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ELASTO-PLASTIC BEHAVIOR OF PLATES AND SHELLS

Weidlinger Associates
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For an isotropic solid shell, a yield condition, a strain-hardening rule, and a flow rule have been formulated in terms of the membrane forces and bending moments and the strains and curvatures of the middle surface. The results of the proposed theory are compared to the results of the through-the-thickness integration of the stress-strain relations of the material.

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PREFACE

The help of Professor G.A. Wempner in this work is most gratefully acknowledged. Professor F.L. DiMaggio, Dr. I.S. Sandler and Dr. D. Rubin offered numerous helpful suggestions.

TABLE OF CONTENTS

LIST OF SYMBOLS

a, b, c		parameters in the equation of the limit yield surface Eq. (13)
h	=	shell thickness (solid shell, Fig 2)
d, t	=	dimensions of a sandwich shell (Fig. 2)
i, j	=	indices, $i = 1, 2$; $j = 1, 2$
e_{ij}	=	strains components of the middle surface
k_{ij}	=	curvature components of the middle surface
e_0, k_0	=	see Fig. 2
$\underset{\sim}{e}$	=	column matrix of shell strains (Eq. (22))
$\underset{\sim}{e}''$	=	plastic part of $\underset{\sim}{e}$
$\underset{\sim}{s}$	=	column matrix of stress resultants, (Eq. (21))
t	=	time
A	=	expression defined by Eq. (31)
$\underset{\sim}{D}$	=	elasto-plastic tangent stiffness matrix
$\underset{\sim}{E}$	=	elastic moduli matrix
F, F_0, F_L	=	subsequent, initial, and limit yield functions
F_s, F_M	=	absolute values of the gradients of the yield function F (see Eqs. (18) and (19))
I_N, I_M, I_{NM}	=	resultant stress invariants (see Eqs. (6), (7), (8))
$N_{ij}, N_{11}, N_{22}, N_{12}$	=	membrane forces in shell
$M_{ij}, M_{11}, M_{22}, M_{12}$	=	moments in shell
M_{ij}^*	=	strain hardening parameters
$M_0 = \sigma_0 h^2/6$, $M_L = \sigma_0 h^2/4$	=	for solid shells
$M_0 = M_L = \sigma_0 td$	=	for ideal sandwich shells

LIST OF SYMBOLS (Continued)

α = a parameter in the yield condition

β = a parameter in the hardening rule

λ = a parameter in the flow rule

ν = Poisson ratio

σ_{ij} = stress components

σ_0 = yield stress in uniaxial tension

ε_{ij} = strain components

I INTRODUCTION

This report discusses the elasto-plastic behavior of thin plates and shells whose material is an elasto-plastic solid, linear in the elastic range, obeying Mises' yield condition, and deforming plastically according to the Prandtl-Reuss flow rule. The mechanical properties of this solid are characterized by two elastic constants and the yield stress in uniaxial tension, σ_0.

The plate or shell is assumed to be a solid layer, with thickness h; occasionally, reference will be made to an ideal sandwich shell (Fig. 1) such that d >> t. The basis assumption of the plate and shell theory used in this paper are summarized in the expressions for the strain components, at any point of the shell in terms of the strain and curvature components of the middle surface

$$\varepsilon_{ij}(z) = e_{ij} + k_{ij} z \tag{1}$$

and in the expressions for the membrane forces and the bending moments (stress resultants)

$$N_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} \, dz \quad , \quad M_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} \, z \, dz \tag{2}$$

It becomes evident from the above equations that, within the accepted approximation, there is no distinction between a plate and a shell, as far as the stress-strain relations are

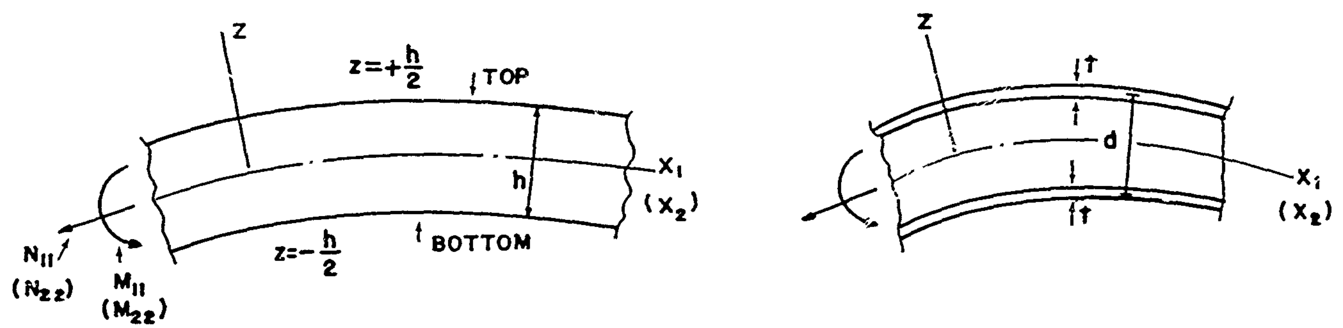

Fig. 1 Solid and ideal sandwich shells, notations and sign conventions.

concerned, and the term "shell" will be used to describe the structure under consideration.

The strain - normal force and the curvature - moment relations under the conditions of uniaxial stress (i.e. beam behavior) are illustrated in Fig. 2. The specific objective of this work is the development of the relations between the stress resultants, N_{ij} and M_{ij}, and the strain components, e_{ij} and k_{ij}, for a general type of loading.

The yield conditions for the simplified behavior in bending have been proposed in several previous investigations; a critical evaluation and comparison of the existing yield surfaces can be found in a paper by M. Robinson (Ref. [1]). In order to take into account the actual moment - curvature relation, M.A. Crisfield (Ref. [2]), applies the ideas of isotropic strain hardening of the theory of plasticity to the elasto-plastic behavior of shells. The approach reported in this paper is a continuation and extension of the above earlier works.

The fact that the stress resultants N_{ij} and M_{ij} of the classical shell theory are not sufficient to describe the state of stress has been recognized by G. Wempner. In Ref. [3], he introduced certain higher - order moments which, together with the classical stress resultants, form the dynamic variables of the problem. In Wempner's most recent work, described in a private communication to this author, the stress distribution through the thickness of the shell is expanded in terms of

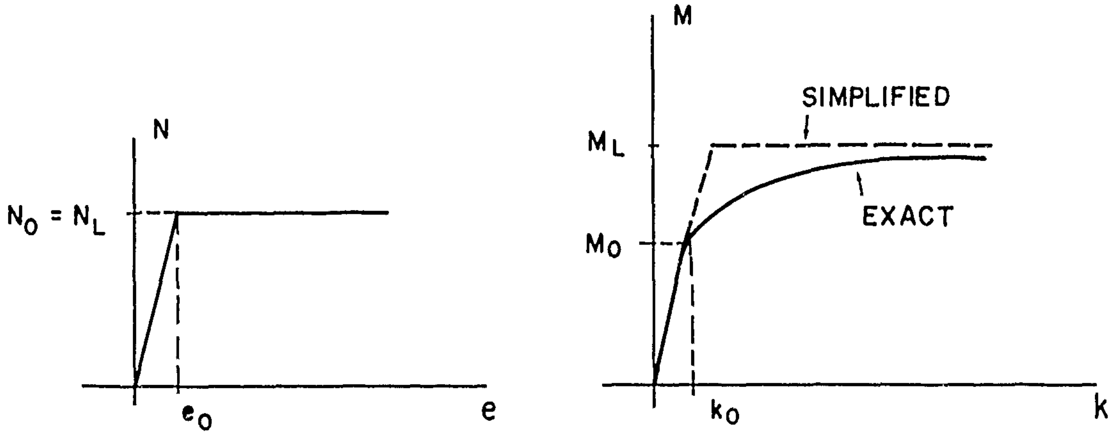

Fig. 2 Normal force vs. strain and moment vs. curvature in uniaxial stress, i.e. beam behavior.

Legendre polynomials, with the coefficients of this expansion being the additional dynamical variables.

It is, of course, possible (and it has been done in many investigations of elasto-plastic shells) to avoid completely the problem of the shell constitutive equations. In the "through-the-thickness-integration" approach, for given increments of e_{ij} and k_{ij}, the increments of strains $\varepsilon_{ij}(z)$ are determined with Eqs. (1); then, from appropriate constitutive equations of the material, the increments of stresses $\sigma_{ij}(z)$ are computed; finally, the increments of shell forces and moment, are determined by numerical evaluation of the integrals in Eqs. (2). Although workable and accurate, the procedure requires sometimes prohibitively large storage capacity of the computer.

II INITIAL AND LIMIT YIELD SURFACES

The initial yield condition, or the initial yield surface in the stress space, can be easily derived from our basic assumptions. The stress components at the top and bottom surfaces of a solid shell are

$$\sigma_{ij} = \frac{N_{ij}}{h} \pm \frac{6M_{ij}}{h^2} \qquad (3)$$

where the plus sign applies to the top and the minus sign to the bottom of the shell. Substitution of expression (3) into Mises' yield condition

$$\frac{1}{\sigma_0^2}(\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\,\sigma_{22} + 3\,\sigma_{12}^2) = 1 \qquad (4)$$

results in

$$F_0 = I_N + I_M \pm 2\,I_{NM} = 1 \qquad (5)$$

where

$$I_N \equiv \frac{1}{N_0^2}(N_{11}^2 + N_{22}^2 - N_{11}\,N_{22} + 3\,N_{12}^2) \qquad (6)$$

$$I_M \equiv \frac{1}{M_0^2}(M_{11}^2 + M_{22}^2 - M_{11}\,M_{22} + 3\,M_{12}^2) \qquad (7)$$

$$I_{NM} \equiv \frac{1}{N_0 M_0}(N_{11}\,M_{11} + N_{22}M_{22} - \frac{1}{2}\,N_{11}M_{22} - \frac{1}{2}\,N_{22}M_{11} + 3\,N_{12}M_{12}) \qquad (8)$$

and

$$N_0 = \sigma_0 h \quad , \quad M_0 = \sigma_0 h^2/6 \qquad (9)$$

When the expression (5) is used, the top or the bottom of the shell should be considered in order to have the larger, i.e., positive, value for $\pm\,2\,I_{NM}$. This is assured by writing the initial yield condition of a solid shell as

$$F_0 \equiv I_N + I_M + 2 \left| I_{NM} \right| = 1 \tag{10}$$

In the case of an ideal sandwich shell the stresses are computed from

$$\sigma_{ij} = \frac{N_{ij}}{2t} \pm \frac{M_{ij}}{dt}$$

and an identical argument leads to the condition (10) except that N_0 and M_0 are now

$$N_0 = \sigma_0 \, 2t \quad , \quad M_0 = \sigma_0 \, td \tag{11}$$

The physical meaning of the quantities N_0 and M_0 is as in Fig. 2. The corresponding strains are e_0 and k_0, respectively. For a solid shell

$$e_0 = \frac{\sigma_0}{E} \quad , \quad k_0 = \frac{\sigma_0 2}{Eh} \tag{12}$$

No direct derivation, of the type used above, is possible for the limit surface. Instead, approaches which are essentially surface-fitting procedures have proven to be useful. Suppose that the limit surface is represented by a linear equation in I_N, I_M, and I_{NM}

$$F_L \equiv a \, I_N + b \, I_M + c \, I_{NM} = 1 \tag{13}$$

In principle, the parameters a, b, and c should be determined in such a way as to minimize the difference between the surface (13) and the yield surface determined by some more precise calculations. For practical purposes, however, the following argument results in sufficiently accurate values of a, b, and c. By considering the case of membrane forces only, we find that with a = 1 Eq. (13) becomes the exact limit condition. Similarly with $b = M_0^2/M_L^2$, Eq. (13) will produce the exact results for the cases of bending moments only. It is now necessary to consider one more loading case, with known exact solution, in order to find the value of c. Such a case may be taken as corresponding to the maximum value of I_{NM}. Upon reflection, it becomes evident that it is $N_{11} = N_{22}$, $M_{11} = M_{22}$, $N_{12} = 0$, and $M_{12} = 0$.

The stress distribution in a section is then as shown in Fig. 3, and the maximum of I_{NM} corresponds to $\eta = h/2\sqrt{3}$. A simple computation results in

$$I_N = \frac{1}{3}\ , \quad I_M = 4M_L^2/9M_0^2, \quad I_{NM} = 2\sqrt{3}\ \ M_L/9M_0$$

Equation (13) will be satisfied with the above values if $c = M_0/M_L\ \sqrt{3}$.

This form of F_L is identical with Iliushin's, (Ref. [4]) yield condition obtained from different considerations.

For an ideal sandwich shell, the functions F_0 and F_L coincide, since there is no distinction between initial yield and limit state.

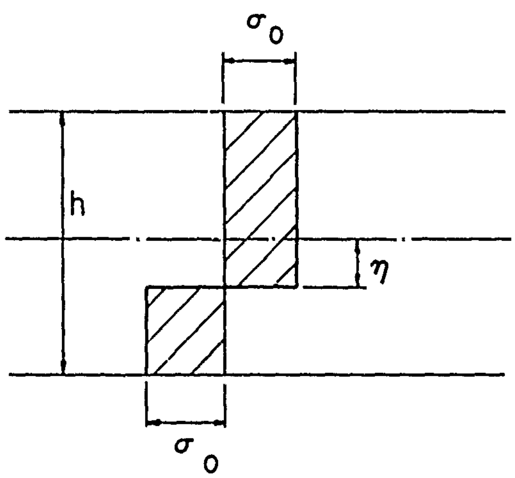

Fig. 3 Stress distribution corresponding to maximum I_{NM} (with $\eta = h/2\sqrt{3}$)

III PROPOSED YIELD CONDITION AND HARDENING RULE

The following yield condition is proposed to describe the "subsequent" yield surfaces as the loading path moves from the initial yield surface towards the limit surface

$$F \equiv I_N + I_M^* + \alpha \left| I_{NM} \right| = 1 \tag{14}$$

where

$$I_M^* = \frac{1}{M_0^2}[(M_{11} - M_{11}^*)^2 + (M_{22} - M_{22}^*)^2 - (M_{11} - M_{11}^*)(M_{22} - M_{22}^*) + 3(M_{12} - M_{12}^*)^2] \tag{15}$$

The quantities M_{ij}^*, which will be referred to as "hardening parameters", are defined by the following

If $F = 1$ and $\frac{\partial F}{\partial N_{ij}} \dot{N}_{ij} + \frac{\partial F}{\partial M_{ij}} \dot{M}_{ij} > 0$:

$$dM_{ij}^* = \beta(1 - F_L) \frac{M_0}{k_0} \frac{F_s^2}{F_M^2} dk_{ij}'' \tag{16}$$

If $F < 0$ or $\frac{\partial F}{\partial N_{ij}} \dot{N}_{ij} + \frac{\partial F}{\partial M_{ij}} \dot{M}_{ij} \leq 0$:

$$d M_{ij}^* = 0 \tag{17}$$

The symbols F_s and F_M are defined as

$$F_s = [(N_0 \frac{\partial F}{\partial N_{11}})^2 + (N_0 \frac{\partial F}{\partial N_{22}})^2 + (N_0 \frac{\partial F}{\partial N_{12}})^2 + (M_0 \frac{\partial F}{\partial M_{11}})^2 + (M_0 \frac{\partial F}{\partial M_{22}})^2 + (M_0 \frac{\partial F}{\partial M_{12}})^2]^{1/2} \tag{18}$$

$$F_M = [(M_0 \frac{\partial F}{\partial M_{11}})^2 + (M_0 \frac{\partial F}{\partial M_{22}})^2 + (M_0 \frac{\partial F}{\partial M_{12}})^2]^{1/2} \qquad (19)$$

F_s is evidently the absolute value of the vector grad F, in a dimensionless formulation; F_M is the part of grad F which corresponds to the bending moments only.

The function F_L which appears in Eq. (16) is defined by Eq. (13). With the values for a, b, and c as determined in the preceding section and with $M_L/M_0 = 3/2$ (for solid shells), it reads

$$F_L = I_N + \frac{4}{9} I_M + \frac{2}{3\sqrt{3}} |I_{NM}| \qquad (20)$$

The above formulation contains two parameters, α and β. The parameter α should be variable. When the loading path is still in the initial elastic range, its value should be $\alpha = 2$ to assure correct predictions of the instant of first yielding. As the loading path approaches the limit surface, the value of α should approach the value of c in Eq. (13). It appears however, that sufficiently close approximations to the exact results can be obtained with a constant value of α. Here, $\alpha = \frac{2}{3\sqrt{3}}$ has been used, i.e. the limit surface is reproduced correctly while an error is accepted in the initial yield surface.

The parameter β controls the moment - curvature relation in the plastic range. Again, a constant value of $\beta = 2$, has been found reasonably satisfactory for solid shells.

The hardening law represented by Eq. (16) is neither isotropic nor kinematic. Its choice is motivated solely by the fact that it reproduces fairly closely the actual behavior of a solid shell in the plastic range. It reproduces also the lowered yield point ("Bauschinger effect") which manifests itself if the bending moment is reversed, the shell unloaded and then loaded in opposite direction.

IV FLOW RULE AND TANGENT STIFFNESS

To complete the formulation of the behavior of elasto-plastic shells, it is necessary to state the elastic law and the flow rule. For this purpose, the stress resultants and the strain components of the shell will be represented by 6 x 1 column matrices

$$\underset{\sim}{s} = \{ N_{11}, N_{22}, N_{12}, M_{11}, M_{22}, M_{12} \} \tag{21}$$

$$\underset{\sim}{e} = \{ e_{11}, e_{22}, 2e_{12}, k_{11}, k_{22}, 2k_{12} \} \tag{22}$$

The following elastic law is assumed

$$\underset{\sim}{s} = \underset{\sim}{E} (\underset{\sim}{e} - \underset{\sim}{e}'') \tag{23}$$

where the elastic matrix $\underset{\sim}{E}$ is the usual shell stiffness matrix relating the membrane forces N_{ij} and the membrane strains e_{ij}, and the moments M_{ij} and the curvatures k_{ij} (its size: 6 x 6).

The associated flow rule is assumed for the plastic strain rates:

$$\underset{\sim}{e}'' = \lambda \frac{\partial F}{\partial \underset{\sim}{s}}$$

if

$$F = 1 \quad \text{and} \quad \left(\frac{\partial F}{\partial \underset{\sim}{s}}\right)^T \dot{\underset{\sim}{s}} > 0 \; ; \tag{24}$$

$$\underset{\sim}{e}'' = 0$$

if

$$F < 1 \quad \text{or} \quad \left(\frac{\partial F}{\partial \underset{\sim}{s}}\right)^T \dot{\underset{\sim}{s}} \leq 0. \tag{25}$$

The symbol $\partial F/\partial \underset{\sim}{s}$ stands for the column matrix

$$\frac{\partial F}{\partial \underset{\sim}{s}} \equiv \left\{ \frac{\partial F}{\partial N_{11}}, \frac{\partial F}{\partial N_{22}}, \frac{\partial F}{\partial N_{12}}, \frac{\partial F}{\partial M_{11}}, \frac{\partial F}{\partial M_{22}}, \frac{\partial F}{\partial M_{12}} \right\} \tag{26}$$

and superscript T indicates the transpose.

The parameter λ in Eq. (24) can be eliminated with the aid of the condition F = 1, or

$$F = \left(\frac{\partial F}{\partial \underset{\sim}{s}}\right)^T \dot{\underset{\sim}{s}} + \left(\frac{\partial F}{\partial \underset{\sim}{s}*}\right)^T \dot{\underset{\sim}{s}}* = 0 \tag{27}$$

where

$$\underset{\sim}{s}* \equiv \{\ 0,\ 0,\ 0,\ M_{11}^*,\ M_{22}^*,\ M_{12}^*\ \} \tag{28}$$

$$\frac{\partial F}{\partial \underset{\sim}{s}*} \equiv \{\ 0,\ 0,\ 0,\ \frac{\partial F}{\partial M_{11}^*},\ \frac{\partial F}{\partial M_{22}^*},\ \frac{\partial F}{\partial M_{12}^*}\ \} \tag{29}$$

both being column matrices. Following a, by now, routine procedure, one obtains

$$\lambda = \frac{\left(\frac{\partial F}{\partial \underset{\sim}{s}}\right)^T \underset{\sim}{E}\ \dot{\underset{\sim}{e}}}{\left(\frac{\partial F}{\partial \underset{\sim}{s}}\right)^T \underset{\sim}{E}\ \frac{\partial F}{\partial \underset{\sim}{s}} - \left(\frac{\partial F}{\partial \underset{\sim}{s}*}\right)^T A\ \frac{\partial F}{\partial \underset{\sim}{s}}} \tag{30}$$

where

$$A = \beta(1 - F_L)\ \frac{M_0}{k_0}\ \frac{F_s^{\,2}}{F_M^{\,2}} \tag{31}$$

Substitution of Eq. (24) into Eq. (23) yields

$$\dot{\underset{\sim}{s}} = \underset{\sim}{E}\left(\dot{\underset{\sim}{e}} - \lambda\ \frac{\partial F}{\partial \underset{\sim}{s}}\right) \tag{32}$$

or, with λ as in Eq. (30),

$$\dot{\underset{\sim}{s}} = \underset{\sim}{D}\ \dot{\underset{\sim}{e}}$$

where the elasto-plastic tangent stiffness $\underset{\sim}{D}$ is given by

$$\underset{\sim}{D} = \underset{\sim}{E}\left[1 - \frac{\left(\frac{\partial F}{\partial \underset{\sim}{s}}\right)^T \underset{\sim}{E}\ \frac{\partial F}{\partial \underset{\sim}{s}}}{\left(\frac{\partial F}{\partial \underset{\sim}{s}}\right)^T \underset{\sim}{E}\ \frac{\partial F}{\partial \underset{\sim}{s}} - \left(\frac{\partial F}{\partial \underset{\sim}{s}*}\right)^T A\ \frac{\partial F}{\partial \underset{\sim}{s}}}\right] \tag{34}$$

It should be noted that the stress-strain relation, Eq. (33) is of rate type. A proper numerical procedure should be used in evaluating Eq. (33) for finite increments of strain, $\Delta \underset{\sim}{e}$, and stress $\Delta \underset{\sim}{s}$. This requirement is, of course, in force for any flow type theory of plasticity.

V EXAMPLES

In order to test the present theory, the effect of some typical loading histories has been applied to the following shell: thickness = 1 in, Young modulus = 29×10^6 psi, Poisson ratio = 0.3, yield stress in uniaxial tension, $\sigma_0 = 30 \times 10^3$ psi.

Figure 4 shows moment - curvature relation for bending in one plane; the curvature k_{11} increases from zero to 0.4×10^{-2} then decreases to -0.4×10^{-2} and increases again to 0.4×10^{-2}. The results of the present theory are shown with continuous line. For the same strain history, the computations have been performed with the use of through-the-thickness integration (trapezoidal integration, 21 points through h); the corresponding results are shown with broken line in Fig. 4.

Figure 5 contains a similar case of bending in one plane, except that the maximum and minimum values of the curvature k_{11} are double of those in Fig. 4, i.e. $+ 0.8 \times 10^{-2}$ and $- 0.8 \times 10^{-2}$, respectively. It is evident that further increase of maximum and minimum of k_{11} would not bring any new aspects of the moment - curvature relation, since both curves approach the values $M_{11} = \pm M_L$ for larger $|k_{11}|$.

The interaction of membrane forces and bending moments is obviously important in various problems of shell analysis. Figures 6 through 9 show the effect of interaction between N_{11} and M_{11}. The loading histories in these figures are: first,

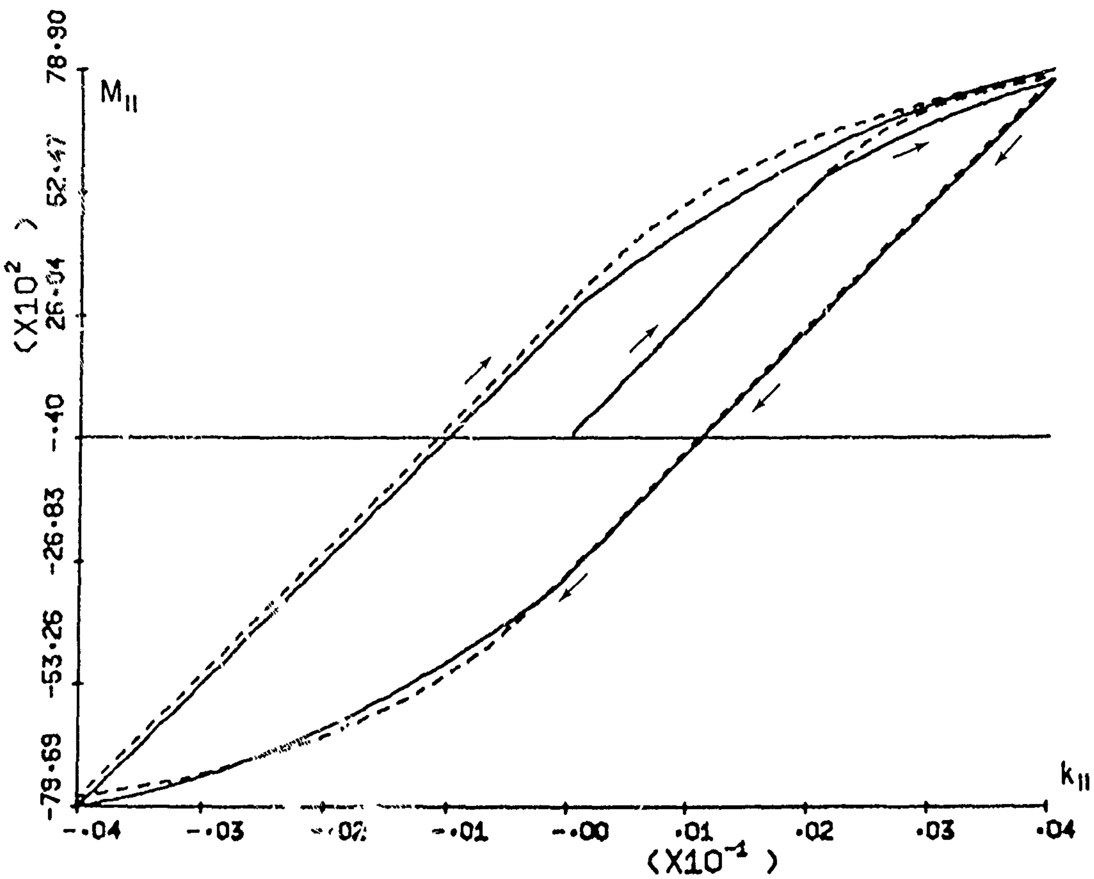

Fig. 4 Moment versus curvature for bending in one plane; $k_{11} \neq 0$, $k_{22} = k_{12} = 0$, $e_{11} = e_{22} = e_{12} = 0$; continuous line: present theory, broken line: through-the-thickness integration

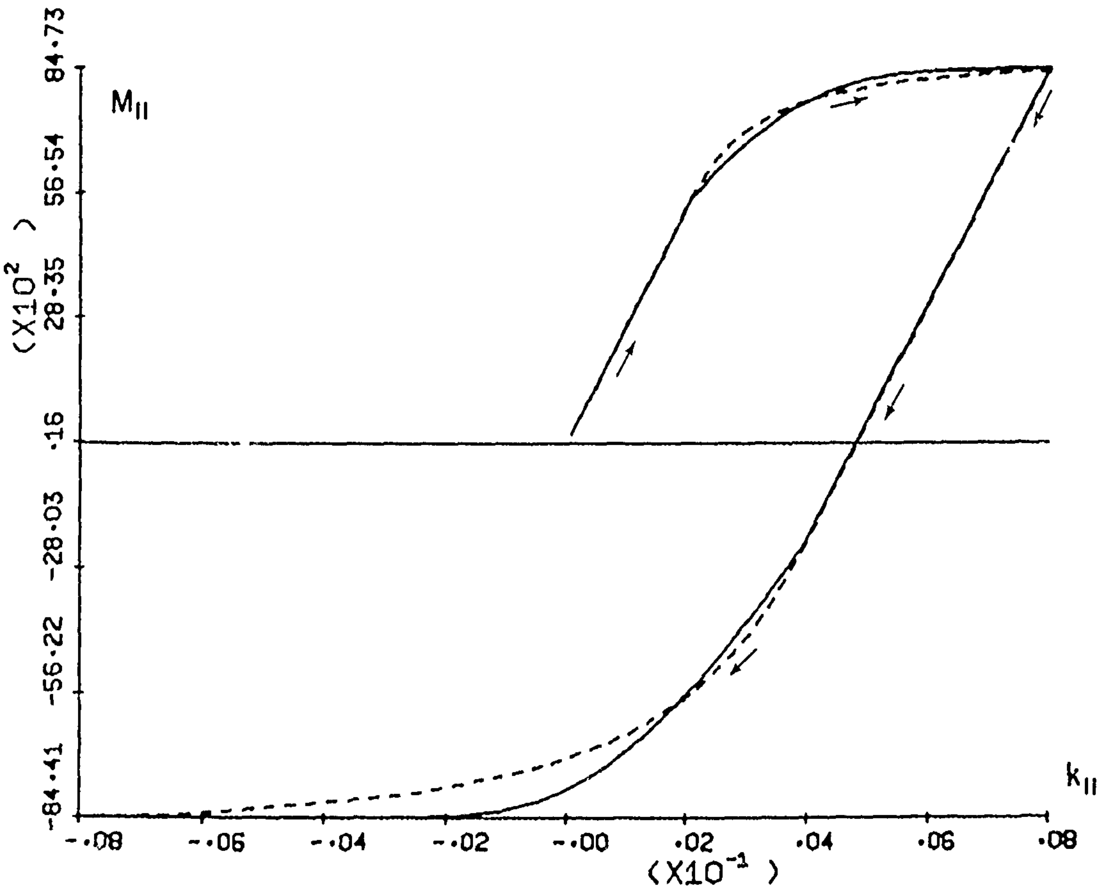

Fig. 5 Moment versus curvature for bending in one plane; $k_{11} \neq 0$, $k_{22} = k_{12} = 0$, $e_{11} = e_{22} = e_{12} = $ ^; continuous line: present theory, broken line; through-the-thickness integration.

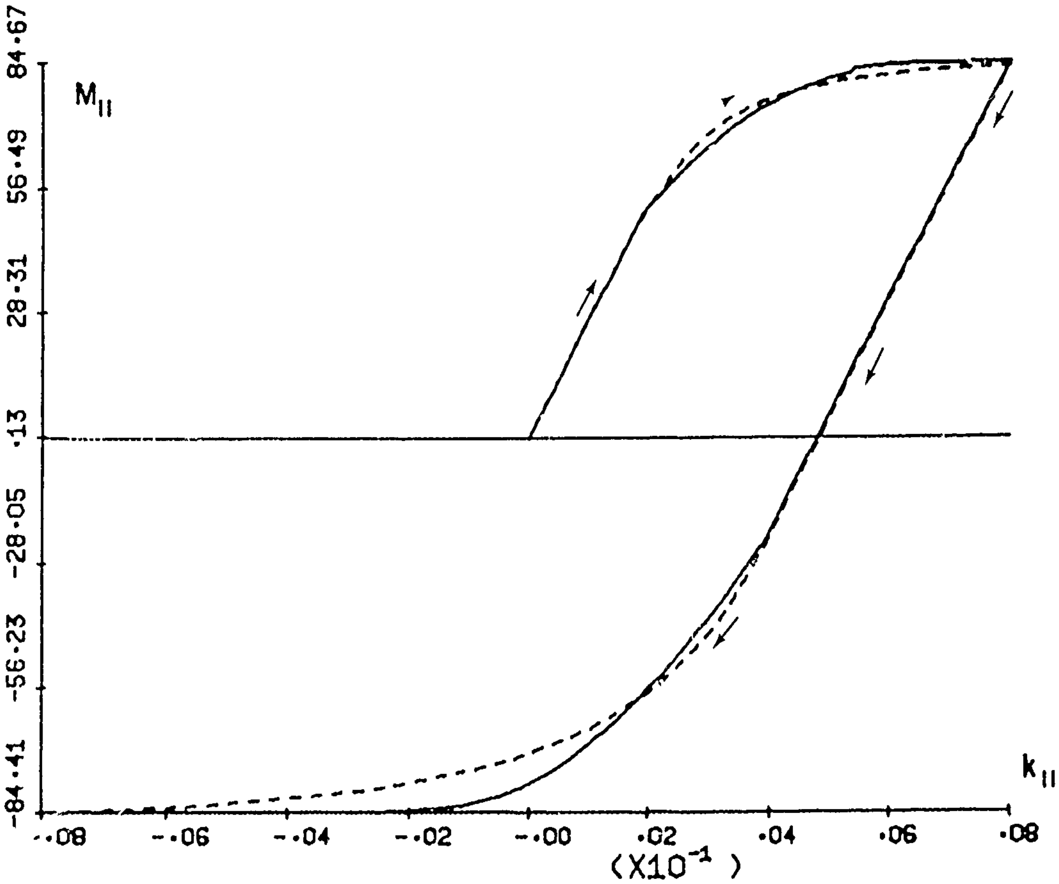

Fig. 6 Moment versus curvature for bending and extension in one plane; $k_{11} \neq 0$, $e_{11} = 0.25\ e_0$, $e_{22} = e_{12} = k_{22} = k_{12} = 0$.

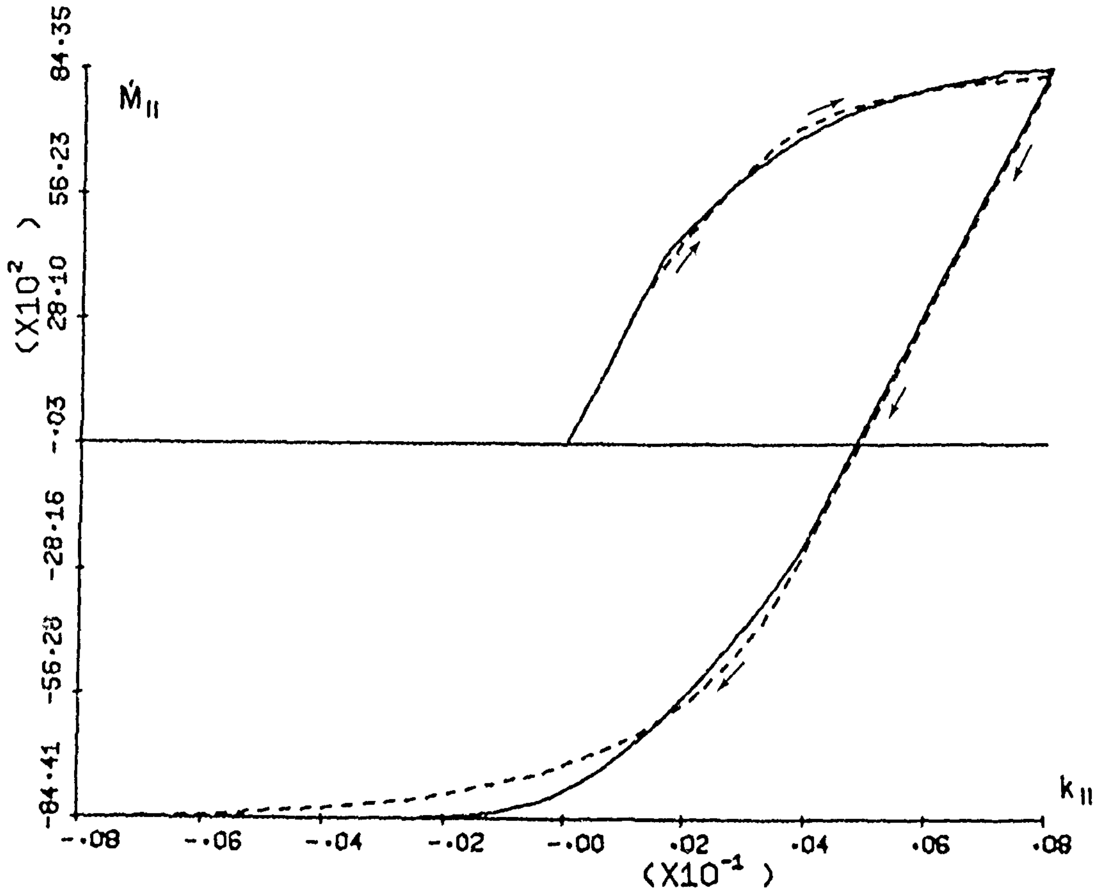

Fig. 7. Moment versus curvature for bending and extension in one plane; $k_{11} \neq 0$, $e_{11} = 0.50\ e_0$, $e_{22} = e_{12} = k_{22} = k_{12} = 0$.

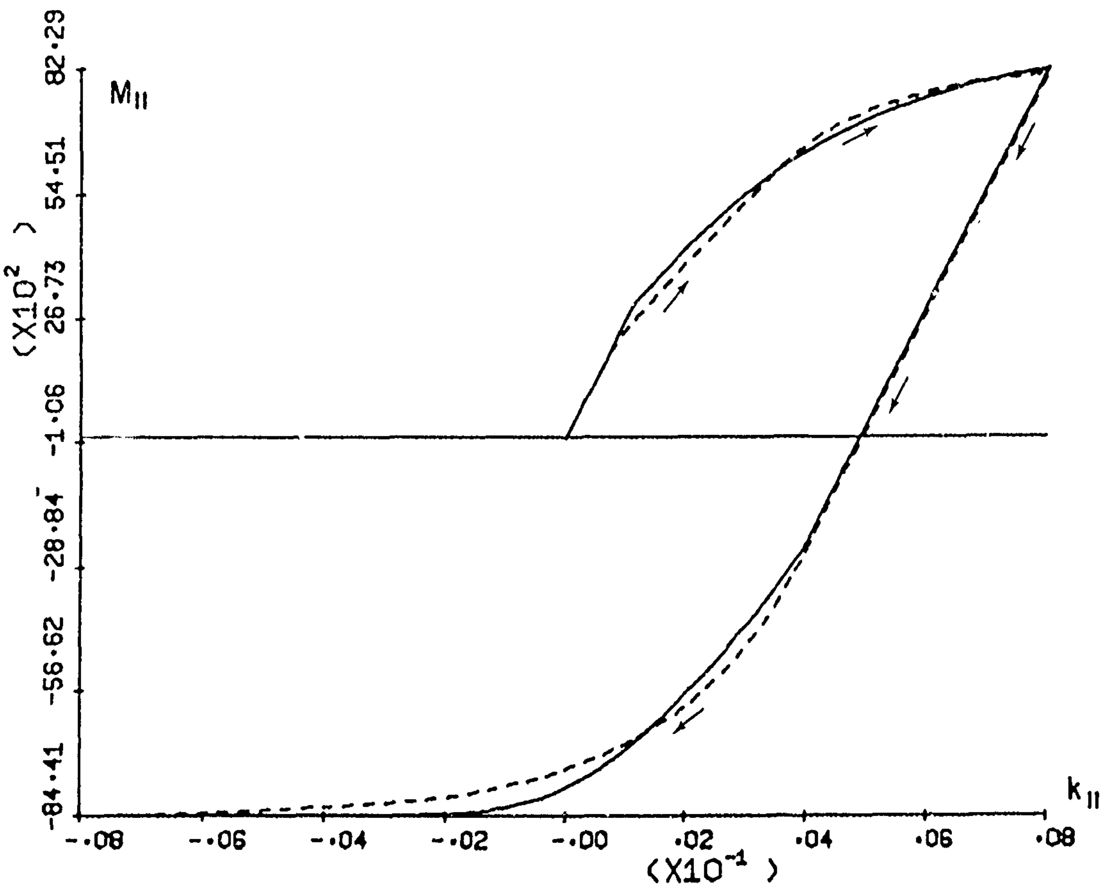

Fig. 8 Moment versus curvature for bending and extension in one plane; $k_{11} \neq 0$, $e_{11} = 0.75\ e_0$, $e_{22} = e_{12} = k_{22} = k_{12} = 0$.

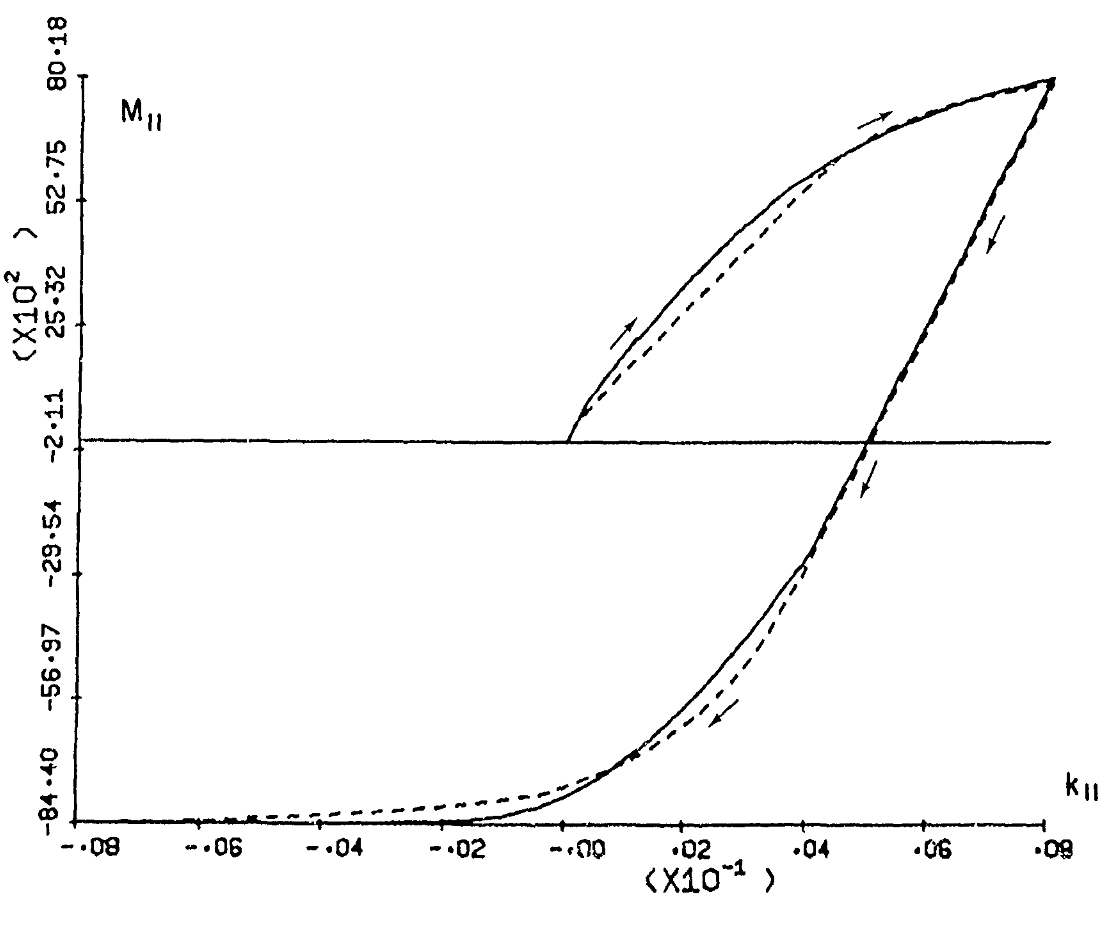

Fig. 9 Moment versus curvature for bending and extension in one plane; $k_{11} \neq 0$, $e_{11} = 1.00\ e_0$, $e_{22} = e_{12} = k_{22} = k_{12} = 0$.

the strain component e_{11} is increased to $e_{11} = 0.25\ e_0$ (Fig. 6), $e_{11} = 0.50\ e_0$ (Fig. 7), $e_{11} = 0.75\ e_0$ (Fig. 8) and $e_{11} = 1.00\ e_0$ (Fig. 9). Then, with e_{11} kept constant, the curvature k_{11} is varied through the cycle from 0 to $+ 0.8 \times 10^{-2}$ and to $- 0.8 \times 10^{-2}$. In Figs. 6 through 9, the results of the present theory are shown in continuous line, with the results of through-the-thickness integration in broken line.

VI CONCLUSIONS

A theory of elasto-plastic behavior of plates and shells has been presented in terms of the membrane forces and moments and the strains and curvatures of the middle surface. The structure of this theory is analogous to the classical theories of plasticity of solids. It consists of a yield condition, a strain hardening rule, and a flow rule. The concept of the yield surface, as known in the classical plasticity, exists here in the stress space of points (N_{11}, N_{22}, N_{12}, M_{11}, M_{22}, M_{12}, M_{21}).

An examination of the test cases presented in this paper indicates that the accuracy of the results of the present theory will probably be acceptable in a large number of engineering applications.

There exists always the possibility of further optimization of the accuracy of this theory. It can be achieved by adjusting the values of the parameters α and β, by introducing integer or fractional powers of the terms $(1 - F_L)$ and F_s/F_M in the strain hardening rule (Eq. 16) etc. Finally, some thought should be given to comparing the predictions of approximate computations not to some other theoretical results (even if they are "exact") but to realistic experimental data.

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