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# An approximate form of the Rayleigh reflection loss and its phase: Application to reverberation calculation

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A useful approximation to the Rayleigh reflection coefficient for two half-spaces composed of water over sediment is derived. This exhibits dependence on angle that may deviate considerably from linear in the interval between grazing and critical. It shows that the non-linearity can be expressed as a separate function that multiplies the linear loss coefficient. This non-linearity term depends only on sediment density and does not depend on sediment sound speed or volume absorption. The non-linearity term tends to unity, i.e., the reflection loss becomes effectively linear, when the density ratio is about 1.27. The reflection phase in the same approximation leads to the well-known “effective depth” and “lateral shift.” A class of closed-form reverberation (and signal-to-reverberation) expressions has already been developed [C. H. Harrison, *J. Acoust. Soc. Am.* **114**, 2744–2756 (2003); C. H. Harrison, *J. Comput. Acoust.* **13**, 317–340 (2005); C. H. Harrison, *IEEE J. Ocean. Eng.* **30**, 660–675 (2005)]. The findings of this paper enable one to convert these reverberation expressions from simple linear loss to more general reflecting environments. Correction curves are calculated in terms of sediment density. These curves are applied to a test case taken from a recent ONR-funded Reverberation Workshop.

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## I. INTRODUCTION

The canonical case for shallow water propagation modeling is the Pekeris waveguide where the seabed is represented as a half-space characterized by density, sound speed, and volume absorption. In more complicated environments variable bathymetry and refraction in the water column are often introduced but still with a half-space seabed. A recent ONR-sponsored pair of workshops<sup>1</sup> on reverberation modeling specified a number of environments such as this. For this reason the behavior of the reflection coefficient (amplitude, phase, and logarithmic loss) for a half-space is of interest, and it can be calculated by the Rayleigh reflection coefficient formula,<sup>2</sup> on the assumption that the roughness does not spoil the outward and return specular reflections (alternative forward scattering laws have been suggested<sup>3</sup>). For sound speeds greater than that of water this predicts a critical angle and a dB loss that is very nearly proportional to angle. In fact an approximate formula for the proportionality constant has been derived by Weston.<sup>4</sup> Consequently there is ducted propagation within the critical angle, and the small linear reflection loss results in an ever narrowing Gaussian angle distribution and hence mode-stripping. This has a profound effect on propagation and particularly reverberation since the propagation favors low angles whereas most scattering laws favor high angles, or at least, reject low angles.

Combining Lambert’s law with mode-stripping, the reverberation behavior can be predicted by closed-form solutions for range-dependent, bistatic, isovelocity, or refracting

environments providing the reflection loss is linear.<sup>5–7</sup> If the reflection loss is not linear then these solutions are still correct at very long range because the narrowness of the Gaussian angle distribution biases the loss to the linear part of the curve. They are also correct at short range because, regardless of the magnitude of the reflection loss, there are very few reflections so there is minimal effect. For intermediate ranges there is clearly some effect, and it would be useful to be able to estimate it. This paper derives an extremely good approximation to the Rayleigh formula applicable between zero grazing and almost up to the critical angle. This clearly shows the non-linear behavior of the reflection loss in very simple closed-form. Using the same approach one can also derive a formula for reflection phase, and from this, one can re-derive Weston’s effective depth<sup>8,9</sup> and clarify its conditions of validity.

In all the closed-form reverberation calculations, whether monostatic, bistatic, range-independent or -dependent, the final stage with the linear reflection loss assumption is to evaluate an angle integral whose integrand consists of a Gaussian times the sine of the angle. A multiplicative correction (i.e., a dB addition) can be tabulated for all ranges and bottom densities in the non-linear case so that the previous solutions can be easily modified. Use of these corrections is demonstrated in Sec. III by application to a test case from the recent ONR Workshop.<sup>1</sup> Solving the isovelocity reverberation angle integral analytically is more difficult, but numerical approaches to this correction provide insight into the magnitude and importance of the non-linear reflection effect. It will be shown that the correction is a function of two parameters (i.e., a family of curves) for all isovelocity range-independent environments.

This work was first reported in Ref. 10.

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## II. THEORY

### A. Reflection loss

The Rayleigh reflection coefficient  $V$  for a half-space is usually written<sup>2</sup> in terms of the impedances on each side of the boundary  $Z_{1,2}$ , but it is more conveniently written in terms of the admittances (the reciprocal of the impedance)  $A_{1,2}$ ,

$$V = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{A_1 - A_2}{A_1 + A_2}, \quad (1)$$

where

$$Z_m = \frac{1}{A_m} = \frac{\rho_m c_m}{\sin \theta_m}, \quad (2)$$

and  $\rho$ ,  $c$ , and  $\theta$  are density, sound speed, and grazing angle with indices  $m=1,2$  representing water and seabed, respectively. In the following, we manipulate these formulae making no apologies for retaining all the steps, and only making approximations where stated. We *assume* that the sea is lossless but that the seabed is an absorbing medium so that

$$A_2 \equiv A_R + iA_I, \quad (3)$$

$$A_1 \equiv A_1 + i0, \quad (4)$$

where  $A_1$ ,  $A_R$ , and  $A_I$  are all real. Thus

$$\begin{aligned} |V|^2 &= \left| \frac{A_1 - A_R - iA_I}{A_1 + A_R + iA_I} \right|^2 = \frac{(A_1 - A_R)^2 + A_I^2}{(A_1 + A_R)^2 + A_I^2} \\ &= \frac{A_1^2 + A_R^2 + A_I^2 - 2A_1A_R}{A_1^2 + A_R^2 + A_I^2 + 2A_1A_R} = \frac{1 - X}{1 + X}, \end{aligned} \quad (5)$$

where

$$X = \frac{2A_1A_R}{A_1^2 + A_R^2 + A_I^2}. \quad (6)$$

Consequently the natural-logarithmic loss (as opposed to the loss in dB) can be written exactly as

$$\begin{aligned} -\log(|V|^2) &= -\{\log(1 - X) - \log(1 + X)\} \\ &= -\{(-X - X^2/2 - X^3/3 - \dots) \\ &\quad - (X - X^2/2 + X^3/3 - \dots)\} \\ &= 2X + 2/3X^3 + 2/5X^5 + \dots \end{aligned} \quad (7)$$

We now define the fractional imaginary part of the wavenumber in the seabed as  $\beta$ , so that the complex wavenumber is  $k(1+i\beta)$  where  $k$  and  $\beta$  are both real. This contrasts with Weston's taking the imaginary part of the sound speed as the starting point.<sup>4</sup> Thus the wave function is

$$\exp(ik(1+i\beta)r) = \exp(ikr)\exp(-k\beta r), \quad (8)$$

and the power decays as  $\exp(-2k\beta r)$ . In terms of volume absorption in the seabed  $a$  dB/wavelength, this same power decay is written as  $10^{-ar(k/2\pi)^{10}}$ . Taking logs to base  $e$  we find

$$\beta = \frac{a}{40\pi \log_{10}(e)} = \frac{a}{54.575}, \quad (9)$$

and we see that since  $a$  is typically of order or less than 1 dB/ $\lambda$  then  $\beta \ll 1$  is a very good approximation since  $\beta$  is always of order 0.02 or less. Nevertheless, at this stage we make no assumptions about  $\beta$ .

We now evaluate the admittances in terms of geoaoustic properties and the sine of the grazing angle in the water for which we use the shorthand  $s$ .

$$A_1 = \frac{s}{\rho_1 c_1}, \quad (10)$$

$$A_2 = \frac{\sqrt{(1+i\beta)^2 - Q}}{\rho_2 c_2} = i \frac{\sqrt{(Q-1+\beta^2) - 2i\beta}}{\rho_2 c_2}, \quad (11)$$

where

$$Q \equiv \left(\frac{c_2}{c_1}\right)^2 (1 - s^2), \quad (12)$$

and  $Q > 1$  always. To separate  $A_2$  into real and imaginary parts we note that it can be written as  $\rho_2 c_2 A_2 / i = \sqrt{a+ib} = c + id$ , where

$$a + ib = (c^2 - d^2) + i2cd, \quad (13)$$

so

$$a = c^2 - d^2; \quad b = 2cd. \quad (14)$$

Solving the quadratic for  $c^2$ , stipulating that  $c^2$  must be positive, we find

$$c^2 = \frac{a + \sqrt{a^2 + b^2}}{2}, \quad (15)$$

so from the definition Eq. (3)

$$A_I = \frac{1}{\rho_2 c_2} \sqrt{\frac{(Q-1+\beta^2) + \sqrt{(Q-1+\beta^2)^2 + 4\beta^2}}{2}}, \quad (16)$$

$$A_R = \frac{\beta\sqrt{2}}{\rho_2 c_2 \sqrt{(Q-1+\beta^2) + \sqrt{(Q-1+\beta^2)^2 + 4\beta^2}}}. \quad (17)$$

Returning to Eqs. (6) and (7) we substitute Eqs. (10), (16), and (17) to obtain

$$X = \frac{2\sqrt{2}s\beta}{X_o \rho_1 c_1 \rho_2 c_2 \sqrt{(Q-1+\beta^2) + \sqrt{(Q-1+\beta^2)^2 + 4\beta^2}}}, \quad (18)$$

where

$$\begin{aligned} X_o &= \frac{s^2}{\rho_1^2 c_1^2} + \frac{2\beta^2}{\rho_2^2 c_2^2 ((Q-1+\beta^2) + \sqrt{(Q-1+\beta^2)^2 + 4\beta^2})} \\ &\quad + \frac{(Q-1+\beta^2) + \sqrt{(Q-1+\beta^2)^2 + 4\beta^2}}{2\rho_2^2 c_2^2}. \end{aligned} \quad (19)$$

Equations (18) and (19) are exact. Neglecting terms in  $\beta^2$  (which are extremely small, in practice  $< (0.02)^2 = 4 \times 10^{-4}$ ) but retaining  $\beta$  and writing  $Q$  as

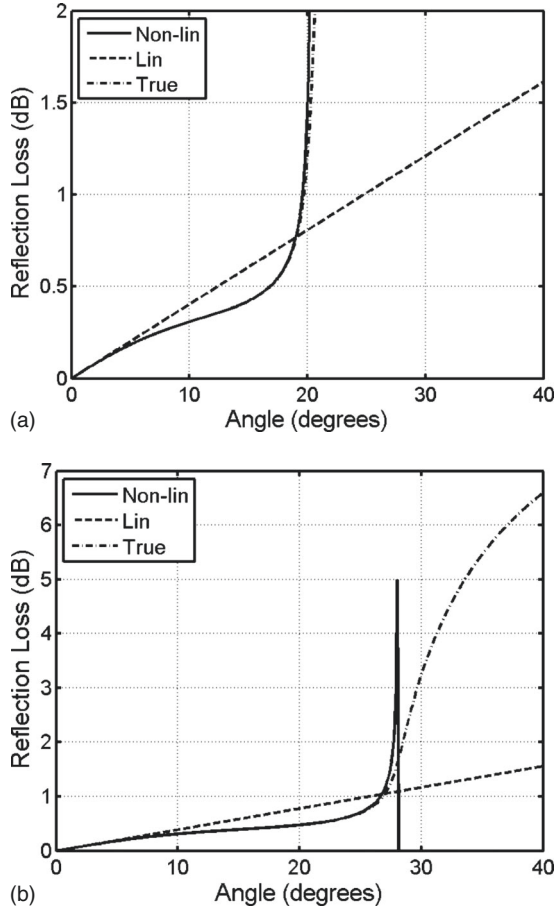


FIG. 1. The non-linear approximation, linear approximation, and exact Rayleigh reflection coefficient for two half-spaces, with density ratio, sound speed, and volume absorption: (a) 1.74, 1600 m/s, 0.2 dB/λ; (b) 1.88, 1700 m/s, 0.5 dB/λ.

$$Q - 1 = \left(\frac{c_2}{c_1}\right)^2 (s_c^2 - s^2), \quad (20)$$

where  $s_c = \sin(\theta_c)$  we have

$$X = \frac{\left(\frac{2s\beta}{\rho_1\rho_2c_2^2\sqrt{s_c^2 - s^2}}\right)}{\left(\frac{s^2}{\rho_1^2c_1^2} + \frac{(s_c^2 - s^2)}{\rho_2^2c_1^2}\right)} = \left\{ \frac{2\rho_2c_1^2\beta}{\rho_1c_2^2s_c^3s} \right\} \times \frac{1}{\sqrt{1 - s^2/s_c^2(1 + ((\rho_2/\rho_1)^2 - 1)s^2/s_c^2)}}. \quad (21)$$

The term in curly brackets on the right hand side of Eq. (21) is recognized as half of Weston's linear coefficient  $\alpha$  multiplied by sine of the grazing angle (rather than the angle itself). He defined  $\alpha$  as<sup>4</sup>

$$\alpha = \frac{4\rho_2c_1^2\beta}{\rho_1c_2^2s_c^3}. \quad (22)$$

The remainder of Eq. (21) is the non-linear factor. This is clearly unity for small  $s$ , tends to infinity as  $s$  approaches  $s_c$ , and otherwise depends only on density ratio.

Returning to the reflection loss as defined in Eq. (7) we see that the first term in the  $X$  expansion is proportional to  $\beta$ ,

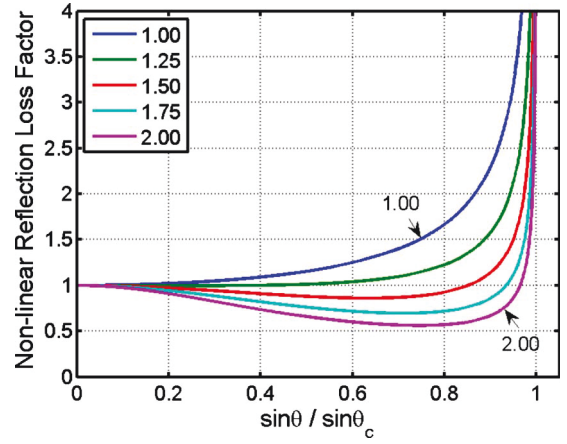


FIG. 2. (Color online) The non-linear reflection loss factor  $[\sqrt{1 - s^2/s_c^2}(1 + ((\rho_2/\rho_1)^2 - 1)s^2/s_c^2)]^{-1}$  [i.e., Equation (23) excluding the numerator] vs. normalized angle with density ratio as parameter (see key).

but the second term is proportional to  $\beta^3$  and is therefore always negligible since their ratio is proportional to  $\beta^2$ . So the reflection loss can be written as

$$-\log(|R|^2) = \frac{\alpha s}{\sqrt{1 - s^2/s_c^2}(1 + ((\rho_2/\rho_1)^2 - 1)s^2/s_c^2)}. \quad (23)$$

It is obvious that Eq. (23) fails when  $s > s_c$ ; it is less obvious that it fails when the grazing angle is extremely close to critical. The reason can be traced to Eq. (20) where  $Q - 1 \rightarrow 0$  as  $s \rightarrow s_c$ . Then the denominator of the exact Eq. (18) contains *only* terms in  $\beta^2$ , and so its neglect begins to have a serious effect. To avoid this we need

$$s^2 \ll s_c^2 - \beta^2 c_1^2 / c_2^2, \quad (24)$$

but since  $\beta^2 < \sim 4 \times 10^{-4}$  this is not a serious problem in normal applications.

Some plots of this function are superimposed on the exact Rayleigh reflection loss for various values of sound speed, density, and volume absorption in Figs. 1(a) and 1(b). Expanding Eq. (23) as a power series we find the approximation

$$\begin{aligned} -\log(|R|^2) &= \alpha s (1 + b_1(s/s_c)^2 + b_2(s/s_c)^4 + \dots) \\ &\approx \alpha s + \alpha s^3 (1.5 - (\rho_2/\rho_1)^2) / s_c^2, \\ b_1 &= 3/2 - (\rho_2/\rho_1)^2; \quad b_2 = 15/8 - (5/2)(\rho_2/\rho_1)^2 \\ &\quad + (\rho_2/\rho_1)^4. \end{aligned} \quad (25)$$

Although this is an approximation it suggests that the reflection loss should approach linearity when the second term is zero, i.e.,  $\rho_2/\rho_1 = \sqrt{3/2} = 1.225$ .

Of course, we cannot expect exact linearity since the function inevitably curves upwards just before the critical angle. However it is clear that densities lower than this value cause upward curvature (more loss than linear) while higher densities cause downward curvature (less loss than linear). This effect is shown through the non-linear factor in Fig. 2, and the closest to linear (i.e., flat) can be seen to be near the density ratio of 1.25. In fact the denominator of Eq. (23) can be written as  $[1 + (2B - 1)v + B(B - 2)v^2 - B^2v^3]^{1/2}$  with

$B=(\rho_2/\rho_1)^2-1$  and  $v=(s/s_c)^2$ . Its gradient is zero near  $v=0$  if  $B=0.5$ , i.e., when the density ratio is  $\sqrt{1.5}$  as before. Forcing the value of the denominator to be unity at  $v=0.25$  (corresponding to approximately  $10^\circ$  in Fig. 2) leads to  $B=0.619$  and a density ratio of 1.27.

## B. Reflection phase

Starting with Eq. (1) and using the same approach we can derive a formula for the phase change on reflection. Again we write the complex reflection coefficient [see Eq. (5)] as

$$V = \frac{A_1 - A_R - iA_I}{A_1 + A_R + iA_I} = \frac{(A_1 - A_R - iA_I)(A_1 + A_R - iA_I)}{(A_1 + A_R)^2 - A_I^2}$$

$$= \frac{A_1^2 - A_R^2 - A_I^2 - i2A_1A_I}{(A_1 + A_R)^2 - A_I^2}. \quad (26)$$

Therefore the phase  $\phi$  is given by

$$\tan \phi = \frac{2A_1A_I}{A_1^2 - A_R^2 - A_I^2}. \quad (27)$$

Substituting Eqs. (10), (16), and (17) we find the exact relation

$$\tan \phi = \frac{\sqrt{2}s}{Y_o \rho_1 c_1 \rho_2 c_2} \times \sqrt{(Q-1+\beta^2) + \sqrt{(Q-1+\beta^2)^2 + 4\beta^2}}, \quad (28)$$

where

$$Y_o = \frac{s^2}{\rho_1^2 c_1^2} - \frac{2\beta^2}{\rho_2^2 c_2^2 ((Q-1+\beta^2) + \sqrt{(Q-1+\beta^2)^2 + 4\beta^2})}$$

$$- \frac{(Q-1+\beta^2) + \sqrt{(Q-1+\beta^2)^2 + 4\beta^2}}{2\rho_2^2 c_2^2}. \quad (29)$$

Note that  $Y_o$  differs from  $X_o$  [Eq. (19)] only in two sign changes. Again neglecting terms in  $\beta^2$  we obtain

$$\tan \phi = \frac{2s}{\rho_1 c_1 \rho_2 c_2} \frac{\sqrt{Q-1}}{\left(\frac{s^2}{\rho_1^2 c_1^2} - \frac{Q-1}{\rho_2^2 c_2^2}\right)}$$

$$= \frac{2s\rho_2}{s_c \rho_1} \frac{\sqrt{1-(s/s_c)^2}}{1 - (1 + (\rho_2/\rho_1)^2)(s/s_c)^2}. \quad (30)$$

The phase is understood to start in the third quadrant and move round to the fourth as  $s$  increases up to critical, i.e.,

$$\phi = -\pi + \operatorname{atan}\left(\frac{2s\rho_2}{s_c \rho_1} \frac{\sqrt{1-(s/s_c)^2}}{1 - (1 + (\rho_2/\rho_1)^2)(s/s_c)^2}\right), \quad (31)$$

or with a four-quadrant tangent

$$\phi = \operatorname{atan}_{4Q}\left(-\frac{2s\rho_2}{s_c \rho_1} \sqrt{1-(s/s_c)^2},\right.$$

$$\left.- (1 - (1 + (\rho_2/\rho_1)^2)(s/s_c)^2)\right). \quad (32)$$

This can be written more elegantly in terms of

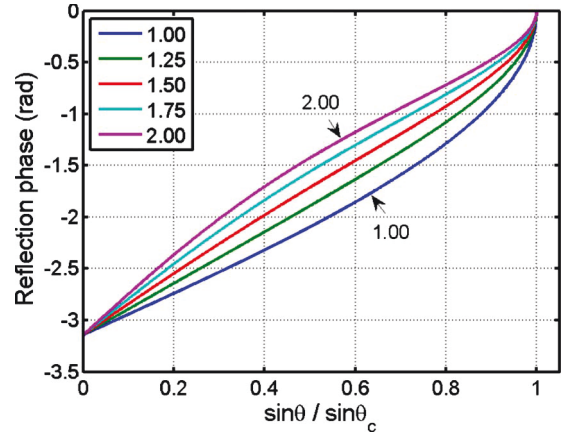


FIG. 3. (Color online) Reflection phase (radians) for normalized grazing angle with density ratio as parameter (see key) using Eq. (31). Note the closeness to linearity particularly near a density of about 1.25.

$$\sin x = s/s_c \quad (33)$$

and

$$\tan y = \rho_2/\rho_1 \quad (34)$$

as

$$\tan \phi = \frac{\sin 2x \sin 2y}{\cos 2x + \cos 2y}. \quad (35)$$

The top line of Eq. (30) can be recognized as the expansion  $\tan 2\varepsilon = 2 \tan \varepsilon / (1 - \tan^2 \varepsilon)$  in which  $\phi = 2\varepsilon$ , and consequently

$$\tan^2 \varepsilon = \left(\frac{\rho_1 c_1}{\rho_2 c_2}\right)^2 \frac{(Q-1)}{s^2} = \left[\left(\frac{\rho_1}{\rho_2}\right) \cot \theta\right]^2$$

$$\times \left[1 - \left(\frac{c_1}{c_2}\right)^2 \sec^2 \theta\right]. \quad (36)$$

This agrees with the formula given by Eq. (2) of Weston<sup>8</sup> (except for a typographical error) which is easily derived directly from Eq. (1) in the loss free case. It also demonstrates that the above formulae are exact for loss free seabed, and that the phase is otherwise independent of  $\beta$  to first order.

There is an interesting special case when  $\rho_1 = \rho_2$ . According to Eq. (34)  $y = \pi/4$ , so  $\tan \phi = \tan 2x$ , i.e.,  $\phi = -\pi + 2 \operatorname{asin}(s/s_c)$ . For small phase angles, phase is proportional to  $s$ , in other words the complex reflection coefficient rotates in a helix between zero grazing angle and the critical angle, starting at  $(-1+i0)$ , passing near  $(0-i1)$ , ending at  $(1+i0)$ . Figure 3 shows this behavior for a number of densities. It can also be seen that the phase remains linear over a larger range of angles when  $\rho_2/\rho_1 \sim 1.25$ , i.e., when the reflection loss is linear (see Fig. 2). Equations (34) and (35) also show that as seabed density tends to infinity (hard bottom) or zero (pressure release) the tangent of the phase tends to zero, independently of angle (in the former case the phase itself is zero, and in the latter it is  $\pi$ ).

### C. Weston's effective depth

In reality we have a single reflection from a simple half-space boundary and we wish to replace this true reflecting surface with a roughly equivalent, slightly deeper, pressure release boundary. This concept was originally developed by Weston<sup>8</sup> and then revisited in Refs. 9, 11, and 12. The effect can easily be understood by inspection of typical normal mode shapes. The first mode (very shallow angle) tends to have a relatively small value just above the seabed, and if extrapolated downwards would approach zero at a point just below the boundary. In contrast the highest mode (at nearly the critical angle) has gradient zero at the boundary, and therefore, if extrapolated would approach zero at one quarter of its vertical wavelength below the boundary. Because this wavelength is smaller for the high order modes than for the first, all modes tend to meet at approximately the same depth, an effective depth where there could have been a pressure release surface. This is the same phenomenon as the end-correction on musical instruments, such as a flute or recorder.<sup>13</sup>

Mathematically we take the equation for the reflection phase and we equate it to the phase difference  $\psi$  imposed on a plane wave by shifting the reflection boundary down a distance  $h$  and reflecting from a pressure release boundary, i.e.,

$$\psi = -\pi + 2hk_o s, \quad (37)$$

where  $k_o$  is the wavenumber in the upper medium.

From Eqs. (31) and (37), using the shorthand of Eqs. (33)–(35) generally we have

$$h = \frac{\lambda_o}{4\pi s} \operatorname{atan}\left(\frac{\sin 2x \sin 2y}{\cos 2x + \cos 2y}\right), \quad (38)$$

where  $\lambda_o$  is the wavelength in the upper medium. For small angles this reduces to

$$h = \frac{\lambda_o \rho_2}{2\pi s_c \rho_1}, \quad (39)$$

which is identical to Eq. (5) of Ref. 8. Also, the special case for any angle, where  $\rho_1 = \rho_2$  leads to

$$h = \frac{\lambda_o}{2\pi s} \operatorname{asin}(s/s_c). \quad (40)$$

If, in addition,  $s$  is not too close to  $s_c$  then

$$h = 1/(k_o s_c) = \lambda_o/(2\pi s_c). \quad (41)$$

This is clearly a distance that depends on frequency and critical angle but is independent of angle (or mode number). This limiting case is obviously independent of density.

Another limiting case of interest is  $\rho_2 \rightarrow \infty$ . From Eq. (34),  $y = \pi/2$ , and so from Eq. (35),  $\phi = 0$ . Under these conditions the concept of an effective angle-independent depth to a pressure release surface certainly does not hold. In fact the surface in this extreme case is already a hard boundary. To investigate the general utility of the effective depth we plot normalized effective depth  $h/\lambda_o$  ( $s_c \rho_1/\rho_2$ ) against  $s/s_c$  for various densities in Fig. 4 using the general formula (38)

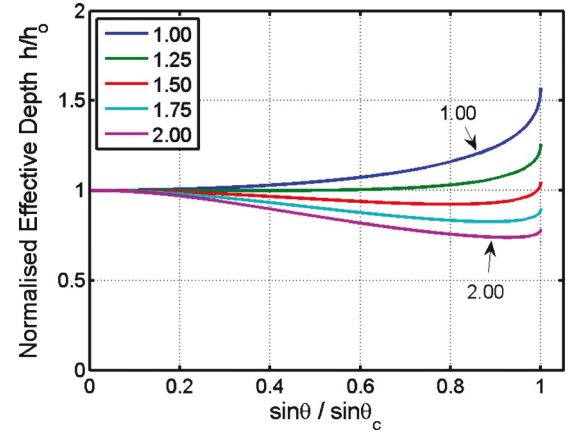


FIG. 4. (Color online) Normalized effective depth ( $h/h_o$  where  $h_o = (\lambda_o/2\pi s_c)(\rho_2/\rho_1)$ ) against normalized grazing angle with density ratio as parameter (see key) using Eq. (38). Note the flatness particularly when density is slightly larger than 1.25.

combined with Eqs. (33) and (34). The effective depth is nearly constant when  $\rho_2/\rho_1 \sim 1.3$ , i.e., when both the reflection loss and its phase are linear.

Note that there are a number of similar “effective/shift/displacement” terms used in the literature and two genuinely different phenomena as well. Weston<sup>12</sup> distinguishes, and shows the mathematical relation between, what he calls a “wave shift” and a “beam shift.” The former results in the effective depth described here and is associated with the modal phase velocity since it is a pure phase effect. The latter results in the lateral shift (or “lateral wave”) associated with point source/receivers, spherical waves,<sup>14</sup> and the modal group velocity since it is seen, for instance, in laboratory experiments where there is a tangible arrival corresponding to the path that runs for part of its course along the boundary. Mathematically the effective depth in this paper corresponds to Eq. (4) of Ref. 12 and Eqs. (2) and (3) of Ref. 8, as already stated. Equations (4.4.5) and (4.4.6) of Ref. 14 (the lateral wave), after minor rearrangement, is the same as Eq. (A3) of Ref. 9 and Eq. (17) of Ref. 12 (except for typographical errors). Finally note that the same effective depth term has been used with different meaning in at least two other contexts.<sup>15,16</sup>

## III. REVERBERATION

### A. Reverberation angle integral

The purpose of searching for an approximate form of the Rayleigh reflection coefficient was to be able to modify the reverberation integral under conditions when the reflection loss was non-linear. The closed-form equations for reverberation have been derived for various types of environments.<sup>5–7</sup> For Lambert’s law

$$S = \mu \sin(\theta_1)\sin(\theta_2), \quad (42)$$

with isovelocity water and flat bottom a general expression is

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<i>Author(s)</i> Harrison, C.H.		
<i>Title</i> An approximate form of the Rayleigh reflection loss and its phase: application to reverberation calculation		
<i>Abstract</i> <p>A useful approximation to the Rayleigh reflection coefficient for two half-spaces composed of water over sediment is derived. This exhibits dependence on angle that may deviate considerably from linear in the interval between grazing and critical. It shows that the non-linearity can be expressed as a separate function that multiplies the linear loss coefficient. This non-linearity term depends only on sediment density and does not depend on sediment sound speed or volume absorption. The non-linearity term tends to unity, i.e., the reflection loss becomes effectively linear, when the density ratio is about 1.27. The reflection phase in the same approximation leads to the well-known “effective depth” and “lateral shift.” A class of closed-form reverberation (and signal-to-reverberation) expressions has already been developed [C. H. Harrison, <i>J. Acoust. Soc. Am.</i> <b>114</b>, 2744–2756 (2003); C. H. Harrison, <i>J. Comput. Acoust.</i> <b>13</b>, 317–340 (2005); C. H. Harrison, <i>IEEE J. Ocean. Eng.</i> <b>30</b>, 660–675 (2005)]. The findings of this paper enable one to convert these reverberation expressions from simple linear loss to more general reflecting environments. Correction curves are calculated in terms of sediment density. These curves are applied to a test case taken from a recent ONR-funded Reverberation Workshop.</p>		
<i>Keywords</i>		
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