

TOTAL PROGENY IN A CRITICAL AGE-DEPENDENT
BRANCHING PROCESS WITH IMMIGRATION

ACCESSION for	
NTIS	White Section ☒
DCS	Staff Section ☐
UNANNOUNCED	☐
JUSTIFICATION	
BY	
DISTRIBUTION/AVAILABILITY CODES	
DISC	APRIL, 1966/1967 SPECIAL
A	

AD-A030 694

by

Howard Weiner

Technical Report No. 234

July 22, 1976

Prepared under Contract N00014-76-C-0475
(NR-042-267)

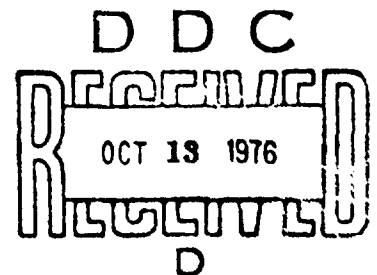
Office of Naval Research

Herbert Solomon, Project Director

Reproduction in Whole or in Part is Permitted for
any Purpose of the United States Government

Approved for public release; distribution unlimited

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA



Total progeny in a critical age-dependent
branching process with immigration

by Howard Weiner

1. Introduction.

At time $t = 0$, a renewal process starts with I.I.D. interarrival times having non-lattice distribution function $G_0(t)$, $G_0(0+) = 0$. These epochs are the arrival times of new-born immigrating cells, where k cells arrive with probability p_{ko} , and let, for $0 \leq s \leq 1$,

$$(1.1) \quad h_0(s) = \sum_{k=0}^{\infty} p_{ko} s^k,$$

and denote

$$(1.2) \quad 0 < \beta = h'_0(1) = \sum_{k=1}^{\infty} k p_{ko} < \infty.$$

Each immigrating cell, independent of any other cells, initiates an age-dependent branching process [3] with cell lifetime distribution $G(t)$, $G(0+) = 0$, and non-lattice. The offspring generating function is $h(s) = \sum_{k=0}^{\infty} p_k s^k$. Assume that each initiated branching process is critical, that is,

$$(1.3) \quad h'(1) = \sum_{k=1}^{\infty} k p_k = 1.$$

Denote by

- (1.4) $Z(t)$ = the total number of cells born by t arising from all cells immigrating by t and their respective initiated critical age-dependent branching processes.

The purpose of this paper is to obtain an explicit limit law for the Laplace transform of $Z(t)$ by using the corresponding result for the Galton-Watson or discrete time process obtained by Pakes [6] and a series of approximations.

2. Integral Equations and Approximations.

Assume that at $t = 0$, there are no cells present due to immigration, and hence no new cells to initiate a branching process.

Let

- (2.1) $N(t)$ = total number of cells born by t in a critical age-dependent branching process as given by (1.3), (1.4).

Define

$$(2.2) \quad \Phi(0, t) = E \exp(-\alpha N(t)).$$

Then

$$(2.3) \quad \Phi(0, t) = e^{-\alpha t} \left\{ 1 - G(t) + \int_0^t h(\Phi(0, t-u)) dG(u) \right\}.$$

Let

$$(2.4) \quad F(\theta, t) = E \exp(-\theta Z(t)),$$

where $Z(t)$ is as in (1.4).

Then for $t > 0$, arguing as in [4],

$$(2.5) \quad F(\theta, t) = 1 - G_0(t) + \int_0^t h_0(\bar{\varphi}(\theta, t-u)) F(\theta, t-u) dG_0(u)$$

and $F(\theta, 0) = 1.$

Let

$$(2.6) \quad \bar{\varphi}_{n+1}(\theta, t) = e^{-\theta} [1 - G(t) + \int_0^t h(\bar{\varphi}_n(\theta, t-u)) dG(u)], \quad n \geq 1,$$

and $\bar{\varphi}_0(\theta, t) \equiv 1.$

Let

$$(2.7) \quad \bar{\varphi}(\theta, n+1) = e^{-\theta} h(\bar{\varphi}(\theta, n)) \quad \text{for } n \geq 1$$

and $\bar{\varphi}(\theta, 0) \equiv 1.$

Let $a > 0$ be a constant. Define the approximants, for $n \geq 1$,

$$(2.8) \quad \gamma_{(n+1)}(\theta, t) = 1 - G_0(t) + \int_0^t h_0(\tilde{\gamma}_{[na]}(\theta, t-u)) F_n(\theta, t-u) dG_0(u)$$

and $F_0(\theta, t) \equiv 1$.

Similarly, let

$$(2.9) \quad P(\theta, (n+1)) = h_0(\tilde{P}(\theta, [na])) P(\theta, n) \quad \text{for } n \geq 1,$$

and $P(\theta, 0) \equiv 1$,

which implies

$$P(\theta, n+1) = \prod_{m=1}^n h_0(\tilde{P}(\theta, [ma])).$$

We note the relationships ([6], lemma 3 and also p. 285)

$$(2.10) \quad \tilde{\gamma}(\theta, t) \downarrow \tilde{\gamma}(\theta) \quad \text{as } t \rightarrow \infty \quad (\text{see also [5], p. 228})$$

$$(2.11) \quad \tilde{P}(\theta, n) \downarrow \tilde{P}(\theta) \quad \text{as } n \rightarrow \infty,$$

where $\tilde{\gamma}(\theta)$ is the transform of a bona fide random variable.

$$(2.12) \quad n(1 - \tilde{\gamma}(\theta/n^2)) < K < \infty, \quad \text{all } n,$$

where K is a constant.

3. Limit Theorems.

Theorem 1. Assume that

$$(3.1) \quad 0 < h(0) < 1, h_0(0) < 1,$$

and that

$$h'(1) = 1, 0 < h''(1) < \infty, h'_0(1) = \beta.$$

Denote $\gamma = \frac{1}{2}h''(1)$, and $\sigma \equiv \beta/\gamma$.

Then for all $\theta > 0, a > 0$,

$$(3.2) \quad \lim_{n \rightarrow \infty} P\left(\frac{\theta}{2}, n\right) = (\operatorname{sech} a\sqrt{\gamma\theta})^{\sigma/a}.$$

Proof. We will indicate the adaptation of the method of proof of Pakes [6] applied to this more general model, as Pakes' proof applies only to the case where the immigration mean interarrival time = mean lifetime = 1.

The model here corresponds to the case where the immigration interarrival time is $a > 0$, and the mean lifetime is 1.

Case I. $a \geq 1$.

Using Pakes' notation ([6] eq. 20, p. 285) for our case,

$$(3.3) \quad P(\theta, (n+1)) = \prod_{m=1}^n h_0(\bar{\varphi}(\theta, [ma])).$$

Since ([6] eq. 21, p. 285)

$$\bar{\varphi}(\theta, k) \downarrow \bar{\varphi}(\theta) \quad \text{as } k \rightarrow \infty,$$

we may write, via ([6] eq. 30, 31, p. 285), where $\theta_n = \theta/n^2$

$$(3.4) \quad \log P(\theta_n, (n+1)) = -\beta \sum_{m=1}^n (1 - \bar{\varphi}(\theta_n, [ma])) + R^{(n)}(\theta)$$

where $R^{(n)}(\theta) \rightarrow 0$ as $n \rightarrow \infty$ if $\sum_{m=1}^n (1 - \bar{\varphi}(\theta_n, [ma]))$ is bounded in n .

Writing

$$(3.5) \quad \begin{aligned} -\beta \sum_{m=1}^n (1 - \bar{\varphi}(\theta_n, [ma])) &= -\beta \sum_{m=1}^n (1 - \bar{\varphi}(\theta_n)) \\ &+ \beta \sum_{m=1}^n (\bar{\varphi}(\theta_n, [ma]) - \bar{\varphi}(\theta_n)), \end{aligned}$$

and by Pakes ([6] eq. 31, p. 287),

$$(3.6) \quad -\beta \sum_{m=1}^n (1 - \bar{\varphi}(\theta_n)) \sim -\beta(\theta/\gamma)^{1/2}.$$

Using the proof of Lemma 4, pp. 286-287 and p. 288 of [6], one obtains that, as $n \rightarrow \infty$

$$(3.7) \quad \beta \sum_{m=1}^n (\varphi(\theta_n, [ma]) - \varphi(\theta_n)) \sim \frac{\beta}{\gamma} \frac{\delta}{n} \sum_{m=1}^n \frac{e^{-\delta[ma]/n}}{1+e^{-\delta[ma]/n}} \rightarrow \frac{\beta}{\gamma} \int_0^1 \frac{e^{-\delta ax}}{1+e^{-\delta ax}} dx,$$

where, following [6], we have denoted

$$(3.8) \quad \delta \equiv 2 \sqrt{\gamma \theta}.$$

Since

$$(3.9) \quad \delta \int_0^1 \frac{e^{-\delta ax}}{1+e^{-\delta ax}} dx = -\frac{1}{a} \log \left(\frac{1+e^{-\delta a}}{2} \right),$$

eq. (3.4) - (3.9) yield the result for the case $a \geq 1$.

Case II. $0 < a < 1$.

Observe first that if $1/a$ is a positive integer, that

$$(3.10) \quad P(\theta, n+1) = \left[\prod_{k=1}^{[na]} h_0(\varphi(\theta, k)) \right]^{1/a}.$$

If $a = l/r$, where $0 < l < r$ are relatively prime positive integers, then it can be seen that

$$(3.11) \quad P(\theta, n+1) = \left(\prod_{k=1}^{[na]} h_0(\varphi(\theta, k)) \right)^{1/a}.$$

$$\prod_{k=1}^{[n/r]} h_0(\varphi(\theta, kl)).$$

Now let $n \rightarrow \infty$ in (3.10), (3.11), replacing θ by θ/n^2 . Apply Pakes' limit result ([6], p. 285) to the right side of (3.10), and to the first

product on the right of (3.11). Apply the result of Case I above to the second product on the right of (3.11). This proves Case II for $0 < a < 1$, a rational.

Suppose $0 < a < 1$ is arbitrary. Choose a sequence of rationals $\{a_r\}$ such that $a_r \rightarrow a$.

By the (uniform) continuity and monotonicity in a of the product for $P(\theta, n+1)$ given on the right side of (2.9), it follows that one can choose the convergent sequence of rationals $\{a_r\}$ such that for all $r > R_0$, $n > N_0$, either (by a slight expansion of notation)

$$(3.12) \quad P(\theta/n^2, n, a_r) \leq P(\theta/n^2, n, a) \leq P(\theta/n^2, n, a_{r+1})$$

or

$$P(\theta/n^2, n, a_{r+1}) \leq P(\theta/n^2, n, a) \leq P(\theta/n^2, n, a_r).$$

Now let $n \rightarrow \infty$, then $r \rightarrow \infty$ in (3.12), using the continuity of the function on the right side of (2.2). This suffices for the result.

Lemma. Let $G_0(t)$ and $G(t)$ be two distribution functions such that

$$G_0(0+) = G(0+) = 0$$

and

$$0 < m_1 = \int_0^\infty t dG(t) < \infty,$$

$$0 < m_2 = \int_0^\infty t dG_0(t) < \infty.$$

Let $a = m_2/m_1$.

Assume

$$0 < \int_0^{\infty} t^2 dG(t) < \infty$$

$$0 < \int_0^{\infty} t^2 dG_0(t) < \infty.$$

Let $G_{or}(t)$ and $G_r(t)$ denote the r -th convolutions of G_0 and G , respectively.

Let $G * H(t)$ denote the convolution of G and H . Then, uniformly in k ,
 $0 \leq k \leq n,$

$$(3.13) \quad \lim_{t \rightarrow \infty} G_{[ka]} * G_{0,n-k}(t) = 0$$

when $n = \left[\frac{t(1+\epsilon)}{m_2} \right]$ for all $\epsilon > 0,$

$$(3.14) \quad \lim_{t \rightarrow \infty} (1 - G_{[ka]} * G_{0,n-k}(t)) = 0$$

when $n = \left[\frac{t(1-\epsilon)}{m_2} \right]$ for all $\epsilon > 0.$

Proof. Let $\{X_\ell\}$ be I.I.D. each with distribution G .

Let $\{Y_\ell\}$ be I.I.D. each with distribution G_0 , and independent of $\{X_\ell\}$.

Define

$$(3.15) \quad S_{n,k} = \frac{1}{n} \left(\sum_{\ell=1}^{[ka]} (X_\ell - EX_\ell) + \sum_{\ell=1}^{n-k} (Y_\ell - EY_\ell) \right).$$

Then to prove (3.13), (3.14) together, it suffices to show that
 (see [2])

$$(3.16) \quad \lim_{n \rightarrow \infty} P(|S_{n,k}| > 3\epsilon m_2) = 0,$$

uniformly in k , $0 \leq k \leq n$.

By Chebyshev's inequality,

$$(3.17) \quad P(|S_{n,k}| > 3\epsilon m_2) \leq \frac{[ka] \text{Var } X_k + (n-k) \text{Var } Y_k}{(3\epsilon m_2 n)^2} \leq \frac{(na) \text{Var } X + n \text{Var } Y}{(3\epsilon m_2 n)^2},$$

which goes to zero independent of k , as $n \rightarrow \infty$.

Note: By using the techniques of [1], it may be possible to relax the second moment conditions and still obtain that $P(|S_{n,k}| > \epsilon)$ goes to zero uniformly in k , as $n \rightarrow \infty$, but this has not been done.

Theorem 2. Assume the conditions of Theorem 1 and the lemma. Then

$$\lim_{t \rightarrow \infty} F\left(\frac{\theta}{t}, t\right) = \left[\text{sech} \frac{\sqrt{y\theta}}{m_1} \right]^{\sigma m_1 / m_2}.$$

Proof. The proof is by use of approximants, writing

$$(3.18) \quad F(\theta, t) - P(\theta, (n+1)) \equiv F(\theta, t) - F_{(n+1)}(\theta, t) + F_{(n+1)}(\theta, t) - P(\theta, (n+1)).$$

$$(3.19) \quad F_{(n+1)}(\theta, t) - F(\theta, t) = \int_0^t [h_0(\phi_{[na]}(\theta, t-u)) F_n(\theta, t-u) \\ - h_0(\phi(\theta, t-u)) F_n(\theta, t-u) + h_0(\phi(\theta, t-u)) F_n(\theta, t-u) \\ - h_0(\phi(\theta, t-u)) F(\theta, t-u)] dG_0(u).$$

The proof will be divided into several claims, using an abbreviated notation. Only the proofs of Claims IV, VIII will be given, since the others either follow similarly, or are similar to those given in [2], [6].

$$(3.20) \quad \underline{\text{Claim I:}} \quad \Phi_n - \Phi \geq 0, \quad \text{all } t, \theta, n.$$

$$(3.21) \quad \underline{\text{Claim II:}} \quad F_{(n+1)} - F \geq 0, \quad \text{all } t, \theta, n.$$

$$(3.22) \quad \underline{\text{Claim III:}} \quad \Phi_{[na]}(\theta, t) - \Phi(\theta, t) \leq (1 - \Phi(\theta)) G_{[na]}(t).$$

$$(3.23) \quad \underline{\text{Claim IV:}} \quad F_{n+1} - F_n \leq m_0(1 - \Phi(\theta)) \cdot \sum_{k=1}^n G_{[ka]} * G_{o, n+1-k}(t) \\ + G_{o, n+1}(t).$$

Proof. By (3.19),

$$(3.24) \quad F_{n+1} - F = \int F_n (h_o(\Phi_{[na]}) - h_o(\Phi)) + \int h_o(\Phi)(F_n - F).$$

Applying the mean value theorem to the first term on the right of (3.24), bounding F_n by 1 in the integrand yields

$$(3.25) \quad F_{n+1} - F \leq \int (F_n - F) dG_o + m_0 C_{[na]} * G_o(t) \cdot (1 - \Phi(\theta)).$$

Iteration of (3.25) and using $F_1 - F = 1 - F \leq 1$ in the last step yields the result.

$$(3.26) \quad \underline{\text{Claim V:}} \quad \Phi_n(\theta, t) - \Phi(\theta, n) \geq 0, \quad \text{all } n, \theta, t.$$

$$(3.27) \quad \underline{\text{Claim VI:}} \quad F_n(\theta, t) - P(\theta, n) \geq 0, \quad \text{all } n, \theta, t.$$

(3.28) Claim VII: $\Phi_{n+1} - \Phi(n+1) \leq (1-G_n(t))(1-\Phi(\theta))$, all $n \geq 0$.

(3.29) Claim VIII:

$$\begin{aligned} F_{n+1}(\theta, t) - P(\theta, n+1) &\leq c(1-\Phi(\theta)) \cdot \sum_{k=1}^{n-1} (1-G_{[ka]}) * G_{0, n-k}(t) \\ &\quad + c(1-G_0) * G_{0, n-1}(t), \end{aligned}$$

where $c = \max(1, m_0)$ and $G_{-\alpha} \equiv 0$ for $\alpha > 0$.

Proof. By the expression

$$\begin{aligned} (3.30) \quad F_{n+1} - P(n+1) &= \int h_0(\Phi_{[na]}) \cdot (F_n - P(n)) dG_0 \\ &\quad + \int P(n) [h_0(\Phi_{[na]}) - h_0(\Phi([na]))] dG_0 \\ &\quad + (1-G_0(t))(1-P(n)), \end{aligned}$$

and the mean value theorem, (3.30) yields

$$(3.31) \quad F_{n+1} - P(n+1) \leq \int (F_n - P(n)) dG_0 + m_0 \int (\Phi_{[na]} - \Phi([na])) dG_0 + (1-G_0(t)).$$

By (3.28) and (3.31),

$$(3.32) \quad F_{n+1} - P(n+1) \leq \int (F_n - P(n)) dG_0 + m_0(1 - G_{[na]-1}) * G_0(t) + (1-G_0(t)).$$

Iteration and use of the facts that $F_1 - P(1) = 0$ and $F_2 - P(2) \leq 1 - G_0(t)$ yield the result.

We may now complete the proof of Theorem 2.

Claims IV, VIII yield the inequalities

$$\begin{aligned}
 (3.33) \quad -G_{0,n+1}(t) - m_0(1-\Phi(\theta,t)) \sum_{k=1}^n G_{[ka]} * G_{0,n+1-k}(t) \\
 \leq F(\theta,t) - P(\theta,n+1) \leq c(1-\Phi(\theta)) \sum_{k=1}^{n-1} (1-G_{[ka]}) * G_{0,n-k} \\
 + c(1-G_0) * G_{0,n-1}.
 \end{aligned}$$

Now, set $n = \left\lceil \frac{t(1+\epsilon)}{m_{G_0}} \right\rceil$ in the right (left) inequalities, respectively.

Substitute θ/n^2 for θ in (3.33). Then, since from (2.12),

$$1 - \Phi(\theta/n^2) < \frac{K}{n}$$

for some constant K , let $t \rightarrow \infty$. The lemma, via uniformity of approach of the summands to zero, yields that the right and left sides of (3.31) go to zero. Now Theorem 1 completes the proof.

REFERENCES

- [1] FRANCK, W. E. and D. L. HANSON (1966). Some results giving rates of convergence in the law of large numbers for weighted sums of independent random variables. Trans. Amer. Math. Soc. 124, 347-359.
- [2] GOLDSTEIN, M. I. (1971). Critical branching processes: single and multitype. Z. Wahr. 17, 74-96.
- [3] HARRIS, T. E. (1963). The Theory of Branching Processes. Prentice-Hall, New York.
- [4] JAGERS, P. (1968). Age-dependent branching processes allowing immigration. Theory of Prob. and Appl. 13, 225-236.
- [5] PAKES, A. G. (1972). Limit theorems for an age-dependent branching process with immigration. Math. Biosciences 14, 221-234.
- [6] PAKES, A. G. (1972). Further results on the critical Galton-Watson process with immigration. Jour. Austral. Math. Soc. 13, 277-290.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 234	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Total Progeny in a Critical Age-Dependent Branching Process with Immigration.	5. TYPE OF REPORT & PERIOD COVERED Technical Report	
7. AUTHOR(s) Howard Weiner	8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0475	
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, Calif. 94305	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (NR-042-267)	
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Statistics & Probability Program Code 436 Arlington, Virginia 22217	12. REPORT DATE 30 Jul 1976	
13. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	14. NUMBER OF PAGES 14	
	15. SECURITY CLASS. (of this report) Unclassified	
16. DECLASSIFICATION/DOWNGRADING SCHEDULE		
17. DISTRIBUTION STATEMENT (of this Report) Approved for Public Release; Distribution Unlimited TR-234		
18. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
19. SUPPLEMENTARY NOTES		
20. KEY WORDS (Continue on reverse side if necessary and identify by block number) branching processes, immigration, stochastic processes		
21. ABSTRACT (Continue on reverse side if necessary and identify by block number) (see reverse side)		

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 68 IS OBSOLETE
S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

332 580

4B