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GOAL PROGRAMMING
AND
MULTIPLE OBJECTIVE OPTIMIZATIONS
(Part I)

by

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Abstract

This is Part I of a survey of recent developments in goal programming and multiple objective optimizations. In this part, attention is directed to goal programming with emphasis on the authors' own work (with others) in a variety of applications. This includes goal and goal interval programming, as well as characterizations which make it possible to obtain alternate representations and explicit solutions from special structural properties. Possibilities for various goal functionals are explored and delineated. One class of examples is developed in detail and an algorithm is supplied which utilizes sequences of ordinary linear programming problems to solve certain nonlinear and non-convex problems involving maxima of ratios of linear forms.

1. Introduction

One might initiate a survey of developments in "goal programming" by enunciating something like a "10-year rule". E.g., some 10 years elapsed from the baby's birth, circa 1952, before the name "goal programming" was affixed to it with the publication, circa 1961-62, of Appendix B in [9].[1] This does not mean that everything remained fixed or unattended in the intervening period. Nevertheless, almost another 10 years elapsed before goal programming began to receive really widespread attention and use in Management Science, Operations Research and other professions and sciences. See [6].

Continuing this brief, and very casual, history we may associate the initial period with extensions of OR from military to civilian (private enterprise) uses. This naturally necessitated the development of new tools as well as new concepts and points of view. The subsequent period, 1961-1971, approximately, was mainly concerned with (a) exploiting these new tools with a variety of new uses that then invited attention[2] and (b) studying alternative formulations

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[1] A synoptic view of some of these early developments as they relate to topic of (inequality) constrained regressions may be found in [8]. See also [6].

[2] See, e.g., Bruno [2], Chisman [19], Rehnius and Wagner [35] and White, Shapiro and Pratt [45] for an idea of the range of applications. Others, including goal programming approaches to advertising media selection, manpower planning and organization design are discussed in [8].

with special reference, perhaps, to their potential value in curve fitting (theory or methods), statistical inference, etc.[1/]

Of course, all of this work has continued with new applications and new results that have opened ways for further uses and results, and so on. Nevertheless, a substantial spurt of developments (and further uses) seems to be apparent from circa 1971-72 on. Doubtless some of this or, in some sense, perhaps all of it, is due to this earlier work. The main impetus for the burst of new applications seems to be associated, however, with the evolution of "public management science" and its very natural orientations toward multi-goal or multi-objective formulations and uses.

In summary, then, we may say that we have one period of invention arising from applications to the then new civilian (private enterprise) section and another arising from the subsequent extensions to other spheres (e.g., governmental applications) which we may designate as the area of "public management science". In short, goal programming was designed as a "work horse" -- strong and rugged and easy to use -- rather than as a "thoroughbred" requiring devoted attention by skilled attendants and used only by specially trained riders, for their own or other's amusement or for, and finally, perhaps, permitted to breed, also under skilled guidance, to produce other thoroughbreds, mainly. Hopefully, the new

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[1/] See e.g., the work by Wagner [43] and [44]. See also [6] for a discussion of relations to much earlier work. Churchill Eisenhart in [23] discusses this, too, in the context of the early development of least squares regression going back at least as far as the work of Roger Boscovich, the Jesuit astronomer, who, circa 1750, was seeking to obtain regression lines for astronomical observations.

developments now emerging (in public management science) will, at least for a while, help to improve some of the work horse properties by pointing toward new uses and what might therefore be required, e.g., in the way of formulation or characterizations as well as improvements in computational efficacy and related computer codes, etc.

It is perhaps a good sign, even if somewhat frustrating (in our present assignment) to observe that the pace of recent applications has reached a point where it is now hopeless, or nearly so, for us to locate and identify each of them in their wide and proliferating variety. In any event we will not try. Instead we will take an alternate course and attempt to delineate new developments in theory and methodology which appear to promise, at least to us, still further openings to still other new applications which might otherwise not be essayed. Hence, in a manner consistent with what has already been said, we shall also try to point up the presentation of these new developments with reference to some of the applied contexts in which they originated. This seems worth doing even if it does tend to restrict us to contexts -- e.g., applications in which we have been personally involved -- so that we can thereby speak with some assurance.

The approach that we are proposing to take has disadvantages. It will prevent us from examining important developments such as uses of goal programming, etc., in multi-dimensional preference representation and analyses, etc., where we shall only be able to list a few references, and, of course, we will not expect to achieve a well balanced presentation, much less a presentation in depth, for all such topics.[1/] On the other hand, we may draw some comfort by arguing that we are at least trying to move upstream in the sense of the following quotation[2/] from the writings of probably the greatest mathematician of the present century and one to whom, in any event, all of OR must be in debt.

[1/] See, e.g., [21] and [38]. The same applies for any attempt to study the related uses in statistical estimation, etc., for which, q.v. [8] and the references cited therein.

[2/] See [40]. We are indebted to O.A. Davis for calling our attention to this quotation from his paper, "Notes on Strategy and Methodology for a Scientific Political Science" in J. L. Bernd, ed., Mathematical Applications in Political Science (Charlottesville: The University of Virginia Press, 1969).

"I think that is a relatively good approximation to truth...that mathematical ideas originate in empirics, although the genealogy is sometimes long and obscure. But, once they are so conceived, the subject begins to live a peculiar life of its own and is better compared to a creative one, governed by almost entirely aesthetical motivations, than to anything else and, in particular, to an empirical science. There is, however, a further point which, I believe, needs stressing. As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from "reality", it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely l'art pour l'art. This need not be bad, if the field is surrounded by correlated subjects, which still have closer empirical connections, or if the discipline is under the influence of men with an exceptionally well developed taste. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities. In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration. At the inception the style is usually classical; when it shows signs of becoming baroque, then the danger signal is up... In any event, whenever this stage is reached, the only remedy seems to me to be the rejuvenating return to the source: the reinjection of more or less directly empirical ideas."

Goal Programming and Multiple Criteria Optimizations

To allow for developments such as have already occurred, and also to allow for future possible courses of development, we may now try to characterize goal programming -- and related multiple criteria optimizations[1/] -- in a manner like the following. In goal programming one evidently encounters certain desired conditions which are characterized as "'goals' to be met 'as closely as possible'". These goals may be specific values or ranges of such values, as in the case of "goal interval programming", and "as closely as possible" may refer to non-metric (e.g., non Archimedean order) as well as metric properties.

In any event each such condition is assigned a functional which penalizes for deviations from the desired goal. These are therefore called "goal functionals." The vector of goal functionals is also submitted to an extremization, which may be of vector of scalar type, and may involve combinations of the individual goal functionals, which are to be "met as closely as possible".

The example which has become typical in the literature involves a weighted sum of absolute values of the individual goal deviations.[2/] These are then replaced by an ordinary linear programming equivalent, as in the earliest example [10], where the theory and interpretations was also set forth in both geometric and algebraic terms.

[1/] "Multiple objectives" is, we think, a better name. See Chapters I and IX in [9].

[2/] It was noted explicitly in [10] that other types of non-linear goal functionals, e.g., any vector norm, could also be used.

Reduction to a linear programming equivalent is convenient, of course, both for computation and interpretation. But restrictions to a weighted sum of absolute value functions is not the only such possibility. The characterization we have just supplied allows for different functional forms in a single model.[1/] Also, the notions of extremality include poly-extremizations[2/] and hence can include concepts of cooperation and compromise such as are involved in n-person games wherein each goal (or set of goals) is associated with a different player. Any of the many possible concepts of solution for such games can thereby be accorded a goal programming interpretation.

At this point, we shall not pursue all of the other possibilities that are also admitted by the characterizations we have just supplied. Instead, we shall turn to recent developments, as promised, and attempt to make our characterizations somewhat more concrete by reference to the applications that gave rise to them.

1/ See, e.g., [15] .

2/ See, e.g, the application of such a poly-extremization to the design of a network of city streets in [9], Chapter XX.

2. Goal Programming, Case I

We might begin with the absolute value format, viz.,

$$\min_{x \in X} \sum_{i \in I} \left| \sum_{j=1}^{n} a_{ij}x_j - g_i \right| \tag{1.1}$$

where X represents a set from which the choices of vectors, x, must be effected. The a_{ij} and g_i are constants and $i \in I$ refers to the index set while the vertical strokes are taken to mean the absolute value for the expression which they enclose.

We can regard the g_i as "goals", since the functionals

$$f_i(x) = \left| \sum_{j=1}^{n} a_{ij}x_j - g_i \right| \tag{1.2}$$

have the properties prescribed for a goal programming formulation -- *viz*., the $f_i(x)$ values increase with discrepancies from g_i for each $x \in X$ choice.

We can also replace the formulation (1.1) with

$$\begin{aligned} &\min \sum_{i \in I} (\delta_i^+ + \delta_i^-) \\ &\text{s.t.} \\ &\sum_{j=1}^{n} a_{ij}x_j - \delta_i^+ + \delta_i^- = b_i \\ &\delta_i^+,\ \delta_i^- \geq 0,\ i \in I \end{aligned} \tag{2.1}$$

where, as in (1.1), the x_j values are restricted via $x \in X$.

A proof of the equivalence between (1.1) and (2.1) would retrace developments that originally gave rise to the ideas of "goal programming". We will not provide such detailed proofs in this paper (when they are available elsewhere) but a sketch like the following should suffice.

Define

(2.2)
$$\delta_i^+ = \frac{\left|\sum_{j=1}^{n} a_{ij}x_j - b_i\right| + \left(\sum_{j=1}^{n} a_{ij}x_j - b_i\right)}{2}$$

$$\delta_i^- = \frac{\left|\sum_{j=1}^{n} a_{ij}x_j - b_i\right| - \left(\sum_{j=1}^{n} a_{ij}x_j - b_i\right)}{2}.$$

Then adding and subtracting the parenthesized expressions to (1.1) and collecting terms we obtain the functional form for (2.1) along with non negativity for each of the δ_i^+, δ_i^- pairs. We can also simplify the constraining relations in (2.2) by subtracting the second set from the first to obtain the thus simplified constraining relations

(2.3)
$$\sum_{j=1}^{n} a_{ij}x_j - \delta_i^+ + \delta_i^- = b_i$$

as in (2.1) without producing ambiguity provided we maintain $\delta_i^+\delta_i^- = 0$ all $i \in I$, as required in (2.2).

The latter (nonlinear) conditions need not be maintained throughout a series of iterations, however, since only equivalence at an optimum is wanted.1/ Moreover, the condition $\delta_i^+\delta_i^- = 0$ is

1/ See [7] and [29].

necessary for optimality so that any optimizing solution of (2.1) produces all that is wanted for an $x = x^*$ which is also optimal for (1.1).

We next observe that <u>any</u> choice of $x \epsilon X$ will also satisfy the constraints of (2.1) since each δ_i^+, δ_i^- choice is independent of every other such pair. In other words we may interpret each such $\delta_i^+, \delta_i^- \geq 0$ as a measure of distance (or discrepancy) relative to the i^{th} goal. Thus, the objective in (2.1), and hence (1.1) as well, is to effect choices of the decision variables, x_j, which minimize the sum of these discrepancies.

Evidently we may weight these discrepancies differently to obtain

$$\min \sum_{i \epsilon I} w_i^+ \delta_i^+ + w_i^- \delta_i^-$$

(3.1) s.t.

$$\sum_{j=1}^{n} a_{ij} x_j - \delta_i^+ + \delta_i^- = g_i$$

$$\delta_i^+, \ \delta_i^- \geq 0$$

where the w_i^+, w_i^- are non negative constants representing the <u>relative</u> weight to be assigned to positive and negative deviations for each relevant goal g_i, $i \epsilon I$.

Instead of such relative weights we can also use "preemptive weights" -- or, more generally, we may use both as in,

$$\min \sum_{i \in I} M_i \sum_{k=1}^{n_i} \left(w_i^+(k)\,\delta_i^+(k) + w_i^-(k)\,\delta_i^-(k) \right)$$

s.t.

$$\sum_{j=1} a_{ij}(k) x_j \quad - \delta_i^+(k) + \delta_i^-(k) = g_i(k) \tag{3.2}$$

$$\delta_i^+(k),\ \delta_i^-(k) \geq 0$$

where $w_i^+(k)$, $w_i^-(k) \geq 0$ represent the relative weights to be assigned to each of the $k=1,\ldots,n_i$ different classes within the i^{th} category to which the non-Archimedean transcendental value M_i is assigned. These constants M_i are defined to produce the desired preemptive properties. These "preemptions" are interpreted to mean that no substitutions across categories can be admitted[1/], and this is accomplished by writing

$$M_i \gg M_{i+s} \tag{4.1}$$

to mean that no real number α, however large, can produce

$$\alpha M_{i+s} \geq M_i \quad . \tag{4.2}$$

Thus, a fortiori, no combination of relative weights and $\delta_i^+(k)$ or $\delta_i^-(k)$ values in (3.2) can produce a substitution across categories in the process of choosing a minimizing $x \epsilon X$.

[1/] See e.g., Jaaskeleinen [30] for application to financial planning. See [12] and [13] for applications to budgeting in public health and drug control programs.

Moving from relative to preemptive priorities via such weighting systems does not end the possibilities. We may also continue to so-called absolute priorities introduced via the constraints. For instance, we may require that $\delta_i^- \leq \delta_r^-$ when we want to assure that deviations below the i^{th} goal will never exceed those for the r^{th} goal. Furthermore we can ensure this absolutely, via the indicated constraint, even if the r^{th} goal is assigned a preemptive value in the functional -- provided we do not violate the condition $\delta_i^+ \delta_i^- = 0$ for any $i \epsilon I$.

We shall not explore these possibilities in depth, as already indicated, but we shall briefly delineate other possibilities and other problems from time to time as in the case, e.g., of goal interval programming, which is developed in the next section.

3. Goal Interval Programs

Figure 1 will provide a start for moving from "goal" to "goal-interval programming". The graph for the "goal programming functional", in the single variable x, reaches its minimum when $x = g$ while for the "goal interval functional" the same minimum value obtains for all $g_1 \leq x \leq g_2$. However, the slopes of the two different functions, given by k_1 and k_3, are the same as those for the corresponding goal program, in this case, for which also $k_2 = 0$.

GRAPHS FOR GOAL PROGRAMMING
AND GOAL INTERVAL FUNCTIONS

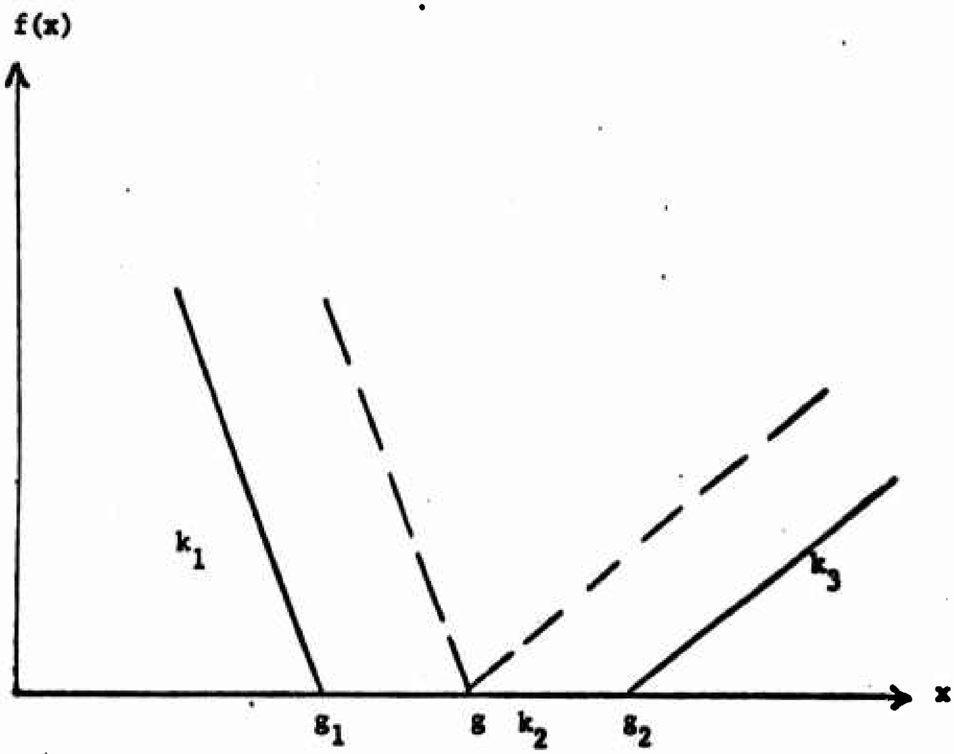
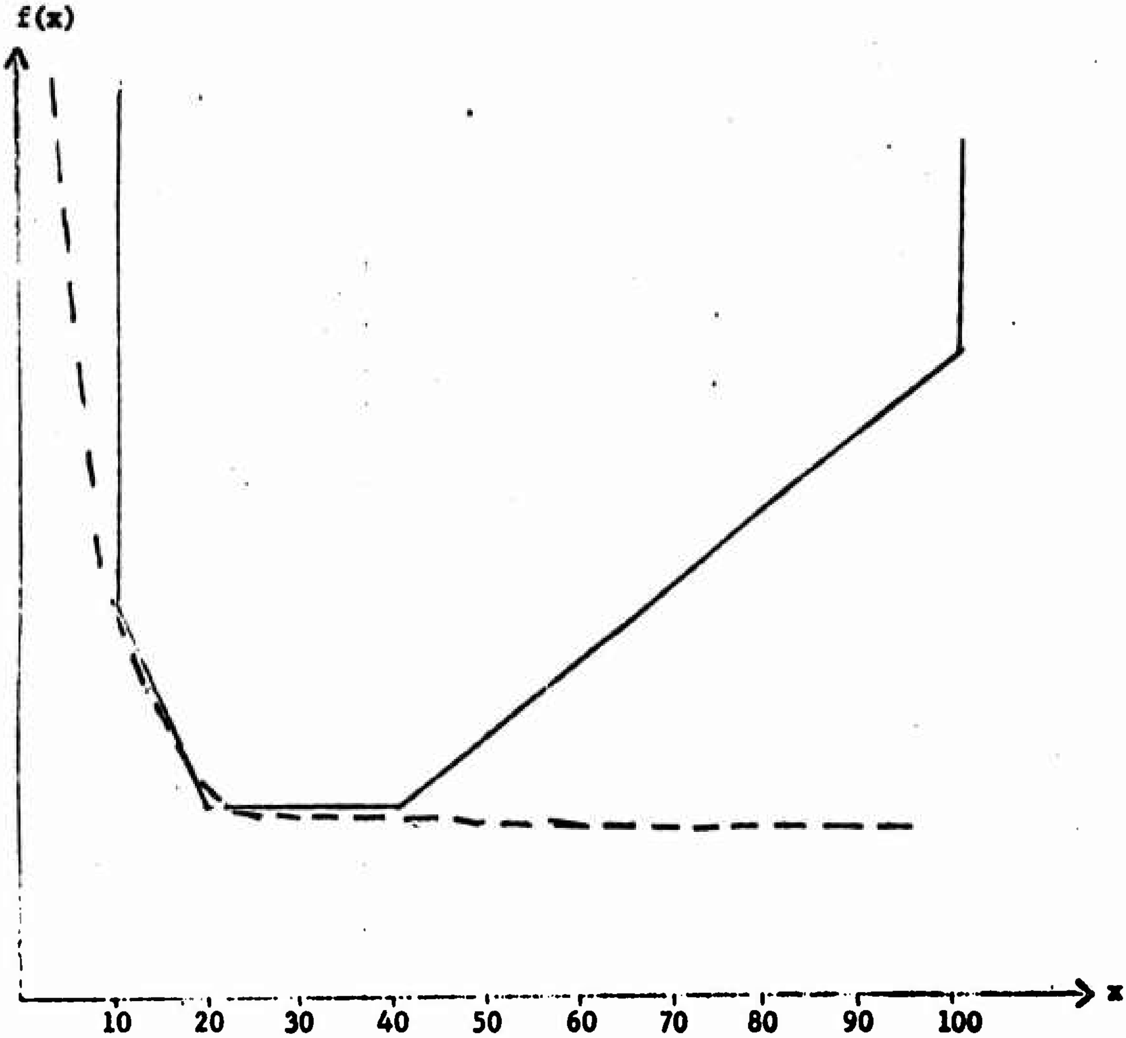


Figure 1

Figure 2, which arose in an application of goal interval programming to the Marine Environmental Protection program of the U. S. Coast Guard provides a concrete application of what may be wanted. Here the variable x refers to the percent of transfer operations (i.e., the transfer of oil from tank, barge or vessel to shore facility or vice versa) which are to be monitored. The broken line (here drawn freely) is intended to represent the relation between spill incidence (total spill volume divided by total volume transferred) and the percentage of transfer operations monitored. The solid (polygonal) curve represents the goal functional -- partly reflecting degree of confidence in the assumed (broken line) relationship in various regions and partly reflecting the subjective judgments and preferences of MEP (Marine Environmental Protection) program management.

APPROXIMATION TO EFFECTIVENESS CURVE
FOR TRANSFER OPERATIONS MONITORED



x = % Transfer Operations Monitored

f(x) = Measure of Ineffectiveness, i.e., Spill Incidence

Source [11].

Figure 2

We may observe that the goal interval, which extends from $x = 20\%$ to $x = 40\%$ in Figure 2 need not be at $f(x) = 0$ -- as was the case in Figure 1. Moreover, this interval and the value assigned for the goal functional may be balanced, along with the "slope preferences", by reference to other competing programs, and so on.

To deal with these and similar problems in a way which maintains contact with the preceding developments in goal programming, we articulate the following:[1/]

Theorem 1: Any polygonal (i.e., piecewise linear and continuous) function, $f(x)$, may be represented

$$f(x) = \sum_{j=1}^{N} \alpha_j \left| x - g_j \right| + \beta x + \gamma$$

where

$$\alpha_j = \frac{k_{j+1} - k_j}{2}$$

$$\beta = \frac{k_{N+1} + k_1}{2}$$

[1/] From [11].

and

$$\gamma = a_1 - \sum_{j=1}^{N} \alpha_j g_j = \ldots$$

$$= a_r + \sum_{j=1}^{r-1} \alpha_j g_j - \sum_{j=r}^{N} \alpha_j g_j = \ldots$$

$$= a_N + \sum_{j=1}^{N-1} \alpha_j g_j - \alpha_N g_N =$$

$$= \frac{a_{N+1} + a_1}{2},$$

in which

$$f(x) = k_r x + a_r, \qquad g_{r-1} \leq x \leq g_r,$$

so that k_r is the slope and a_r the intercept constant for the corresponding linear function in the indicated section of the curve initiated at $x = g_{r-1}$, where $k_{r-1}x + a_{r-1}$ intersects $k_r x + a_r$, and terminated at $x = g_r$ where the latter intersects $k_{r+1}x + a_{r+1}$.

Turning to Figure 3 we see a situation such as might be encountered for any $f_i(x_i)$. Here the slopes are indicated by the constants k_{ij}, $j=1,\ldots,N+1$, with the goal interval for this $f_i(x_i)$ located somewhere between the abscissa at g_{i3} and g_{iN-1}.

GRAPH FOR GOAL INTERVAL FUNCTIONAL
WITH VARYING SLOPES

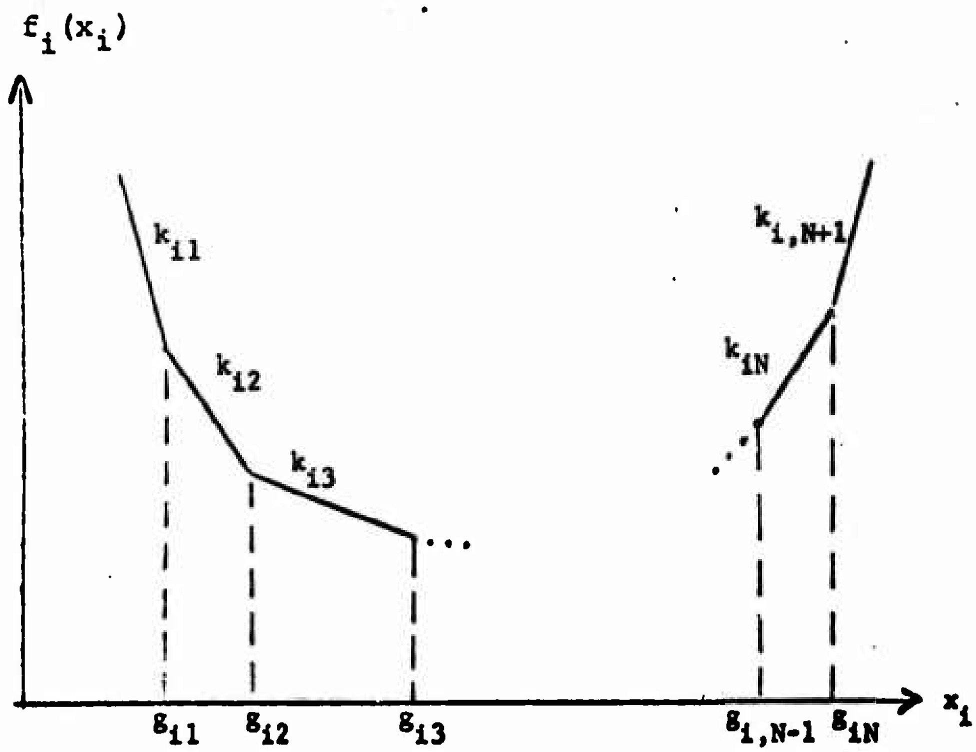


Figure 3

Since the constants γ_i do not enter into the optimizing choices we can simplify matters by omitting them from explicit consideration. Therefore, we define

(5) $$y_i = f_i(x_i) - \gamma_i$$

and write our general goal interval programming formulation as

$$\min z = \sum_{i=1}^{m} w_i y_i = \sum_{i=1}^{m} w_i \sum_{j=1}^{n_i} \alpha_{ij}(\delta_{ij}^{-} + \delta_{ij}^{+}) + \beta_i x_i$$

(6) $$\text{with } g_{ij} = \delta_{ij}^{-} - \delta_{ij}^{+} + x_i, \text{ all } i, j.$$

Here g_{ij}, a constant, is assigned to the j^{th} intervalized goal segment, $j=1,\ldots,n_i$, for y_{ij} controlled by the decision variable x_i. The values for δ_{ij}^{+}, $\delta_{ij}^{-} \geq 0$ represent deviations from g_{ij} with, of course, $\delta_{ij}^{+}\delta_{ij}^{-} = 0$, as before. The choices, also as before, are to be effected from a set X, defined by additional linear inequality constraining relations which we do not (for the moment) write explicitly.

The point to be emphasized is that the theory used in the preceding section to provide access to an ordinary linear programming problem is also applicable here. This means that the computational power of linear programming is also available along with the very sharp duality relations and the rich array of interpretative possibilities that such access brings with it.

Now we might observe that the weights $w_i \geq 0$ provide relative priorities between the goal functionals, $f_i(x_i)$. Here, however, the slope differences given by

(7) $$\alpha_{ij} = \frac{k_{ij+1} - k_{ij}}{2}$$

must also be considered. Evidently the differences between these slopes may alter the relative priorities between the corresponding goal functionals in various ranges so that (as in the Coast Guard's MEP program) these possibilities must also be considered when choosing these w_i values.

The goal interval functional in Figure 3 is convex, but this need not be the case. The functionals represented in Theorem 1, however, are not necessarily suitable for use as goal functionals unless they satisfy the conditions set forth in the following.[1]

Theorem 2: A necessary and sufficient condition for f(x) to be a goal functional is

$$\sum_{j=1}^{N} |\alpha_j| \geq |\beta| \quad ,$$

since then the choice of the variable x will not be oriented away from the goals by the values assumed by βx in the indicated extremizations. In concluding this section in a manner consistent with the preceding one, we might therefore say that here the α_j and β values are interpreted to include the weights as well as the slope coefficients for the functionals which are represented by the corresponding linear segments.

[1] See [14] .

4. Alternate and Combined Representations

Discovering that functions of the above varieties could be dealt with by the adjacent extreme point methods of linear programming very naturally awakened interest in exploring the "nonlinear power" of these methods.[1] That is, this interest was directed to ascertaining the extent to which these methods, devised for solving ordinary linear programming problems, could be used to deal with nonlinear problems as well. Precise delineations were wanted as to the character of such problems and the alterations, if any, that might be needed in the ordinary algorithms such as, e.g., the simplex method.

We are here concerned only with functions, $f(x)$, which are suited to goal (and goal interval) programming. Since these functions will generally be convex, we may represent them as

(7) $$f(x) = \sum_{j=1}^{N+1} k_j x_j + d$$

where, as before, k_r is the slope of the r^{th} line segment corresponding to $k_r x + a_r$, and

(8.1) $$x = \sum_{j=1}^{r} x_j$$

for

$$x_1 \leq g_1$$

(8.2 $$0 \leq x_j \leq g_j - g_{j-1},\ j=2,\ldots,N$$

$$0 \leq x_{N+1}$$

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[1] See Chapter X in [9]. See also [34].

with

$$(8.3) \qquad x = x_r + g_{r-1}$$

when $g_{r-1} \leq x \leq g_r$. That is, we utilize the convexity property to obtain the order of entry, under minimization[1/], which produces

$$(8.4) \qquad x = g_1 + \sum_{j=2}^{r-1} (g_j - g_{j-1}) + x_r$$

$$= g_{r-1} + x_r$$

for any $r=2,\ldots,N$. For $x \leq g_1$ we have, $x=x_1$ and, similarly, at the other end $x = g_N + x_{N+1}$.

We now want to compare this with the absolute value function

$$(9) \qquad f^a(x) = \sum_{j=1}^{N} \alpha_j \left| x - g_j \right| + \beta x + \gamma$$

which, as we saw, via Theorem 1 of the preceding section, provides an alternate representation that can be used for this same (piece-wise linear) function.

We want now to develop formulas that will enable us to move from (7) to (9), and <u>vice versa</u>, whenever this might suit our needs. Once again, however, we have recently published work (or work which is soon to be published [14]) to call on and hence we may content ourselves with a sketch as follows. For

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[1/] The pertinent theorem, if wanted, may be found in the Appendix to [16].

$$x = x_1 \leq g_1$$

$$f^a(x) = \sum_{j=1}^{N} \alpha_j \, |x-g_j| + x + \gamma$$

$$= \sum_{j=1}^{n} \alpha_j (g_j - x_1) + \beta x_1 + \gamma$$

$$= (\beta - \sum_{j=1}^{n} \alpha_j)\, x_1 + \sum_{j=1}^{n} \alpha_j g_j + \gamma$$

and

$$f^s(x) = k_1 x_1 + d.$$

Over the entire interval $x \leq g_1$ equality holds between these expressions for $f^a(x)$ and $f^s(x)$. Hence we may appeal to the property of analyticity and obtain

$$k_1 = \beta - \sum_{j=1}^{n} \alpha_j$$

$$d = \sum_{j=1}^{N} \alpha_j g_j + \gamma .$$

Using Theorem 1 of the preceding section we find that $d = a_1$ the intercept value for the first linear segment, viz., $k_1 x + a_1$, which applies for $x \leq g_1$.

Continuing our sketch we proceed to $g_{r-1} \leq x \leq g_r$ and obtain

$$f^s(x) = k_r x_r + k_1 g_1 + \sum_{j=2}^{r-1} k_j (g_j - g_{j-1}) + a_1$$

as well as

$$f^a(x) = \sum_{j=1}^{r-1} \alpha_j (x-g_j) - \sum_{j=r}^{N} \alpha_j (x-g_j) + \beta x + \gamma$$

$$= \left(\sum_{j=1}^{r-1} \alpha_j - \sum_{j=r}^{n} \alpha_j + \beta \right) x_r + \left(\sum_{j=1}^{r-1} \alpha_j - \sum_{j=r}^{n} \alpha_j + \beta \right) g_{r-1} +$$

$$+ \left(\sum_{j=r}^{n} \alpha_j g_j - \sum_{j=1}^{r-1} \alpha_j g_j + \gamma \right).$$

Thus, reasoning as before, we have

$$k_r = \sum_{j=1}^{r=1} \alpha_j - \sum_{j=r}^{N} \alpha_j + \beta .$$

Similarly

$$k_{r+1} = \sum_{j=1}^{r} \alpha_j - \sum_{j=r+1}^{N} \alpha_j + \beta$$

and therefore

(10.2) $$k_{r+1} - k_r = 2\alpha_r, \quad r=2,\ldots,N,$$

as we saw in the preceding section.

Given knowledge of the α_j, g_j, β and γ we are evidently in a position to determine the k_j and $d = a_1$ values for (7). Conversely, given the latter values we are evidently in a position to determine the α_j, β and γ values for (9) via

$$\alpha_j = \frac{k_{j+1}-k_j}{2}$$

(10.3)
$$\beta = k_1 + \sum_{j=1}^{N} \alpha_j = \frac{k_{N+1}+k_1}{2}$$

$$\gamma = d - \sum_{j=1}^{N} \alpha_j g_j = a_1 - \sum_{j=1}^{N} \left(\frac{k_{j+1}-k_j}{2}\right) g_j.$$

We now have a way of relating the developments in this and the immediately preceding section. In particular, we can move between the representations for $f^s(x)$ and $f^a(x)$ as given in (7) and (9) at our pleasure. Evidently we can immediately extend all parts of this analysis to functions which are sums of functions $f_i^s(x_i)$ and $f_i^a(x_i)$ with further segmentation and weighting, as was done in the preceding section, to relate them to slopes and goals g_{ij}, $j=1,\ldots,n_i$ for each $i\epsilon I$.

Before proceeding to the next section, however, it may be useful to relate what has now been accomplished to the earlier discussion of goal programming in Section 2 as well. For this we recall the definition of "separability"[1] -- viz., $F(y_1,\ldots,y_n)$ is separable if

$$F(y_1,\ldots,y_n) = \sum_{j=1}^{n} F_j(y_j).$$

[1] See pp. 351 ff. in [9] for a discussion of the use of this property in securing approximations to a variety of nonlinear functions.

In other words it can be represented as a sum of functions each involving only one variable in its argument. Then we extend this to "weak separability" which comprehends cases in which $F(y_1,\ldots,y_n)$ may be brought into separable form by suitable linear transformations. That is, via transformations of the form

$$x_i = \sum_{j=1}^{n} a_{ij} y_j, \qquad i \varepsilon I$$

we can represent $F(y_1,\ldots,y_n)$ via the expressions

$$\sum_{i \varepsilon I} f_i(x_i)$$

when $F(y_1,\ldots,y_n)$ is weakly separable.

As a case in point consider the expression (1.1) in Section 2 which we represent

$$F(y_1,\ldots,y_n) = \sum_{i \varepsilon I} \left| \sum_{j=1}^{n} a_{ij} y_j - g_i \right| .$$

This may be replaced by

$$\sum_{i \varepsilon I} f_i(x_i) = \sum_{i \varepsilon I} \left| x_i - g_i \right|$$

(11) with

$$x_i = \sum_{j=1}^{n} a_{ij} y_j, \qquad i \varepsilon I$$

so that we can in this way bring the functionals of Section 2 into the same form as those which we have been examining in this and the immediately preceding section. We may therefore hereafter assume that the linear transformations needed to do this, as in (11), are comprehended by the set X defined by the restrictions for the admissible choices of vectors, x.

5. Explicit Solutions

We may again utilize relatively recent results (see [14]) to obtain certain simplifications and improved solution precedures. Indeed, for certain classes of problems we can write the solutions explicitly via a development which proceeds as follows. First, we formulate the functions to be considered as $f(x) = \sum_i f_i(x_i)$ in the problem

$$\min f(x) = \sum_i f_i(x_i)$$

with

$$b_o \geq \sum_i x_i \tag{12}$$

$$b_i \geq x_i \geq a_i,$$

where the $f_i(x_i)$ are goal functionals, and so are monotone decreasing for $x_i \leq g_i$ and monotone increasing for $x_i \geq g_i$. Such functions need not be convex, although we shall restrict our examples to this class in the discussion that follows.[1] Similarly the results we shall obtain also extend to a wider class of constraints than those given in (12) as we shall also show.

To proceed with our development we first state[2]

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[1] Cf. [14] fcr a discussion of the kinds of functions comprehended and ways in which they might be utilized.

[2] From [14].

Theorem 3: If x satisfies the constraints in (12) with some of its components $x_{i_o} > g_{i_o}$ then $\bar{x}$ with components

$$\bar{x}_i = \begin{cases} x_i, & i \neq i_o \\ \max(a_{i_o}, g_{i_o}), & i = i_o \end{cases}$$

also satisfies the constraints in (12) with

$$f(\bar{x}) \leq f(x).$$

In other words, this theorem asserts that we do not worsen the value of the functional and we continue to satisfy the constraints when we replace x by $\bar{x}$.

The proof of the above theorem is relatively straightforward and so we do not reproduce it here.[1] Instead we illustrate its use in the case where each

(13.1) $$f_i(x_i) = \mu_i | x_i - g_i |$$

and these $\mu_i \geq 0$ represent weighted α_i as in (5) and (6). Furthermore, we assume without loss of generality that these constants are indexed so that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_i \geq \ldots \geq \mu_n$, say. By

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[1] See [14] .

virtue of Theorem 3 and the minimizing objective we can modify the interval constraints via

(13.2) $$x_i \leq \max (a_i, g_i) \equiv \hat{b}_i.$$

Also if $g_i \leq a_i$ for any i then we set $x_i = a_i$ and reduce the problem by omitting these variables. We thus attain the following problem

$$\min \quad \sum_i \mu_i \mid x_i - g_i \mid$$

with

(14) $$\hat{b}_o \geq \sum_i x_i$$

$$\hat{b}_i \geq x_i \geq a_i$$

in place of (12).

The modifications noted in (13.2) ff. are assumed to be incorporated in (14). Because of these modifications, however, we now have $x_i \leq g_i$, all i, and this implies

$$\mid x_i - g_i \mid = g_i - x_i.$$

Hence we can replace this already reduced non-linear problem (14) with an ordinary interval linear programming problem, viz.,

$$\max \sum_i \mu_i x_i$$

with

(15)
$$\hat{b}_o \geq \sum_i x_i$$
$$\hat{b}_i \geq x_i \geq a_i$$

without any additional constraints.

If we assume that $\sum_i a_i \leq \hat{b}_o$, as required for consistency, then we may immediately write the optimum solution to (15.1) as

(16.1)
$$x_i^* = \hat{b}_i, \quad i=1,\ldots,k-1$$
$$x_k^* = \hat{b}_o - \sum_{i=1}^{k-1} \hat{b}_i - \sum_{i=k+1}^{n} a_i$$

where k is the smallest positive integer such that

(16.2)
$$\hat{b}_o - \sum_{i=k+1}^{n} a_i \geq \sum_{i=1}^{k-1} \hat{b}_i$$
and
$$\hat{b}_o - \sum_{i=k+1}^{n} a_i < \sum_{i=1}^{k} \hat{b}_i \quad .$$

A proof that this is the optimizing solution is obtained by considerations of duality and recourse to the "regrouping principle,"[1] as explained in [14]. Here we need only make it plausible by

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[1] See [16] for a discussion of this principle.

observing that the ordering is $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$. Hence one starts with the consistency condition $\hat{b}_o - \sum_{i=1}^{n} a_i \geq 0$ and replaces a_1 with $\hat{b}_1$, the upper limit for x_1, if possible. This process is continued until the second condition in (16.2) is encountered. This first encounter determines the integer k with the optimizing solution (16.1) then resulting. Hence, starting with (12) and applying Theorem 3, as has just been indicated, we are able to write the solution explicitly, as in (16.1) and (16.2) along with the values $x_i = a_i$ obtained in the manner shown immediately after (13.2)

Of course, no great practical interest per se attaches to problems such as (12) where all save one constraint is in an interval form which bounds the variables, one at a time. Extensions are possible, however. For instance, Theorem 3 evidently applies when the constraints which involve more than one variable at a time have only non-negative coefficients.

Other extensions which are also possible are treated in detail in [14]. We therefore conclude this section only with a specific (but truncated) problem encountered in the development of goal programming models for use by the U. S. Navy's Office of Civilian Manpower Management.[1/] We write this example problem as

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[1/] See [17], p. II-18.

$$\min \quad f(x,y) \equiv |x_1-30| + |x_2-200| + |y_1-70| + |y_2-300|$$

with

$$
(17.1)\qquad
\begin{array}{rlllllll}
\infty & \geq & x_1 & & & & \geq 0 \\
\infty & \geq & & x_2 & & & \geq 0 \\
3{,}000 & \geq & 15x_1 & + 13x_2 & & & \\
\infty & \geq & & & y_1 & & \geq 44 \\
\infty & \geq & & & & y_2 & \geq 143 \\
4{,}000 & \geq & & & 15y_1 & + 13y_2 &
\end{array}
$$

where the subscripts refer to two different types of manpower which are further distinguished by x and y according to the period being considered.

By virtue of Theorem 3,

$$|x_i-g_i| = g_i-x_i, \quad |y_j-g_j| = g_j-y_j$$

and $x_i \leq \max\ (a_i,g_i)$, $y_j \leq \max\ (a_j,g_j)$ for i, j=1,2. Hence we may replace (17.1) by

$$\max \quad x_1 + x_2 + y_1 + y_2$$

with

$$
(17.2)\qquad
\begin{array}{rlllllll}
30 & \geq & x_1 & & & & \geq 0 \\
200 & \geq & & x_2 & & & \geq 0 \\
3{,}000 & \geq & 15x_1 + & 13x_2 & & & \\
70 & \geq & & & y_1 & & \geq 44 \\
300 & \geq & & & & y_2 & \geq 143 \\
4{,}000 & \geq & & & 15y_1 + & 13y_2 &
\end{array}
$$

which is an ordinary linear programming problem.

In fact we may split (17.2) into two smaller linear programming problems -- *viz.*,

$$\max \quad x_1 + x_2$$

with

$$\begin{aligned} 30 &\geq x_1 \geq 0 \\ 200 &\geq x_2 \geq 0 \\ 3{,}000 &\geq 15x_1 + 13x_2 \end{aligned} \tag{18.1}$$

and

$$\max \quad y_1 + y_2$$

with

$$\begin{aligned} 70 &\geq y_1 \geq 44 \\ 300 &\geq y_2 \geq 143 \\ 4{,}000 &\geq 15y_1 + 13y_2 \end{aligned} \tag{18.2}$$

The explicit solutions of these problems -- and hence of (17.1) -- are immediately at hand as

$$x_1^* = 400/15 \quad , \quad x_2^* = 200$$

$$y_1^* = 44 \quad , \quad y_2^* = \frac{4{,}000-660}{13} \doteq 257,$$

Alternatively we may put these problems in the form of (12) and thereby provide access to (16.1) and (16.2) by inserting

$$\hat{x}_1 = 15x_1, \quad \hat{x}_2 = 13x_2$$
$$\hat{y}_1 = 15y_1, \quad \hat{y}_2 = 13y_2$$

in (18.1) and (18.2), respectively.

In either case we have obtained the illustration which was wanted to show how Theorem 3 may be used to obtain an explicit solution for the goal programming problem (17.1). As observed earlier, a specific delineation of other extensions and how they can be effected is given in [14] and will not be developed here.

6. Ratio Forms

We now turn to other metrics and other functionals for the alternatives they can provide. For instance, JHS Kornbluth in [32] considers the problem

$$(19.1) \qquad \min \sum_{i=1}^{m} \left| \frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i} - \rho_i \right|$$

where α_i, β_i and ρ_i are scalars and the superscript T represents transposition on the column vectors c_i and d_i. The set

$$(19.2) \qquad X = \left\{ x: Ax=b, \ x \geq 0 \right\}$$

which defines the admissible choices is assumed to be non-empty with A an mxn matrix and x and b as column vectors which are nx1 and mx1, respectively.

As Kornbluth indicates, utilizing the developments of Section 3, above, produces a set of nonlinear constraining relations which may require recourse to special algorithms and specially arranged computer codes.[1/] But other possibilities and other metrics are available and they should not be overlooked when they might offer clear advantages. The Chebychev metric is one such metric which, as we shall see, provides access to a series of ordinary linear

[1/] This may be mitigated -- also as Kornbluth notes -- when the fractional functionals in (19.1) have special features or when they are accorded special properties such as those associated with non-Archimedean order, etc.

programming problems with related algorithmic advantages, including ready access to available computer codes. In addition to the algorithms that we shall provide via this metric, it also seems to have a certain natural appeal for problems of equity (or equality) as observed by Vogt in his development of an "Equal Employment Opportunity Index" for use by corporate management or as observed by Charnes, Cox and Lane in their development of a model [18] for allocating state funds to educational institutions.

To bring the problem (using the Chebychev, or C, metric)[1/] into the form in which we want to deal with it, we replace (19.1) with

$$\text{(20)} \qquad \min \lambda$$

$$\text{with} \quad \lambda - \left| \frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i} - \rho_i \right| \geq 0 , \qquad i=1,\ldots,m,$$

where the minimizing value of λ, a scalar, is determined by the choices of x in accordance with (19.2). Assuming, with Kornbluth, that $d_i^T x + \beta_i > 0$ for all admissible value of x we may evidently write the $i=1,\ldots,m$ goal constraints as

$$\text{(21)} \qquad \lambda (d_i^T x + \beta_i) \geq \left| c_i^T x + \alpha_i - \rho_i \, (d_i^T x + \beta_i) \right| .$$

The resulting model is nonlinear and nonconvex but, as we shall show, it can be submitted to a linear programming development which provides a new and simpler alternative to those that have

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1/ See Appendix A in [9].

heretofore been available. First, however, we show how bounding conditions can be used to obtain other simplifications and reductions. For, in many (if not most) applications one will have among the set $\{x : Ax=b,\ x \geq 0\}$ lower and upper bound constraints for the denominator in (20) -- viz.,

$$L_i \leq d_i^T x + \beta_i \leq U_i. \tag{22.1}$$

(In fact, in the manpower planning problems noted at the close of our last section, the range for the difference between the constants, $U_i - L_i$, is characteristically fairly small for a particular job, or site.) In any case, replacing $d_i^T x + \beta_i$ on the left side of (21) by L_i and U_i, respectively, clearly results in a linear programming problem[1/] with optimal values satisfying

$$\lambda_L^* \geq \lambda^* \geq \lambda_U^* \tag{22.2}$$

where λ^* is the global optimum of the original (exact) non-convex problem and λ_L^*, λ_U^* are the optima associated with the replacements for L_i and U_i, respectively.

We can do better than this, however. We can, in fact, generate a sequence of improving approximations which converge to the optimum. To do this we look at the problem from another point of view. Suppose, for instance, that one wishes to be no more than a

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1/ The remaining absolute value terms in the constraints can be replaced by pairs of linear inequalities as described on p. 460 of [9] .

predetermined value, $\bar{\lambda}$, away from the indicated goals. One could then set up the problem.

$$\max \ \mu$$

with

$$\mu \leq \bar{\lambda}\,(d_i^T x + \beta_i) - \left| c_i^T x + \alpha_i - \rho_i\,(d_i^T x + \beta_i) \right| \tag{23}$$

for $\{x: \ Ax=b, \ x \geq 0\}$ as in (20) but μ otherwise unrestricted. Associating solutions $\bar{x}$ with (23) and λ, x with (20), and using an asterisk to denote optimal values, we would then have the following possibilities.

(i) $\mu^* < 0$; $\bar{\lambda}, \bar{x}^*$ is not feasible for (20)

(ii) $\mu^* = 0$; $\bar{\lambda} = \lambda^*$ and $\bar{x}^*$ is optimal for (20)

(iii) $\mu^* > 0$; $\bar{\lambda}, \bar{x}^*$ is feasible for (20) with

$\lambda^* \leq \lambda^*(\bar{x}^*) < \bar{\lambda}$, an improvement over $\bar{\lambda}$

where $\lambda^*(\bar{x}^*) = \max\limits_i \left\{ \left| \dfrac{c_i^T \bar{x}^* + \alpha_i}{d_i^T \bar{x}^* + \beta_i} - \rho_i \right| \right\}$ and λ^* is optimal for (20).

In case (i) $\bar{\lambda}$ is too small to be attained. The predetermined value must be recast higher and the new linear program solved.

In case (ii), we are done.

In case (iii), we replace $\bar{\lambda}$ by $\lambda^*(\bar{x}^*)$ and proceed as before. Continuing in this manner we set up a simple sequence of linear programming problems which converges to a solution of the nonconvex programming problem (20). Q.E.D.

7. Conclusion

The completion of this survey will extend the discussion to other types of multiple objective optimizations. This will bring to the fore ideas like "functional efficiency" and "solution concepts to n-person non-zero sum games" which are necessarily more recondite than the topics covered here. It therefore seems best to treat them separately and in more detail -- with proofs supplied, as required -- in the paper that will form Part II of this survey.

In the present paper, i.e., Part I, we have tried to provide immediately useful (and easily used) results in ways which are consistent with the objectives set forth in the opening section of the present paper. The references that are herewith appended are intended to flush this out, wherever further detail is required, either to extend the present state of the art in research or to bring the ideas of goal programming to bear on problems of application which are within the states described in these references.

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13. ABSTRACT

This is Part I of a survey of recent developments in goal programming and multiple objective optimizations. In this part, attention is directed to goal programming with emphasis on the authors' own work (with others) in a variety of applications. This includes goal and goal interval programming, as well as characterizations which make it possible to obtain alternate representations and explicit solutions from special structural properties. Possibilities for various goal functionals are explored and delineated. One class of examples is developed in detail and an algorithm is supplied which utilizes sequences of ordinary linear programming problems to solve certain nonlinear and non-convex problems involving maxims of ratios of linear forms.

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