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LATE STAGE OF RAYLEIGH-TAYLOR INSTABILITY

David H. Sharp, et al

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CONTENTS

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David H. Sharp* and John A. Wheeler+

ABSTRACT

When a nearly flat fluid surface z = o is subject to a small sinusoidal distrubance of the form $\delta z = \delta z_0$ cos kx and when gravity - or an equivalent acceleration field - acts on the fluid in such a sense as to destabilize the surface, then the irregularities grow exponentially with time in the regime of small amplitudes where linear theory applies (Rayleigh). At larger amplitudes linear theory does not apply and the gas or magnetic field on the other side of the interface penetrates linearly with time into the fluid in the form of fingers or bubbles (G.I. Taylor). Considered here is the subsequent stage of successive mergers between these fingers to make larger and faster moving fingers. A primitive and tentative analysis of this merger process suggests the conclusion that

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*California Institute of Technology

+Princeton University

the advance of the leading - and growing - fingers relative to the position of the ideal interface is described by an acceleration which is of the order of 1/50 of the acceleration responsible for the instability. Observations are not available to check this conclusion, which however gives some indication of what effects to look for when experiments like those of Lewis and Allred and Blount are pushed into the regime of very high accelerations. To the extent that the statistical analysis is valid, it suggests that under the conditions assumed, a free slab of fluid cannot be accelerated to more than roughly 50 times its own thickness without suffering breakthrough.

I. INTRODUCTION AND SUMMARY

Magnetic fields can be envisaged today with a strength so great (much more than 10^5 gauss) that the resulting magnetic pressure can accelerate a metal object to a high velocity. We are concerned here with a situation where the field is so intense (field energy per unit volume of the order of heat of fusion per unit volume or $B \sim 10^6$ gauss) that the metal becomes liquid and the conditions are at hand for the development of Rayleigh-Taylor[1,2] instability with perturbations in the interface between magnetic field and liquid of ever increasing magnitude. Under the extreme conditions that can be envisaged no way presents itself to hold down this instability.

The surface tension of metal (Section II) is high but the accelerations considered are so extreme ($g \sim 10^9$ (or 10^{13}) cm/sec^2) that surface tension prevents only the growth of those instabilities whose reduced wavelength $\lambda\!\!\!^{-} = \lambda/2\pi$ is of the order of $\sim 10^{-4}$ (or 1c cm or less. The idea is discussed to increase the effective surface tension by covering the surface with an alternating laminar structure composed of two fluids of high mutual surface tension, thus enhancing the effective surface tension by a factor proportional to the number of lamina. However the factor of stabilization achievable in this way is not great enough to prevent the growth of interface irregularities with reduced wavelengths of micrometer and greater magnitudes.

An alternative idea is discarded in the present context without discussion: to reduce the rate of growth of interface disturbances with $\lambda\!\!\!^{-}$ less than some specified value $\lambda\!\!\!^{-}_1$, by interposing between the principal metal and the magnetic field a layer with thickness $\sim \lambda\!\!\!^{-}_1$, in which the density falls off smoothly from that of the principal metal to very small values(3). We limit attention to metal sheets so thin that this approach is not feasible.

Viscosity is a third effect which acts to inhibit the growth of Taylor instability, but a simple analysis shows that, in the extreme circumstances here considered, viscosity is ineffectual in diminishing the rate of growth of instabilities of reduced wavelength $\lambda\!\!\!^{-} = 10^{-4}$ cm (or 10^{-6} cm) or less.

In the absence of any effective way to inhibit them Taylor instabilities will grow beyond the small amplitude regime of exponential rise[1, 2] and beyond the regime of formation of spikes and fingers - and linear rise of these fingers or bubbles- considered by Taylor[4] and will come into a third regime where larger bubbles grow by capture of smaller ones (Fig. 1). An attempt is made in Section III at an order of magnitude statistical analysis of this late phase of Taylor instability. In default of relevant experimental information about the details of this late stage the theoretical analysis is necessarily uncertain. In so far as the simplified assumptions and approximations used in Section III are valid, they would suggest that the heads of the bubbles, or fingers, eat their way through the liquid with an effective acceleration -relative to the ideal interface- of the order of 1/50 of the bulk acceleration of the fluid itself. In other words under the conditions assumed it is concluded that it is impossible to propel a sheet of fluid metal to more than roughly fifty times its thickness without breakthrough. This estimate should be taken not as a reliable guide for the design of a propulsion system but as an indicator for the kind of observations and measurements that might be attempted in future experiments.

In so far as the statistical analysis of Taylor instability has any validity it provides a new approach to an old question: How far does it pay to go in trying to produce an ideal surface in an effort to reduce Taylor instability? The analysis of this point (Section IV) suggests that it is impossible ever to approach so close to ideality that further improvements are without

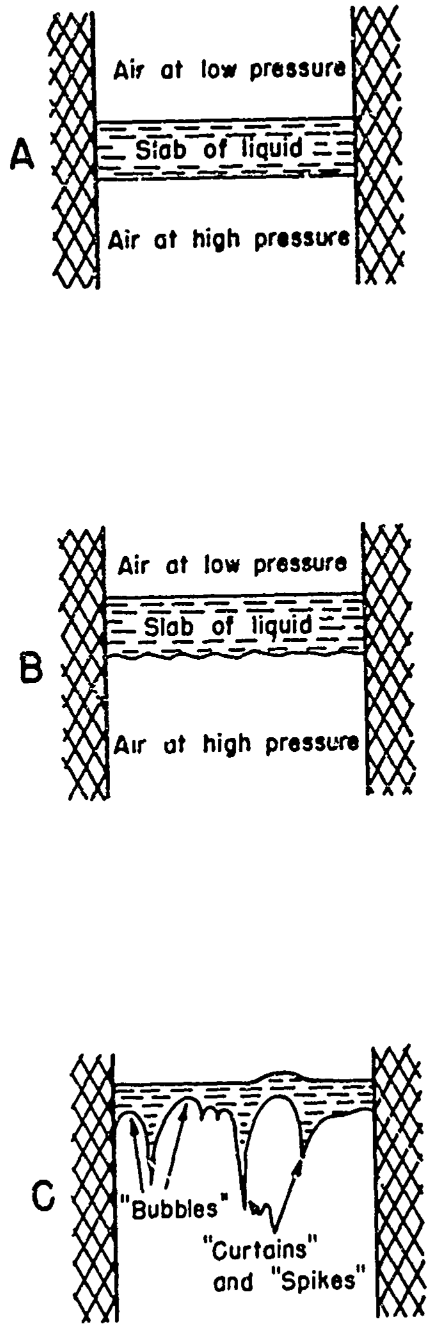

Fig. 1. Later or "non-linear" phase of Taylor instability (B $\rightarrow$ C) compared and contrasted with the earlier "linear" phase (A $\rightarrow$ B). In the linear phase, the shorter is the reduced wave length $\bar{\lambda}$ of a disturbance, the greater is its exponential rate of rise: $\alpha = (g/\bar{\lambda})^{\frac{1}{2}}$. This law of growth ceases to be valid, however, when the amplitude of the disturbance has become comparable with $\bar{\lambda}$. Also the shape of an initially exactly sinusoidal disturbance of the surface ceases to be sinusoidal when it has grown into the non-linear regime.

significant pay off. On this account electro-polishing is briefly recalled as one technique by which to improve the smoothness of a surface.

II. THE MINOR EFFECTS OF SURFACE TENSION AND VISCOSITY IN INHIBITING TAYLOR INSTABILITY AT HIGH ACCELERATION

When a fluid of density ρ and surface tension σ is plastered smoothly on the ceiling of a room a small disturbance in the surface

$$\delta z = \delta z_0 \cos kx \tag{1}$$

will grow exponentially in time in accordance with the formula

$$\delta z = \delta z_{00} \, e^{\alpha t} \tag{2}$$

or

$$\delta z = \delta z_{00} \cosh \alpha t \quad . \tag{3}$$

In a more complete discussion one allows for the effect of surface tension σ (erg / cm^2):

$$\alpha^2 = gk - (\sigma/\rho)k^3 \tag{4}$$

or the effect of viscosity (dynamic, η, g/cm sec; kinematic, $\nu = \eta/\rho$, cm^2/sec) or both in inhibiting the growth of the disturbances[5,6].

Experiments have confirmed the expected effect of surface tension. There is no reason to doubt that the expected effect of viscosity will appear when appropriate experiments are performed. Both effects one would like to increase dramatically if by so doing the growth of Taylor instability were effectively inhibited.

TABLE I. Reduced wavelength $\lambda\!\!\!^-_{CRIT}$ above which disturbances on water and copper are unstable against exponential growth of small amplitude disturbances. From Eq (14) of text, $K_{CRIT} = (g\rho/\sigma)^{1/2} = \lambda\!\!\!^-_{CRIT}{}^{-1}$

Acceleration	H_2O	Cu
Surface tension (g/sec^2)	73 (20°C)	1103 (1131°C)
Acceleration in Case I	10^9 cm/sec^2	10^9 cm/sec^2
$\lambda\!\!\!^-_{CRIT,1}$	2.7×10^{-4} cm	3.6×10^{-4} cm
Acceleration in Case II	10^{13} cm/sec^2	10^{13} cm/sec^2
$\lambda\!\!\!^-_{CRIT,2}$	2.7×10^{-6} cm	3.6×10^{-6} cm

The effective surface tension is non-isotropic when the push comes from a magnetic field with lines of force running in the y-direction parallel to the surface (yz-plane). From Faraday's picture of tensions acting along the lines of force it is clear (Chandrasekhar(6)) that this tension augments the natural surface tension,

$$\sigma_{eff} = \sigma_{METAL} + \left\{ \begin{matrix} 1/4\pi & cgs \\ 1/\mu_0 & mks \end{matrix} \right\} B^2 , \qquad (5)$$

for surface irregularities

$$\delta z = \delta z_0 \cos(k_x x + k_y y) \qquad (6)$$

whose circular wave number $\underset{\sim}{k} = (o, k_y)$ points exclusively along the magnetic lines of force. Under the conditions of interest here the magnetic fields are so strong, the augmentation of surface tension is so great and the stabilization thereby achieved is so marked that surface disturbances running in this direction will be disregarded. On the other hand, when the disturbance is described by a vector $\underset{\sim}{k}$ running in the x-direction the magnetic lines of force in effect behave like separate bundles of rubber tubing, every other bundle being raised and intermediate bundles depressed by the perturbation. In this case the magnetic field provides no assistance in stabilizing the surface and the effective surface tension is to be identified with the surface tension of the metal alone. To this extent it is irrelevant that the acceleration is brought about by a magnetic field instead of a gas. In another respect the difference is significant. In the magnetic case the pattern of spikes and bubbles will be drawn out along the magnetic lines of force in such a way as to create not spikes and bubbles but crests and troughs. However, the cross section of these topologically different types of irregularities we envisage to be not greatly different. Consequently, the factor $\sim 1/50$ in the statistical analysis of bubble growth is thought to be changed somewhat in absolute value but not in general import when one goes from the bubble case to the magnetic case. Therefore the bubble theory developed in Section III will not be revised on this account, and in most of the ensuing discussion the anisotropy will be disregarded, that is the Taylor instabilities will be treated as if they arose from pressure by a gas. This is presumably the case in which experiments can be carried out, to judge from the work of Lewis(7) and Allred and Blount(8) where, however, the accelerations were too small and the surface tension too great to allow opportunity for the later amalgamative phase of Taylor instability to develope. In contrast, for the accelerations listed in Table I the natural scale of dimensions for the first bubbles is so small that macroscopic observations will reveal, not these first bubbles, but only the much larger ones that arise from them by many generations of merger.

In principle, one can greatly increase the effective surface tension operative at the interface between two fluids, A (magnetic field) and C (molten copper), by replacing the simple interface by a composite interface (Fig. 2) built out of many thin lamina of liquid C and another liquid B (a plastic, for example), according to the pattern...AAAACBCB......CBCCCC. Ordinarily such a structure is difficult to produce and even after being produced will disappear through the mechanism of formation of drops and rearrangement into a structure such, for example, as

....AAAABBBCCCCCCC.... .

Fabrication and preservation of the laminar structure would seem much simpler in the case of a solid metal sheet which is destined to be converted into a liquid only at the time of sudden buildup of a powerful magnetic field just above its surface. Let the homogeneous metal sheet be replaced by a metal sheet of laminar

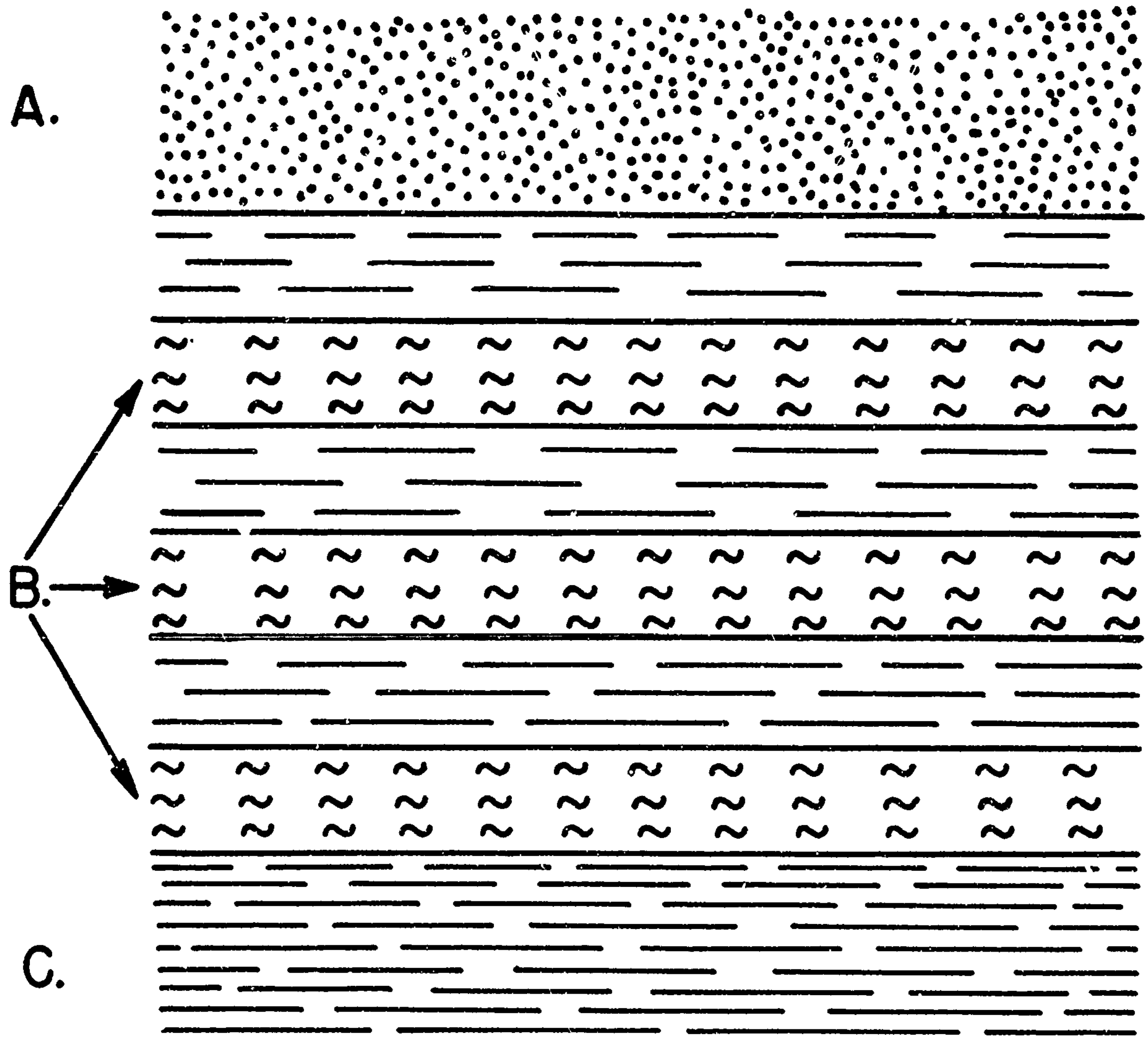

Fig. 2. Increase of effective surface tension at interface between C and A by interposition of many thin lamina of a third substance, B.

structure produced by alternate electroplating in two electrolytes of very different constitution or otherwise. Let the two substances be so chosen that their interfacial tension is as large as possible. Then the effective surface tension of the entire molten metal sheet with respect to deformations of reduced wavelength $\lambda\!\!\!^{-}$ of the order of the foil thickness or greater has been increased by a factor of the order of the number of lamina.

Consider for example (Table II) a metal sheet of thickness 10^{-2} cm^2 made of alternate sheets of metal and plastic each $\sim$ 100 atoms or $\sim$ 10^{-6} cm thick. The effective surface tension for small amplitude disturbances of reduced wavelength $\lambda\!\!\!^{-} >$ 10^{-2} cm will be increased above its normal value by a factor of the order $\sim N = 10^{-2}$ cm$/10^{-6}$ cm $= 10^4$; thus

$$\sigma_{effective} \sim N\sigma \sim 10^4\sigma \quad . \qquad (7)$$

TABLE II. Surface tension for certain liquid metal-organic compound and liquid metal-air interfaces; also estimates of the critical point temperatures for these metals (at which the surface tension goes to zero). The fourth column gives the calculated critical reduced wavelength $\lambda\!\!\!^{-}_{crit} = \lambda/2\pi$ (transition from stability for shorter wavelengths to instability for longer wavelengths) for a small amplitude disturbance in a molten metal sheet 10^{-2}cm thick of laminar construction with $N = 10^4$ lamina, under conditions where the acceleration is $\sim 10^{13}$ cm/sec^2. The column $\lambda\!\!\!^{-}_{fastest\ growth} = 3^{\frac{1}{2}}\ \lambda\!\!\!^{-}_{crit}$ gives the reduced wavelength for which small amplitude theory predicts the greatest growth constant, $\alpha = \alpha_{max} =$ (maximum value of $(gk - \sigma k^3/\rho\ )^{\frac{1}{2}} = (2g\ /\ 3\,\lambda\!\!\!^{-}_{fastest\ growth})^{\frac{1}{2}}$. This growth constant itself is tabulated in the next column. Consider the initial surface, prepared however so well, and Fourier analyze the irregularities in this surface, and enquire as to the typical amplitude associated with disturbances of the reduced wavelength $\lambda\!\!\!^{-}_{fastest\ growth}$. Let it be required, for the sake of illustration, that these disturbances not grow during a time interval of the order of $\sim 1\,\mu sec$ (for example) to the point (amplitude $\sim \lambda_{f.g.}$) where the non-linear effects of Taylor instability take hold. Then the initial amplitude A must be less than $\sim A_{max\ allowable} = \lambda_{f.g.} \exp[-\alpha_{f.g.} t]$. This preposterous number is tabulated in the last column. The impossibility to attain any such perfection in the surface shows the hopelessness of counting on surface tension to inhibit Taylor instability in the present circumstances.

Interface	T_{crit} (°K)	T(°K)	σ (erg/cm²)	λ_{crit} (cm)	λ fastest growth (cm)	α (sec⁻¹) fastest growth	A_{max} (cm) allowable
Mercury-Benzene		293	357	$\sim 1.7\times10^{-4}$	$\sim 2.9\times10^{-4}$	$\sim 1.5\times10^{8}$	$\sim 2.9\times10^{-4}\times e^{-[illegible]}$
Mercury-Air	> 1823	288	487	$\sim 1.9\times10^{-4}$	$\sim 3.3\times10^{-4}$	$\sim 1.4\times10^{8}$	$\sim 3.3\times10^{-4}\times e^{-1}$
Copper-H_2		1404	1103	$\sim 3.6\times10^{-4}$	$\sim 6.2\times10^{-4}$	$\sim 1.0\times10^{8}$	$\sim 6.2\times10^{-4}\times e^{-10}$
Aluminum-Air		973	840	$\sim 5.9\times10^{-4}$	$\sim 10.2\times10^{-4}$	$\sim 8.1\times10^{7}$	$\sim 1.0\times10^{-3}\times e^{-8}$

It is evident from the numbers for λ_{CRIT} in the table calculated for a number of laminations as great as $N = 10^4$ that it has not been of much use to laminate the metal sheet to inhibit Taylor instability! In addition, the metal sheet is thought of as driven by magnetic fields so great and comes to temperatures so high that it is carried from the liquid state past the critical temperature--where the surface tension vanishes--into the gaseous state where the normal concept of surface tension has to be abandoned.

The natural scale of dimensions at the beginning of the merger process - as distinguished from the scale of fabricational imperfections (which are overlooked here, see section IV) - is so small (Table I) and the accelerations so high, 10^9 (or 10^{13}) cm/sec^2 that it can well be asked (Table III) whether viscosity may not be more important than surface tension in governing the precise value of this natural scale of sizes.

TABLE III. Critical acceleration $g_{crit} = (\sigma/\rho)^3/\nu^4$.

Here σ is the surface tension (erg/cm^2), ρ is the density (g/cm^3) and $\nu = \eta/\rho$ is the kinematic viscosity (cm^2/sec). For accelerations much higher than g_{crit}, viscosity dominates over surface tension in determining the circular wave number $k_{f.g.}$ and wavelength $\lambda_{f.g.} = 2\pi/k_{f.g.}$ of the fastest growing small amplitude disturbances and the exponential growth constant α_{max} (sec^{-1}) of these Taylor instabilities. For accelerations less than g_{crit}, the surface tension dominates over viscosity in determining the circular wave number and the growth constant of the small amplitude disturbances which multiply at the highest rate: $k_{f.g.} \sim (g\rho/\sigma)^{1/2}$ and $\alpha_{max} \sim (g^3\rho/\sigma)^{1/4}$. When surface tension dominates it stabilizes small amplitude disturbances whose reduced wavelengths $\bar{\lambda}$ are of the order of ½ the reduced wavelength of the fastest growing disturbances, $\bar{\lambda}_{f.g.}$, and less. When viscosity dominates it does not stabilize small amplitude disturbances of any wavelength, but it does reduce the rate of multiplication of disturbances with $\bar{\lambda} < \bar{\lambda}_{f.g.}$ relative to what the growth constant would be in the absence of viscosity, $\alpha = (g/\bar{\lambda})^{1/2}$. Each material in the table is considered under circumstances where it is pushed by a medium (gas or magnetic field or radiation) of density negligible in comparison with the density of the pushed medium. The temperatures in the table can be converted to electron volts by dividing the conversion factor 1.15×10^4 °K/electron volt.

Pushed Medium	T(°K)	ρ	σ	ν	g_{CRIT} (cm/sec^2)
H_2O, Liq	293	1.00	73	0.010	3.9×10^{13}
Cu, liq.	1500	8.2	1103	0.004	9.5×10^{15}
Cu, gas	15000	1.0	—		—
D-T, fully ionized	2×10^7	2.0	—		—
H, fully ionized	dependent upon circumstances	1.6×10^{-24}	—		—

*The viscosity listed here for fully ionized hydrogen under conditions of interstellar density refers to the conditions of ideal laminar flow such as are assumed in conventional definitions of viscosity. In other words $\nu' = \eta/\rho$ has been estimated from the formula[9]:

$\nu = (\eta/\rho) = D \sim \frac{1}{4}(\pi m k T)^{1/2}/\pi\sigma^2\rho$, T is the temperature ($^{\circ}$K), k the Boltzman constant, $\pi\sigma^2$ is the collision cross section for a rigid spherical molecule, D is the coefficient of diffusion.

In actuality, the interstellar medium, like the earth's atmosphere, will typically be in turbulent motion already before it is subject to pressure from magnetic fields or radiation such as can generate Taylor instability. Under these conditions, the exchange of momentum between one part of the gas and another is brought about very little by kinetic theory motion of individual particles, such as are considered in the elementary theory of viscosity. The momentum exchange comes about to an enormously greater extent through eddies and vortices that carry whole masses of the medium from a layer with one average velocity to a layer with another average velocity. The effective viscosity to be used under such conditions is discussed, for example, in Landau and Lifshitz(10), and Hirschfelder, Curtiss and Bird(9).

Viscosity, like surface tension, decreases at first as the temperature of a molten metal is raised toward the critical temperature. However, contrary to the surface tension which goes to zero at the critical temperature, the viscosity goes only to a reduced value somewhere in the neighborhood of the critical temperature and pressure provided that the path pursued in the PT-plane leads near the critical point. As the temperature is increased to a figure much higher than the critical temperature the viscosity of the metal, like the viscosity of other gases, approaches Maxwell's law of proportionality to the square root of the temperature T (here and in the following expressed directly in energy units to eliminate the purely conventional

Boltzmann factor k and its possible confusion with the wave number k):

$$\eta \sim \sum_i \frac{M_i v_i}{\sigma_{TRANS,i}} \sim \sum_i \frac{(M_i T)^{1/2}}{\sigma_{TRANS,i}} \quad . \qquad (8)$$

Here M_i is the mass of an ion (i = 1) or electron (i = 2), v_i is a velocity typical of thermal agitation and $\sigma_{trans, i}$ is an effective cross section for transport such as is defined in diffusion theory(9). For very high temperatures it is no longer possible to treat this transport cross section as independent of velocity - and therefore as independent of temperature. Its fall with temperature makes the viscosity rise with a power of T higher than $\frac{1}{2}$. Table IV gives a very rough estimate of the viscosity of copper at a temperature $T \sim 1.3$ ev (15000 °K).

TABLE IV. Order of magnitude estimate of contributions to the viscosity of copper at $T \sim 1.3$ ev. It is assumed that the typical atom at this temperature has been dissociated into one Cu^{++} ion and two electrons.

Particle	Momentum $\sim (M_i T)^{\frac{1}{2}}$ (gcm/sec)	σ transport (cm^2)	Contribution to η (g/cm sec)
Electron	$\sim 0.5 \times 10^{-19}$	$\sim 10^{-17}$	$\sim$ 0.005
Cu^{++} ion	$\sim 150 \times 10^{-19}$	$\sim 10^{-16}$	$\sim$ 0.15

Total $\eta \sim$ 0.16 g/cm sec

Kinematic viscosity (at $\rho \sim 8 g/cm^2$) estimated as $\nu = \eta/\rho \sim 0.02$ cm^2/sec

Consider a sheet of copper vapor with the kinematic viscosity $\nu \sim 2 \times 10^{-2}$ cm^2/sec subject to acceleration by a pushing fluid of negligible density at a rate of $\sim 10^{13}$ cm/sec^2. Then in the small amplitude theory of Taylor instability those Fourier components are calculated to grow most rapidly which have a wave number of the order

$$k_{\text{fastest growth}} \sim (g/\nu^2)^{1/3} \sim \left(\frac{10^{13}\ \text{cm/sec}^2}{4\times10^{-4}\ \text{cm}^4/\text{sec}^2}\right)^{1/3} \sim 3\times10^5\ \text{cm}^{-1}. \quad (9)$$

They have a calculated exponential growth constant

$$\alpha_{max} \sim (g^2/\nu)^{1/3} \sim \left(\frac{10^{26}\ \text{cm}^2/\text{sec}^4}{2\times10^{-2}\ \text{cm}^2/\text{sec}}\right)^{1/3} \sim 1.7\times10^9\ \text{sec}^{-1}. \quad (10)$$

Here as in the case of surface tension, the scale of wavelengths at which the inhibiting effects are significant are far too short to make viscosity of any relevance at the scale of instabilities which are of concern.

III. PRIMITIVE STATISTICAL ANALYSIS OF REGIME OF BUBBLE AMALGAMATION

So far it has been enough to consider the rise of a single bubble because the starting assumption was complete periodicity with all bubbles identical. In actuality there will be small differences in size from bubble to bubble due to slight departures from ideality at the start of the acceleration of the interface. These differences will occur whether natural causes or machining determines the dominant wavelength. In the one case, the spacing is determined by the circumstance that the growth constant has a smooth maximum as a function of the wavelength. Therefore, even a few extra atoms here or there on the interface will favor at the start a scale of dimensions which here is a little less than the size for optimum growth and there a little greater. Similarly, in the case where machinery or other techniques have built into the initial surface irregularities with a unique characteristic length, there will nevertheless again occur small variations above and below this unique size from one incipient bubble to another due to the presence or absence of a few more atoms here or there if due to no more mundane cause!

A small fractional difference in size

$$\delta = (R_2 - R_1) / R_1 \tag{11}$$

between one growing bubble and another will cause a corresponding fractional difference $\delta/2$ in the rate of rise $v_i \sim (g R_i)^{1/2}$ of the two bubbles when they are fully developed. In consequence of this difference in rate of rise, one bubble will get ahead of its neighbor and dominate it. The two bubbles will come to constitute in effect a single bubble with some minor irregularities in shape (Fig. 3). This process of merger of two bubbles is a decisive feature of the later stages of Taylor instability. Clear evidence of repeated mergers is seen in typical photographs of Taylor instability, such as those taken by Lewis [7] and Allred and Blount [8].

It is necessary to examine in more detail the conditions for merger and the consequences of merger in order to arrive at anything approaching a rational estimate of the time required for

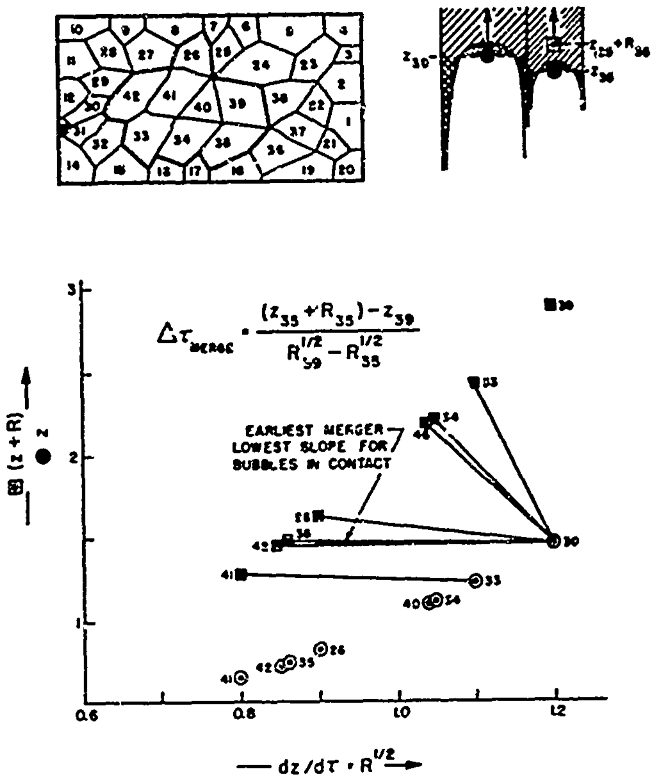

Fig. 3. Factors governing bubble merger in idealized model. Upper left: Representative bubble pattern as viewed from above. The cross sectional area associated with one bubble in this diagram is identified with πR^2 to define the effective radius R. Upper right: Bubble or finger No. 39 is idealized as engulfing finger No. 35 as soon as the effective height z_{39} of the larger one exceeds the effective height of the smaller one by the radius of the smaller one; that is, as soon as z_{39} (point ⊡) comes to equality with $z_{35} + R_{35}$ (point ⊡). The arrows recall that the rate of rise of the larger bubble is greater than the rate of rise of the smaller finger, as is required for capture of the smaller one: $dz/dt = 0.48\ (gR)^{\frac{1}{2}}$. Instead of the true time t it is convenient to use the "reduced time", $\tau = 0.48\ g^{\frac{1}{2}}t$, so that the rate of rise becomes $dz/d\tau = R^{\frac{1}{2}}$. Lower diagram: Graphical procedure to determine which pair of bubbles merge first. For simplicity, attention is limited to the system of eight fingers outlined by a heavy line in the upper left hand diagram. The calculated value of the reduced time interval $\Delta\tau$ required for No. 39 to capture No. 35 is determined by the slope (with sign reversed) of the designated line in the diagram. The line from No. 39 to No. 42 is irrelevant because it corresponds to two bubbles which are not in contact. For fingers in contact, the line with the lowest slope corresponds to the earliest merger. A new diagram of this type was constructed for each successive stage in the process of bubble merger in the process of working out the illustrative numbers in Table V.

bubbles to break through a slab of liquid of finite thickness.

No detailed hydrodynamic analysis of the merger process is available. Therefore, order of magnitude considerations are demanded. For this purpose, the actual situation will be replaced by an idealized model, here called the "Maine bubble model" because the first illustrative calculations on the model were made in Maine. In this model, each bubble will be considered to stake out for itself a certain area when it is projected on the ideal initial interface. This area will be assumed to remain constant during the time of rise of the bubble. The area will typically be irregular in outline. However, the rate of rise will be calculated as if the shape were that of an ideal circle with the same cross sectional area πR^2 (definition of R!). For the rate of rise, Taylor's formula will be adopted,

$$v = (0.48)\ (gR)^{\frac{1}{2}} \tag{12}$$

appropriate for the case of a pushing fluid of negligible density. Suitable changes can be made to correct the analysis when the pushing fluid has a density appreciable in comparison with that of the pushed fluid.

The volume of gas which fills the bubble above the ideal initial surface, divided by the cross sectional area of the bubble, defines an effective height z for the bubble which is somewhat lower than the vertex.

Two bubbles will be assumed in the model to merge into one when the larger one is sufficiently far ahead of the smaller one.

The lead required for merger will be very little when the smaller bubble is very small compared with its winning neighbor. The necessary lead will be greater when the effective radius of the smaller bubble constitutes a larger fraction of the radius of the larger one. This qualitative aspect of the merger process will be sharpened up in the model to the following rule:

1st rule. Two bubbles will merge when the larger one of them has a lead on the smaller as great as or greater than the radius of the smaller one:

$$z_+ - z_- \geqslant R_- \quad . \tag{13}$$

The other two rules adopted in the model for the merger process are:

2nd rule. Conservation of cross sectional area,

$$\pi R_m^2 = \pi R_+^2 + \pi R_-^2 \quad . \tag{14}$$

This rule provides a way to calculate the effective radius, R_m of the merged bubble.

3rd rule. Conservation of volume,

$$\pi R_m^2 z_m = \pi R_+^2 z_+ + \pi R_-^2 z_- \quad . \tag{15}$$

From rules 2 and 3 follows an expression for the effective height of the new bubble at the moment of formation:

$$z_m = \frac{R_+^2 z_+ + R_-^2 z_-}{R_+^2 + R_-^2} \quad . \tag{16}$$

As starting conditions for the model one has to specify the quantities R and z for each bubble at the time t = o and has to give, in addition, the <u>connectivity</u> of the bubble pattern, either by way of a qualitative diagram like Fig. 3, or by way of a listing of all bubble-to-bubble contacts which exist at the time t = o. In the example in Fig. 3, these connections are 1-2, 1-22, 1-21, 1-20; 2-3, 2-23, 2-22; · · · ; 37-38.

Table V illustrates the evolution in time of an arbitrary initial pattern of eight bubbles calculated according to the rules of the foregoing model.

TABLE V. Development of a sample pattern of eight bubbles (Fig. 3) as computed from rules 1,2,3 of the Maine bubble model. The initial conditions were arbitrarily adopted to correspond to (1) a fractional variation of the order of 50 percent in bubble cross sections and (2) an initial elevation for each bubble equal to the effective radius of that bubble. This condition is meant to correspond qualitatively to the circumstance that the larger disturbances have typically risen to a greater height than have the smaller ones at the stage when the bubbles have come into the non-linear regime of linear rise with time. However, the here assumed proportionality of z_i to the first power of R_i has no basis in theory nor any special significance. The assumption is made only to give definite and simple starting conditions for this particular problem. In the table, the quantity "reduced time" τ is an abbreviation for the quantity $(2/3)\, g^{\frac{1}{2}} t$. In terms of τ the formula for the velocity of rise becomes $dz/d\tau = R^{\frac{1}{2}}$. At each stage of the calculation, every contacting pair of bubbles is considered in turn, and for each such pair the reduced time to merger is calculated in each phase of development that one merger is listed which is calculated to occur earliest.

Bubble	26	33	34	35	39	40	41	42	Reduced time
R	0.81	1.21	1.10	0.74	1.44	1.09	0.64	0.72	for merger
z	0.81	1.21	1.10	0.74	1.44	1.09	0.64	0.72	$\Delta\tau$, $\tau = \sum \Delta\tau$
$dz/d\tau$	0.90	1.10	1.05	0.86	1.20	1.04	0.80	0.85	

Bubble	26	33	34	35-39	40	41	42	
R	0.81	1.21	1.10	1.62	1.09	0.64	0.72	
z	0.81	1.21	1.10	1.29	1.09	0.64	0.72	$\Delta\tau_{min.} = 0.23$
$dz/d\tau$	0.90	1.10	1.05	1.27	1.04	0.80	0.85	$\tau = 0.23$

Bubble	26	33-41	34	35-39	40	42
R	0.81	1.37	1.10	1.62	1.09	6.72
z	0.81	1.08	1.10	1.29	1.09	0.72
$dz/d\tau$	0.90	1.17	1.05	1.27	1.04	0.85

$\Delta\tau_{min} = 0.89$

$\tau = 1.12$

Bubble	26-35-39	33-41	34	40	42
R	1.81	1.37	1.10	1.09	0.72
z	0.80	1.08	1.10	1.09	0.72
$dz/d\tau$	1.34	1.17	1.05	1.04	0.85

$\Delta\tau_{min} = 1.13$

$\tau = 2.25$

Bubble	26-35-39	33-41-42	34	40
R	1.81	1.55	1.10	1.09
z	0.80	1.00	1.10	1.09
$dz/d\tau$	1.34	1.24	1.05	1.04

$\Delta\tau_{min} = 1.27$

$\tau = 3.52$

Bubble	26-35-39-40	33-41-42	34
R	2.12	1.55	1.10
z	0.88	1.00	1.10
$dz/d\tau$	1.46	1.24	1.05

$\Delta\tau_{min} = 3.21$

$\tau = 6.73$

Bubble	26-34-35-39-40	33-41-42
R	2.38	1.55
z	0.93	1.00
$dz/d\tau$	1.54	1.24

$\Delta\tau_{min} = 5.40$

$\tau = 12.13$

Bubble	26-34-35-39-40-33-41-42
R	2.84
z	0.95
$dz/d\tau$	1.68

The average velocity of the system of fingers increases step by step as the mergers proceed until finally the entire cross sectional area is occupied by a single finger rising at a constant velocity. Here it is assumed that the slab of fluid is thick enough so that breakthrough does not occur until finger amalgamation is complete. A thinner slab of heavy fluid will be penetrated by a finger of the lighter fluid which is still not merged with all other fingers.

It follows from this analysis that a slab of fluid very thick in comparison with its effective radius will almost always be penetrated by a single finger, independently of the original pattern of the fingers. Moreover, the time for merger of the fingers in this case will constitute only a small fraction of the much longer time taken for the final big rapidly moving finger to break through. Therefore, an estimate of the time for breakthrough can be made in this case, from the formula

$$\begin{pmatrix}\text{breakthrough time}\\ \text{for a slab of fluid}\\ \text{thick in comparison}\\ \text{with its effective}\\ \text{radius}\end{pmatrix} \sim \frac{(\text{thickness of slab})}{(2/3)(\text{acceleration})^{1/2}(\text{effective radius})^{1/2}} \tag{17}$$

In this time the slab has advanced through the distance

$$\begin{pmatrix}\text{distance}\\ \text{of advance}\end{pmatrix} \sim (1/2)(\text{acceleration})(\text{time})^2$$

$$\sim (9/8)(\text{thickness of slab})^2/(\text{effective radius}), \tag{18}$$

a quantity which is large when compared either with the effective radius or with the thickness of the slab.

An estimate of the breakthrough time in the case of a thinner slab demands an analysis of the details of merger. In the case of the eight fingers considered in Table V, the velocities change with time as indicated in Fig. 4.

The spread in velocities throughout most of the stage of cannibalization is roughly of the order of one quarter to one half of the average velocity itself at each instant. This average velocity increases in a stepwise fashion with time. However, if the increase is represented in idealized form by a curve, then a straight line curve is not out of place to describe the results of the computations. The rate of increase of velocity with time read off of such a curve lies between 0.02 g and 0.05 g in the example.

Abstracting from this special example one is led to consider a pattern of fingers which develops in time according to a similarity or scaling transformation, as follows:

(1) An average rate of rise for the fingers which are present at any given time which is given by the expression v_{av} = (average velocity) = $k_1\,gt$, where k_1 is a small dimensionless numerical factor of the order of

$$0.02 \text{ to } 0.05. \qquad (19)$$

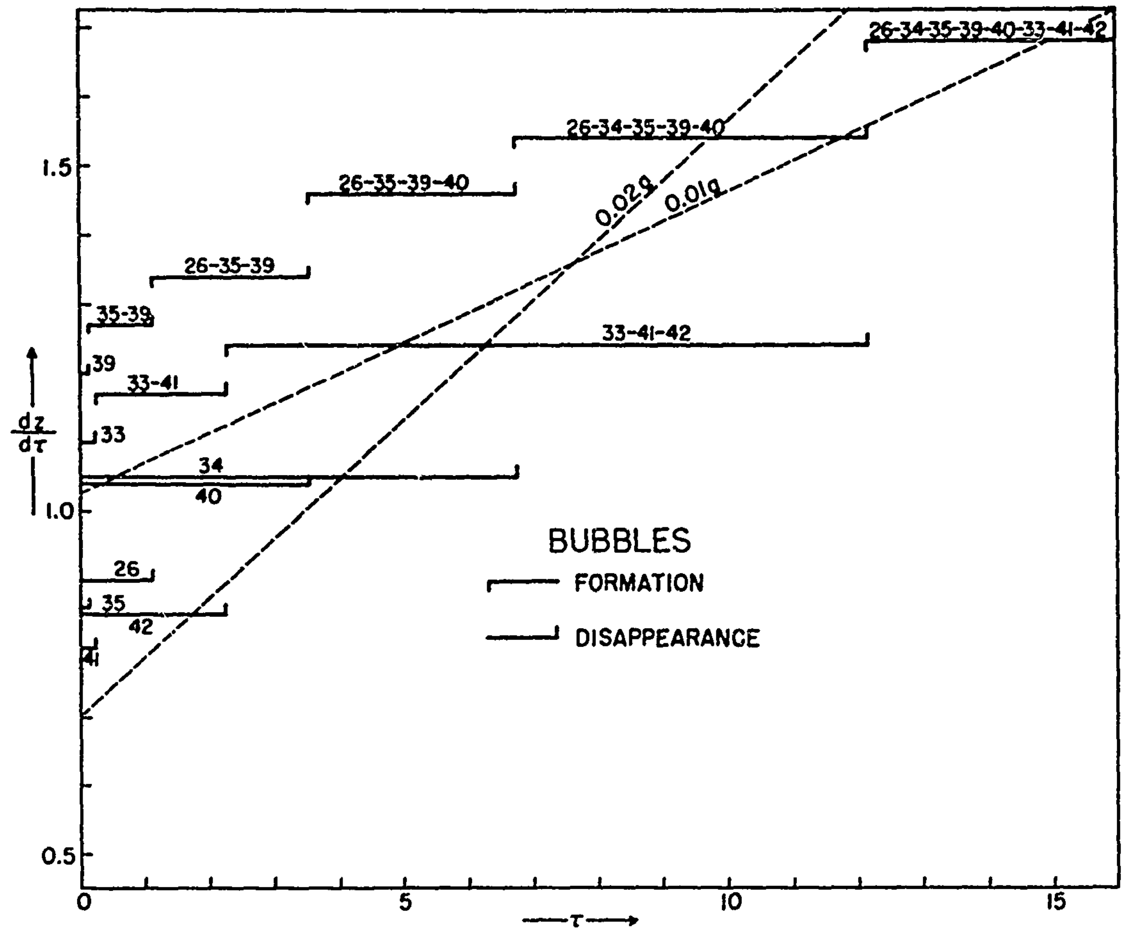

Fig. 4. Velocity as a function of time for the fingers listed in Table V. Each horizontal line corresponds to one finger. The line starts when the finger is formed and disappears when the finger merges with another. Some foreshadowing can be seen in this diagram of the central points of a statistical analysis:

(1) A statistical distribution in rise velocities at any given time. This distribution in the example, neither clearly narrows up as time advances nor clearly broadens out when expressed in terms of the fractional variation of the velocity about the average velocity which obtains at that instant.

(2) A roughly linear increase of this average rise velocity with time, leading to the concept of an <u>effective acceleration</u> associated with bubble rise during the regime of continued merger. The two dashed lines correspond to values of this effective acceleration of 0.02 g and 0.05 g, which, therefore, serve in the particular example as lower and upper limits for this effective acceleration.

(2) An average effective finger radius

$$R_{AV} \sim v_{AV}^2/g \quad , \text{ OR } \quad R_{AV} = k_2\, g t^2 \tag{20}$$

where k_2 is another dimensionless factor.

(3) A statistical distribution of finger radii about the average radius given by something qualitatively like a Gaussian curve, with a fractional spread in radii which does not change in time as the fingers merge:

$$[(R - R_{AV})^2]_{AV} / R_{AV}^2 = \delta^2 = \text{CONSTANT} .$$

If a distribution as special as the Gaussian distribution were to be assumed, one would have the formula

$$d \begin{pmatrix} \text{number of fingers} \\ \text{with radii between} \\ R \text{ and } R+dR \end{pmatrix} = \begin{pmatrix} \text{total number of} \\ \text{fingers at the} \\ \text{given time} \end{pmatrix} \frac{dR}{(2\pi)^{\frac{1}{2}} R_{AV}\delta} \exp\left\{ - \frac{(R-R_{AV})^2}{2\,(R_{AV}\,\delta)^2} \right\} \tag{21}$$

not only for the radii themselves, but also for the cross section area. For the rise velocity and for the effective height distributions are assumed---expressed in fractional deviations from the average---which do not change with time. This statistical assumption is taken to apply throughout the regime of continuing mergers, apart from the first few generations of mergers where the specialities of the starting mechanism show themselves, and apart from the final stage (if breakthrough has not already taken place) where only a few fingers are left.

It is possible to make these assumptions plausible on dimensional grounds. In the regime under consideration, the fingers have grown in size to dimensions so great compared to the characteristic λ set by surface tension and acceleration (or by viscosity and acceleration) that surface tension (and viscosity) have nothing to do with the main features of the rise phenomenon. At the same time the fingers are small enough in comparison with the dimensions of the slab of fluid that these macroscopic dimensions also can have no significant effect on what is going on. Therefore, the only physical quantities left to dominate the situation are the acceleration g and the elapsed time t. Out of these magnitudes there is only one way to construct a quantity with the dimensions of an average finger radius (Eq. 19) and a quantity with the dimensions of an average rise velocity (Eq. 20).

These dimensional arguments do not justify formulas of the type (19) and (20) with <u>universal</u> coefficients k_1 and k_2, independent of the original spread in finger sizes. It is even possible to point to a perfectly conceivable situation in which the distribution in finger sizes deviates enormously from a Gaussian curve, not only in the first few generations of mergers, but also right up to the stage where only a few bubbles are left. (Fig. 5). Nevertheless, even in this extreme case a simple analysis makes it reasonable to think of the effective rate of rise as being unaffected by the anomaly in the distribution of finger sizes.

Rather than the big finger in Fig. 5 helping the little ones to move forward faster, the little fingers hold back the big one, if one can take the Maine model as a guide to the true state of

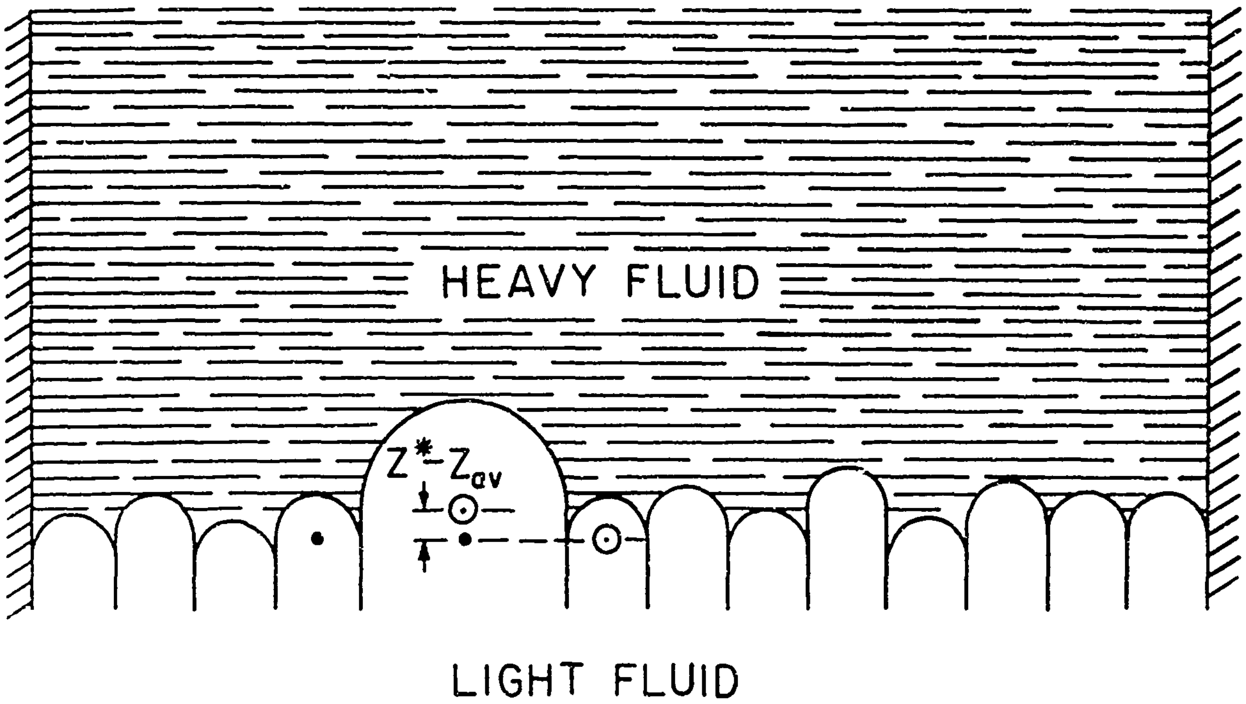

Fig. 5. Case of anomalous distribution in finger sizes. The larger finger rises much more rapidly than the small ones and engulfs them one by one until it spans the entire cross sectional area of the slab of fluid.

affairs. According to one of the rules of that model, a large bubble of effective height z^* engulfs smaller ones of average effective height z^* as soon as the difference $z^* - z_{av}$ rises to a value as great as the average effective radius R_{av} of these fingers. The consequence is this, that the big finger grows in radius faster than the little ones, but does not get out of step with them in vertical elevation. In each interval of time dt it eats whatever number of little bubbles, dN, is required to dilute its rise back to parity with its fellows. The new effective height is found by adding the volume of the old finger and the newly assimilated ones and dividing by the cross-sectional area of this collection. If too many small bubbles have been taken in during the interval dt, this effective height will be reduced too much and new cannibalization cannot go on until the faster rate of rise of the larger bubble once more allows it to get ahead. The converse also applies--if the big bubble assimilates too few small ones, its elevation will not be diluted back enough to make up for its faster rate of rise. Therefore, it will get ahead and in the next time interval take in a more than normal number of fingerlets. Thus, in the model, there is a stability about the mechanism through which the little bubbles govern the rate of rise of the big one.

This reasoning can be expressed in mathematical terms. Let the average rate of rise of the small fingers be expressed in the form (19) expected from similarity arguments,

$$v_{av}(t) = k_1 gt \tag{22}$$

and let their average height and average radius be written in a form consistent with this expression:

$$z_{av} = (k_1 /2)\, gt^2 \tag{23}$$

$$R_{av}\,(t) = (k_1 /0.48)^2\, gt^2 \tag{24}$$

The big finger is always at the point where the smallest addition to its height makes the difference between more cannibalization and none at all; or, according to the model,

$$z^* - z_{av} = R_{av} \tag{25}$$

Thus one finds

$$z^*\,(t) = \left[\,(k_1/2) + (k_1/0.48)^2\right] gt^2 \tag{26}$$

The rate of rise of the big finger calculated from this expression is

$$v^*(t) = dz^*/dt = \left[\,k_1 + (k_1^2/0.12)\right] gt \tag{27}$$

Any value of the effective radius R^* of the large bubble is compatible with this expression. However, the larger R^* happens to be, the greater is the natural rate of rise associated with a finger so big,

$$\begin{pmatrix}\text{natural rate of}\\ \text{rise of big finger}\end{pmatrix} = (0.48)(gR^*)^{\frac{1}{2}} \tag{28}$$

and the more little bubbles it must assimilate per second to hold its rate of rise back to the figure (27). The "drop back" due to eating out from the effective radius R^* to the effective radius

$R^* + dR^*$ in the time dt is

$$z^*_{\text{before eating}} - z^*_{\text{after eating}} = z^* - \frac{z^* R^{*2} + z_{av}\, 2R^*\, dR^*}{R^{*2} + 2R^*\, dR^*}$$

$$= (2dR^*/R^*)(z^* - z_{av})$$

$$= (2dR^*/R^*)\, R_{av} \quad . \qquad (29)$$

The rate of drop back on account of assimilating smaller bubbles has to satisfy in the model, the equation

$$\begin{pmatrix}\text{rate of drop back}\\ \text{due to assimilation}\\ \text{of smaller bubbles}\end{pmatrix} = \begin{pmatrix}\text{natural rate of}\\ \text{rise of big finger}\\ \text{in absence of growth}\end{pmatrix} - \begin{pmatrix}\text{slower rate of rise}\\ \text{required to remain in}\\ \text{step with smaller}\\ \text{bubbles}\end{pmatrix} \qquad (30)$$

or

$$(2\, R_{av}/R^*)(dR^*/dt) = (0.48)(gR^*)^{\frac{1}{2}} - (0.48)(gR_{av})^{\frac{1}{2}} \; . \qquad (31)$$

Under the conditions contemplated in Fig. 5, the radius R^* is large compared to R_{av}. Therefore, it is appropriate to neglect the last term in (31). Then Eq. (31) for the growth of the big finger may be integrated to give the effective radius

$$R^*(t) = R^*(t_0) \left\{ \frac{9k_1}{9k_1 + [R^*(t_0)/0.52\, R_{av}(t)] - [R^*(t_0)/0.52\, R_{av}(0)]} \right\}^2 . \qquad (32)$$

Here the radius $R_{av}(t)$ of the small fingers has been taken to follow Eq. (24).

To fix ideas, assume, for example

$$k_1 = \frac{\begin{pmatrix}\text{advance of bubbles}\\ \text{relative to liquid}\end{pmatrix}}{\begin{pmatrix}\text{advance of}\\ \text{slab of liquid}\end{pmatrix}} = \begin{pmatrix}\text{"breakthrough}\\ \text{constant"}\end{pmatrix} = 0.02;$$

$$g = 2 \times 10^{13} \ \text{cm/sec}^2;$$

$$R_{av}(t_o) \sim \begin{pmatrix}\text{dimension fixed}\\ \text{by small amplitude}\\ \text{theory allowing}\\ \text{for surface tension}\end{pmatrix} \sim 3.8317 \ (3\sigma/g\rho)^{\frac{1}{2}} \tag{33}$$

$$\sim 6.6 \left[\frac{(800g/\text{sec}^2)}{(2 \times 10^{13} \ \text{cm/sec}^2)(10 \ \text{g/cm}^3)} \right]^{1/2} = 1.3 \times 10^{-5} \ \text{cm};$$

$$v_{av}(t_o) = (0.48)(gR_{av})^{\frac{1}{2}} = 7.7 \times 10^3 \ \text{cm};$$

$$z_{av}(t_o) = (0.23/2k_1)\, R_{av} \ (\text{Eqs. 23, 24}) = 7.8 \times 10^{-5} \ \text{cm}.$$

These "initial" values correspond to a time coordinate $t = 2.0 \times 10^{-8}$ sec in Eqs. (23) and (24). At any later t the calculated effective radius of the small fingers is

$$R_{av}(t) = (3.3 \times 10^{10} \ \text{cm/sec}^2)\, t^2. \tag{34}$$

Assume that a flaw somewhere in the original surface gives rise to a single disturbance--in the midst of these smaller bubbles--with an effective radius

$$R^*(t_o) = 10^{-3} \ \text{cm}. \tag{35}$$

Then the calculated radius of this dominating finger at any later time is

$$R^*(t) = 10^{-3} \ \text{cm} \left\{ \frac{0.18}{0.18 + \frac{2 \times 10^{-3}\text{cm}}{R_{av}(t)} - \frac{2 \times 10^{-3}\text{cm}}{1.3 \times 10^{-5}\text{cm}}} \right\}^2. \tag{36}$$

The denominator vanishes and the finger radius goes to infinity when R_{av} has increased by the very small amount

$$\Delta R_{av} = (0.18)^2 \; (1.3 \times 10^{-5} \text{ cm})^2 \; / 4 \times 10^{-3} \text{cm} = 1.4 \times 10^{-9} \text{cm}$$

to the value

$$R_{av}(\text{new}) = R_{av}\,(t_o) + \Delta R = 1.3 \times 10^{-5} + 1.4 \times 10^{-9} \text{cm}. \qquad (37)$$

At this instant, according to the model, the big finger has grown so much that it engulfs the entire fluid. Until this instant, the little ones govern the rate of rise; thereafter, the big one does, and the velocity keeps the constant value

$$v = 0.48 \left(g R_{\substack{\text{effective for} \\ \text{entire slab}}} \right)^{1/2}$$
$$= (0.48) \; g^{\frac{1}{2}} (\text{area of slab}/\pi)^{\frac{1}{4}}. \qquad (38)$$

The example just discussed would make it appear that one finger substantially larger than its fellows will quickly come to dominate the flow pattern of the entire mass of fluid. How much validity should be ascribed to this consequence of a rather idealized model of bubble merger? (1). Available experimental information (Davies and Taylor[4]; Lewis[7]; Allred and Blount[8]; Chang[11]) does not show any such sudden rise of one finger to dominance. However, these experiments are not relevant anyway to the point at issue. The original size of all fingers--and the thickness of the slab of fluid--were not great enough compared to the characteristic dimension $6.6(\sigma / g\rho)^{\frac{1}{2}}$ set by surface tension and by the acceleration. Therefore, surface tension greatly changed the phenomena which inertia and pressure would otherwise have brought into evidence--and which are the object of concern here.

(2) A qualitative look at the hydrodynamics of bubble merger in the absence of surface tension would suggest that a finite time is required to complete the process of bubble merger. The debris from the smaller finger has to be washed down the surface of the newly enlarged finger before this object can be regarded as a unit. Only then can it be expected to live up to its new cross-sectional dimensions in its ability to engulf a new bubble. Consequently, one would expect a certain limitation on the rate at which a larger bubble can spread its influence sideways to take over one smaller bubble after another. No such limitation of rate was built into the Maine model of finger merger. Therefore, it is permissible to doubt the prediction of the model that a significantly larger than average bubble will take over the whole fluid before the other fingers have had time even to double in height.

(3) It is conceivable that the distribution of finger sizes in Rayleigh-Taylor instability has a certain analogy to the distribution of electron velocities in a gas subject to an electric field in this respect, that there is a sharp distinction between a statistically stable situation and a runaway situation. In the electrical case, the critical parameter is the applied electric field. In the hydrodynamic case, the difference between (1) runaway and (2) approach to a standard distribution is governed--on this view--not by the acceleration (regarded as fixed) but by

the initial distribution of finger sizes itself. In other words, a distribution which departs not too greatly from the canonical distribution is conceived to approach in time the canonical one. The characteristic scale R_{av} of this canonical distribution is, of course, itself increasing at the same time. On the other hand, a distribution of finger sizes which departs too greatly (Fig. 5) from the canonical one is conceived to depart more and more from that distribution and to lead to one or more runaway bubbles.

In summary, present information does not suffice to distinguish between the following three possibilities:

(1) One bubble always quickly runs away.

(2) Depending upon the degree of anomaly in the original distribution of sizes, either one bubble runs away, <u>or</u> a standard statistical distribution in sizes is attained, with a corresponding standard law of growth.

(3) A standard distribution and rate of rise are almost always reached.

Possibilities with a light fluid to push a thin slab of heavy fluid a great distance would seem to depend very heavily upon the correctness of alternative (2) or (3).

How rapid will be the rate of breakthrough if (2) or (3) apply? The following order of magnitude analysis of this question is assumed for simplicity:

(1) something like the Maine model for bubble merger <u>plus</u>

(2) stability of a canonical distribution of finger sizes.

In addition, for simplicity and definiteness, this distribution will be taken to follow

(3) a Gaussian distribution law, with a parameter δ of fractional spread in radii equal to 0.5:

$$\frac{d(\text{number of fingers})}{(\text{total number of fingers})} = \frac{dR}{(2\pi)^{\frac{1}{2}} R_{av}\delta} \exp\left\{ - \frac{(R-R_{av})^2}{2(R_{av}\delta)^2} \right\} ;$$

$$\delta = 0.5. \qquad (39)$$

Expression (39) predicts a non-zero probability for fingers of negative radius. However, the absolute value given for this probability is so low that this minor drawback of the Gaussian formula can be disregarded in favor of mathematical simplicity. Now to estimate the rate of evolution of such a canonical distribution assuming its self-perpetuating character!

Let $R_1 \sim R_{av}\,(1-\delta/2)$ represent the effective radius of one bubble which is soon to be eaten. The larger one which eats it has a radius of the order of $R_2 \sim R_{av}\,(1+\delta/2)$. The velocities of rise of the two fingers,

$\sim 0.48\,(gR_{av})^{\frac{1}{2}}(1-\delta/2)^{\frac{1}{2}}$ and $\sim 0.48\,(gR_1)^{\frac{1}{2}}\,(1+\delta/2)^{\frac{1}{2}}$,

will differ by an amount

$$\Delta v \sim 0.48\,(gR_{av})^{\frac{1}{2}}\,(\delta/2). \qquad (40)$$

One must get ahead of the other by an amount Δz of the order R_{av}

to merge with it. The time required for this lead is of the order

$$\Delta t \sim (2/0.48\,\delta)(R_{av}/g)^{\frac{1}{2}} . \qquad (41)$$

In this time, two bubbles have amalgamated to give a single one of roughly twice the area. The same effect has been happening everywhere else through the fluid. Thus, πR_{av}^{2} has been increased by a factor 2; and R_{av} has been magnified by $2^{\frac{1}{2}}$; and the average velocity of rise has gone up by a factor $2^{\frac{1}{4}}$, from

$$0.48\,(gR_{av})^{\frac{1}{2}} \text{ to } 2^{\frac{1}{4}} \times 0.48\,(gR_{av})^{\frac{1}{2}}. \qquad (42)$$

The increase in velocity, divided by the time Δt required to bring it about, defines the <u>effective acceleration</u> (see Fig. 4, 6) <u>associated</u>--by the statistical model--<u>with finger penetration</u>:

$$\begin{pmatrix}\text{average acceleration}\\ \text{of fingers with}\\ \text{respect to the base line}\\ \text{which would be associated}\\ \text{with acceleration of}\\ \text{ideally flat interface}\end{pmatrix} \sim \Delta v/\Delta t$$

$$\sim \frac{(2^{\frac{1}{4}}-1)\,0.48\,(gR_{av})^{\frac{1}{2}}}{(2/0.48\,\delta)(R_{av}/g)^{\frac{1}{2}}} = (0.022\,\delta)\,g$$

or

$$g_{av} = k_1\,g \sim 0.01g . \qquad (43)$$

This <u>estimate of a breakthrough coefficient</u> k_1, even if the ideas

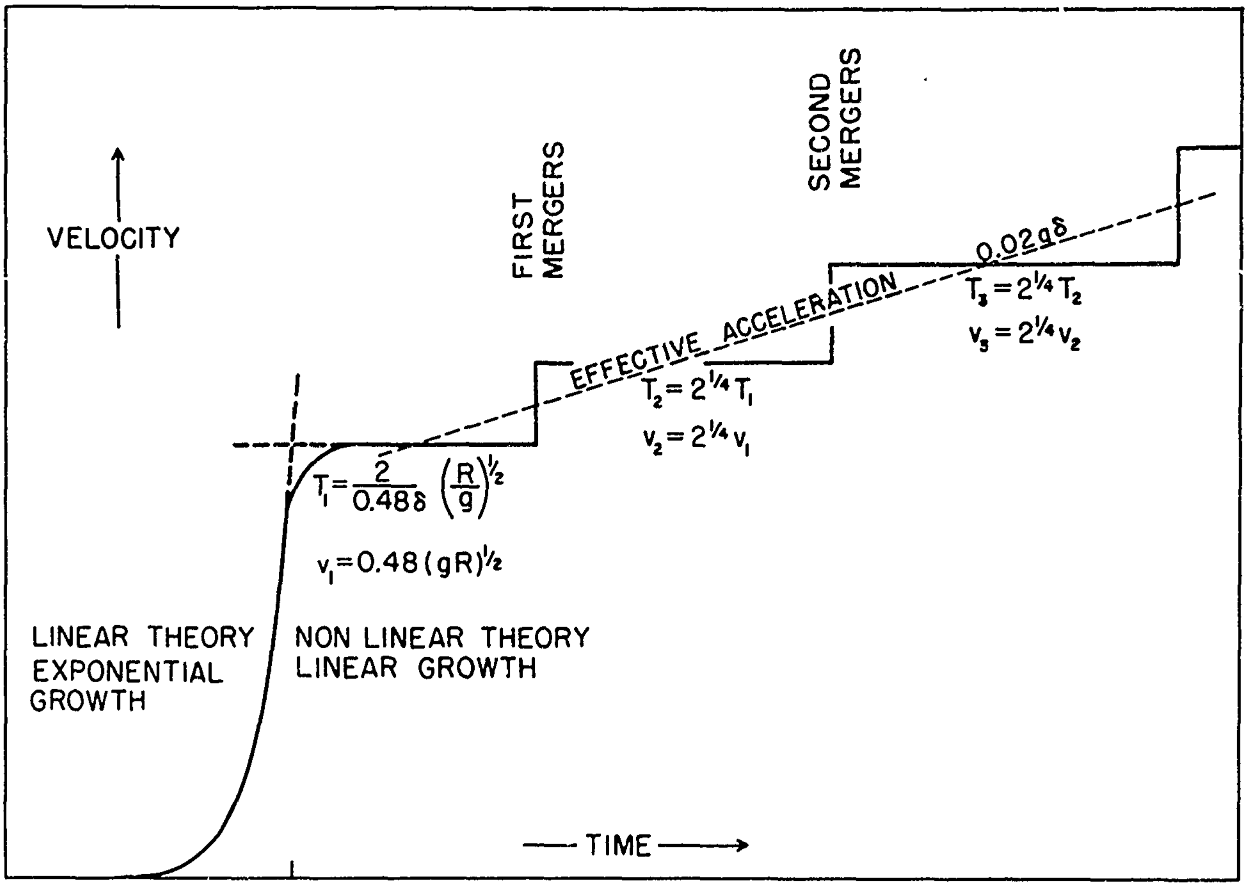

Fig. 6. Increase of velocity with time as a consequence of bubble merger according to the primitive idealized statistical model of the text. In each step the velocity increases by a factor $2^{\frac{1}{4}}$ and the length of the step increases by a factor $2^{\frac{1}{4}}$ compared to the time occupied in the previous step. Compare this regular increase of the velocity with that shown in Fig. 4.

behind it have qualitative validity, is evidently uncertain by one or more powers of two.

The present model of breakthrough predicts that a slab of heavier liquid cannot be accelerated by gas to a distance greater than $(1/k_1) \sim$ (25 or) 50 or 100 or 200 (or 400) times its own thickness without suffering finger penetration. It can be called an optimistic model of Rayleigh - Taylor instability in this sense, that it is hard--short of drastic measures, like freezing the liquid, etc.--to think of a way for a light fluid to accelerate a heavy fluid to a <u>greater</u> distance. On the other hand, rapid growth of larger than normal fingers could well drastically <u>reduce</u> the distance over which the liquid holds together. Optimistic as it is, the model makes it hard to see how a magnetic field could ever propel a 0.01 cm slab of molten metal or plasma for a distance as great as 10 cm.

Statistical mechanics as applied to gases teaches that the number of molecules or the energy of one extended region is the closer percentagewise to the energy or number of molecules in another extended region of the same volume, the larger this volume happens to be chosen. Therefore on a surface covered over with an enormous number of small scale bubbles or fingers-one might be inclined to reason-<u>large</u> scale differences between one extended region and another can never arise. Therefore the statistical picture of bubble merger might be considered to be complete nonsense. However, this reasoning would appear tacitly to assume that there is some conserved quantity - like total energy, or total number of particles- which cannot be supplied in surplus to one region except at the expense of another region. There is no such conserved quantity for bubble advance. The sooner bubbles merge, the faster they can go forward - and this not at all by depriving <u>other</u> regions of the chance to get ahead. Bubble merger like hurricane growth is a typical example of a divergent process in the sense of Langmuir[12].

IV. HOW FAR DO IMPROVEMENTS IN THE SURFACE SMOOTHNESS PAY OFF IN REDUCED TAYLOR INSTABILITY

The concept of a breakthrough coefficient k_1--*if* such a concept is valid--would seem to provide some rational grounds for deciding at what degree of perfection to stop trying to improve the original surface.

The disturbances which have evolved by merger out of the inevitable initial small scale perturbations ($\lambda \sim \lambda_{\text{fastest growing}}$) are taken in the statistical model to follow the law of growth,

$$\begin{aligned} z_{av} &= (k_1/2)\, gt^2 \\ v_{av} &= k_1\, gt = 0.48\, (gR_{av})^{\frac{1}{2}} \qquad (44) \\ R_{av} &= (k_1/0.48)^2\, gt^2 \end{aligned}$$

with a continually increasing scale of sizes. In the meantime the irregularities, if any, due to machining--here assumed to have a scale λ_{mach} much greater than $\lambda_{\text{fastest growing}}$ --have themselves been developing. Their normal rate of increase will be little affected by the presence of the merging and remerging smaller scale bubble pattern until the scale of that pattern, R_{av}, has reached the scale,

$$R_{mach} = 3.83 \; \bar{\lambda}_{mach} \tag{45}$$

(of Section III) of the machining irregularities.. The two scales come into concordance, according to the model, at a time (Eq. 44)

$$t_{concordance} = (0.48/k_1)(3.83 \; \bar{\lambda}_{mach}/g)^{\frac{1}{2}}. \tag{46}$$

At this time the calculated advance of the fingers which have resulted from merger is

$$\begin{aligned} z_{av} &= \left((0.48)^2 /2k_1 \right) R_{av} \\ &= \left((0.48)^2/2k_1 \right) R_{mach} \\ &= 3.83 \left((0.48)^2 /2k_1 \right) \; \bar{\lambda}_{mach} \end{aligned} \tag{47}$$

$$\left(\sim 22 \; \bar{\lambda}_{mach} \text{ or } 3.5 \; \lambda_{mach} \quad \text{for } k_1 = 1/50 \right)$$

Three possibilities present themselves at this time:

(1) The disturbances which originated from machining irregularities have already grown to an amplitude large compared to (47). In this case they have developed well beyond the regime of the small amplitude theory, as may be seen from the circumstance that in (47) z_{av} is large compared to R_{av}. Therefore, it will be expected that fingers have developed out of the initial fabricational imperfections

and even that these fingers have made substantial progress towards merger with one another. Evidently the instabilities from this source are well ahead of those that came from $\lambda_{\text{fastest growing}}$. They dominate, not only at $t_{\text{concordance}}$ (Eq. 46) but thereafter. Consequently, it can be said that the machining irregularities are decisive in determining the time of breakthrough in case (1).

(2) The disturbances which originate from imperfections in the surface of reduced wave length λ_{mach} have grown by the time $t_{\text{concordance}}$ of Eq. (46) to an amplitude which is still small compared to the z_{av} of (47). In this case the instabilities of natural origin dominate over those which have their source in machining irregularities. They dominate not only at $t = t_{\text{concordance}}$, but thereafter. Consequently---provided that the statistical model is valid---the imperfections in the fabrication can be neglected in the analysis of Taylor instability in case (2). In this case there would seem to be little point in trying to improve the perfection of manufacturing. Money can be saved by loosening up on the machining tolerances!

(3) The disturbances which have grown from initial irregularities of reduced wave lengths $\lambda_{\text{fastest growing}}$ and $\lambda_{\text{machining}}$ have comparable amplitudes at $t = t_{\text{concordance}}$.

From this circumstance it follows that the machining tolerances have been widened to the critical limit,

$$(\delta z)_{machining} = (\delta z)_{critical} \tag{48}$$

beyond which any further decrease in the quality of the surface will shorten the time to breakthrough. This critical limit is the fabrication criterion sought from the present analysis.

The concept of a critical magnitude for fabricational irregularities, while easy to describe in qualitative terms, is difficult to make precise in the present state of knowledge. To illustrate this point, it is sufficient to make one very literal minded calculation of $(\delta z)_{crit}$ along the lines illustrated in Fig. 7, and then to look at the uncertainties that are associated with this calculation. In the figure, the fingers of natural origin are taken to increase as in the provisional statistical theory of bubbler merger.

$$z = (k_1/2)\, gt^2$$
$$dz/dt = k_1\, gt \tag{49}$$

The rate of rise given by (49) at the time

$$t_c = (0.48/k_1)(3.83\, ƛ_{mach}/g)^{\frac{1}{2}}$$
$$(=47\ (ƛ_{mach}/g)^{\frac{1}{2}} \text{ for } k_1 = 0.02) \tag{50}$$

has the value

$$(dz/dt)_c = 0.48\ (3.83\ ƛ_{mach}\ g)^{\frac{1}{2}}. \tag{51}$$

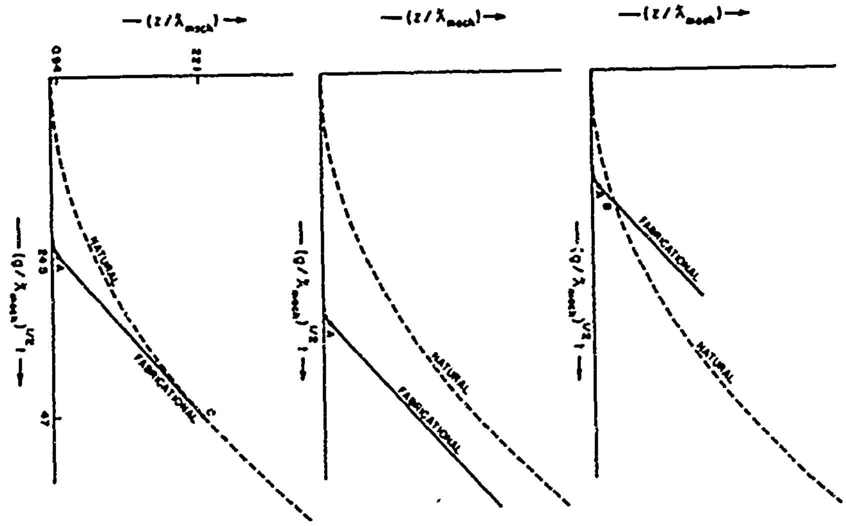

Fig. 7. Competition between irregularities of fabricational origin and of natural origin for control of Rayleigh-Taylor instability. Top: Fabricational irregularities dominate. Middle: Departures from ideality of natural origin--governed by surface tension or viscosity--dominate. Bottom: Irregularities of fabricational and natural origin contribute comparably to the later stages of instability. The conditions in this case determine the critical magnitude, down to which it is desirable to reduce the fabricational irregularities, but below which any further reduction gives a decreased or even negligible payoff. Though the general ideas are clear, and the curves in the diagram have been calculated on the basis of reasonably well defined assumptions, the details of these assumptions are not determined at the present time with any precision by either available observations or existing theory. Moreover, small changes in these assumptions make a big change in the order of magnitude of the critical magnitude $(\delta z)_{crit}$ for the irregularities $(\delta z)_{mach}$ of fabricational origin at the start of the acceleration. Therefore, the particular assumptions used in computing the curves in the diagram are recapitulated here: (1) The scale $\lambda_{fastest\ growing}$ of the natural irregularities at the start is negligible compared to the scale λ_{mach} of fabricational marks. (2) The natural instabilities subsequently have a canonical distribution in sizes, of which the scale but not the form changes with time due to finger mergers, with an average height $z_{av} = (k_1/2)\, g\, t^2$ given by a "breakthrough coefficient" k_1 which is arbitrarily equated to 0.02 (3) The fabricational irregularities grow exponentially following the Rayleigh-Taylor formula $z = (\delta z)_{mach} \exp (g/\lambda_{mach})^{\frac{1}{2}}\, t$ up to $z = 0.48(3.83)^{\frac{1}{2}} \lambda_{mach}$ and thereafter (with continuity of z and dz/dt at point A) grow linearly at the rate $(dz/dt) = 0.48(3.83 g\, \lambda_{mach})^{\frac{1}{2}}$. This simplifying assumption neglects merger of the fingers resulting from fabrication until these fingers have been overtaken by the fingers which have grown from instabilities of natural origin.

The number agrees with the rate of linear rise of fingers of radius 3.83 λ_{mach} or of reduced wave length λ_{mach} (point C in the diagram). Calculate back along this linear curve of rise of the fingers of fabricational origin to the place (point A in diagram) where the straight line agrees in magnitude and slope with the Rayleigh-Taylor exponential curve:

$$(dz/dt)_A = 0.48\ (3.83\ \lambda_{mach}\ g)^{\frac{1}{2}} \qquad (52)$$

$$z_A = (1/\alpha)(dz/dt)_A = (\lambda_{mach}/g)^{\frac{1}{2}}\ (dz/dt) \qquad (53)$$
$$= (0.48)(3.83)^{\frac{1}{2}}\ \lambda_{mach} = 0.94\ \lambda_{mach};$$

$$t_A = t_c - (z_c - z_A)/(dz/dt)_{\text{STRAIGHT LINE}}$$
$$= (\lambda_{mach}/g)^{\frac{1}{2}}\ [\ 1 + (0.48)(3.83)^{\frac{1}{2}}/2k_1\ ] \qquad (54)$$
$$[\ = 24.5\ \ (\lambda_{mach}/g)^{\frac{1}{2}}\ \text{ for } k_1 = 0.02\ ]$$

Follow the exponential curve back to the time $t = o$ to find the critical amplitude for fabricational irregularities:

$$(\delta z)_{crit} = (0.48)(3.83)^{\frac{1}{2}}\ \lambda_{mach}\ e^{-\alpha t_A}$$
$$= 0.94\ \lambda_{mach}\ e^{-1-\ (0.48)(3.83)^{\frac{1}{2}}/2k_1} \qquad (55)$$

$$(\ = 0.94\ \lambda_{mach}\ e^{-24.5} = 2.2 \times 10^{-11}\ \lambda_{mach} \text{ for } k_1 = 0.02)$$

If a value $k_1 = 0.04$ is adopted for the breakthrough coefficient (slab of fluid accelerable to only 25 times its thickness before the fingers of natural origin penetrate) then the calculated value of the critical amplitude is increased from (55) to

$$(\delta z)_{crit} = 0.94\ \lambda_{mach}\ e^{-12.75} = 10^{-6}\ \lambda_{mach} \quad (k_1 = 0.04) \qquad (56)$$

A comparably drastic change in the order of magnitude of the calculated $(\delta z)_{crit}$ will be made by allowing--as Fig.7 does not--for some preliminary merger of the fingers which grow out of disturbances of reduced wave length $\lambda = \lambda_{mach}$. Evidently it is not possible from present information to state a reliable figure for the critical amplitude $(\delta z)_{crit}$ or for the corresponding dimensionless ratio $(\delta z)_{crit}/\lambda_{mach}$. Experiments to test the validity of the concept of a critical amplitude and to measure its value would therefore seem in order. They would be a natural extension of any observational program to search for a breakthrough coefficient.

In Section II electrolysis was mentioned as a technique to build up the proposed laminar structure. It may not be out of place to think again of electrolysis as a general technique to produce surfaces of very high perfection. Such perfection--according to analysis of Sections III and IV of this report--will not prevent Taylor instability, and could, at the worst, be a form of "abacadabra" which costs much time and money; but it could also help to postpone the breakthrough of spikes and bubbles. Electro-polishing today is so greatly developed as a technology for the manufacture of the chrome parts of automobiles and for many other purposes that it is hardly necessary to recall the principle of the polishing:

(1) Alternate cycles of current with different and programmed magnitudes for the product of (current density) · (time) in the forward cycle and the reverse cycle.

(2) Deposition of a very thin layer of material in each forward cycle.

(3) Inevitable formation of small bumps and spikes in this process.

(4) Concentration of lines of force and field gradient in the electrolytic fluid near these projections.

(5) Cessation of the deposition before this concentration effect has succeeded in magnifying substantially these projections.

(6) Reversal of the current in the next phase of the program to take advantage of this concentration of current density selectively to remove the projections.

(7) A new cycle of buildup (one sign of the product of (current)·(time) and electro-deposition (smaller magnitude and opposite sign for the product of (current) · (time).

The technology of electro-polishing, however fully developed for making shiny fenders, is not known by us ever to have been used for scientific purposes by anyone with the talents and passion for precision which Henry A. Rowland(13) showed in the design and construction of an almost perfect screw for his engine to rule gratings. Among the factors that might be considered if Rowland's model were to be followed, this time to construct an almost perfect surface, are these:

(1) Preparation of an almost ideal substrate upon which to start the electro-deposition. In this connection one thinks particularly of surface tension plus the pressure of a thin

filling gas as a mechanism to produce a thin cylinder of fluid connecting two separated hemispherical metal caps.

(2) Design of a system of electrodes, or a system of automatic control (scanning by light source plus photo electric cell feeding into circuit which controls electrolytic current) or both so as to maintain uniformity of thickness as closely over large distances as over the very small distances at which electro-polishing is relevant.

(3) Control of temperature and convection currents to a completeness sufficient to prevent imperfections from these sources greater than a preassigned magnitude.

(4) Method of mounting foil and inserting it in final device which preserves its near perfection to the moment of use.

Surfaces prepared in this way might be expected (a) to be most suitable for the experimental investigation of the statistical bubble theory of Section III, which focusses attention on irregularities at a microscopic - not fabricational - scale (b) to be the most suitable for those technical applications where it is important to minimize in any way that is feasible the effects of Taylor instability.

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