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Nonsymmetric ballistic range, height, time-of-flight and optimal flight path angle computations with programs for a Hewlett-Packard 65 calculator

Shudde, Rex H.

Monterey, California. Naval Postgraduate School

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NAVAL POSTGRADUATE SCHOOL

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NONSYMMETRIC BALLISTIC RANGE, HEIGHT, TIME-OF-FLIGHT AND OPTIMAL FLIGHT PATH ANGLE COMPUTATIONS WITH PROGRAMS FOR A HEWLETT-PACKARD 65 CALCULATOR

by

Rex H. Shudde

March 1976

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The purpose of this report is to provide the equations and HP-65 Programmable Calculator programs for computing ballastic range, height, time-of-flight, and particularly the flight path angle which maximizes ballistic ranges for non-symmetric launch and target positions. A no-atmosphere, non-rotating, spherical Earth is assumed.

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NONSYMMETRIC BALLASTIC RANGE, HEIGHT, TIME-OF-FLIGHT AND OPTIMAL FLIGHT PATH ANGLE COMPUTATIONS WITH PROGRAMS FOR A HEWLETT-PACKARD 65 CALCULATOR

I. The Introduction and Apology.

The range of a ballistic missile over the surface of the Earth is a topic which is discussed in almost every text on astronautics (Reference 1, for example). Unfortunately, none of the available texts contained a needed procedure for the exact solution of the unsymmetric ballistic problem (defined in Section II); all procedures found were approximate and were based on a symmetric approximation [See References 2 and 3, for example]. The author of this report faced the dilemma of either (1) taking the time to solve the problem afresh, or (2) taking an inordinate longer time to do a thorough search to find a solution which might not be the one desired. The former course of action was taken and all due apologies are hereby extended to those that have published more elegant solutions.

All of the computational methods are outlined in Section III. Instructions and a sample problem for the Hewlett-Packard 65 Calculator (HP-65) are given in Section IV. This HP-65 programs are in Section V, and the development of the β_{opt} procedure is in the Appendix. Familiarity with Newtonian two-body theory is assumed.

II. The Problem.

Basically, the problem is to compute the range, S, and time-of-flight, t_f, of a ballistic missile given the booster cutoff at height h_E after a vertical ascent, a cutoff velocity v_L, and a subsequent flight path angle β (Fig. 1). Given that a solution to the stated problem is available, the next natural question to ask is, "What flight path angle β_{opt} will obtain the maximum range S_{max}?" The solution is fairly simple in the symmetric ballistic problem, i.e. when $h_E = 0$, and called symmetric because the launch point L and target point T are symmetric with the line of apses of the Keplerian orbit (i.e. $\phi_1 = \phi_2$ in Fig. 1). The unsymmetric ballistic problem is that in which $h_E \neq 0$. Here we assume that h_E is given, the Earth with radius r_E is spherical, non-rotating, and has no atmosphere. The effect of the Earth's rotation may be calculated figuring how far the target has moved during the time of flight of the missile. Explicitly, we assume that Keplerian orbit is obtained at height h_E with terminal booster cutoff velocity v_L and flight path angle β(or β_L), and the target T is at the Earth's surface.

The computation of range and time-of-flight for given h_E, r_E, v_L, and β can easily be obtained from the equations of elliptical orbits. The determination of β_{opt} is somewhat complicated, however, and requires an iterative procedure. In all of the developments we assume that L and T are separated by the line of apses (which implies that $0 < \beta \leq \frac{\pi}{2}$).

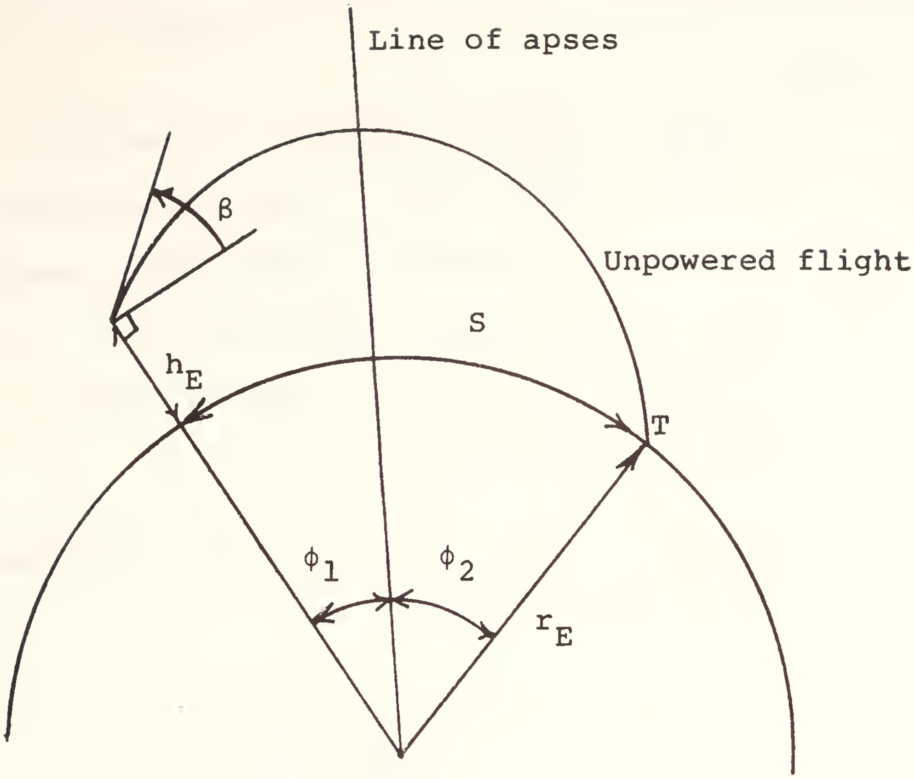

FIGURE 1

III. The Computational Procedures.

A. Range

1. Let r_L and r_T denote the distance of the launch point L and target point T from the center of the Earth. Usually we let $r_L = r_E+h_E$ and $r_T = r_E$ although r_L and r_T need not be so defined in general.

2. From r_L and v_L, compute the semimajor axis of the ellipse,

$$a = \frac{\mu}{\left(\frac{2\mu}{r_L}\right) - v_L^2} ,$$

where $\mu = k_E^2$ and k_E is the (Gaussian) gravitational constant for the Earth. We have assumed that the mass of the missile is negligible with respect to the mass of the Earth.

3. Using the flight path angle β, compute the elliptical semiparameter

$$p = (r_L v_L \cos\beta)^2/\mu .$$

4. Compute the eccentricity

$$e = \sqrt{1 - \frac{p}{a}} .$$

5. Compute the true anomaly of the launch point

$$f_L = \cos^{-1}\left[\frac{1}{e}\left(\frac{p}{r_L} - 1\right)\right]$$

where $0 \le f_L \le \pi$. Note that the line of apses can be said to have true anomaly $f = \pi$.

6. Compute the true anomaly of the target point

$$f_T = 2\pi - \cos^{-1}\left[\frac{1}{e}\left(\frac{p}{r_T} - 1\right)\right]$$

7. The surface range of the missile is then

$$S = r_E(f_T - f_L) \; .$$

B. Time-of-Flight

1. Compute the eccentric anomaly of the launch point,

$$E_L = \cos^{-1}\left[\frac{1}{e}\left(1 - \frac{r_L}{a}\right)\right]$$

where $0 \le E_L \le \pi$.

2. Compute the eccentric anomaly of the target point,

$$E_T = 2\pi - \cos^{-1}\left[\frac{1}{e}\left(1 - \frac{r_T}{a}\right)\right]$$

where $\pi \le E_T \le 2\pi$.

3. Using Keplers' equation, compute the time-of-flight

$$t_f = \frac{a^{3/2}}{\sqrt{\mu}}\left[(E_T - E_L) - e(\sin E_T - \sin E_L)\right] .$$

C. Height of Trajectory & Circular Velocity

1. The maximum height of the trajectory above the Earth's surface for any h_E, v_L, and β is given by

$$h_{max} = h = a(1 + e) - r_E$$

2. Circular Velocity

The velocity required for a circular orbit at r_L is

$$v_c = \sqrt{\mu/r_L} .$$

v_c will be used in Section III D.

D. Optimum Flight Path Angle & Maximum Range

For a specified h_E and v_L, a value of β can be found such that

$$S_{max} = S(\beta_{opt}) = \max_{\beta} S(\beta) .$$

In this unconstrained optimization problem we set $\left.\frac{\partial S}{\partial \beta}\right|_{\beta = \beta_{opt}} = 0$ and solve for β_{opt}. The relationship between S and β is given by the computational formulas for the range in part A. Although an explicit formula for β is unobtainable, the following exact relationship is useful:

(1) $$\cos^2\beta_{opt} = \frac{1}{(2-\alpha^2)}\left\{1+\frac{\left[1-\left(\frac{r_L}{r_T}\right)(2-\alpha^2)\cos^2\beta_{opt}-\frac{2\mu}{v_L^2}\left(\frac{1}{r_L}-\frac{1}{r_T}\right)\right]\sin\beta_{opt}}{\sqrt{\left[1-\left(\frac{r_L}{r_T}\right)\cos^2\beta_{opt}-\frac{2\mu}{v_L}\left(\frac{1}{r_L}-\frac{1}{r_T}\right)\right]^2}}\right\}$$

where $\alpha = \frac{v_L}{v_c}$.

In the symmetric ballistic problem $r_L = r_T$; the equation for β_{opt} then reduces to

(2) $$\cos^2\beta_{opt} = \frac{1}{2-\alpha^2},$$

which is the result given in References 1, 2, and 3.

In the unsymmetric case Equation (1) is used to find β_{opt} as follows:

1. Solve (2) for β_{opt}.
2. Substitute β_{opt} into the right hand side of (1) to obtain a new value for β_{opt}.

3. Average the last two values of β_{opt} to obtain a new value of β_{opt} and return this value of β_{opt} to step 2.

4. It has been found that five iterations are sufficient to obtain a reliable value for β_{opt}.

5. Using β_{opt}, compute the maximum range $S_{max} = S(\beta_{opt})$ using procedure A.

IV. The Illustrative Example.

The HP-65 program consists of four magnetic cards which are used as follows:

Card 1: This card must be entered first. It is used to set the two constants r_E and μ for any choice of four distance units and any choice of four time units. The distance units are kilometers, meters, statute miles, and nautical miles; the distance unit is entered by pressing key A, B, C, or D, respectively. The display shows the equatorial Earth radius r_E in the respective unit. The time units are seconds, minutes, hours, and days; the time unit is entered by pressing key A, B, C, or D, respectively. The display shows the constant μ in the appropriate $(\text{distance})^3/(\text{time})^2$ unit. Note - after entering program Card 1, the distance unit must be entered before the time unit is entered since the keys A through D perform a dual function. The basic definitions used are:

1 statute mile = 1.609344 kilometers(km)

1 nautical mile = 1.852 km

r_E = 6378.160 km

μ = 398603 km^3/sec^2

Card 2: This card computes S, h_{max}, and v_c given h_E, v_L and β. It must be used after the execution of Card 1 and it may be used before and/or after the execution of Cards 3 and 4. If it is used after Card 4 to compute S_{max}, then β_{opt} as computed by Card 4 need not be re-entered. Otherwise, in any order: enter h_E and press A, enter v_L and press B, enter β and press C. To find the range S, press E. Then to find h_{max}, press R/S; and then to find v_c, press R/S again. Note: If h_E, v_L, and β have been entered using Card 3, then they do not have to be reentered when using Card 2. Also, any new value of h_E, v_L, or β can be entered and the range computed without having to re-enter unchanged values.

Card 3: This card computes the time of flight, t_f. It must be used after the execution of Card 1 and it may be used before and/or after the execution of Cards 2 and 4. If it is used after Card 4, then β_{opt} as computed by Card 4 need not be re-entered; if it is used after Card 2, then only new quantities need be entered. Otherwise, in any order: enter h_E and press A, enter v_L and press B, enter β and press C. To find the time of flight, press E.

Card 4: This card computes β_{opt}. It must be used after h_E and v_L have been entered from either Card 2 or Card 3. No entries are required. To compute β_{opt}, press any key A through E; five iterations will be performed and the resulting value of β_{opt} will be displayed. To perform five more iterations, press R/S. To monitor the intermediate iterations press keys f SF2. Each intermediate result will be displayed. To continue, press R/S. To disable the monitoring, press keys f^{-1} SF2.

Using the four program cards as described above find the range and time-of-flight for a ballistic missile that has a velocity of 10000 knots at a cut-off altitude of 50 nautical miles for a flight path angle of 30° and 40°. Also, find the maximum possible range and time-of-flight for the same missile.

Proceed as follows:

tep	Instruction	Data	Key	Display
1	Enter Card 1			
2	Distance unit is n.mi.		D	r_E=3443.93n.mi.
3	Time unit is hours		C	$\mu=8.13247\times10^{11}$
4	Enter Card 2			
5	Enter cutoff altitude	50	A	r_L=3493.93n.mi.
6	Enter cutoff velocity	10000	B	
7	Enter flight path angle	30	C	
8	Compute range		E	S=1928.08n.mi.

Step	Instruction	Data	Key	Display
8a	Optional: Display h_{max}		R/S	h_{max}=344.74n.mi
8b	Optional: Display v_c		R/S	v_c=15256.47knot
9	Enter flight path angle	40	C	
10	Compute range		E	S=1956.60n.mi.
11	Enter Card 3			
12	Compute time-of-flight		E	t_f=0.3043hrs.
	Note: $h_E = 50$, $v_L = 10000$, and $\beta = 40^o$ since no further entries have been made.			
13	Enter flight path angle	30	C	
14	Compute time-of-flight		E	0.2504hrs.
15	Enter Card 4			
16	Compute β_{opt} (5 iterations)		E	$\beta_{opt}=36^o.0895$
17	Make 5 more iterations to be sure		R/S	$\beta_{opt}=36^o.0895$
18	Enter Card 2			
19	Compute S_{max}		E	S_{max}=1973.95n.m
20	Enter Card 3			
21	Compute time-of-flight for maximum range.		E	$t_f(\beta_{opt})$=0.2846

V. The Program Listings.

CARD 1

Set units of distance and time display r_E and μ

Key Entry	Code Shown	Key Entry	Code Shown
LBL	23	3	03
B	12	7	07
EEX	43	8	08
3	03	•	83
CHS	42	1	01
GTO	22	6	06
0	00	X	71
LBL	23	STO 4	3304
C	13	R/S	84
1	01	LBL	23
•	83	B	12
6	06	6	06
0	00	0	00
9	09	GTO	22
3	03	1	01
4	04	LBL	23
4	04	C	13
GTO	22	3	03
0	00	6	06
LBL	23	0	00
D	14	0	00
1	01	GTO	22
•	83	1	01
8	08	LBL	23
5	05	D	14
2	02	8	08
GTO	22	6	06
0	00	4	04
LBL	23	0	00
A	11	0	00
1	01	GTO	22
LBL	23	1	01
0	00	LBL	23
g	35	A	11
1/x	04	1	01
ENTER	41	LBL	23
ENTER	41	1	01
ENTER	41	ENTER	41
X	71	X	71
X	71	RCL 1	3401
3	03	X	71
9	09	STO 1	3301
8	08	R/S	84
6	06	g NOP	3501
0	00	g NOP	3501
3	03	"	3501
X	71	"	3501
STO 1	3301	"	3501
CL x	44	"	3501
6	06	"	3501

CARD 2

Compute range, h_{max}, v_c

Key Entry	Code Shown	Key Entry	Code Shown
RCL 4	3404	X	71
+	61	1	01
STO 3	3303	-	51
R/S	84	CHS	42
LBL	23	f	31
B	12	√	09
STO 2	3302	STO 6	3306
R/S	84	RCL 8	3408
LBL	23	RCL 3	3403
C	13	D	14
STO 7	3307	RCL 8	3408
R/S	84	RCL 4	3404
LBL	23	D	14
D	14	+	61
g	35	g	35
RAD	42	π	02
÷	81	ENTER	41
1	01	+	61
-	51	g x↔y	3507
RCL 6	3406	-	51
÷	81	RCL 4	3404
f-1	32	x	71
COS	05	R/S	84
RTN	24	RCL 6	3406
LBL	23	1	01
E	15	+	61
g	35	RCL 5	3405
DEG	41	÷	81
RCL 7	3407	RCL 4	3404
f	31	-	51
COS	05	R/S	84
RCL 2	3402	RCL 1	3401
X	71	RCL 3	3403
RCL 3	3403	÷	81
X	71	f	31
ENTER	41	√	09
X	71	R/S	84
RCL 1	3401	GTO	22
÷	81	E	15
STO 8	3308	g NOP	3501
2	02	g NOP	3501
RCL 3	3403	"	3501
÷	81	"	3501
RCL 2	3402	"	3501
ENTER	41	"	3501
X	71	"	3501
RCL 1	3401	"	3501
÷	81	"	3501
-	51	"	3501
STO 5	3305	"	3501

NOTE: Before putting a program into memory, after switching to W/PGM mode, press f and then PRGM to clear memory.

CARD 3

Compute time-of-flight

Key Entry	Code Shown	Key Entry	Code Shown
RCL 4	3404	-	51
+	61	STO 5	3305
STO 3	3303	X	71
R/S	84	1	01
LBL	23	-	51
B	12	CHS	42
STO 2	3302	f	31
R/S	84	√	09
LBL	23	STO 6	3306
C	13	g	35
STO 7	3307	π	02
R/S	84	ENTER	41
LBL	23	+	61
D	14	RCL 4	3404
g	35	D	14
RAD	42	-	51
RCL 5	3405	ENTER	41
X	71	f	31
1	01	SIN	04
-	51	RCL 6	3406
CHS	42	X	71
RCL 6	3406	-	51
÷	81	RCL 3	3403
f^{-1}	32	D	14
COS	05	ENTER	41
RTN	24	f	31
LBL	23	SIN	04
E	15	RCL 6	3406
g	35	X	71
DEG	41	-	51
RCL 7	3407	-	51
f	31	RCL 5	3405
COS	05	÷	81
RCL 2	3402	RCL 1	3401
X	71	RCL 5	3405
RCL 3	3403	X	71
X	71	f	31
ENTER	41	√	09
X	71	÷	81
RCL 1	3401	R/S	84
÷	81	GTO	22
STO 8	3308	E	15
2	02	g NOP	3501
RCL 3	3403	g NOP	3501
÷	81	g NOP	3501
RCL 2	3402	"	3501
ENTER	41	"	3501
X	71	"	3501
RCL 1	3401	"	3501
÷	81	"	3501

CARD 4

Compute maximizing flight path angle β_{opt}.

Key Entry	Code Shown	Key Entry	Code Shown
2	02	SIN	04
RCL 2	3402	X	71
ENTER	41	RCL 5	3405
X	71	RCL 3	3403
STO 5	3305	RCL 4	3404
RCL 3	3403	÷	81
X	71	RCL 7	3407
RCL 1	3401	f	31
÷	81	COS	05
-	51	X	71
STO 6	3306	ENTER	41
RCL 4	3404	X	71
g	35	-	51
1/x	04	f	31
RCL 3	3403	√	09
g	35	÷	81
1/x	04	1	01
-	51	+	61
RCL 5	3405	RCL 6	3406
÷	81	÷	81
RCL 1	3401	f	31
X	71	√	09
ENTER	41	f^{-1}	32
+	61	COS	05
1	01	f	31
+	61	TF 2	81
STO 5	3305	R/S	84
g	35	g NOP	3501
DEG	41	RCL 7	3407
0	00	gx↔y	3507
STO 7	3307	STO 7	3307
5	05	0	00
STO 8	3308	g x=y	3523
LBL	23	GTO	22
0	00	0	00
RCL 5	3405	+	61
RCL 3	3403	+	61
RCL 4	3404	2	02
÷	81	÷	81
RCL 6	3406	STO 7	3307
X	71	g	35
RCL 7	3407	DSZ	83
f	31	GTO	22
COS	05	0	00
X	71	R/S	84
gLSTx	3500	5	05
X	71	STO 8	3308
-	51	GTO	22
RCL 7	3407	0	00
f	31	g NOP	3501

NOTE: Before putting a program into memory, after switching to W/PGM mode, press f and then PRGM to clear memory.

VI. The References.

1. Berman, Arthur I., The Physical Principles of Astronautics, John Wiley and Sons, Inc. 1961.

2. Seifert, Howard (editor), Space Technology, John Wiley and Sons, Inc. 1959.

3. Wyckoff, Robert C., Private Communication, 1975.

VII. The Appendix.

As indicated in section IIID, the maximum range for a specified cutoff altitude h_E and cutoff velocity v_L is found by the following unconstrained optimization:

$$S_{max} = S(\beta_{opt}) = \max_{0<\beta\leq \frac{\pi}{2}} S(\beta)$$

Using the equations in section IIIA, we want to find β_{opt} such that

$$\left.\frac{\partial S}{\partial \beta}\right|_{\beta = \beta_{opt}} = r_E \frac{\partial}{\partial \beta} (f_T - f_L)\Big|_{\beta = \beta_{opt}} = 0 .$$

We find that

$$\frac{\partial}{\partial \beta} (f_T - f_L) = (\gamma_T - \gamma_L) \frac{\partial p}{\partial \beta} ,$$

where

$$\gamma_i = \frac{1}{\sqrt{1 - \frac{1}{e^2}\left(\frac{p}{r_i}\right) - 1}} \left[\frac{1}{er_i} + \frac{1}{2ae^3}\left(\frac{p}{r_i} - 1\right)\right], \quad i=T,L .$$

For the derivative of $f_T - f_L$ with respect to β to vanish at $\beta = \beta_{opt}$, we must have either:

1. $$(\gamma_T - \gamma_L)_{\beta = \beta_{opt}} = 0$$

or

2. $$\left.\frac{\partial p}{\partial \beta}\right|_{\beta = \beta_{opt}} = 0$$

We find that

$$\frac{\partial p}{\partial \beta} = - \frac{r_L^2 v_L^2 \sin 2\beta}{\mu} = 0$$

for $\beta = 0, \pm \pi/2, \pm \pi$, etc. Since we require that the launch point and the target point be separated by the line of apses, we require that $\theta < \beta \leq \frac{\pi}{2}$. This requirement is imposed because we must properly interface the discontinuity caused by the principal angle in the arc cosine used in steps 5 and 6 of section III A. Thus $\beta = \pi/2$ is the only angle of interest for which $\partial p/\partial \beta = 0$, and this angle clearly gives rise to a minimum in range since the trajectory of the missile is vertical (straight up, and then straight down).

The value of β which maximizes S is found from the requirement that

$$(\gamma_T - \gamma_L)_{\beta = \beta_{opt}} = 0 .$$

This equation has defied attempts to find an analytic solution, but it can be rearranged and displayed as equation 1 in section III D, where the solution procedure is detailed.

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