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SOME ASYMPTOTIC RESULTS FOR OCCUPANCY PROBLEMS

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SOME ASYMPTOTIC RESULTS FOR OCCUPANCY PROBLEMS

Lars Holst

1. Introduction.

Suppose that balls are thrown independently of each other into N cells, so that each ball has the probability p_k of falling into the kth cell, $p_1 + \ldots + p_N = 1$. Let Y_n denote the number of empty cells after n throws and let T_b denote the throw on which for the first time exactly b cells remain empty, $0 \leq b < N$. The symmetrical case $p_1 = \ldots = p_N = 1/N$ is discussed in e.g. Feller (1968), see occupancy or waiting time problems.

Depending on how $b, n, N \to \infty$, different asymptotic distributions for Y_n and T_b can be obtained, see e.g. Holst (1971) and for the symmetric case see e.g. Samuel-Cahn (1974). In this paper some remaining problems are investigated for the nonsymmetrical case.

To give precise meanings of the limits obtained, double sequences e.g. $(p_{kN})_N, (Y_{nN})_N$ are considered. But in order to simplify the notation the extra index N will usually be omitted.

2. A bounded number of empty cells.

The following limit theorem for Y_n, the number of empty cells after n throws, was proved by Sevastyanov (1972).

Theorem 1. If the p's are such that

$$\max_{1 \leq k \leq N} (1 - p_k)^n \to 0 \tag{2.1}$$

and

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(2.2) $$E(Y_n) = \sum_{k=1}^{N} (1 - p_k)^n \to m < \infty ,$$

then

(2.3) $$P(Y_n = y) \to m^y \cdot e^{-m}/y! ,$$

or equivalently

(2.4) $$Y_n \Rightarrow Po(m) , \text{ when } N \to \infty .$$

Remark. When the p's are equal an expression for $P(Y_n = y)$ can be obtained from which (2.3) can be derived by elementary methods, see e.g. Feller (1968). In this case (2.1) and (2.2) are replaced by

(2.5) $$N \cdot \exp(-n/N) \to m < \infty$$

or

(2.6) $$n/N - \log N \to -\log m > -\infty .$$

For T_b, the number of balls until b empty cells remain, the limit distribution is given by:

Theorem 2. If b is a fixed integer and for some fixed numbers C and D,

(2.7) $$0 < C \leq Np_k \leq D < \infty , \text{ for all } k \text{ and } N ,$$

then, when $N \to \infty$,

(2.8) $$\sum_{k=1}^{N} (1 - p_k)^{T_b} \Rightarrow \tfrac{1}{2} \chi^2(2(b+1)) ,$$

and

(2.9) $$\sum_{k=1}^{N} \exp(-T_b p_k) \Rightarrow \tfrac{1}{2} \chi^2(2(b+1)) .$$

Before proving the theorem the following functions are considered:

(2.10) $$f(t) = f_N(t) = \sum_{k=1}^{N} (1-p_k)^t , \quad t > 0 ,$$

and

(2.11) $$g(t) = g_N(t) = \sum_{k=1}^{N} \exp(-tp_k).$$

Lemma 1. If Condition (2.7) is satisfied, $y > 0$ is a fixed number, and $t = t_N = t(y)$ is defined by the equation

(2.12) $$f(t) = y ,$$

then

(2.13) $$0 < C \le \liminf_{N\to\infty} N \log N/t_N \le \limsup_{N\to\infty} N\log N/t_N \le D < \infty$$

and when $N \to \infty$

(2.14) $$f([t]) \to y ,$$

(2.15) $$\max_{1 \le k \le N} (1 - p_k)^{[t]} \to 0 ,$$

(2.16) $$g(t) \text{ and } g([t]) \to y .$$

where $[t]$ denotes the integer part of t.

Lemma 2. If f is replaced by g and g by f in Lemma 1, then the same conclusions hold.

Proof of Lemma 1. From Condition (2.7), it follows that

(2.17) $$y = \sum_{k=1}^{N} (1 - p_k)^t \ge N \cdot (1 - D/N)^t .$$

Hence for $\varepsilon > 0$ and N sufficiently large

(2.18) $$\log y \ge \log N - t \cdot (D+\varepsilon)/N$$

and therefore

(2.19) $$D+\varepsilon = (D+\varepsilon) \lim_{N\to\infty} (1/(1-\log y/\log N)) \ge \limsup_{N\to\infty} N \log N/t_N ,$$

which proves the right inequality of (2.13).

To prove the left inequality of (2.13) the following estimate follows from (2.7):

(2.20) $$y = \sum_{1}^{N} (1-p_k)^t \le N \cdot (1 - C/N)^t ,$$

or

(2.21) $$\log y \leq \log N - t \log(1 - C/N) \leq \log N - tC/N .$$

From this it follows that

(2.22) $$C = C \lim_{N\to\infty} (1 - \log y/\log N)^{-1} \leq \liminf_{N\to\infty} N \log N/t_N .$$

To prove (2.14) we observe that

(2.23) $$(1 - p_k)^{t-1} \geq (1 - p_k)^{[t]} \geq (1 - p_k)^t ,$$

and using (2.7)

(2.24) $$(1 - D/N)^{-1} \sum_1^N (1-p_k)^t \geq \sum_1^N (1-p_k)^{[t]} \geq \sum_1^N (1 - p_k)^t ,$$

or from (2.12)

(2.25) $$(1 - D/N)^{-1} y \geq f([t]) \geq y .$$

From which (2.14) follows.

Combining (2.7) and (2.13) give for some $K_1 > 0$ and N sufficiently large that

(2.26) $$\max(1 - p_k)^{[t]} \leq (1 - C/N)^{[t]} \leq (1- C/N)^{K_1 N \log N} \to 0, \; N \to \infty$$

which proves (2.15).

Using (2.7) and (2.13) it follows that for some constant K

(2.27) $$|1 - e^{-tp_k}/(1 - p_k)^t| \leq K \cdot \log N/N ,$$

and therefore

(2.28) $$|f(t) - g(t)| \leq \sum_1^N (1 - p_k)^t \cdot |1 - e^{-tp_k}/(1-p_k)^t|$$

$$\leq K \sum_1^N (1-p_k)^t \log N/N = K y \log N/N \to 0 ,$$

which proves (2.16). ∎

Proof of Lemma 2. The proof is essentially the same as that for Lemma 1. ∎

Proof of Theorem 2. From the definitions it follows that

$$(2.29) \qquad Y_n \leq b \iff T_b \leq n ,$$

and therefore

$$(2.30) \qquad P(Y_n \leq b) = P(T_b \leq n) = P(f(T_b) \geq f(n)) .$$

Let $y > 0$ be fixed and define $n = [t]$ with $t = t(y)$ as in Lemma 1. According to Lemma 1 the assumptions of Theorem 1 are satisfied. Hence

$$(2.31) \qquad P(f(T_b) \geq y) = P(Y_n \leq b) \to P(Y \leq b) ,$$

where Y is $Po(y)$. Furthermore it is well-known that

$$(2.32) \qquad P(Y \leq b) = P(\tfrac{1}{2}\chi^2(2(b+1)) \geq y) .$$

(2.31) and (2.32) prove (2.8). Using Lemma 2, the assertion (2.9) follows. ∎

Remark. When the p's are equal the theorem can be written

$$(2.33) \qquad N \cdot (1 - 1/N)^{T_b} \Rightarrow \tfrac{1}{2}\chi^2(2(b+1)) ,$$

and therefore

$$(2.34) \qquad T_b/N - \log N \Rightarrow \log(\tfrac{1}{2}\chi^2(2(b+1))) .$$

This result was found by Baum and Billingsley (1965) using complicated calculations. Using the result in Feller (1968) and the method of proof of Theorem 2, (2.33) and (2.34) follows. A consequence of (2.34) is

$$(2.35) \qquad T_b/N \log N \to 1 , \text{ in probability, as } N \to \infty .$$

Now (2.35) will be generalized. First introduce the distribution function

$$(2.36) \qquad H_N(x) = \#(p_k ; Np_k \leq x)/N .$$

Lemma 3. If $t = t_N = t(y)$ is defined by

$$g(t) = g_N(t_N) = y > 0 , \tag{2.37}$$

and there exists a distribution function $H(x)$ on $[C, D]$ such that

$$H_N(x) \to H(x) , \quad N \to \infty , \tag{2.38}$$

and

$$0 < C = \inf\{x ; H(x) > 0\} , \tag{2.39}$$

then for $1/C > \varepsilon > 0$, when $N \to \infty$,

$$g_N((\varepsilon + 1/C)(N \log N)) \to 0 , \tag{2.40}$$

and

$$g_N((-\varepsilon + 1/C)(N \log N)) \to +\infty . \tag{2.41}$$

Proof. From the definitions it follows that

$$0 < y = g_N(t_N) = N \cdot \int_C^D \exp(-t_N x/N) dH_N(x) = \tag{2.42}$$

$$= \int_C^D \exp((1 - t_N x/N \log N) \log N) \, dH_N(x) .$$

Consider

$$g_N((\varepsilon + 1/C) N \log N) = \int_C^D \exp((1 - x(1+\varepsilon C)/C) \log N) dH_N(x) . \tag{2.43}$$

Now for $C \leq x \leq D$ it is true that $1 - x(1+\varepsilon C)/C < 0$ and therefore the exponent in (2.43) is negative so the integral tend to 0 when $N \to \infty$, which proves (2.40).

For proving (2.41) consider

$$g_N((-\varepsilon + 1/C) N \log N) = \int_C^D \exp((1 - x(1 - \varepsilon C)/C) \log N) dH_N(x) . \tag{2.44}$$

For $C \leq x \leq C/(1-C\varepsilon)$ the exponent is positive and as the integrand is positive (2.44) could be estimated by

$$\int_C^{C/(1-C\varepsilon)} \exp((1 - x(1-\varepsilon C)/C) \log N) dH_N(x) \to +\infty \tag{2.45}$$

by Condition (2.39). ∎

Corollary to Theorem 2. If the Conditions (2.38) and (2.39) are satisfied then

(2.46) $T_b/N \log N \to 1/C$, in probability, $N \to \infty$.

Proof. Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be given. Take a $\delta > 0$ so that

(2.47) $P(\frac{1}{2}\chi^2(2(b+1)) < \delta) < \varepsilon_2/2$.

For N sufficiently large it follows from Theorem 2 that

(2.48) $P(g_N(T_b) < \delta) < \varepsilon_2/2$

and from Lemma 3 that

(2.49) $g_N((\varepsilon_1 + 1/C)(N \log N)) < \delta$.

Hence

(2.50) $P(T_b/N \log N > \varepsilon_1 + 1/C) =$

$P(g_N(T_b) < g_N((\varepsilon_1 + 1/C)(N \log N)) \leq$

$P(g_N(T_b) < \delta) < \varepsilon_2/2$.

In a similar way it is proven that

(2.51) $P(T_b/N \log N < -\varepsilon_1 + 1/C) < \varepsilon_2/2$.

Hence for N sufficiently large

(2.52) $P(|T_b/N \log N - 1/C| > \varepsilon_1) < \varepsilon_2$.

Thus the assertion is proved. ■

3. A small fraction of empty cells.

As above, Y_n denotes the number of empty cells after n throws.

Theorem 3. If

(3.1) $0 < C \leq Np_k \leq D < \infty$, for all k and N ,

(3.2) $n/N \to \infty$,

and

(3.3) $$f(n) = E(Y_n) = \sum_{k=1}^{N} (1 - p_k)^n \to +\infty ,$$

then, when $n \to \infty$,

(3.4) $$(Y_n - f(n))/(f(n))^{\frac{1}{2}} \Rightarrow N(0,1) ,$$

and

(3.5) $$(Y_n - g(n))/(g(n))^{\frac{1}{2}} \Rightarrow N(0,1) ,$$

where

(3.6) $$g(n) = \sum_{k=1}^{N} \exp(-np_k) .$$

<u>Proof.</u> Using (3.1) and (3.3) it follows that

(3.7) $$\sum_{1}^{N}(1 - p_k)^n \leq N \cdot (1 - C/N)^n \to +\infty ,$$

hence

(3.8) $$n/N \log N = O(1) .$$

Using (3.1), (3.2), and (3.8) give

(3.9) $$|f(n) - g(n)| \leq \sum_{1}^{N} \exp(-np_k) \cdot$$
$$\cdot \; |\exp(n \log (1-p_k) + np_k) - 1| \leq$$
$$\leq \sum_{1}^{N} \exp(-np_k) \cdot K \cdot n/N^2 \leq$$
$$\leq K \cdot (n/N) \cdot \exp(-C\, n/N) \to 0 .$$

Hence it is sufficient to prove (3.5). This will be established using convergence of characteristic functions.

In Holst (1971) p. 1672 the characteristic function of Y_n is given by

(3.10) $$E(\exp(i\, t\, Y_n)) = (n\,! / 2\pi i\, N^n) \cdot$$
$$\cdot \oint_{|z| = n/N} (e^{Nz}/z^{n+1}) \prod_{k=1}^{N} (1 + (e^{it}-1)\exp(-Np_k z))dz .$$

Using Stirling's formula and changing to polar coordinates it follows that

(3.11) $$E(\exp(it(Y_n - \mu)/\sigma)) = (1 + o(1)) \cdot$$

$$\cdot \int_{-\pi}^{\pi} (n/2\pi)^{\frac{1}{2}} \cdot \exp(n(e^{i\theta} - 1 - i\theta)) \cdot$$

$$\cdot \prod_{1}^{N} (\exp(-it\, e^{-np_k}/\sigma) \cdot (1 + (e^{it/\sigma} - 1)\exp(-np_k\, e^{i\theta})))d\theta$$

$$= (1 + o(1)) \cdot \int_{-\pi}^{\pi} h_n(\theta, t)d\theta ,$$

where

(3.12) $$\mu = \sigma^2 = g(n) = \sum_{1}^{N} \exp(-np_k), \quad \sigma > 0 .$$

The integral will be studied by the same method as in Holst (1971).

Take $0 < a < 1/6$ and split the interval $-\pi \leq \theta \leq \pi$ into

(3.13) $$A = \{\theta ; a \leq |\theta| \leq \pi \},$$

(3.14) $$B = \{ \theta; n^{a-\frac{1}{2}} \leq |\theta| < a \} ,$$

and

(3.15) $$C = \{\theta ; |\theta| < n^{a-\frac{1}{2}} \} .$$

From Lemmas 4-6 below it follows that

(3.16) $$E(\exp(it(Y_n - \mu)/\sigma) = (1 + o(1)) \cdot$$

$$(\int_A h_n + \int_B h_n + \int_C h_n) \to 0 + 0 + \exp(-t^2/2), \quad n \to \infty .$$

By the continuity theorem for characteristic functions assertion (3. 5) is proved, and thus the theorem. ∎

With the same conditions as in Theorem 3 the following lemmas hold.

Lemma 4. For every fixed real number t

$$\int_A h_n(\theta,t)d\theta \to 0 \;, \quad n \to \infty \;. \tag{3.17}$$

Proof. As $n/N \to \infty$ and $\sigma \to \infty$ it follows that

$$\begin{aligned} \Big|\int_A\Big| &\le K_1 \cdot n^{\frac{1}{2}} e^{-n} \cdot \int_A \prod_1^N |\exp(np_k e^{i\theta}) + e^{it/\sigma} - 1| d\theta \\ &\le K_2 n^{\frac{1}{2}} e^{-n} \prod_1^N (\exp(np_k \cos a) + o(1)) \\ &\le K_2 n^{\frac{1}{2}} e^{-n} 2^N e^{n\cos a} \to 0 \;. \quad \blacksquare \end{aligned} \tag{3.18}$$

Lemma 5 For ever· ·ixed real number t

$$\int_B h_n(\theta,t)d\theta \to 0 \;, \quad n \to \infty \;. \tag{3.19}$$

Proof. From the assumptions, it follows that there exist positive numbers $K_3 - K_9$ such that

$$\begin{aligned} \Big|\int_B\Big| &\le K_3 n^{\frac{1}{2}} e^{-n} \int_B \prod_1^N (\exp(np_k \cos\theta) + O(1/\sigma))d\theta \\ &\le K_4 n^{\frac{1}{2}} e^{-n} \prod_1^N \exp(np_k \cos n^{a-\frac{1}{2}}) \cdot \\ &\quad \cdot (1 + K_5 \cdot \exp(-K_6 n/N)/\sigma) \\ &\le K_7 n^{\frac{1}{2}} e^{-n} \exp(n(1 - K_8 n^{2a-1})) \\ &\le \exp(-K_9 n^{2a}) \to 0 \;, \quad n \to \infty \;. \quad \blacksquare \end{aligned} \tag{3.20}$$

Lemma 6. For every fixed real number t,

$$\int_C h_n(\theta,t)d\theta \to \exp(-t^2/2), \quad n \to \infty \;. \tag{3.21}$$

Proof. Expanding in series gives

$$\begin{aligned} \log h_n(\theta,t) &= -n\,\theta^2/2 + o(1) \\ &+ \sum_1^N (\log(1 + \exp(-np_k e^{i\theta})(e^{it/\sigma} - 1)) - it \exp(-np_k)/\sigma) + \tfrac{1}{2}\log(n/2\pi). \end{aligned} \tag{3.22}$$

Now, when $n \to \infty$,

$$(3.23) \qquad \sum_1^N |\exp(-2np_k e^{i\theta})(e^{it/\sigma} - 1)^2|$$

$$= o(1) \cdot \sum_1^N \exp(-np_k)/\sigma^2 = o(1) ,$$

and therefore

$$(3.24) \qquad \sum_1^N (\log (1 + \ldots) - \ldots)$$

$$= \sum_1^N (\exp(-np_k e^{i\theta})(e^{it/\sigma} - 1) - it \exp(-np_k)/\sigma) + o(1) .$$

Furthermore, using (3.8), (3.9) and the assumptions, it follows that

$$(3.25) \qquad \sum_1^N \exp(-np_k e^{i\theta})/\sigma^2 \to 1 ,$$

and therefore (3.24) can be written

$$(3.26) \qquad \sum_1^N (\ldots) = \sum_1^N (\exp(-np_k e^{i\theta})(it/\sigma - t^2/2\sigma^2)$$

$$- it \exp(-np_k)/\sigma) + o(1)$$

$$= it \sum_1^N (\exp(-np_k(e^{i\theta} - 1)) - 1) \exp(-np_k)/\sigma$$

$$- t^2/2 + o(1) .$$

Now, when $n \to \infty$,

$$(3.27) \qquad \sum_1^N (np_k)^2 \theta^2 \exp(-np_k)/\sigma \le$$

$$\le K_1 (n/N)^2 n^{2a-1} N^{\frac{1}{2}} \exp(-K_2 n/N) \to 0 .$$

From this it follows that

$$(3.28) \qquad \sum_1^N (\ldots) = \theta t \sum_1^N np_k \exp(-np_k)/\sigma - t^2/2 + o(1) .$$

Hence for θ in C,

(3.29) $$\log h_n(\theta,t) - \tfrac{1}{2}\log(2\pi/n) = -n\theta^2/2 + \theta t \sum_1^N np_k \exp(-np_k)/\sigma$$

$$- t^2/2 + o(1) = -(n^{\frac{1}{2}}\theta - t\sum_1^N n^{\frac{1}{2}} p_k \exp(-np_k)/\sigma)^2/2$$

$$-t^2(1-(\sum_1^N n^{\frac{1}{2}} p_k \exp(-n p_k)/\sigma)^2)/2 + o(1).$$

Now, when $n \to \infty$,

(3.30) $$\sum_1^N n^{\frac{1}{2}} p_k \exp(-n p_k)/\sigma \le K_3 n^{\frac{1}{2}} N^{-1} \cdot N^{\frac{1}{2}}$$

$$\cdot \exp(-K_4 n/N) \to 0.$$

Thus with $\psi = n^{\frac{1}{2}}\theta$ the integral (3.21) can be written

(3.31) $$\int_C h_n = \int_{|\psi| \le n^a} (2\pi)^{-\frac{1}{2}}$$

$$\cdot \exp(-(\psi - o(1))^2/2 - t^2/2 + o(1))\, d\psi,$$

which converges to $\exp(-t^2/2)$ when $n \to \infty$. ∎

4. The waiting time for a small fraction.

As above let T_b denote the number of balls thrown until exactly $b = b_N$ cells remain empty. Let t_b be the unique solution of the equation

(4.1) $$b = g(t_b) = \sum_{k=1}^N \exp(-t_b p_k).$$

Theorem 4. If, when $N \to \infty$,

(4.2) $$b_N \to +\infty,$$

(4.3) $$b_N/N \to 0,$$

and

(4.4) $$0 < C \le N p_k \le D < \infty, \text{ for all } k \text{ and } N,$$

then

(4.5) $$b_N^{-\frac{1}{2}}(T_b - t_b)\sum_{k=1}^N p_k \exp(-t_b p_k) \Rightarrow N(0,1).$$

Proof. From the assumptions it follows that

(4.6) $$C\, b/N \leq \Delta = \sum_{1}^{N} p_k \exp(-t_b p_k) \leq D\, b/N .$$

Thus for N sufficiently large

(4.7) $$0 < C \leq \Delta \cdot N/b \leq D < \infty .$$

As in the proof of Theorem 2 the following relation holds

(4.8) $$P((T_b - t_b)\, \Delta/b^{\frac{1}{2}} \leq x) = P(Y_n \leq b) ,$$

where

(4.9) $$n = [t_b + x\, b^{\frac{1}{2}}/\Delta] .$$

It is seen that

(4.10) $$g(n)\,(1 + o(1)) = g(t_b + x\, b^{\frac{1}{2}}/\Delta)$$

$$= \sum \exp(-t_b p_k) \cdot (1 - x\, p_k\, b^{\frac{1}{2}}/\Delta + O(1/b)),$$

$$= b - x \cdot b^{\frac{1}{2}} + O(1) ,$$

and thus

(4.11) $$g(n) \to +\infty ,$$

and from (3.9) it follows that

(4.12) $$f(n) \to +\infty .$$

Furthermore,

(4.13) $$b = g(t_b) \geq N \exp(-D\, t_b/N),$$

implying that

(4.14) $$t_b /N \to +\infty ,$$

and therefore

(4.15) $$n/N \to +\infty .$$

Hence the assumptions of Theorem 3 are fulfilled and (4.8) and (4.10) give

$$(4.16) \qquad P((T_b - t_b)/b^{\frac{1}{2}} \leq x) = P(Y_n \leq b) =$$

$$= \Phi((b - g(n)) / (g(n))^{\frac{1}{2}}) + o(1) =$$

$$= \Phi((x\, b^{\frac{1}{2}} + O(1))/(b(1 + o(1)))^{\frac{1}{2}}) + o(1) \to \Phi(x) ,$$

where $\Phi(x)$ is the standardized normal distribution function. This proves the theorem. ∎

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