so the expansion functions remain finite as $\rho \to \infty$) into Eq. (E11), and evaluating the result (Erdélyi, ref. 41, p. 50), one obtains

$$J_n(k_\rho \rho) \left\{ 1 - e^{-2ikL} e^{+\frac{ik_\rho^2 L}{k}} \right\} = 0 \tag{E12}$$

This condition, which must hold for all ρ on the mirrors, will be satisfied if

$$k = \frac{\pi q}{L} + \frac{k_\rho^2}{2k} \tag{E13}$$

where q is an integer.

The exact results for this problem (specialized to the lossless case) are reported by Harrington (ref. 32, p. 209). For real k_ρ, and with $\rho = 0$ included, these results correspond to waveguide current modes of the form,

$$u_n(\rho,\theta) = J_n(k_\rho \rho)\, e^{+in\theta} \tag{E14}$$

where k' is specified by

$$k' = \sqrt{\left(\frac{\pi q}{L}\right)^2 + k_\rho^2} \tag{E15}$$

At optical or infrared wavelengths, k' is a large number. Thus, if k_ρ is not large, q must be a large integer. Under these conditions, the square root is accurately approximated by using the first two terms of its binomial expansion. That is,

$$k' \approx \frac{\pi q}{L} + \frac{Lk_\rho^2}{2\pi q} \tag{E16}$$

which is very nearly equal to

$$k' \approx \frac{\pi q}{L} + \frac{k_\rho^2}{2k'} \tag{E17}$$

Then, specializing Eq. (E13) to the case where $k'' \rightarrow 0$ and comparing the result to Eq. (E17), one can see that for paraxial modes (modes for which $k_\rho/k' << 1$), the procedure for approximately solving the integral equation yields essentially the same results obtained by other authors.

<u>Stable Resonator Calculations</u>

The systems considered in this section are symmetric, stable resonators ($o<g<1$) satisfying both the paraxial approximation and the condition shown in Eq. (E2). For these resonators, which are formed by two perfectly conducting mirrors of either rectangular or circular cross section, the spatial dependence of the modes separates to yield two independent governing equations. Each of these equations is written in operator notation below,

$$u_1 = \gamma \hat{K}_{12}\hat{K}_{21}u_1 \tag{E18}$$

where u_1 represents the current distribution on mirror #1, and the $\hat{K}_{ij}$ involve the coordinates of both mirrors.

As the resonators being considered are symmetric, the integral operators $\hat{K}_{12}$ and $\hat{K}_{21}$ are identical in form. Therefore, the eigenvectors of the operator $\hat{K}_{21}$

$$u_1 = \pm\, \gamma_1 \hat{K}_{21}u_1 \tag{E19}$$

are also eigenvectors of the operator $\hat{K}_{12}\hat{K}_{21}$, with $\gamma_1^2 = \gamma$. As a result, the following analysis of these stable resonators is based on relatively simple equations of the general form shown in Eq. (E19).

Rectangular Mirror Resonators. For resonators with rectangular mirrors, the set of equations corresponding to Eq. (E19) is shown below (Eqs. (D11) through (D13)),

$$u_n(\xi_2) = \gamma_n \sqrt{\frac{i}{2\pi}} \int_{-H_x}^{H_x} u_n(\xi_1)\, e^{-\frac{ig}{2}(\xi_1^2 + \xi_2^2) + i\xi_1\xi_2}\, d\xi_1 \qquad \text{(E20)}$$

$$u_m(\eta_2) = \gamma_m \sqrt{\frac{i}{2\pi}} \int_{-H_y}^{H_y} u_m(\eta_1)\, e^{-\frac{ig}{2}(\eta_1^2 + \eta_2^2) + i\eta_1\eta_2}\, d\eta_1 \qquad \text{(E21)}$$

$$\gamma_n\gamma_m = e^{+i\{\pi q - kL\}} \qquad \text{(E22)}$$

where $\xi = \sqrt{k/L}\, x$, $\eta = \sqrt{k/L}\, y$, $H_x = \sqrt{k/L}\, a$, $H_y = \sqrt{k/L}\, b$, and q is an integer.

To obtain the oscillation condition, the general form of the expansion functions for this resonator (see appendix D) is substituted into Eq. (E20). As that form is negligible for large $|x|$, the finite limits of integration are replaced by infinite limits to obtain the equation shown below,

$$D_n(\delta'\xi_2) = \gamma_n \sqrt{\frac{i}{2\pi}} \int_{-\infty\iota}^{\infty\iota} D_n(\delta'\xi_1)\, e^{-\frac{ig}{2}(\xi_1^2 + \xi_2^2) + i\xi_1\xi_2}\, d\xi_1 \qquad \text{(E23)}$$

where $D_n(\delta'\xi_i)$ is a parabolic cylinder function of order n, $\delta'^2 = 2\sqrt{(1 - g^2)}$, and ι is a complex constant such that $0 \leq \arg\iota \leq \pi/4$.

Next, the following contour integral representation for $D_n(\delta'\xi)$ (Whittaker and Watson, ref. 39, p. 350)

$$D_n(\delta'\xi) = -\frac{n!e^{-\frac{\delta'^2\xi^2}{4}}}{2\pi i}\int^{(o+)} \frac{e^{-\delta'\xi t - \frac{1}{2}t^2}}{(-t)^{n+1}}\,dt \qquad \text{(E24)}$$

is substituted into Eq. (E23) and the order of integration is interchanged. Then, the integral over ξ_1 is evaluated by completing the square in the exponent, and the result is simplified to yield the following equation,

$$D_n(\delta'\xi_2) = -\frac{\gamma_n n!}{2\pi i}\sqrt{\frac{i}{\frac{\delta'^2}{2} + ig}}\; e^{-\frac{i}{2}\sqrt{g^2 - 1}\,\xi_2^2}$$

$$\int^{(o+)} \frac{e^{-\frac{\Lambda^2 t^2}{2}}}{(-t)^{n+1}}\, e^{-\Lambda\delta'\xi_2 t}\,dt \qquad \text{(E25)}$$

where $\Lambda = g + i\sqrt{1 - g^2}$. Changing variables and applying Eq. (E24) leads to the relation

$$\gamma_n = \Lambda^{-(n + \frac{1}{2})} \qquad \text{(E26)}$$

An identical procedure involving the y-variation yields a similar result for γ_m. By substituting these expressions into Eq. (E22), the desired oscillation condition is obtained.

$$e^{+i\{\pi q - kL\}} = \Lambda^{-(n + m + 1)} \qquad \text{(E27)}$$

Finally, assuming that k is real, this equation is manipulated to yield,

$$f_{mnq} = \frac{c}{2L}\left\{q + \frac{(n + m + 1)}{\pi} \cos^{-1} g\right\} \qquad \text{(E28)}$$

which is identical to the result reported by Pressley (ref. 43, p. 433).

Circular Mirror Resonators. The appropriate set of equations for these resonators is (see Eq. (D44))

$$u_n(\xi_2) = \gamma_n \int_0^{\sqrt{\frac{k}{L}}a} u_n(\xi_1)\, K_n(\xi_1|\xi_2)\, d\xi_1 \qquad \text{(E29)}$$

where

$$K_n(\xi_1|\xi_2) = e^{-\frac{ig}{2}(\xi_1^2 + \xi_2^2)} J_n(\xi_1\xi_2)\xi_1 \qquad \text{(E30)}$$

and

$$\gamma_n = i\, e^{+i\{\pi q + \frac{n\pi}{2} - kL\}} \qquad \text{(E31)}$$

As before, the general form of the expansion functions (see Eq. (D64))

$$u_n(\xi_2) = \xi_2{}^n\, e^{-\frac{\alpha'}{2}\xi_2^2}\, F_m{}^n(\alpha'\xi_2^2) \qquad \text{(E32)}$$

with $\alpha' = \sqrt{1 - g^2}$, is substituted into the integral equation. Then using infinite limits of integration, the integral is converted to a known form (Erdélyi, ref. 41, vol. 2, p. 43), and the result is simplified to yield the following expression.

$$e^{+i\{\pi q + \frac{(n + 1)}{2} - kL\}} = (-1)^m (\alpha' + ig)^{2n + m + 1} \qquad \text{(E33)}$$

For real k, Eq. (E33) is manipulated to yield the following oscillation condition.

$$f_{mnq} = \frac{c}{2L} \left\{ q + \frac{(2n + m + 1)}{\pi} \cos^{-1} g \right\} \qquad \text{(E34)}$$

As with rectangular mirror resonators, this result is identical to the one reported by Pressley (ref. 43, p. 433).

APPENDIX F

Additional Numerical Results

The purpose of this appendix is to present the circular mirror results not included in chapter VI as well as all of the rectangular mirror results. The circular mirror results are presented first, followed by those for rectangular mirror resonators.

Circular Mirror Resonator Results

The only results for circular mirror resonators not covered in chapter VI are six mode plots in the series for a plane parallel resonator with N = 10. These six plots include the third and fourth azimuthally symmetric (n = 0) modes and the first four n = 1 modes. As was the case for the two modes covered in chapter VI, there is excellent agreement between the distributions obtained by this author and those obtained by Fox and Li (ref. 25, p. 464, figs. 5 and 6). This agreement is especially good for the relative phase distributions (in degrees), where the only disagreements of any significance occur in regions where the field magnitude is quite small.

These mode distributions (denoted by +) as well as the comparative distributions from Fox and Li (denoted by *) are given in figures 16 through 21 on the following pages.

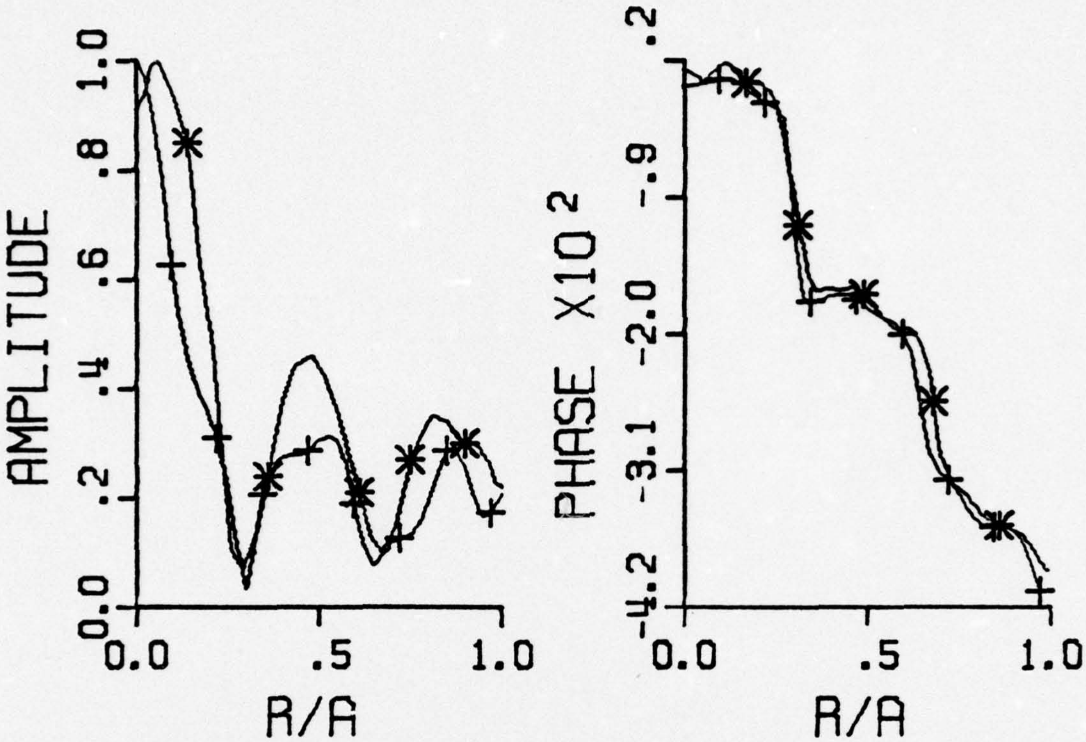

Figure 16. Relative Amplitude and Phase Distributions for the Third Lowest Loss, n = 0 Mode

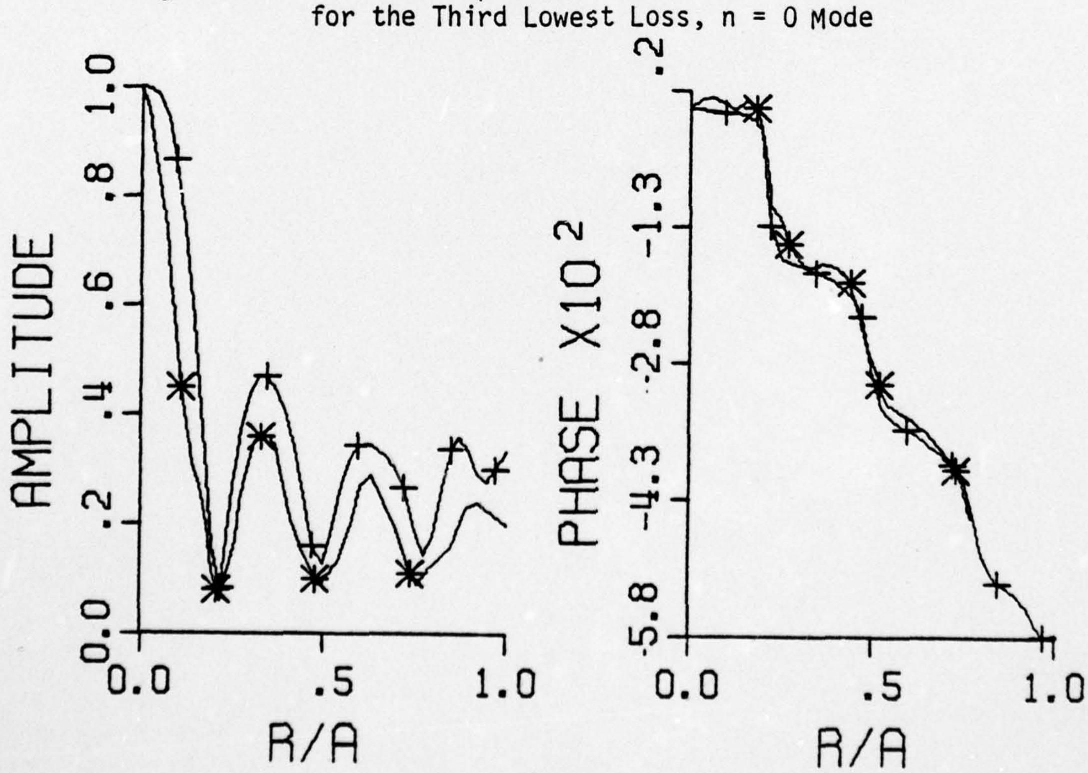

Figure 17. Relative Amplitude and Phase Distributions for the Fourth Lowest Loss, n = 0 Mode

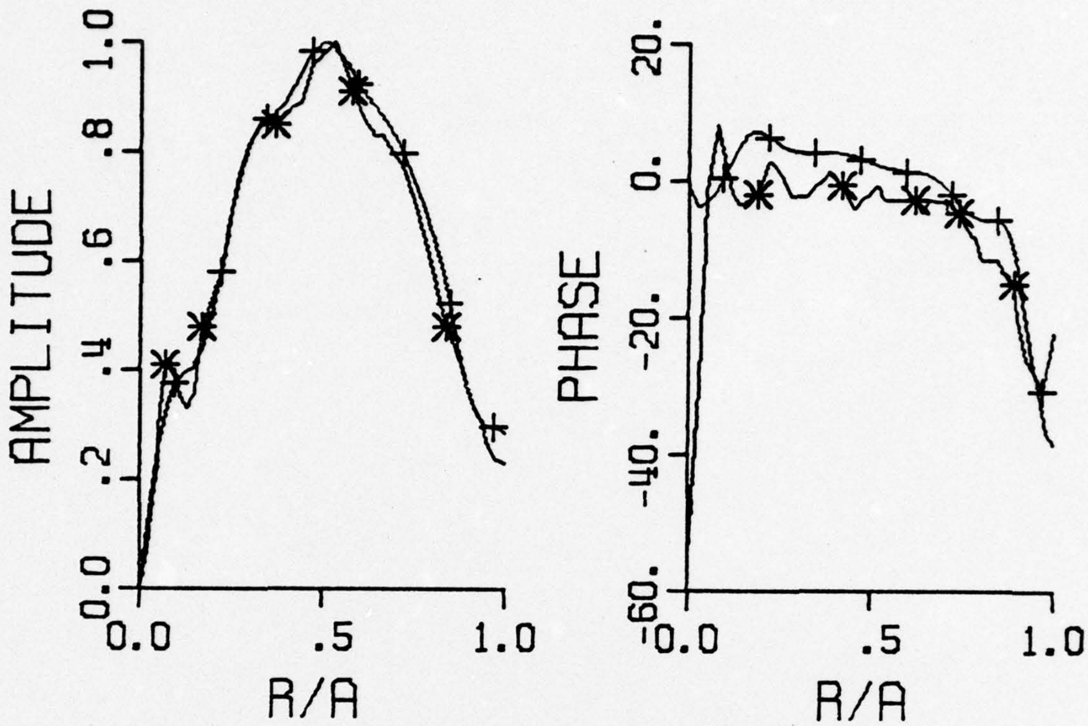

Figure 18. Relative Amplitude and Phase Distributions for the Lowest Loss, n = 1 Mode

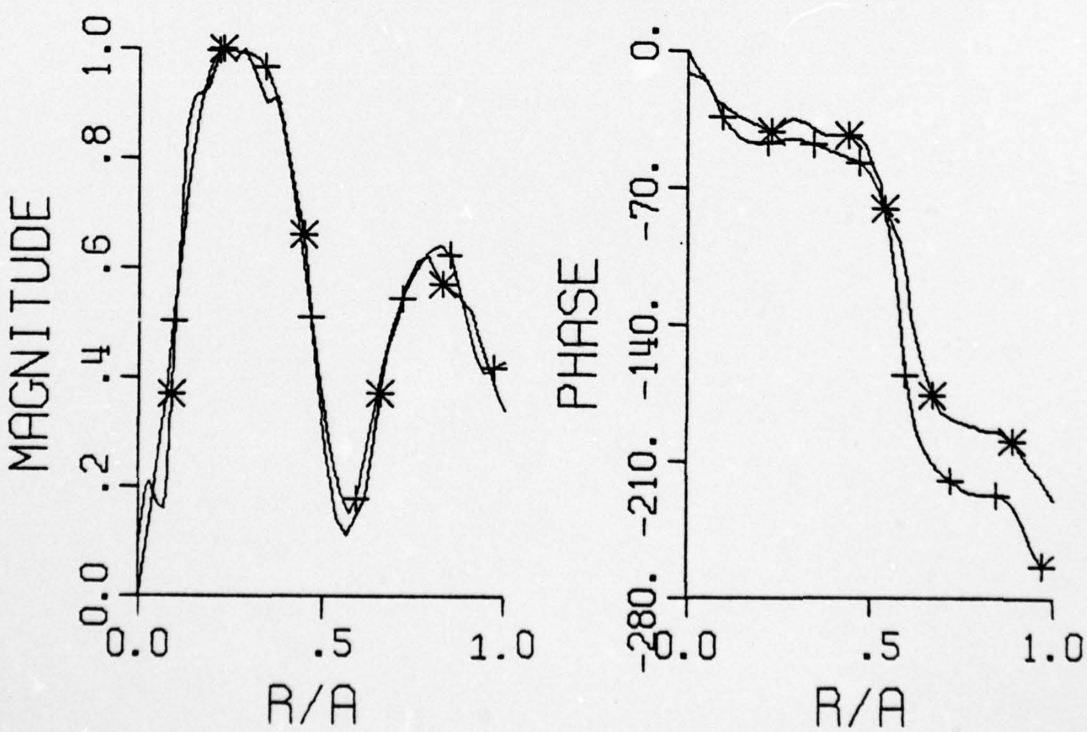

Figure 19. Relative Magnitude and Phase Distributions for the Second Lowest Loss, n = 1 Mode

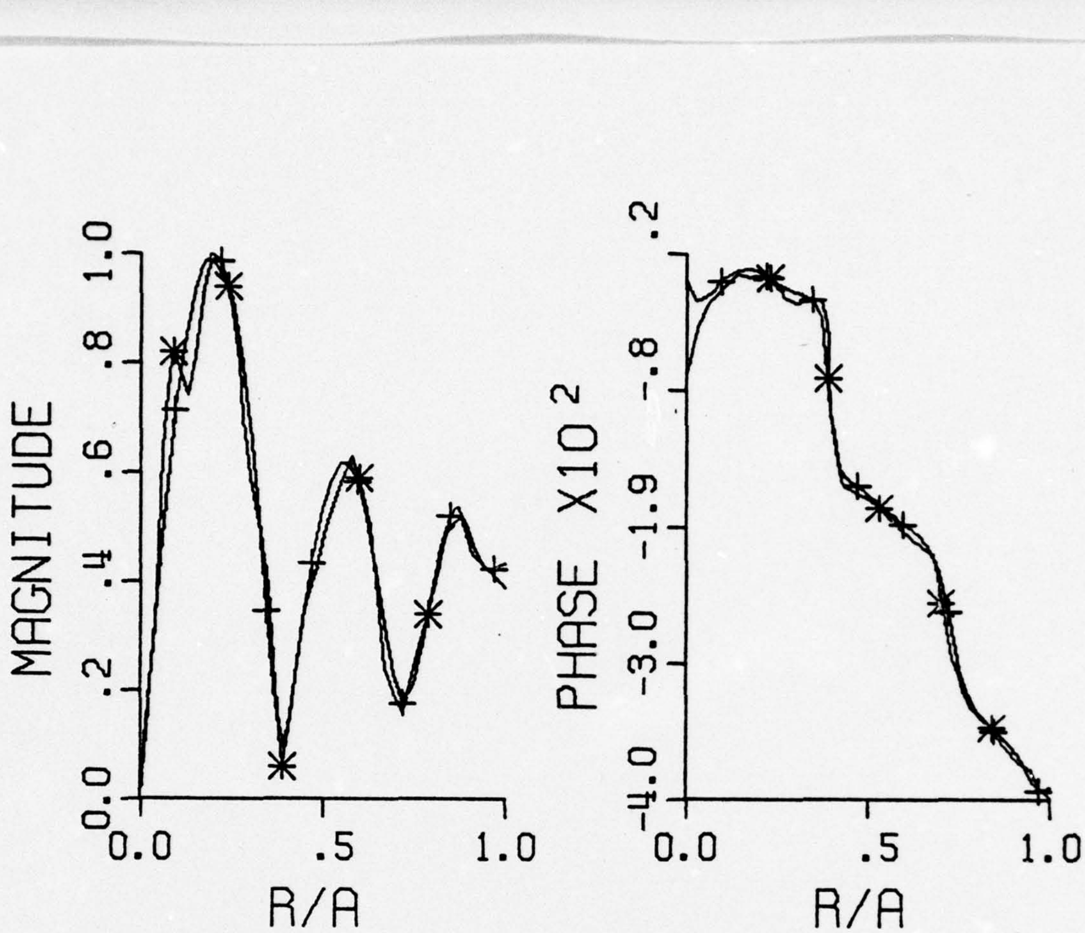

Figure 20. Relative Magnitude and Phase Distributions for the Third Lowest Loss, n = 1 Mode

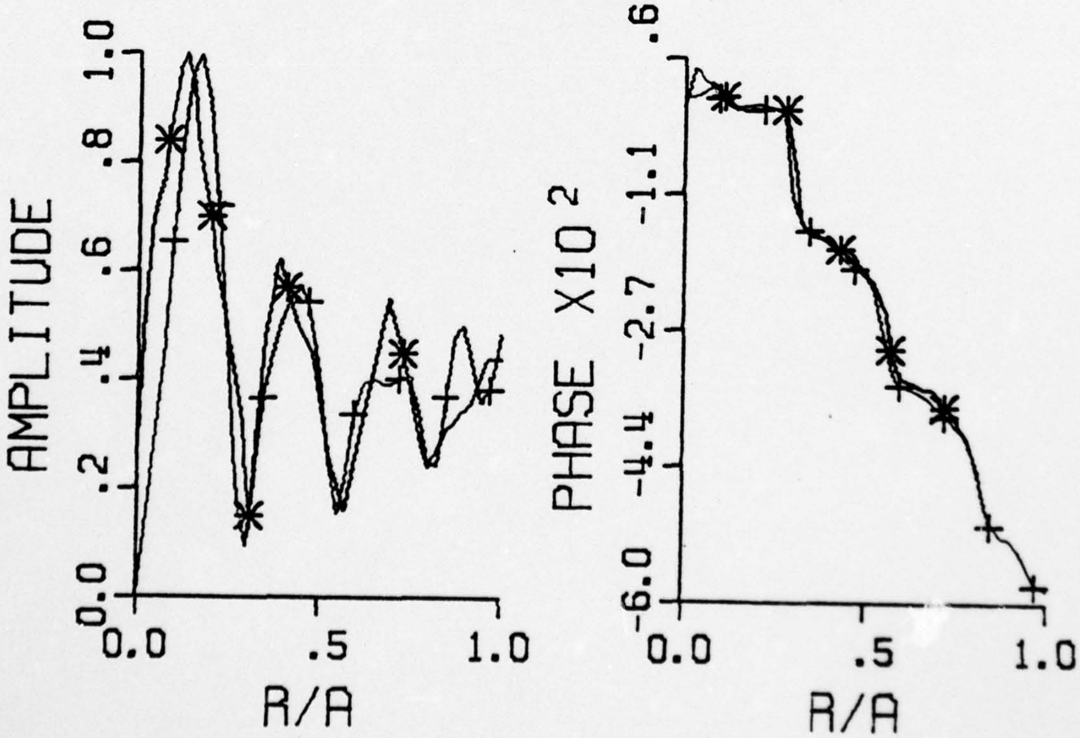

Figure 21. Relative Amplitude and Phase Distributions for the Fourth Lowest Loss, n = 1 Mode

Rectangular Mirror Resonator Results

As with circular mirror resonators, there are two types of data to be presented. The eigenvalue data, which correspond to a single pass through the resonator, are covered first. These data are followed by a series of mode plots for rectangular mirror resonators with g = 1.8.

Eigenvalue Data. The first cases for which eigenvalue data were obtained were two plane parallel resonators with N = 0 and N = 8/π. The percentage power loss for the first several even symmetric modes of these two resonators is presented below in tables X and XI. The comparative data were taken from Sanderson and Streifer (ref. 9, p. 135, tables IV and V).

Table X

Percentage Power Loss for Rectangular Mirror Resonator with g = 1.0, N = 10 for Even Symmetric Modes

DOUGHTY	FOX & LI	VAINSHTEIN	GAUSSIAN QUADRATURE
0.355	0.36	0.35	0.35
3.16	----	3.18	3.19
8.44	----	8.84	8.45

Table XI

Percentage Power Loss for Rectangular Mirror Resonator with g = 1.0, N = 8/π for Even Symmetric Modes

DOUGHTY	GAUSSIAN QUADRATURE	KERNEL EXPANSION
2.217	2.273	2.347
19.746	19.923	20.462
47.457	47.397	48.445
74.93	74.440	73.242

As reported in chapter VI, the difference for any of the first three modes is less than 2.5%.

The next cases considered were symmetric rectangular mirror resonators with g = 1.2. The percentage power loss for the first three odd symmetric modes for three values of N_e is displayed in table XII. For this series, the comparative data were taken from Sanderson and Streifer (ref. 42, p. 2131, fig. 7).

Table XII

Percentage Power Loss for Rectangular Mirror Resonator with g = 1.2 for Odd Symmetric Modes

DOUGHTY	SANDERSON & STREIFER	N_e
0.542	0.54	1.0
0.681	0.68	
0.913	0.92	
0.562	0.57	1.5
0.677	0.68	
0.793	0.78	
0.600	0.60	2.0
0.668	0.68	
0.773	0.72	

Finally, eigenvalue data for a wide range of equivalent Fresnel numbers for symmetric resonators with g = 1.8 are presented in tables XIII, XIV, and XV. The data listed correspond to the percentage power loss for several even symmetric modes (Sanderson and Streifer, ref. 42, p. 2132, fig. 10).

Table XIII

Percentage Power Loss With g = 1.8 for Even Symmetric Modes at Three Non-integral Values of $2N_e$

DOUGHTY	SANDERSON & STREIFER	$2N_e$
0.639	0.64	4.5
0.864	0.90	
0.923	0.92	
0.997	1.0	
0.717	0.75	7.5
0.909	0.86	
0.924	0.94	
0.947	0.95	
0.794	0.72	$28/\pi$
0.794	0.77	
0.927	0.96	
0.979	0.98	

Table XIV

Percentage Power Loss with g = 1.8 for Even Symmetric Modes at Three Integral Values of $2N_e$

DOUGHTY	SANDERSON & STREIFER	$2N_e$
0.712	0.72	6.0
0.866	0.81	
0.933	0.98	
0.968	0.98	
0.647	0.66	9.0
0.916	0.90	
0.954	0.91	
0.959	0.96	
0.738	0.74	12.0
0.858	0.86	
0.869	0.91	
0.937	0.94	

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A THEORETICAL ANALYSIS OF RESONATOR MODES IN THE PRESENCE OF HO--ETC(U)
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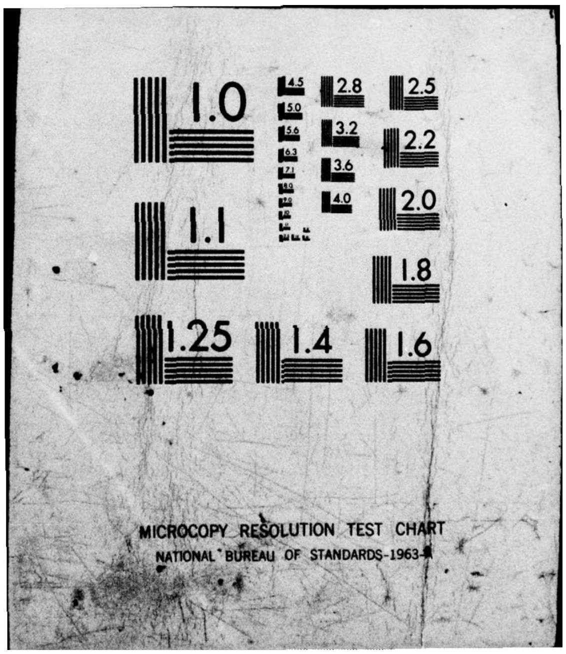
1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0
5.6
6.3
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

Table XV

Percentage Power Loss with g = 1.8 for Even Symmetric Modes at Large Values of $2N_e$

DOUGHTY	SANDERSON & STREIFER	$2\pi N_e$
0.652	0.65	34.0
0.899	0.91	
0.931	0.93	
0.709	0.68	36.0
0.899	0.91	
0.924	0.92	
0.651	0.65	40.0
0.861	0.89	
0.905	0.91	

Examination of the results in these three tables reveals that the differences are typically 4 percent or less. However, the maximum difference was 7 percent, which occurred for the second mode with $2N_e = 6.0$.

Mode Distributions. To further check the analysis presented in this paper, a series of mode plots was made for resonators with g = 1.8. These mode plots (figs. 22 through 28) include the relative intensity and phase (radians) for the first two even symmetric modes for $2\pi N_e$ = 34, 36, and 40 and the relative intensity distributions for $2\pi N_e$ = 18. For this last case, relative phase data were not included as they were not available in the paper from which the comparative data were taken (Sanderson and Streifer, ref. 42, p. 2133, figs. 13 through 20).

After comparing the results for these cases, several general remarks are offered in the next few pages. First, the intensity distributions obtained in this analysis are rather slowly varying.

Second, the relative values of some of the peaks and troughs are somewhat different. Although this is especially true at points where the intensity is low, it is certainly not uncommon at other points. These two facts are likely the result of

1. The tendency of a variational method to average the field distributions when the convergence criterion is based only on the eigenvalues, and
2. Computational errors in determining the expansion functions. This last fact also causes errors in the expansion coefficients, which might further explain any errors or differences.

However, despite these differences, sharp disagreement does not exist for any of these intensity distributions. In fact, the basic behavior or nature of the intensity distribution is correctly predicted in every case.

Further examination of these plots reveals that, except for the second mode with $2\pi N_e = 34$, there is basic agreement between the phase distributions for all modes. For the one case, the disagreement occurs over a region in which the phase is changing quite rapidly so that the difference could result from

1. Inability of the variational method to follow such rapid changes with the number of functions used,
2. Computational errors, or
3. 2π ambiguities in the phase data.

Interestingly, for regions where the phase is not changing so rapidly, elimination of these 2π ambiguities results in excellent agreement.

In any case, improved computer routines would allow the use of more expansion functions and would likely result in excellent agreement for these modes.

Since these and other results in this paper are discussed in the last section of chapter VI, no discussion is included with the mode plots presented in this appendix. In these plots, the results obtained by Sanderson and Streifer are denoted by an * while those obtained by this author are denoted by a +.

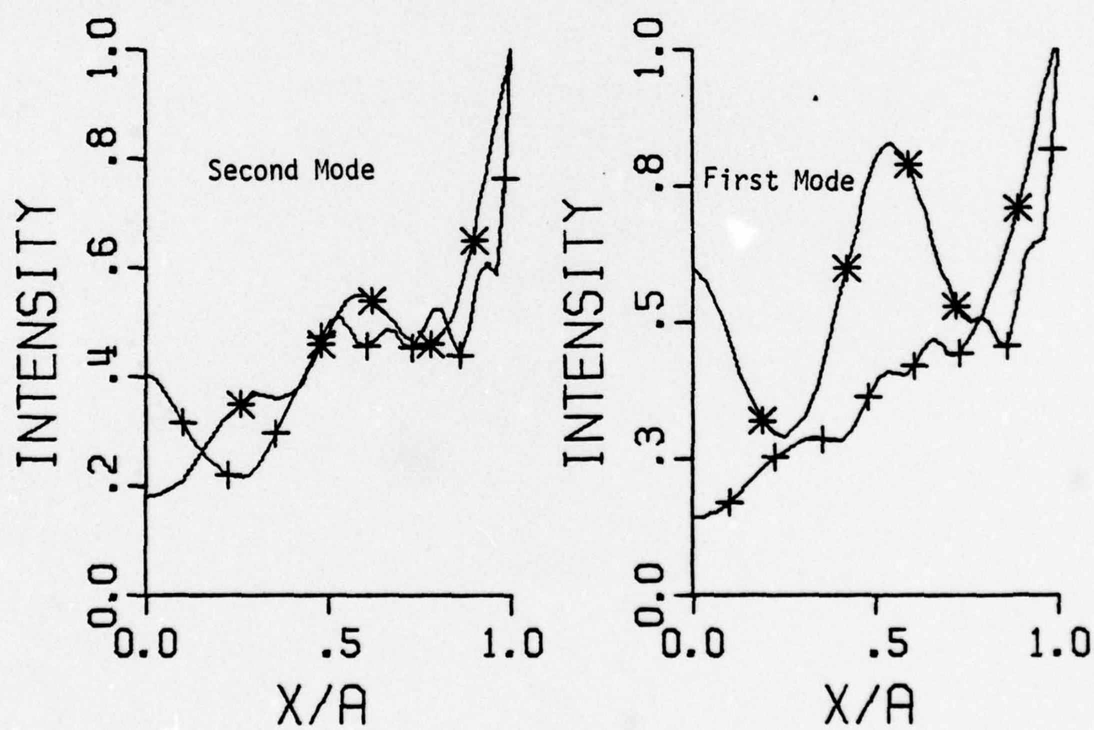

Figure 22. Relative Distributions for the First Two Modes with g = 1.8 and $2\pi N_e$ = 18

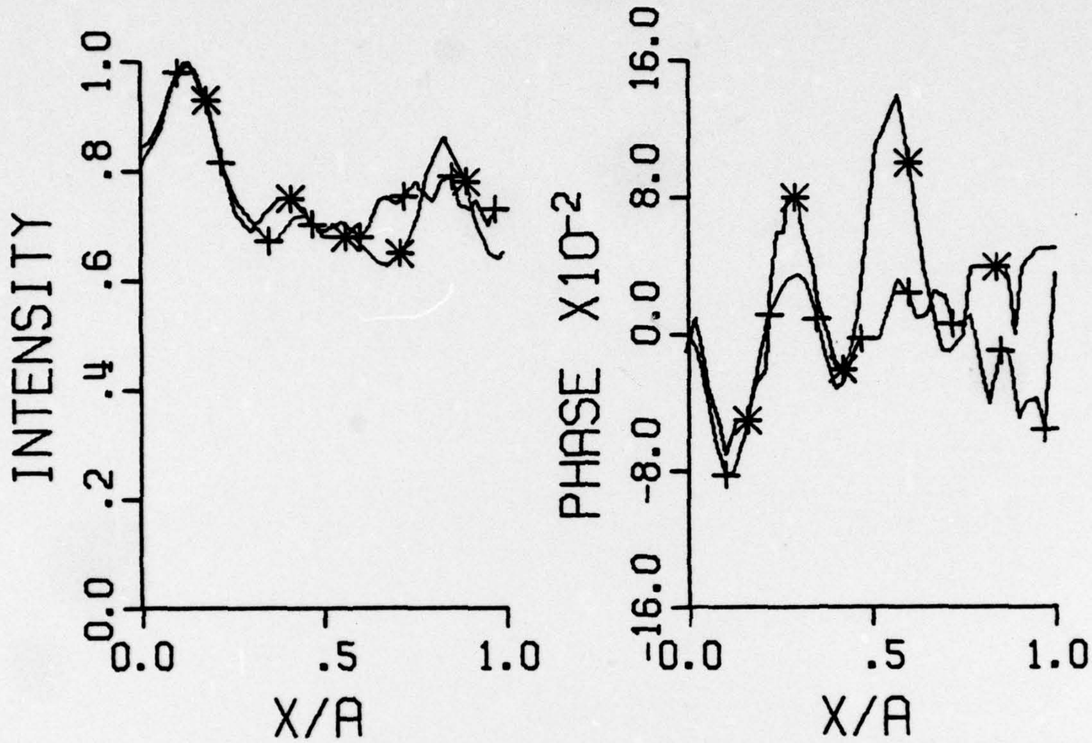

Figure 23. Relative Distributions for the Lowest Loss Mode with g = 1.8 and $2\pi N_e$ = 34

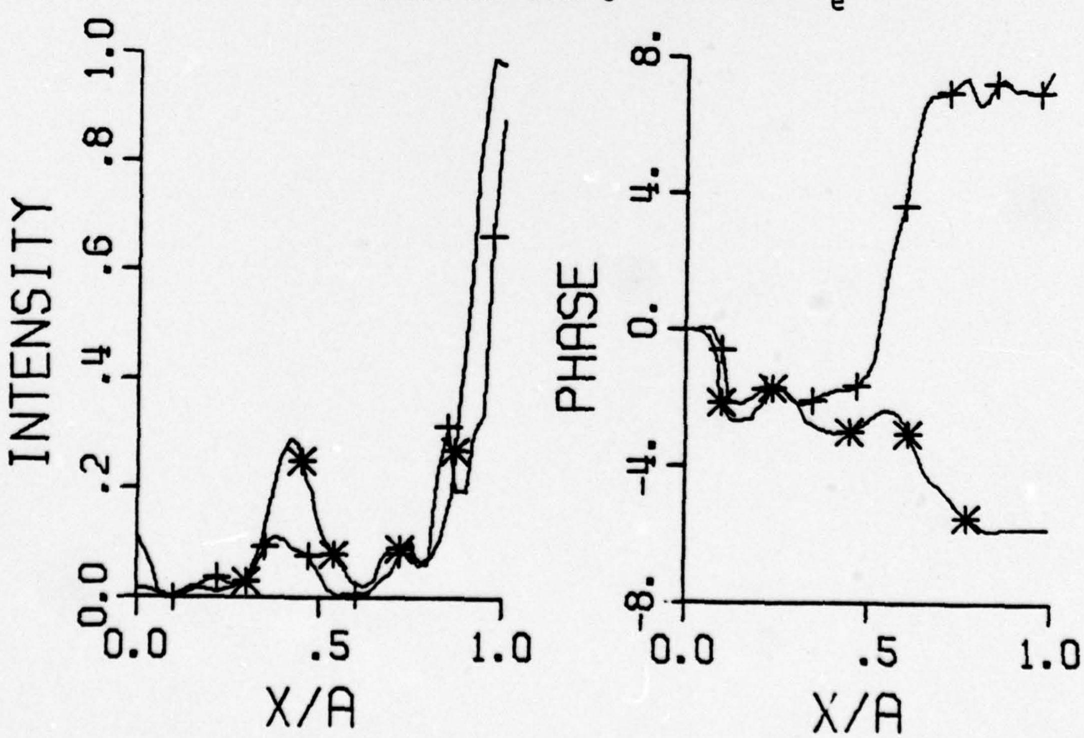

Figure 24. Relative Distributions for the Second Lowest Loss Mode with g = 1.8 and $2\pi N_e$ = 34

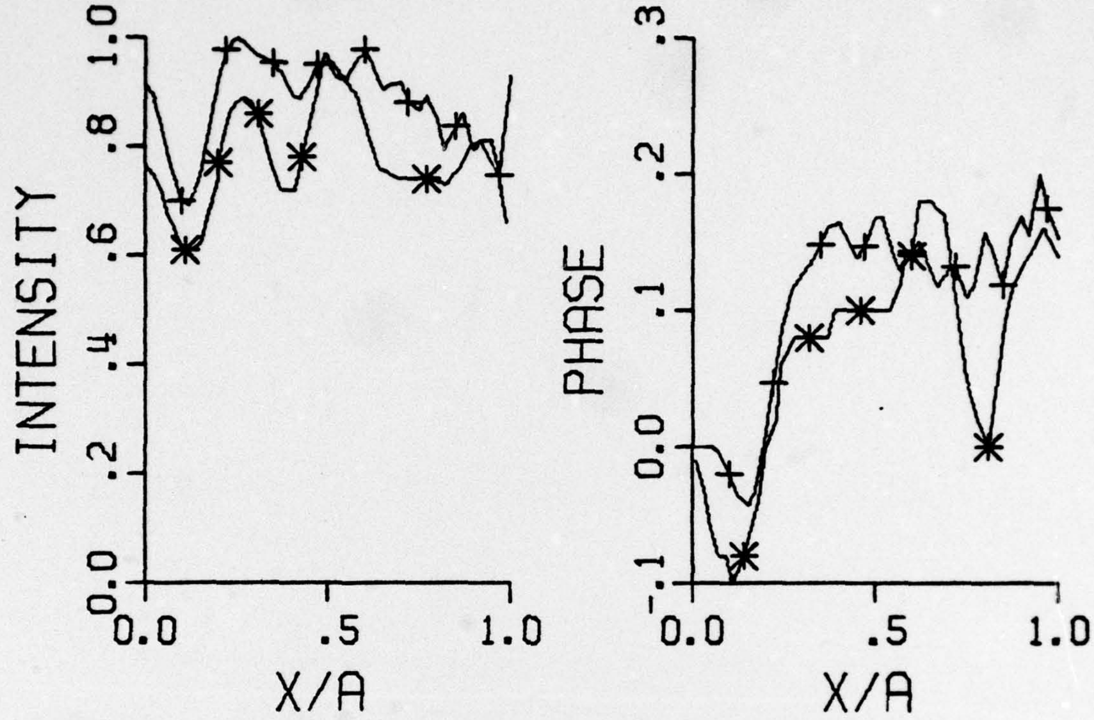

Figure 25. Relative Distributions for the Lowest Loss Mode with g = 1.8 and $2\pi N_e$ = 36

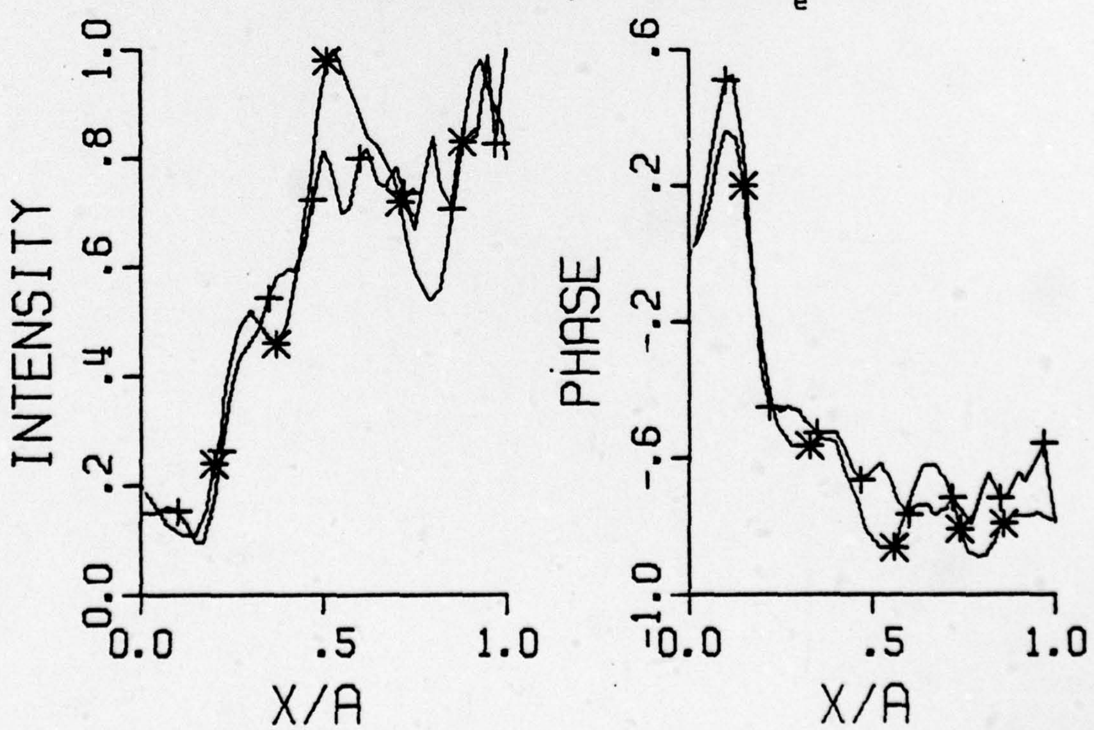

Figure 26. Relative Distributions for the Second Lowest Loss Mode with g = 1.8 and $2\pi N_e$ = 36

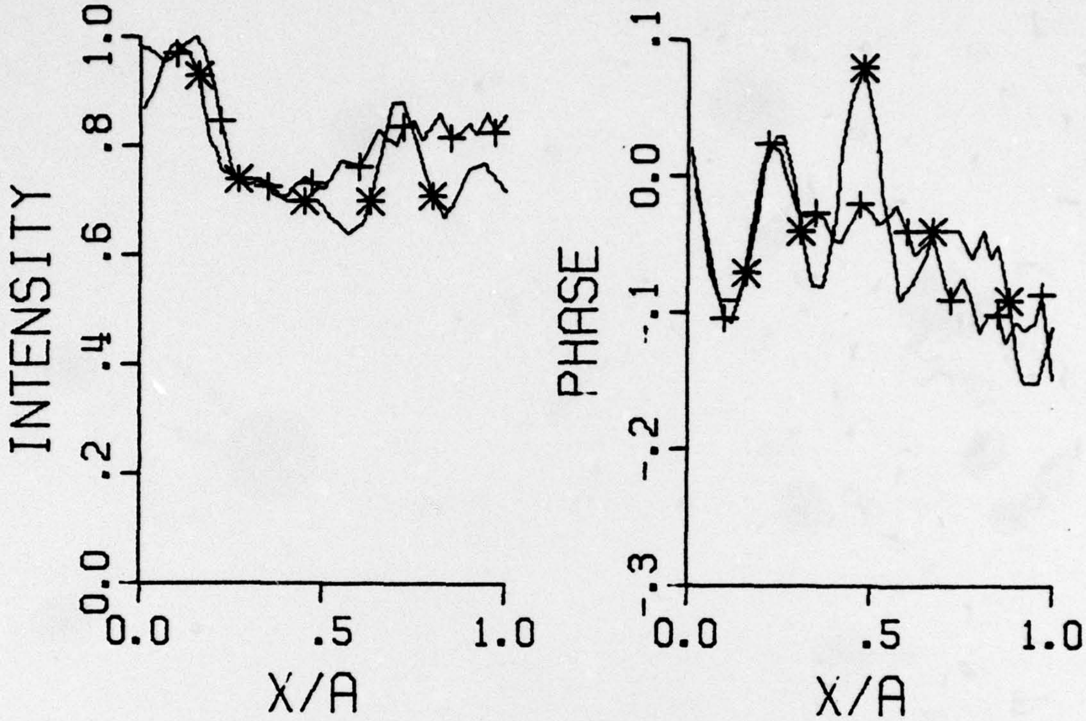

Figure 27. Relative Distributions for the Lowest Loss Mode with g = 1.8 and $2\pi N_e = 40$

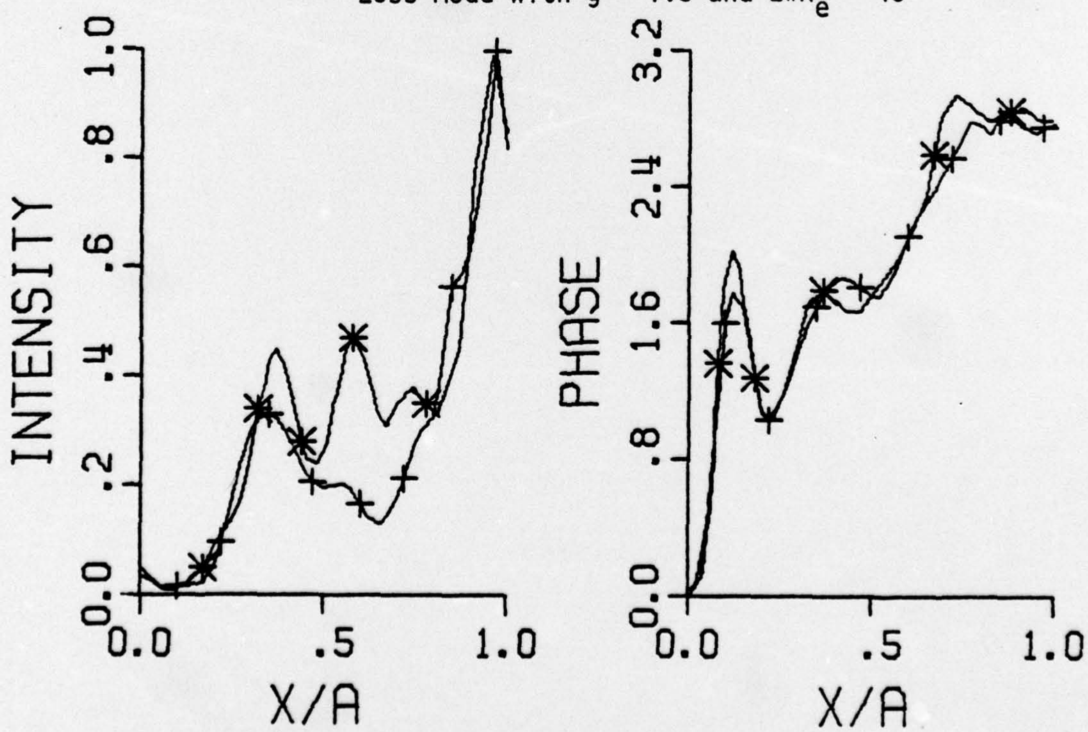

Figure 28. Relative Distributions for the Second Lowest Loss Mode with g = 1.8 and $2\pi N_e = 40$

APPENDIX G

Mathematical Approach for Solving the Matrix Problem

In chapter IV, the problem of determining the modes of paraxial resonators was reduced to solving a particular matrix equation. The equation, which involves the symmetric matrices K and Ψ and the column matrix A, is shown below.

$$KA = \gamma \Psi A \tag{G1}$$

The purpose of this appendix is to describe the procedure for solving Eq. (G1) such that the resulting approximations to the modes, $u_j(\zeta)$, obey the orthogonality condition

$$(u_j(\zeta)|u_k(\zeta)) = \delta_{jk} \tag{G2}$$

where

$$(u_j(\zeta)|u_k(\zeta)) = \int_\zeta u_j(\zeta)\, u_k(\zeta)\, d\zeta \tag{G3}$$

However, before describing the procedure, the method used to obtain Eq. (G1) is briefly reviewed.

The original problem of solving for the resonator modes, $u(\zeta)$, was formulated in terms of the integral equation

$$\hat{K}u(\zeta) = \bar{\gamma}u(\zeta) \tag{G4}$$

where $\hat{K}$ denotes the appropriate integral operator, and $\bar{\gamma} = \gamma^{-1}$. To solve this equation, the modes were expanded in terms of a set of known functions $\{\psi_q(\zeta)\}$ with coefficients A_q,

$$u(\zeta) = \sum_{q=1}^{n} A_q \psi_q(\zeta) \tag{G5}$$

and the Rayleigh-Ritz procedure was applied. That procedure led to Eq. (G1) with the elements of the matrices K and Ψ given by

$$K_{qm} = (\psi_q(\zeta)|\hat{K}|\psi_m(\zeta)) \tag{G6}$$

$$\Psi_{qm} = (\psi_q(\zeta)|\psi_m(\zeta)) \tag{G7}$$

The first step in solving this matrix problem is to reduce Eq. (G1) to an eigenvalue problem involving a symmetric matrix B. To do this, one decomposes the symmetric matrix Ψ using a Cholesky decomposition (Wilkinson, ref. 44, p. 229). This decomposition yields the form,

$$\Psi = L\,L^T \tag{G8}$$

where L is a lower left triangular matrix and L^T is the transpose of L.

To determine the matrix L, it is helpful to write Eq. (G8) in the expanded form shown below.

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} -- & \Psi_{1n} \\ \Psi_{12} & \Psi_{22} -- & \Psi_{2n} \\ \vdots & \vdots & \vdots \\ \Psi_{1n} & \Psi_{2n} -- & \Psi_{nn} \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 -- & 0 \\ & & \vdots \\ \ell_{12} & \ell_{22} & 0 \\ \vdots & \vdots & \\ \ell_{1n} & \ell_{2n} -- & \ell_{nn} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{12} -- & \ell_{1n} \\ 0 & \ell_{22} & \\ \vdots & & \\ 0 & 0 -- & \ell_{nn} \end{bmatrix} \tag{G9}$$

Then, performing the multiplications involving the first row of L, one obtains the relation

$$\Psi_{1i} = \ell_{11}\, \ell_{1i} \tag{G10}$$

for i = 1,2,--n. These equations can be solved for the elements of the first row of L^T. To compute the (n - 1) unknown elements of row 2, one uses the (n - 1) known elements, ℓ_{1i}, and the relation

$$\Psi_{k2} = \sum_{i=1}^{2} \ell_{i2}\, \ell_{ik} \tag{G11}$$

This process, which involves the general relation

$$\Psi_{kr} = \sum_{i=1}^{r} \ell_{ir}\, \ell_{ik} \tag{G12}$$

is continued until all of the ℓ_{ik} have been determined.

Using that decomposition, one premultiplies Eq. (G1) by L^{-1} and makes the substitution

$$A = (L^T)^{-1}F \tag{G13}$$

With these manipulations, one obtains the eigenvalue problem,

$$BF = \gamma F \tag{G14}$$

where B is a symmetric matrix given by $B = L^{-1}\, K\, (L^T)^{-1}$.

To solve this eigenvalue problem, B is reduced to a tri-diagonal matrix using Householder's method (Acton, ref. 45, p. 324). That method, which preserves symmetry and eigenvalues, involves the use of symmetric, orthogonal transformations as described on the following page.

In the first step of the procedure to tridiagonalize the matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{12} & b_{22} & & \vdots \\ \vdots & \vdots & & \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{bmatrix} \tag{G15}$$

one finds a symmetric, orthogonal matrix P_1, such that

$$P_1 \begin{bmatrix} b_{12} \\ \vdots \\ b_{1n} \end{bmatrix} = f_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{G16}$$

where $f_1^2 = \sum_{i=2}^{n} b_{1i}^2$. The matrix P_1 is then applied in the manner shown below,

$$\begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & [\;P_1\;] \\ 0 & \end{bmatrix} [B] \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & [\;P_1\;] \\ 0 & \end{bmatrix} = \begin{bmatrix} b_{11} & f_1 \quad 0 \cdots 0 \\ f_1 & \\ 0 & [\;C\;] \\ \vdots & \\ 0 & \end{bmatrix} \tag{G17}$$

where C is the $(n - 1)^{th}$ order matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1,n-1} \\ C_{12} & C_{22} & \cdots & C_{2,n-1} \\ \vdots & \vdots & & \vdots \\ C_{1,n-1} & C_{2,n-1} & \cdots & C_{n-1,n-1} \end{bmatrix} \tag{G18}$$

This procedure is then repeated to yield a matrix P_2 such that

$$P_2 \begin{bmatrix} C_{12} \\ \cdot \\ \cdot \\ \cdot \\ C_{1,n-1} \end{bmatrix} = f_2 \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad \text{(G19)}$$

where $f_2^2 = \sum_{i=2}^{n-1} C_{1i}^2$. The matrix P_2 is then applied to yield the result shown below,

$$\begin{bmatrix} [I] & [0] \\ [0] & [P_2] \end{bmatrix} \begin{bmatrix} b_{11} & f_1 \;--\; 0 \\ f_1 & \\ \cdot & [C] \\ \cdot & \\ \cdot & \\ 0 & \end{bmatrix} \begin{bmatrix} [I] & [0] \\ [0] & [P_2] \end{bmatrix} = \begin{bmatrix} b_{11} & f_1 & 0 & --0 \\ f_1 & C_{11} & f_2 & --0 \\ 0 & f_2 & & \\ \cdot & \cdot & [D] & \\ \cdot & \cdot & & \\ 0 & 0 & & \end{bmatrix} \qquad \text{(G20)}$$

where I is the identity matrix. This procedure of applying symmetric, orthogonal transformations P_i, is continued until a tridiagonal matrix T is obtained. The method used to obtain the P_i is covered in Acton (ref. 45, pp. 326-329).

Once the matrix T is known, the eigenvalues are determined using the "LR" algorithm (Acton, ref. 45, p. 350). This algorithm, which is not normally used for a general matrix, was chosen because the tri-diagonal form is preserved at every step of the iteration procedure. This preservation of form makes it possible to accomplish each iteration using only (n - 1) multiplications and (n - 1) divisions.

To apply this algorithm, one first decomposes the matrix T such that

$$T = LR \tag{G21}$$

where L and R have the following characteristics:

1. L is a lower left triangular matrix,
2. R is an upper right triangular matrix, and
3. The elements on the main diagonal of L have the value one.

Then, the matrix T_1 is computed according to

$$T_1 = RL \tag{G22}$$

and the procedure is repeated. After the $(s + 1)^{th}$ iteration, one obtains

$$T_{s+1} = R_s L_s \tag{G23}$$

such that

$$T_{s+1} = L_s^{-1} T_s L_s \tag{G24}$$

This process is continued until the set $\{T_s\}$ converges to an upper right triangular matrix with the eigenvalues on the main diagonal.

In the next step, these eigenvalues are used to compute the eigenvectors of T by employing a procedure which utilizes all n equations of the homogeneous system

$$(T - \gamma_k I)\, w_k = 0 \tag{G25}$$

where w_k is a column matrix with n elements. This inverse iteration procedure (Acton, ref. 45, p. 357) is based on the iteration scheme,

$$(T - \gamma_k I)\, v_{i+1}^k = v_i^k \tag{G26}$$

where v_i^k (the i^{th} approximation to the k^{th} eigenvector of T) is determined using Gaussian elimination with interchanges. As the eigenvalues γ_k have already been determined, this iteration procedure rapidly converges to the desired eigenvectors. These solutions are then used to obtain the eigenvectors F_k (of B) by applying the same similarity transformation used to obtain T from B. The A_k, which are obtained by applying the relation shown below,

$$A_k = (L^T)^{-1} F_k \qquad (G27)$$

are then substituted into Eq. (G5) to obtain the resonator modes.

To show that the eigenfunctions $u_j(\zeta)$ are orthogonal (see Eq. (G2)), one computes the scalar product

$$(u_j(\zeta)|u_k(\zeta)) = \sum_{m,q} A_{jq} (\psi_q(\zeta)|\psi_m(\zeta)) A_{km} \qquad (G28)$$

which can be written in matrix form as shown below.

$$(u_j(\zeta)|u_k(\zeta)) = A_j^T \Psi A_k \qquad (G29)$$

One then applies Eq. (G27) to obtain the following result

$$(u_j(\zeta)|u_k(\zeta)) = F_j^T L^{-1} \Psi(L^T)^{-1} F_k \qquad (G30)$$

Then using Eq. (G8), one obtains

$$(u_j(\zeta)|u_k(\zeta)) = F_j^T F_k \qquad (G31)$$

Since the eigenvectors F_j of a symmetric matrix B obey the relation,

$$F_j^T F_k = \delta_{jk} \qquad (G32)$$

the modes $u_j(\zeta)$ obey Eq. (G2) as asserted.

APPENDIX H

The Confluent Hypergeometric Functions

As was shown in chapter V, the expansion functions for the modes of both rectangular and circular mirror resonators are directly related to the confluent hypergeometric functions of the first and second kind. The purpose of this appendix is to present some of the more important characteristics and relationships involving those functions. For more detailed discussions, the reader is referred to the sources of this appendix (Lebedev, ref. 40, chaps. 9 and 10; Erdélyi, ref. 41, chaps. 6, 8, and 10).

As the name implies, these functions are solutions of the confluent hypergeometric equation which is shown below (Lebedev, ref. 40, p. 262).

$$z \frac{d^2u(z)}{dz^2} + (\gamma - z) \frac{du(z)}{dz} - \alpha u(z) = 0 \tag{H1}$$

The solution of the first kind, denoted $\Phi(\alpha, \gamma, z)$, is an entire function of z, but it does have simple poles at the points $\gamma = 0,-1,-2,---$. For values of $\gamma \neq 0,-1,2,---$, this function is defined by the series (Lebedev, ref. 40, p. 260),

$$\Phi(\alpha, \gamma, z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k z^k}{(\gamma)_k k!} \tag{H2}$$

where $(T)_0 = 1$, $(T)_k = \frac{\Gamma(T + k)}{\Gamma(T)}$, and $T = \alpha$ or γ.

Unfortunately, the solution of the second kind, denoted $\Psi(\alpha, \gamma, z)$, is somewhat more complex. For values of γ such that

$\gamma \neq 0, \pm 1, \pm 2, ---$ this function is defined in terms of the $\Phi(\alpha, \gamma, z)$ as shown below (Lebedev, ref. 49, p. 263),

$$\Psi(\alpha, \gamma, z) = \frac{\Gamma(1 - \gamma)}{\Gamma(1 + \alpha - \gamma)} \Phi(\alpha, \gamma, z)$$

$$+ \frac{\Gamma(\gamma - 1)}{\Gamma(\alpha)} z^{(1 - \gamma)} \Phi(1 + \alpha - \gamma, 2 - \gamma, z) \qquad \text{(H3)}$$

with $|\arg z| < \pi$. However, for $\gamma = n + 1$, where n is zero or a positive integer, this expression is indeterminant and $\Psi(\alpha, n + 1, z)$ must be evaluated using a limiting procedure. The procedure, which involves the substitution of the series representations for Φ and Γ into Eq. (H3) as well as the application of L'Hospital's rule, leads to the following equation (Lebedev, ref. 40, p. 264),

$$\Psi(\alpha,n+1,z) = \frac{(-1)^{n+1}}{\Gamma(\alpha-n)} \sum_{k=0}^{\infty} \frac{(\alpha)_k \, z^k}{(n+k)!k!} \left\{ \psi(\alpha+k) - \psi(1+k) - \psi(n+1+k) + \ln(z) \right\}$$

$$+ \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \frac{(-1)^k (n-k-1)! \, (\alpha-n)_k}{k!} z^{k-n} \qquad \text{(H4)}$$

where $\psi(x)$ is the logarithmic derivative of the gamma function (Erdélyi, ref. 41, p. 44) and $|\arg z| < \pi$. In addition, Eq. (H4) can be used in conjunction with a recurrence relation (Lebedev, ref. 40, p. 265) to define $\Psi(\alpha, \gamma, z)$ for all values of γ. As a result, $\Psi(\alpha, \gamma, z)$ is not only an analytic function of z in the plane cut along $\{-\infty, 0\}$, but it is also an entire function of α and γ.

For values of γ such that $\gamma \neq 0, -1, -2, ---$ and for values of z such that $|\arg z| < \pi$, one can calculate the Wronskian of these two solutions to obtain the result shown below (Lebedev, ref. 40, p. 265).

$$W\{\Phi(\alpha,\gamma,z);\ \Psi(\alpha,\gamma,z)\} = -\frac{\Gamma(\gamma)}{\Gamma(\alpha)} z^{-\gamma} e^{z} \tag{H5}$$

From this expression, one can see that if $\alpha \neq 0,-1,-2,---$, then $\Phi(\alpha, \gamma, z)$ and $\Psi(\alpha, \gamma, z)$ are linearly independent solutions of Eq. (H1).

In addition to the series representations shown above, Φ and Ψ are also represented by asymptotic expansions as $|z|$ (or one of the parameters) becomes large. For example, if $|z|$ is large, $\gamma \neq 0,-1,-2,---$, and $|\arg z| \leq \pi - \delta$, then (Lebedev, ref. 40, p. 271),

$$\Phi(\alpha,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} z^{-\alpha} e^{\pm i\pi\alpha} \left\{ \sum_{k=0}^{n} \frac{(-1)^k(\alpha)_k(1+\alpha-\gamma)_k}{k!} z^{-k} + 0\left(|z|^{-n-1}\right) \right\} + \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^{z} z^{-(\gamma-\alpha)} \left\{ \sum_{k=0}^{n} \frac{(\gamma-\alpha)_k(1-\alpha)_k}{k!} z^{-k} + 0\left(|z|^{-n-1}\right) \right\} \tag{H6}$$

where the plus sign is chosen if Im $(z) > 0$ and the minus sign is chosen if Im $(z) < 0$. Similarly, one can show that for the identical conditions on z (Lebedev, ref. 40, p. 270),

$$\Psi(\alpha,\gamma,z) = z^{-\alpha} \left\{ \sum_{k=0}^{n} \frac{(-1)^k(\alpha)_k(1+\alpha-\gamma)_k}{k!} z^{-k} + 0\left(|z|^{-n-1}\right) \right\} \tag{H7}$$

Another asymptotic relation involving $\Phi(\alpha, \gamma, z)$ can be obtained by letting the magnitude of α become large. For that case, which is applied several times in this report, one lets $\kappa = \frac{\gamma}{2} - \alpha$. Then if γ and κz are bounded and if $|\arg z - \arg \kappa| \leq \pi$ (Erdélyi, ref. 41, p. 280),

$$\Phi(\alpha,\gamma,z) \underset{|\kappa|\to\infty}{\to} \Gamma(\gamma)\ (\kappa z)^{\frac{1-\gamma}{2}}\ e^{\frac{z}{2}}\ J_{\gamma-1}(2\sqrt{\kappa z}\ ) + O\ (|\kappa|^{-1}) \qquad \text{(H8)}$$

In addition to these expressions, there are many important relationships between the confluent hypergeometric functions which have played a key role in this analysis. Several of these relationships are listed in the following paragraph.

The first two relationships involve the Whittaker functions of the first and second kind (Erdélyi, ref. 41, p. 264),

$$M_{\kappa,\mu}(z) = z^{\mu+\frac{1}{2}}\ e^{-\frac{z}{2}}\ \Phi(\tfrac{1}{2}-\kappa+\mu, 2\mu+1, z) \qquad \text{(H9)}$$

$$W_{\kappa,\mu}(z) = z^{\mu+\frac{1}{2}}\ e^{-\frac{z}{2}}\ \Psi(\tfrac{1}{2}-\kappa+\mu, 2\mu+1, z) \qquad \text{(H10)}$$

where $|\arg z| < \pi$. Next, an important relation involving the parabolic cylinder function of order ν, $D_\nu(z)$ is shown in Eq. (H11) (Erdélyi, ref. 41, vol. 2, p. 117).

$$D_\nu(z) = 2^{\frac{\nu}{2}}\ e^{-\frac{z^2}{4}} \left\{ \frac{\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1-\nu}{2}\right)}\ \Phi\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{z^2}{2}\right) + \frac{z}{\sqrt{2}}\ \frac{\Gamma(\frac{1}{2})}{\Gamma\left(-\frac{\nu}{2}\right)}\ \Phi\left(\frac{1-\nu}{2}, \frac{3}{2}, \frac{z^2}{2}\right) \right\} \qquad \text{(H11)}$$

Finally, the generalized Laguerre polynomial, $F_m^n(z)$, is related to $\Phi(\alpha, \gamma, z)$ in the following manner (Lebedev, ref. 40, p. 273),

$$F_m^n(z) = \frac{(n+1)_m}{m!} \Phi(-m,n+1,z) \qquad \text{(H12)}$$

where $(n+1)_m = \frac{(n+m)!}{m!}$

Vita

Glenn Roy Doughty was born on 5 October 1942 in San Antonio, Texas. He graduated from W.B. Ray High School in Corpus Christi, Texas in May 1961. Two months after graduation, he entered the United States Military Academy at West Point, New York. Following graduation in 1965, he entered the Air Force as a second lieutenant with a Bachelor of Science degree. He was then assigned to the Laser Division of the Air Force Weapons Laboratory at Kirtland Air Force Base, New Mexico where he served for five years. Following that assignment, he entered the Air Force Institute of Technology to obtain a Master of Science degree in Engineering Physics and a Doctor of Philosophy degree in Aerospace Engineering. Upon completion of his studies in August 1973, he was reassigned to the Air Force Weapons Laboratory.