THE GENERALIZED VARIANCE OF A STATIONARY AUTOREGRESSIVE PROCESS

BY

T. W. ANDERSON and RAÚL P. MENTZ

TECHNICAL REPORT NO. 27
OCTOBER 1976

PREPARED UNDER CONTRACT N00014-75-C-0442
(NR-042-034)
OFFICE OF NAVAL RESEARCH

THEODORE W. ANDERSON, PROJECT DIRECTOR

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

THE GENERALIZED VARIANCE OF A STATIONARY AUTOREGRESSIVE PROCESS

By

T. W. Anderson and Raúl P. Mentz

Technical Report No. 27
October 1976

PREPARED UNDER CONTRACT N00014-75-C-0442
(NR-042-034)
OFFICE OF NAVAL RESEARCH

Theodore W. Anderson, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

The Generalized Variance of a Stationary Autoregressive Process

by

T. W. Anderson and Raúl P. Mentz
Stanford University and University of Tucumán, Argentina

An autoregressive process $\{y_t\}$ of order p with mean 0 is defined by

$$(1) \qquad y_t + \beta_1 y_{t-1} + \dots + \beta_p y_{t-p} = u_t , \quad t = \dots,-1,0,1,\dots ,$$

where the u_t are independent random variables with $\mathcal{E} u_t = 0, \mathcal{E} u_t^2 = \sigma^2$, $0 < \sigma^2 < \infty$. The stochastic process is stationary and y_t is independent of $u_{t+1}, u_{t+2}, \dots$ if and only if the β_j are such that the associated polynomial equation

$$(2) \qquad b(w) = \sum_{j=0}^{p} \beta_j w^{p-j} = 0 ,$$

where $\beta_0 = 1$, has roots $w_1, \dots, w_p$ less than 1 in absolute value. The purpose of this paper is to show that the generalized variance of the process is a power of the variance of u_t times $\Pi_{i,j=1}^{p}(1-w_i w_j)^{-1}$.

The covariance sequence of the process is composed of $\sigma_s = \mathcal{E} y_t y_{t+s} = \sigma_{-s}$, $s = 0,1,\dots$. Consider a sequence of observations $y_1, \dots, y_T$ for $T \geq p$ constituting a sample vector $\underset{\sim}{y}_T = (y_1,\dots,y_T)'$.

The covariance matrix of the sample vector is

$$\mathcal{E} \underset{\sim}{y}_T \underset{\sim}{y}_T' = (\sigma_{i-j}) \equiv \underset{\sim}{\Sigma}_T \equiv \sigma^2 \underset{\sim}{Q}_T . \tag{3}$$

The determinant $|\underset{\sim}{\Sigma}_T| = (\sigma^2)^T |\underset{\sim}{Q}_T|$ is the generalized variance of $\underset{\sim}{y}_T$.

In the Gaussian case the joint density of $\underset{\sim}{y}_p$ and $u_{p+1}, \ldots, u_T$ $(T \geq p)$ is

$$\frac{|\underset{\sim}{Q}_p^{-1}|^{\frac{1}{2}}}{(2\pi)^{T/2} \sigma^T} \exp\left\{-\frac{1}{2\sigma^2}\left[\underset{\sim}{y}_p' \underset{\sim}{Q}_p^{-1} \underset{\sim}{y}_p + \sum_{t=p+1}^{T} u_t^2\right]\right\} . \tag{4}$$

Since the Jacobian of the transformation from $\underset{\sim}{y}_p, u_{p+1}, \ldots, u_T$ to $\underset{\sim}{y}_T$ is 1 , the constant of the density of $\underset{\sim}{y}_T$ is the same as of (4) and hence $|\underset{\sim}{Q}_p^{-1}| = |\underset{\sim}{Q}_T^{-1}|$, $T \geq p$. (See Walker (1961) and Siddiqui (1958).) Since $|\underset{\sim}{Q}_T| = |\underset{\sim}{Q}_p|$ for $T \geq p$, we call $|\underset{\sim}{Q}_p|$ the normalized generalized variance of the process.

For $T \geq p$ substitution for u_t from (1), $t = p+1, \ldots, T$, into (4) to obtain the density of $\underset{\sim}{y}_T$ yields the quadratic form $\underset{\sim}{y}_T' \underset{\sim}{Q}_T^{-1} \underset{\sim}{y}_T$, showing that every element of $\underset{\sim}{Q}_T^{-1}$ is a second degree polynomial in $\beta_1, \ldots, \beta_p$ except possibly elements of $\underset{\sim}{Q}_p^{-1}$. However, since the density of $y_1, \ldots, y_T$ is identical to the density of $y_T, \ldots, y_1$, the elements of $\underset{\sim}{Q}_p^{-1}$ must be second degree polynomials in $\beta_1, \ldots, \beta_p$. The components of $\underset{\sim}{Q}_p^{-1}$ are therefore polynomials in the roots $w_1, \ldots, w_p$ of degree at most $2p$. Hence, the determinant $|\underset{\sim}{Q}_p^{-1}|$ is a polynomial in $w_1, \ldots, w_p$ of degree at most $(2p)^p$.

Lemma 1. *If*

$$\text{(5)} \qquad \underset{\sim}{C}=(c_{ij}) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ h_1 & h_2 & \cdots & h_p \\ h_1^2 & h_2^2 & \cdots & h_p^2 \\ \vdots & \vdots & & \vdots \\ h_1^{p-1} & h_2^{p-1} & \cdots & h_p^{p-1} \end{pmatrix} ,$$

then

$$\text{(6)} \qquad |\underset{\sim}{C}| = \prod_{i<j} (h_j - h_i)$$

and

$$\text{(7)} \qquad C_{1k} = (-1)^{k-1} \left(\prod_{i \neq k} h_i \right) \prod_{\substack{i<j \\ i \neq k \neq j}} (h_j - h_i) ,$$

where C_{1k} *denotes the cofactor of* c_{1k} *in* $\underset{\sim}{C}$.

Proof. $\underset{\sim}{C}$ is a Vandermonde matrix, and $|\underset{\sim}{C}|$ and $\underset{\sim}{C}^{-1}$ are given, for example, by Hamming (1962), Sections 8.2 and 10.3. A direct proof of (7) using (6) is as follows: to form C_{1k} delete row 1 and column k of $|\underset{\sim}{C}|$; in the cofactor, factor h_i out of the i-th column $(i \neq k)$ to obtain a Vandermonde determinant of order p-1 . Q.E.D.

Lemma 2. *The determinant of order* p ,

$$\text{(8)} \qquad D_p \equiv \left| \frac{1}{a_i + b_j} \right| = \frac{\prod_{i<j} (a_j - a_i)(b_j - b_i)}{\prod_{i,j=1}^{p} (a_i + b_j)} .$$

Proof. This is Cauchy's determinant; see, for example, Bellman (1960), Section 11.6, Exercise 1. A direct proof is as follows: To convert into 0 each element in the first column, except for that in the first row, we subtract from each row an appropriate multiple of its first row. The i,j-th element is thus converted into

$$\frac{1}{a_i+b_j} - \frac{1}{a_1+b_j}\frac{a_1+b_1}{a_i+b_1} = \frac{a_i-a_1}{a_i+b_1}\frac{b_j-b_1}{a_1+b_j}\frac{1}{a_i+b_j}\ , \qquad i,j = 2, \ldots, p\ . \tag{9}$$

The first factor on the right-hand side is common to the i-th row and the second to the j-th column. Hence,

$$D_p = \frac{1}{a_1+b_1}\left(\prod_{i=2}^{p}\frac{a_i-a_1}{a_i+b_1}\right)\left(\prod_{j=2}^{p}\frac{b_j-b_1}{a_1+b_j}\right)D_{p-1}\ , \tag{10}$$

and the result follows. Q.E.D.

Theorem. For $\beta_1, \ldots, \beta_p$ such that the roots of (2) are less than 1 in absolute value for $T \geq p$

$$|\underset{\sim}{\Sigma}_T| = (\sigma^2)^T \prod_{i,j=1}^{p}(1-w_i w_j)^{-1}\ . \tag{11}$$

Proof. We first consider the case where $w_1, \ldots, w_p$ are different and different from 0 . If $h_i = w_i^{-1}$, then in (5) $|\underset{\sim}{C}| \neq 0$. Anderson (1971) in (24) of Section 5.3 gives an expression for the elements of $\underset{\sim}{\Sigma}_p$ in terms of $w_1, \ldots, w_p$. (See also Problem 19 of Chapter 5.) Then

$$(12)\qquad |\underset{\sim}{Q}_p| = \frac{c_{11}^2 \cdots c_{1p}^2}{|\underset{\sim}{C}|^{2p-2}} |\underset{\sim}{V}| ,$$

where

$$(13)\qquad |\underset{\sim}{V}| = \left|\frac{1}{1 - w_i w_j}\right| = \left|\frac{w_i^{-1}}{w_i^{-1} + (-w_j)}\right| = \frac{1}{\prod_{i=1}^{p} w_i} \left|\frac{1}{w_i^{-1} + (-w_j)}\right|$$

$$= \frac{1}{\prod_{i=1}^{p} w_i} \frac{\prod_{i<j} (w_j^{-1} - w_i^{-1})(w_i - w_j)}{\prod_{i,j=1}^{p} (w_i^{-1} - w_j)}$$

$$= \left(\prod_{i=1}^{p} w_i\right)^{p-1} \frac{\prod_{i<j} (w_i - w_j)^2 w_i^{-1} w_j^{-1}}{\prod_{i,j=1}^{p} (1 - w_i w_j)} = \frac{\prod_{i<j} (w_i - w_j)^2}{\prod_{i,j=1}^{p} (1 - w_i w_j)} .$$

Further,

$$(14)\qquad \frac{c_{11}^2 \cdots c_{1p}^2}{|\underset{\sim}{C}|^{2p-2}} = \left[\frac{\left(\prod_{i=1}^{p} h_i^{p-1}\right) \prod_{i<j} (h_j - h_i)^{p-2}}{\prod_{i<j} (h_j - h_i)^{p-1}}\right]^2$$

$$= \frac{\prod_{i=1}^{p} h_i^{2p-2}}{\prod_{i<j} (h_j - h_i)^2} = \frac{1}{\prod_{i<j} (w_i - w_j)^2} ,$$

and (11) follows for the roots different and nonzero. The determinant $|\underset{\sim}{Q}_p^{-1}|$ is the polynomial $\Pi_{i,j=1}^{p}(1 - w_i w_j)$, which holds for all w_j such that $|w_j| < 1$, $j = 1, \ldots, p$. Q.E.D.

Discussion

1. If the process is Gaussian, the normalizing constant in the normal density of $\underset{\sim}{y}_T$ is $(2\pi)^{-T/2}$ times

$$(15) \qquad |\underset{\sim}{\Sigma}_T|^{-\frac{1}{2}} = (\sigma^2)^{-T/2} \prod_{i,j=1}^{p} (1 - w_i w_j)^{\frac{1}{2}} .$$

2. If one or more of the roots approaches 1 in absolute value, $|\underset{\sim}{\Sigma}_T^{-1}| \to 0$ and $|\underset{\sim}{\Sigma}_T| \to \infty$. These facts agree with the nonexistence of a nontrivial stationary process satisfying (1) if one or more roots are equal to 1 in absolute value.

3. Grenander and Szegö (1958) in effect showed that $\lim_{T\to\infty} |\underset{\sim}{Q}_T| = \Pi_{i,j=1}^{p} (1 - w_i w_j)^{-1}$ by use of an integral of $|b(w)|^{-2}$, though they did not relate this result to the generalized variance of the autoregressive process. Walker (1961) noted that $|\underset{\sim}{Q}_T| = |\underset{\sim}{Q}_p| = 1/|\underset{\sim}{Q}_p^{-1}|$ for $T \geq p$. An alternate proof of the theorem can be assembled from the results of Grenander and Szegö and Walker. (See also Finch (1960).)

4. A moving average model of order q is defined by

$$(16) \qquad x_t = v_t + \alpha_1 v_{t-1} + \ldots + \alpha_q v_{t-q} ,$$

where the v_t are independent random variables with $\mathcal{E}v_t = 0$, $\mathcal{E}v_t^2 = \tau^2$, $0 < \tau^2 < \infty$. The associated polynomial equation

$$z^q + \alpha_1 z^{q-1} + \ldots + \alpha_q = 0 \tag{17}$$

has roots $z_1, \ldots, z_q$. Durbin (1959) conjectured that if $\tau^2 \underset{\sim}{N}_T$ is the covariance matrix of $x_1, \ldots, x_T$ generated by (16), and if all roots of (2) are less than 1 in absolute value, then

$$\lim_{T\to\infty} |\underset{\sim}{N}_T| = |\underset{\sim}{Q}_n| , \tag{18}$$

for some n sufficiently large compared with $p=q$ when $\alpha_j = \beta_j$, $j = 1, \ldots, p$. Finch (1960) showed that

$$\lim_{T\to\infty} |\underset{\sim}{N}_T| = \lim_{T\to\infty} |\underset{\sim}{Q}_T| \tag{19}$$

by use of some results of Grenander and Szegö (1958) and gave explicitly the limiting value of the generalized variance for an autoregressive moving average process. Walker (1961) used more algebraic methods to show (18) for $n = p = q$. As an example, these results for $p = q = 1$ are $Q_1 = 1/(1-\beta_1^2)$ and $|\underset{\sim}{N}_T| = (1-\alpha_1^{2T+2})/(1-\alpha_1^2) \to 1/(1-\alpha_1^2)$ as $T \to \infty$. Durbin (1959) considered the case $p = q = 2$ in detail.

For further discussion, see the recent paper by Shaman (1976).

REFERENCES

Anderson, T. W. (1971), The Statistical Analysis of Time Series, John Wiley and Sons, Inc., New York.

Bellman, R. (1960), Introduction to Matrix Analysis, McGraw-Hill Book Co., Inc., New York.

Durbin, J. (1959), "Efficient estimation of parameters in moving-average models", Biometrika, **46**, 306-316.

Finch, P. D. (1960), "On the covariance determinants of moving-average and autoregressive models", Biometrika, **47**, 194-196.

Grenander, Ulf, and Szegö, Gabor (1958), Toeplitz Forms and Their Applications, University of California Press, Berkeley and Los Angeles.

Hamming, R. W. (1962), Numerical Methods for Scientists and Engineers, McGraw-Hill Book Co., Inc., New York.

Shaman, P. (1976), "Approximations for stationary covariance matrices and their inverses with application to ARMA models", The Annals of Statistics, **4**, No. 2, 292-301.

Siddiqui, M. M. (1958), "On the inversion of the sample covariance matrix in a stationary autoregressive process", The Annals of Mathematical Statistics, **29**, 585-588.

Walker, A. M. (1961), "On Durbin's formula for the limiting generalized variance of a sample of consecutive observations from a moving-average process", Biometrika, **48**, 197-199.

TECHNICAL REPORTS

OFFICE OF NAVAL RESEARCH CONTRACT N00014-67-A-0112-0030 (NR-042-034)

1. "Confidence Limits for the Expected Value of an Arbitrary Bounded Random Variable with a Continuous Distribution Function," T. W. Anderson, October 1, 1969.

2. "Efficient Estimation of Regression Coefficients in Time Series," T. W. Anderson, October 1, 1970.

3. "Determining the Appropriate Sample Size for Confidence Limits for a Proportion," T. W. Anderson and H. Burstein, October 15, 1970.

4. "Some General Results on Time-Ordered Classification," D. V. Hinkley, July 30, 1971.

5. "Tests for Randomness of Directions against Equatorial and Bimodal Alternatives," T. W. Anderson and M. A. Stephens, August 30, 1971.

6. "Estimation of Covariance Matrices with Linear Structure and Moving Average Processes of Finite Order," T. W. Anderson, October 29, 1971.

7. "The Stationarity of an Estimated Autoregressive Process," T. W. Anderson, November 15, 1971.

8. "On the Inverse of Some Covariance Matrices of Toeplitz Type," Raul Pedro Mentz, July 12, 1972.

9. "An Asymptotic Expansion of the Distribution of "Studentized" Classification Statistics," T. W. Anderson, September 10, 1972.

10. "Asymptotic Evaluation of the Probabilities of Misclassification by Linear Discriminant Functions," T. W. Anderson, September 28, 1972.

11. "Population Mixing Models and Clustering Algorithms," Stanley L. Sclove, February 1, 1973.

12. "Asymptotic Properties and Computation of Maximum Likelihood Estimates in the Mixed Model of the Analysis of Variance," John James Miller, November 21, 1973.

13. "Maximum Likelihood Estimation in the Birth-and-Death Process," Niels Keiding, November 28, 1973.

14. "Random Orthogonal Set Functions and Stochastic Models for the Gravity Potential of the Earth," Steffen L. Lauritzen, December 27, 1973.

15. "Maximum Likelihood Estimation of Parameters of an Autoregressive Process with Moving Average Residuals and Other Covariance Matrices with Linear Structure," T. W. Anderson, December, 1973.

16. "Note on a Case-Study in Box-Jenkins Seasonal Forecasting of Time Series," Steffen L. Lauritzen, April, 1974.

17. "General Exponential Models for Discrete Observations," Steffen L. Lauritzen, May, 1974.

18. "On the Interrelationships among Sufficiency, Total Sufficiency and Some Related Concepts," Steffen L. Lauritzen, June, 1974.

19. "Statistical Inference for Multiply Truncated Power Series Distributions," T. Cacoullos, September 30, 1974.

Office of Naval Research Contract N00014-75-C-0442 (NR-042-034)

20. "Estimation by Maximum Likelihood in Autoregressive Moving Average Models in the Time and Frequency Domains," T. W. Anderson, June 1975.

21. "Asymptotic Properties of Some Estimators in Moving Average Models," Raul Pedro Mentz, September 8, 1975.

22. "On a Spectral Estimate Obtained by an Autoregressive Model Fitting," Mituaki Huzii, February 1976.

23. "Estimating Means when Some Observations are Classified by Linear Discriminant Function," Chien-Pai Han, April 1976.

24. "Panels and Time Series Analysis: Markov Chains and Autoregressive Processes," T. W. Anderson, July 1976.

25. "Repeated Measurements on Autoregressive Processes," T. W. Anderson, September 1976.

26. "The Recurrence Classification of Risk and Storage Processes," J. Michael Harrison and Sidney I. Resnick, September 1976.

27. "The Generalized Variance of a Stationary Autoregressive Process," T. W. Anderson and Raul P.Mentz, October 1976.

Unlimited

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 27	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE GENERALIZED VARIANCE OF A STATIONARY AUTOREGRESSIVE PROCESS		5. TYPE OF REPORT & PERIOD COVERED Technical Report 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Theodore W. Anderson and Raúl Pedro Mentz		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0442
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, Calif. 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (NR-042-034)
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Statistics & Probability Program Code 436 Arlington, Virginia 22217		12. REPORT DATE October 1976 13. NUMBER OF PAGES 8
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Generalized variance, autoregressive process, covariance matrix.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For a stationary autoregressive process of order p and disturbance variance σ^2 it is shown that the determinant of the covariance of $T(\geq p)$ consecutive random variables of the process is $(\sigma^2)^T \Pi_{i,j=1}^{p}(1 - w_i w_j)$, where $w_1,\ldots,w_p$ are the roots of the associated polynomial equation.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)