NAVAL SURFACE WARFARE CENTER
PANAMA CITY DIVISION
PANAMA CITY, FL 32407-7001

TECHNICAL REPORT
NSWC PCD TR-2017/009

THEORY BEHIND COIL-GEOMETRY PARAMETERS FOR FANSELAU/BRAUNBECK MAGNET COILS

John T. Bono
Advanced Signal Processing & ATR Branch (Code X23)
Unmanned Systems, Automation & Processing Division
Science & Technology Department

John W. Purpura
Intelligent Sensing & Irregular Warfare Branch (Code X12)
Sensing Sciences & Systems Division
Science & Technology Department

August 2017

REQUEST FOR PUBLIC RELEASE OF UNCLASSIFIED TECHNICAL INFORMATION Print Form

From (Author; if author is a contractor, Contracting Officer's Representative (COR)): JOHN W. PURPURA

LOG ______ (For record keeping purposes only)

To: Public Affairs Office

Reference (a) NSWCPCINST 5720.4F

Enclosure (1) Material for release (include one copy for Public Affairs file)

Enclosure (2) Sponsor Approval

NOTE: If NAVSEA sponsor, must be submitted to NAVSEA Public Affairs for release. (Ref (a), para 3c.) See NSWCPCWeb for NAVSEA instruction and local guidance.

IN ACCORDANCE WITH REF (a), ENCL (1) IS SUBMITTED FOR REVIEW AND APPROVAL FOR PUBLIC RELEASE

MATERIAL TITLED:

THEORY BEHIND COIL-GEOMETRY PARAMETERS FOR FANSELAU/BRAUNBECK MAGNET COILS

TYPE OF MATERIAL (presentation/speech/conference paper; magazine/journal/newspaper article; contractor report/release);	PURPOSE (i.e., conference name/location)	REQUIRED PUBLIC RELEASE DATE
Technical Report		

SPONSOR NAME/ORGANIZATION (Attach sponsor approval prior to fowarding. See note above if NAVSEA sponsor.)	CONTRACT NUMBER & COR (if applicable):

REQUIRED APPROVALS: Signature below certifies that this material has been reviewed and is technically accurate; contains no controlled unlcassified information, classified, intelligence, or proprietary information; and conforms with public affairs policy.

Sponsor approval attached? [X] Yes [ ] No

SIGNATURE (Author/COR) John W. Purpura DATE 12-12-2019 | CODE X12 | PHONE 235-5140 | JON

Release? [X] Yes [ ] No

SIGNATURE (Br. Hd.) [signature] DATE 2/5/20 | COMMENTS

Timely release? [X] Yes [ ] No | COMMENTS

(Per Ref (a), justification required if not released from department/directorate 2 weeks prior to required public release date (above); 5 weeks, if higher echelon review needed.)

If no, justification attached? [ ] Yes [ ] No

Release? [X] Yes [ ] No

SIGNATURE (Dept./Dir. Hd.) [signature] DATE 2/21/20

Release? [X] Yes [ ] No | COMMENTS

SIGNATURE (Security) Linda A. Bland DATE 2/28/2020

Release? [X] Yes [ ] No | COMMENTS

SIGNATURE (STILO, Intel only) [signature] DATE 3/2/20

IS HIGHER ECHELON REVIEW REQUIRED? (Ref (a), para. 3d) [ ] Yes [ ] No

If yes, submit six copies and provide the following information:

SYSCOM AND CODE ______ LOCATION ______

CONTACT'S NAME ______ TELEPHONE ______

[X] **Approved for Public Release; distribution is unlimited.**

SIGNATURE (Public Affairs) [signature] DATE 2 Mar 20

- NOTE -
Ensure Distribution Statement A, "Approved for Public Release; distribution is unlimited," is affixed to all media prior to release.

REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 0704-0188*

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From - To)*
31-08-2017	Technical	Apr 2016-Aug 2017

4. TITLE AND SUBTITLE Theory Behind Coil-Geometry Parameters for Fanselau/Braunbeck Magnet Coils	5a. CONTRACT NUMBER N0002417WX00563
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S) John T. Bono John W. Purpura	5d. PROJECT NUMBER
	5e. TASK NUMBER 6050
	5f. WORK UNIT NUMBER WBS 1.1.22.14
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) NAVAL SURFACE WARFARE CENTER, PANAMA CITY DIVISION 110 VERNON AVENUE PANAMA CITY FL 32407-7001	8. PERFORMING ORGANIZATION REPORT NUMBER NSWC PCD TR-2017-009
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) NAVAL SEA SYSTEMS COMMAND EXPLOSIVE ORDNANCE DISPOSAL PROGRAM MANAGEMENT OFFICE 1333 ISSAC HULL AVE SE STOP WASHINGTON NAVY YARD DC 20376	10. SPONSOR/MONITOR'S ACRONYM(S) PMS 408
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

(A): Approved for public release: distribution unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

The Naval Surface Warfare Center, Panama City Division Fanselau Magnetic Test Facility includes a large, three-axis, magnet coil system that provides a highly uniform magnetic field over a large spatial volume. Each axis consists of an arrangement of four circular coils aligned along a common axis, whose relative dimensions and spacing are optimized for field uniformity. The design of the coil arrangement is based on an analysis originally published by Gerhard Fanselau in 1929, and later improved by Werner Braunbeck in 1934. Since the Fanselau and Braunbeck journal articles were published in German, an examination of their theoretical analyses, along with some additional supporting material on electromagnetic theory to introduce the problem, are presented here for the convenience of the reader. The essence of their approach centers on expressing the magnetic scalar potential of the coil system as an infinite series and adjusting the coil diameters and spacing to zero-out the largest terms contributing to nonuniformity of the magnetic field.

15. SUBJECT TERMS

Fanselau Coil; Braunbeck Coil; Helmholtz Coil; Uniform Magnetic Field; Magnet Coil Geometry Parameters; Classical Electromagnetic Theory; Maxwell's Equations; Magnet Coil Systems

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			John W. Purpura
U	U	U	SAR	36	19b. TELEPHONE NUMBER *(Include area code)* (850) 235-5140

FOREWORD

This report has been prepared, reviewed, and approved by the Science and Technology Department.

Prepared by JOHN T. BONO
Advanced Signal Processing and
ATR Branch (X23)

Co-Author JOHN W. PURPURA
Intelligent Sensing and Irregular
Warfare Branch (X12)

DARSHAN BRYNER
Head, Advanced Signal Processing and
ATR Branch (X23)

ANN MARIE SHOVER
Head, Intelligent Sensing and
Irregular Warfare Branch (X12)

FRANK J. CROSBY
Unmanned Systems, Automation and
Processing Division (X20)

KEITH M. ALIBERTI
Head, Sciences & Technology
Department (X)

CONTENTS

ILLUSTRATIONS

ABSTRACT

The Naval Surface Warfare Center, Panama City Division Fanselau Magnetic Test Facility includes a large, three-axis, magnet coil system that provides a highly uniform magnetic field over a large spatial volume. Each axis consists of an arrangement of four circular coils aligned along a common axis, whose relative dimensions and spacing are optimized for field uniformity. The design of the coil arrangement is based on an analysis originally published by Gerhard Fanselau in 1929, and later improved by Werner Braunbeck in 1934. Since the Fanselau and Braunbeck journal articles were published in German, an examination of their theoretical analyses, along with some additional supporting material on electromagnetic theory to introduce the problem, are presented here for the convenience of the reader. The essence of their approach centers on expressing the magnetic scalar potential of the coil system as an infinite series and adjusting the coil diameters and spacing to zero-out the largest terms contributing to nonuniformity of the magnetic field.

INTRODUCTION

The Naval Surface Warfare Center, Panama City Division (NSWC PCD) Fanselau Magnetic Test Facility features a large, three-axis, magnet coil system having a coil arrangement based on a design originally developed by Gerhard Fanselau[1] and later refined by Werner Braunbeck.[2] Figure 1 shows the Fanselau/Braunbeck coil arrangement of the facility. The coil design provides a magnetic field having a high degree of uniformity over a large spatial volume. Since the original papers by Fanselau and Braunbeck were published in German, in 1929 and 1934, respectively, the basic theory behind the derivation of the coil-geometry parameters, based on the original works, will be presented in this document for the reader's convenience. We have supplemented the content of their original analyses with sufficient background material on electromagnetic theory to provide a complete treatment of the problem.

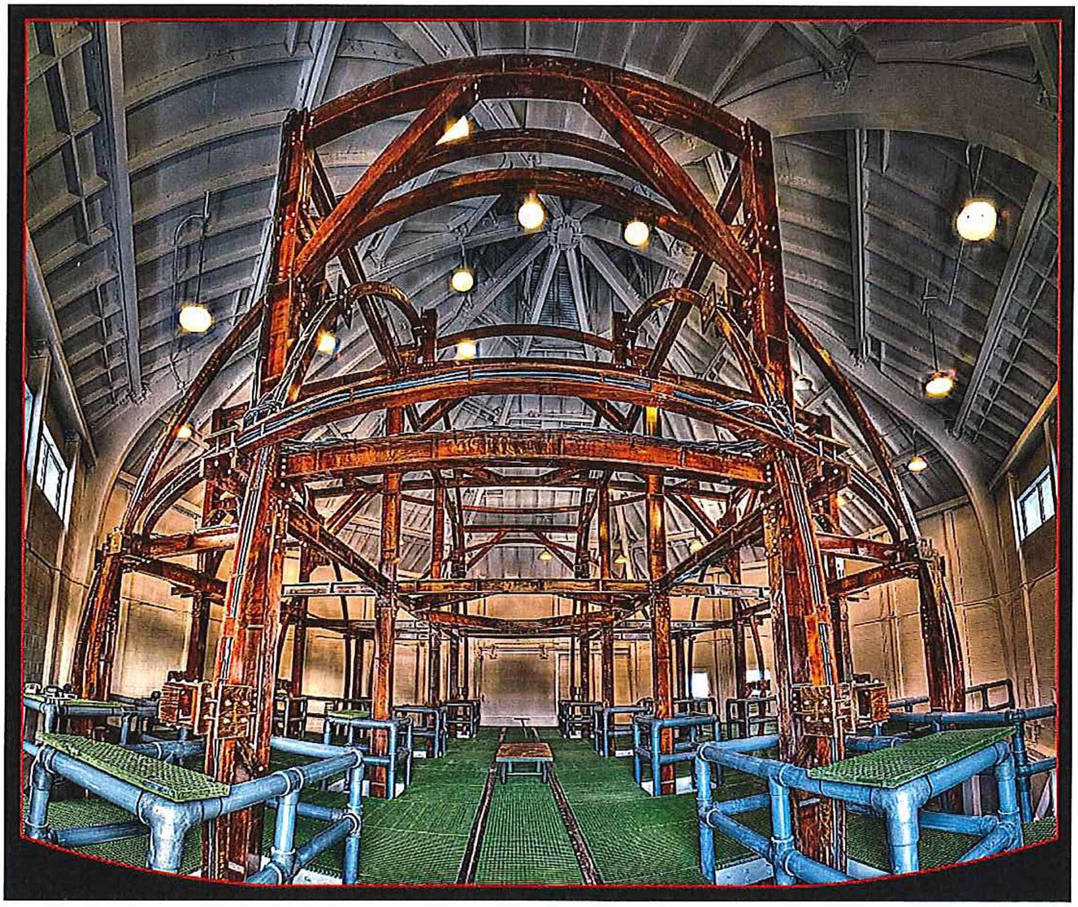

Figure 1. NSWC PCD Fanselau Magnetic Test Facility 3-axis magnet coil

THEORY

Fanselau performed a theoretical analysis to optimize the magnetic field uniformity of magnet coil systems consisting of various numbers of circular coil pairs, all aligned along a common axis, with each coil carrying the same amount of total electric current. The system geometry based on a single coil pair gives the well-known Helmholtz coil configuration, and the system composed of two coil pairs (providing even greater field uniformity) is known as a Fanselau coil. Additional coil pairs further increase the field uniformity, but may not be practical or necessary for all situations.

Fanselau based his analysis upon classical electromagnetic theory. In the case of static magnetic fields generated by constant-current coils, Maxwell's equations pertaining to the magnetic field are given by:

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1}$$

and

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \ , \tag{2}$$

where, in terms of the International System of Units (abbreviated SI), B is the magnetic flux density in tesla, H is the magnetic field intensity in A/m, and J is the current density in A/m^2.

In any region free of current, such as the empty volume of space surrounding magnet coils, Equation (2) becomes

$$\nabla \times \boldsymbol{H} = 0 \ . \tag{3}$$

From vector calculus, Equation (3) infers that H can be expressed as the gradient of a scalar potential (since the curl of the gradient of any scalar field is always zero):

$$\boldsymbol{H} = -\nabla\Phi_M \ , \tag{4}$$

where Φ_M is introduced as the magnetic scalar potential. The minus sign indicates the convention that $\boldsymbol{H}$ points towards lower levels of magnetic potential. With the assumption that air is the medium in the region surrounding the magnet coils, the relation between B and H is linear and given by

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} \ , \tag{5}$$

where $\mu_0 = 4\pi \times 10^{-7}$H/m is the magnetic permeability of free space.

Substituting Equation (5) into Equation (4) and applying Equation (1) gives:

$$\nabla \cdot \mu_0 \nabla\Phi_M = \mu_0 \nabla^2 \Phi_M = 0 \ . \tag{6}$$

Dividing both sides of Equation (6) by μ_0, we have

$$\nabla^2 \Phi_M = 0 \quad , \tag{7}$$

which is Laplace's equation. Therefore, the expression for the magnetic scalar potential of the magnetic coil system must satisfy Laplace's equation in regions of space surrounding the coils.

Since Fanselau's analysis made use of a spherical coordinate system, let us define the nomenclature and angular direction conventions that we will be using in our discussion. Figure 2 shows the relationship between spherical (r, θ, ϕ) and Cartesian (x, y, z) coordinates in three-dimensions for a position vector $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, where $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ are unit vectors pointing along the positive orthogonal directions of the x-, y-, and z- axes, respectively. In spherical coordinates, the point P located by the vector $\boldsymbol{r}$ is specified by the radial distance r of the point from the origin, the polar angle θ measured from the positive z-axis to r, and its azimuth angle ϕ measured from the positive x-axis in a counterclockwise direction (as shown) about the z-axis to the orthogonal projection of r onto the xy-plane. The coordinates ranges are: $0 \le r < \infty, 0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$. The equations necessary to obtain Cartesian components of r from spherical coordinates are:

$$x = r \sin\theta \cos\phi \tag{8}$$

$$y = r \sin\theta \sin\phi \tag{9}$$

and

$$z = r \cos\theta \quad . \tag{10}$$

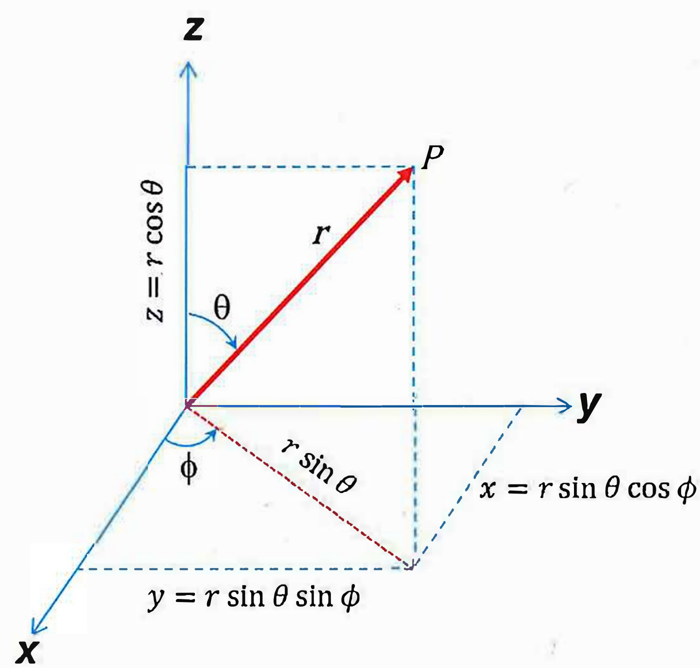

Figure 2. Comparison of Cartesian and spherical coordinates

Fanselau's approach to optimize the field uniformity of a given coil system involved constructing the magnetic scalar potential for the system and adjusting the relative geometrical dimensions of the system to make the largest terms contributing to a nonuniform field vanish. We can explain his approach by referring to the simple case of a single pair of circular coils as illustrated in Figure 3. Here we have two identical, horizontal coils of radius a with their axes co-aligned with the z-axis and separated from each other by a distance, $2d$. Each coil carries an electrical current, I, flowing counterclockwise when viewed from above, as indicated in the figure.

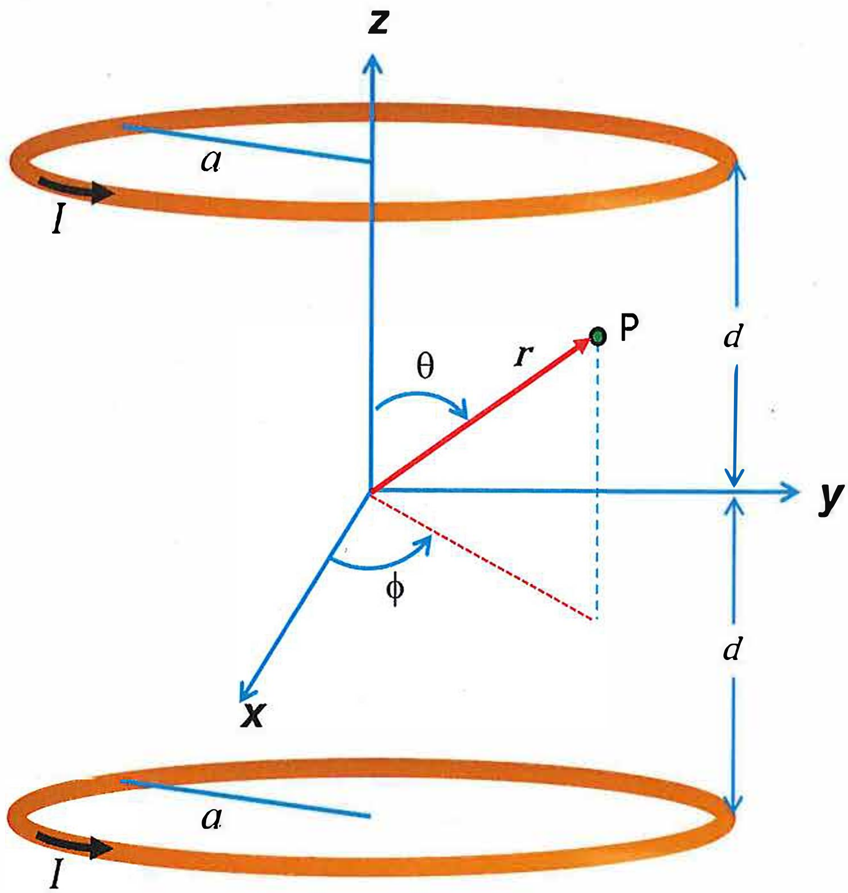

Figure 3. Single pair of coils aligned along the z-axis

We have shown that the magnetic scalar potential at any point P located in the region surrounding the coil pair (specified in the figure by the position vector $\boldsymbol{r}$) must be a solution to Laplace's equation. In spherical coordinates, Laplace's equation is expressed as:

$$\nabla^2\Phi_M(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi_M}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi_M}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi_M}{\partial\phi^2} = 0\ . \tag{11}$$

The two-coil system under discussion exhibits axial symmetry, since the coil geometry is independent of the azimuthal angle ϕ. Therefore, the last term of the Laplacian in Equation (11) vanishes. With this simplification, the general solution for the potential is well known[3] and takes the form:

$$\Phi_M(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + C_l r^{-l-1})\, P_l(\cos\theta)\,, \tag{12}$$

where the A_l's and C_l's are arbitrary constants and the $P_l(\cos\theta)$ are the Legendre polynomials. For reference, the first eight Legendre polynomials are given in Table 1 and plotted in Figure 4. Equation

(12) simplifies further if we restrict the solution to the region between the coils, such that $z = r\cos\theta \leq \pm d$. Since this region includes the origin, at $r = 0$, the terms with inverse powers of r (the C_l terms) will blow up; and thus, all the C_l's must be set equal to zero. Therefore, the general solution of the magnetic scalar potential in the region between the coils becomes:

$$\Phi_M(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)\,. \tag{13}$$

Table 1. First Eight Legendre Polynomials

l	$P_l(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{1}{2}(3\cos^2\theta - 1)$
3	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$
4	$\frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3)$
5	$\frac{1}{8}(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)$
6	$\frac{1}{16}(231\cos^6\theta - 315\cos^4\theta + 105\cos^2\theta - 5)$
7	$\frac{1}{16}(429\cos^7\theta - 693\cos^5\theta + 315\cos^3\theta - 35\cos\theta)$

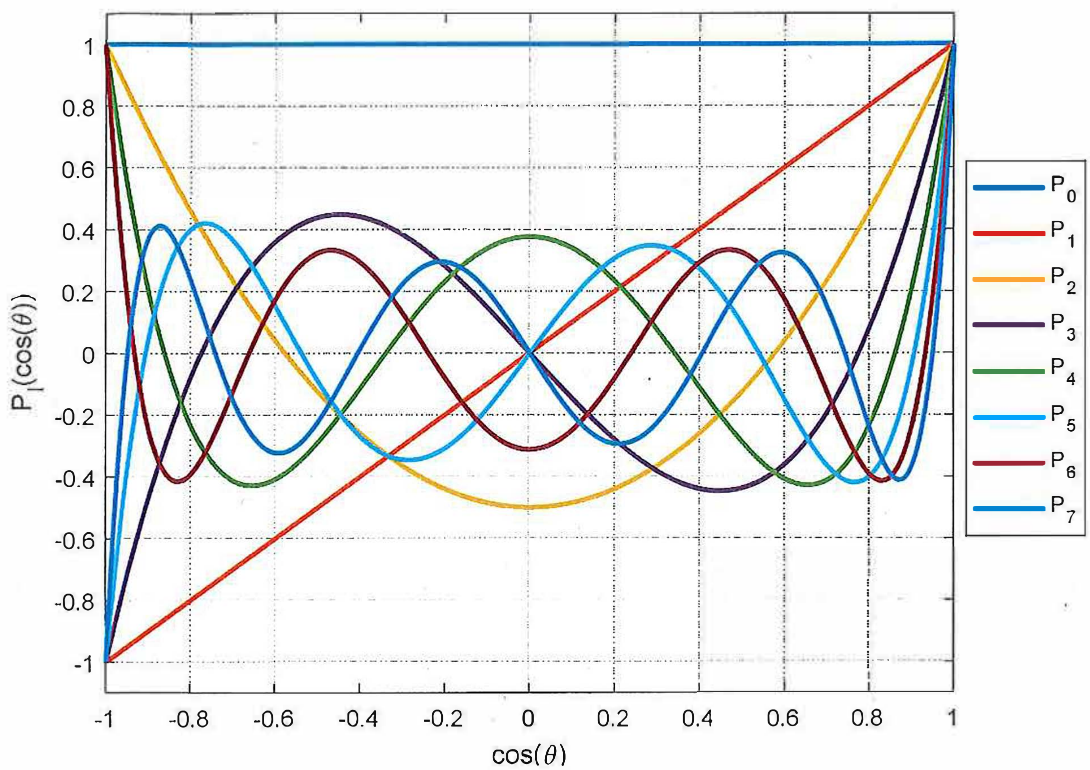

Figure 4. Plots of the first eight Legendre polynomials

To determine the coil diameters and spacings of a given coil system that maximize its field uniformity, Fanselau started by generating the magnetic scalar potential of the system expressed as an infinite sum having the form of Equation (13). He then analytically solved for the coil system's dimensional parameters by setting as many of the largest terms in the sum contributing to a nonuniform field to zero as the number of degrees of freedom of the system allowed. Any term in Equation (13) with $l > 1$ contributes to the field's nonuniformity.

We will now begin the derivation of the magnetic scalar potential of a circular coil system. We start by deriving the expression for the potential along the *z*-axis of a coil pair and then generalize it later to be applicable to all space between the coils based on the known general form of the solution to Laplace's equation given by Equation (13). The solution for the magnetic scalar potential will then be adapted to include coil systems containing an arbitrary number of coil pairs.

Consider a circular loop of radius a carrying a current I, located in the *xy*-plane of a Cartesian coordinate system, as shown in Figure 5 below. The center of the loop is located at the origin, with its axis along the *z*-direction. The law of Biot and Savart is used to calculate the magnetic field at a point located on the loop axis. The infinitesimal field dH, a distance z from the center of the loop along the *z*-axis to the field measurement point P, associated with the current flowing in differential length element $d\boldsymbol{l}$ is given by

$$dH = \frac{I}{4\pi} \frac{dl \times R}{R^3} , \tag{14}$$

where R is the displacement vector from dl to P. The differential vector dl, shown in the figure for a cylindrical coordinate azimuthal angle $\varphi = \pi/2$, is tangent to the loop and points along the current direction into the page. Its cross product with R gives the direction of dH, making the magnetic field direction perpendicular to both dl and R, lying in the plane of the page as shown. The field contribution dH makes an angle α with the z-axis and can be broken up into a component perpendicular to the loop axis, dH_{perp}, and one along the axis, dH_z.

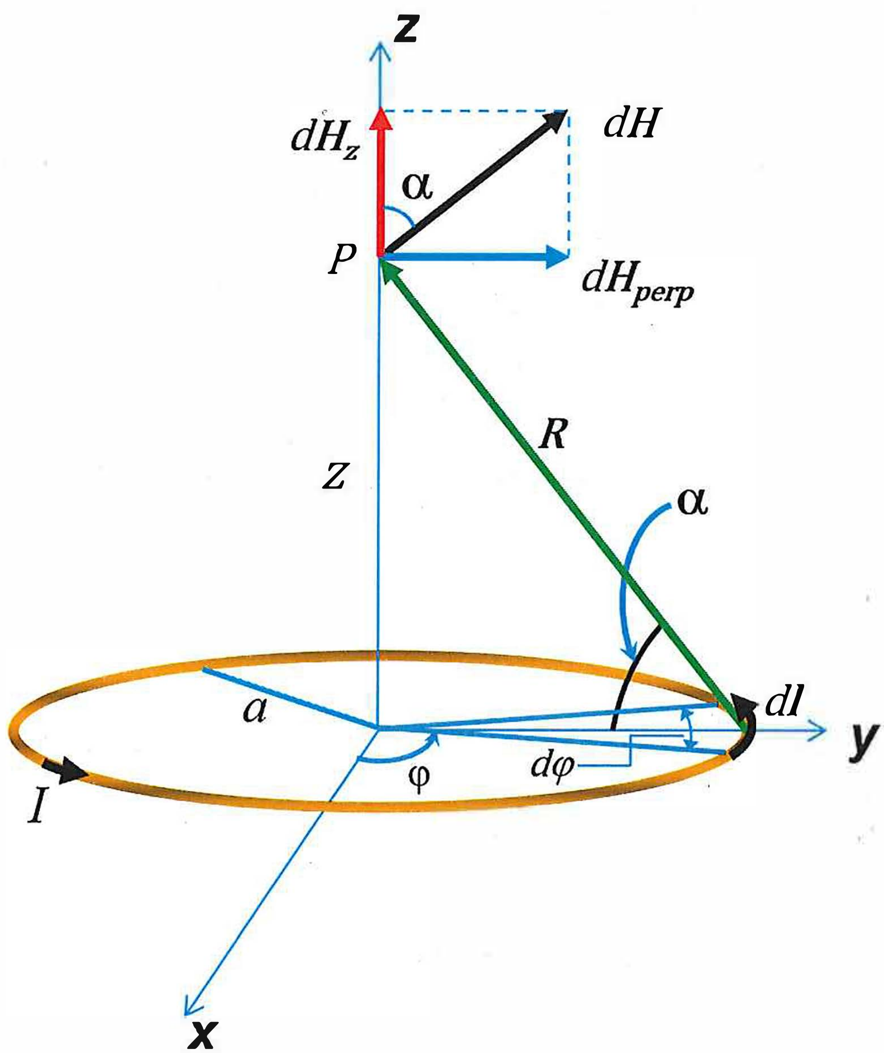

Figure 5. Current loop in the xy-plane

Integrating Equation (14) along the current elements around the circumference of the loop gives the magnetic field at P. Given the symmetry of the loop, the contributions to the integral from dH_{perp} cancel out, as each $\boldsymbol{dl}$ has a balancing current element counterpart, pointing in the opposite direction, half of a circumference away. The dH_z contributions to the field all point in the same direction along the loop axis and add together to produce the entire field. As we see in Figure 5,

$$d\boldsymbol{H_z} = d\boldsymbol{H}\cos\alpha \tag{15}$$

and

$$\cos\alpha = \frac{a}{R} = \frac{a}{\sqrt{a^2 + z^2}} \quad . \tag{16}$$

Since $\boldsymbol{dl}$ is perpendicular to $\boldsymbol{R}$, the magnitude of the differential field component along the axis obtained using Equations (14), (15), and (16) is

$$dH_z = \frac{I}{4\pi}\frac{R\,dl\sin(\pi/2)}{R^3}\frac{a}{R} = \frac{I}{4\pi}\frac{a^2\,d\varphi}{(a^2+z^2)^{3/2}} \quad , \tag{17}$$

where the substitution $dl = ad\varphi$ relating the differential length element and the differential cylindrical azimuthal angle $d\varphi$ was inserted. Therefore, integrating Equation (17) around the loop circumference, the magnetic field at point P can be expressed as

$$H(z) = \frac{I}{4\pi}\frac{a^2}{(a^2+z^2)^{\frac{3}{2}}}\int_0^{2\pi} d\varphi = \frac{Ia^2}{2(a^2+z^2)^{\frac{3}{2}}} \quad . \tag{18}$$

The magnetic scalar potential along the positive axis of the current loop is obtained by inverting Equation (4) as follows:

$$\Phi_M(z>0) = -\int_{\infty}^{z} H(z')dz' = \frac{Ia^2}{2}\int_z^{\infty}\frac{dz'}{\left(a^2+z'^2\right)^{\frac{3}{2}}} = \frac{I}{2}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right). \tag{19}$$

For points along the z-axis below the loop, given by $z < 0$, the potential can be computed by evaluating the above integral from $z' = -\infty$ to $z' = z$. Below the loop, the magnetic field still points upward (positive z-direction) and the potential goes to zero at $z = -\infty$ along the axis. Upon evaluation of the integral, the potential for negative z takes the form:

$$\Phi_M(z<0) = -\frac{I}{2}\left(1 + \frac{z}{\sqrt{a^2+z^2}}\right) \quad . \tag{20}$$

Figure 6 is a plot of the magnetic scalar potential along the z-axis computed using Equations (19) and (20) for a 1-m-diameter coil carrying a current of 1 ampere. Note that the units of Φ_M are amperes, and thus its gradient computed in the expression given by Equation (4) gives the usual units for the magnetic field intensity as amperes per meter (A/m).

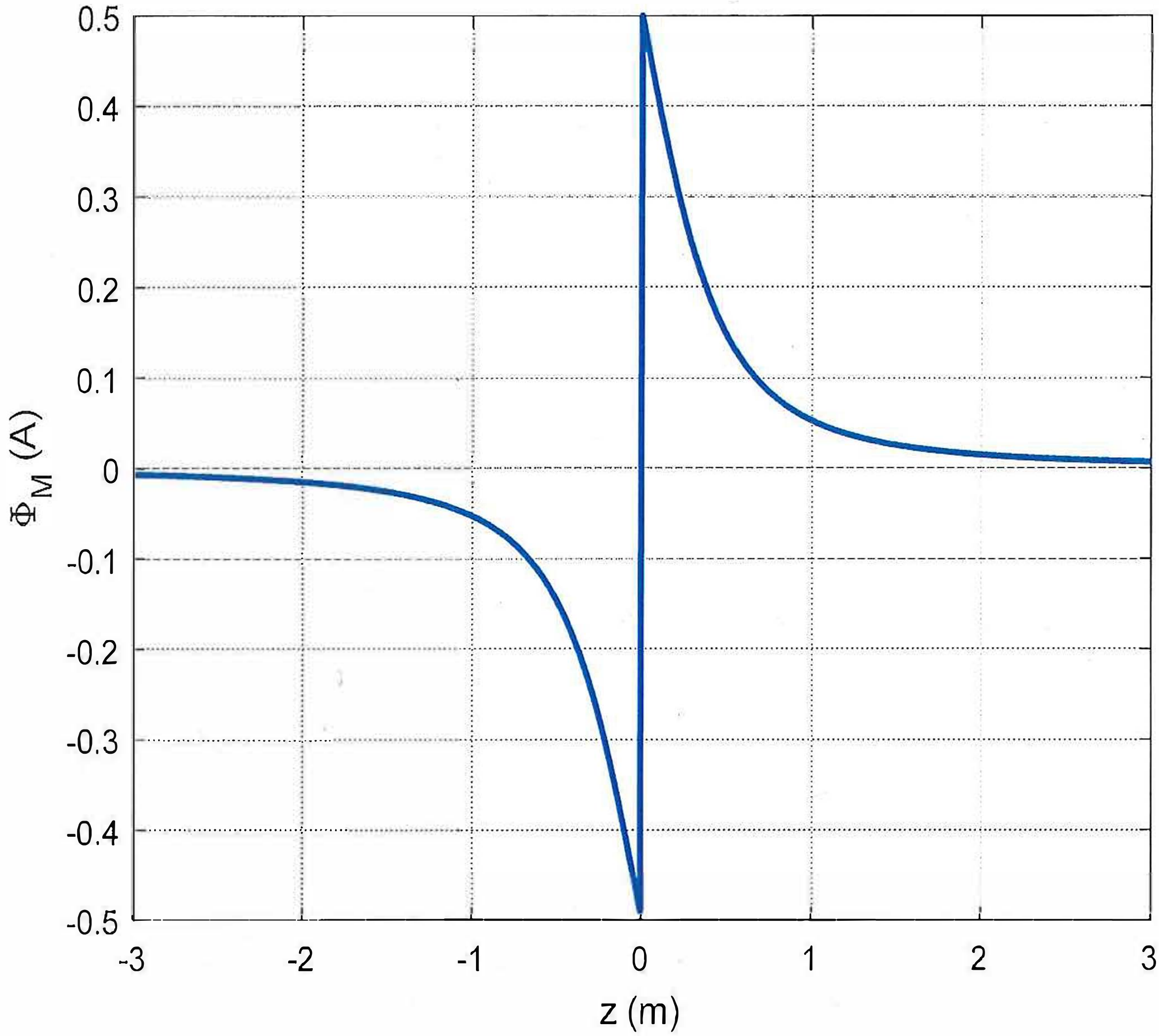

Figure 6. Magnetic scalar potential of a current loop

We now want to determine the magnetic scalar potential for a pair of coaxial magnet coils like those shown earlier in Figure 3. Consider a system of two identically sized coils (shown in blue), each having a current *I* and a radius *a*, with their axes aligned along the z-axis, as illustrated in the diagram of Figure 7. The upper coil is located a distance d above the *xy*-plane containing the origin, and the lower coil is an equal distance below, such that the coils are separated by $2d$. An arbitrary point *P*, used to specify the computation position of the potential, is located a distance z (shown in red) along the *z*-axis. The distance from the origin to the edge of either coil (indicated in green) is labeled *R*, and the polar angle associated with R will be referred to as θ_1. Finally, the distances from point *P* to the upper and lower coil edges (brown lines) are labeled R_- and R_+, respectively.

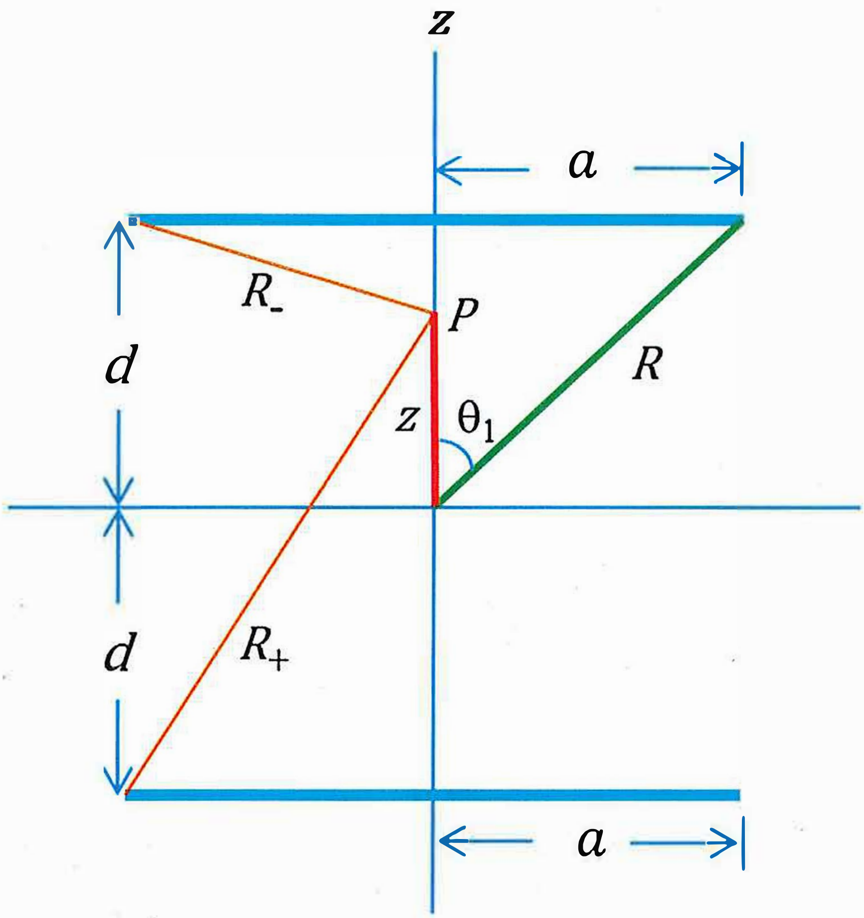

Figure 7. Diagram of two vertical-axis coils

The magnetic scalar potential in the region between the two coils is the sum of the potentials from each coil. For the upper coil, using Equation (20) and accounting for the fact that the coil is offset a distance d from the origin, we make the change of variable $z \xrightarrow{becomes} z - d$, and get:

$$\Phi_M^{upper}(z) = -\frac{I}{2}\left[1 + \frac{z-d}{\sqrt{(z-d)^2 + a^2}}\right]. \tag{21}$$

Similarly, the potential between the coils contributed by the lower coil, obtained using Equation (19) with $z \xrightarrow{becomes} z + d$, is expressed as

$$\Phi_M^{lower}(z) = \frac{I}{2}\left[1 - \frac{z+d}{\sqrt{(z+d)^2 + a^2}}\right]. \tag{22}$$

Adding Equations (21) and (22) gives us the total magnetic scalar potential $\Phi_M(z)$ between the coils for points along the axis:

$$\Phi_M(z) = -\frac{I}{2}\left[\frac{z-d}{\sqrt{(z-d)^2+a^2}} + \frac{z+d}{\sqrt{(z+d)^2+a^2}}\right]. \quad (23)$$

Plots of Equations (21), (22), and (23) are shown in Figure 8 for the arbitrary case of one ampere flowing through coils having a 1-meter diameter (a = 0.5 meters) and a 2-meter spacing (d = 1 meter). The positive polarity current circulates in a counterclockwise direction when viewed looking vertically downward. Since the slope of the total potential (green trace) is negative along the axis of the coil system at all points between the two coils, according to Equation (4) the magnetic field is positive and appears to have its minimum value at z = 0 m for this case.

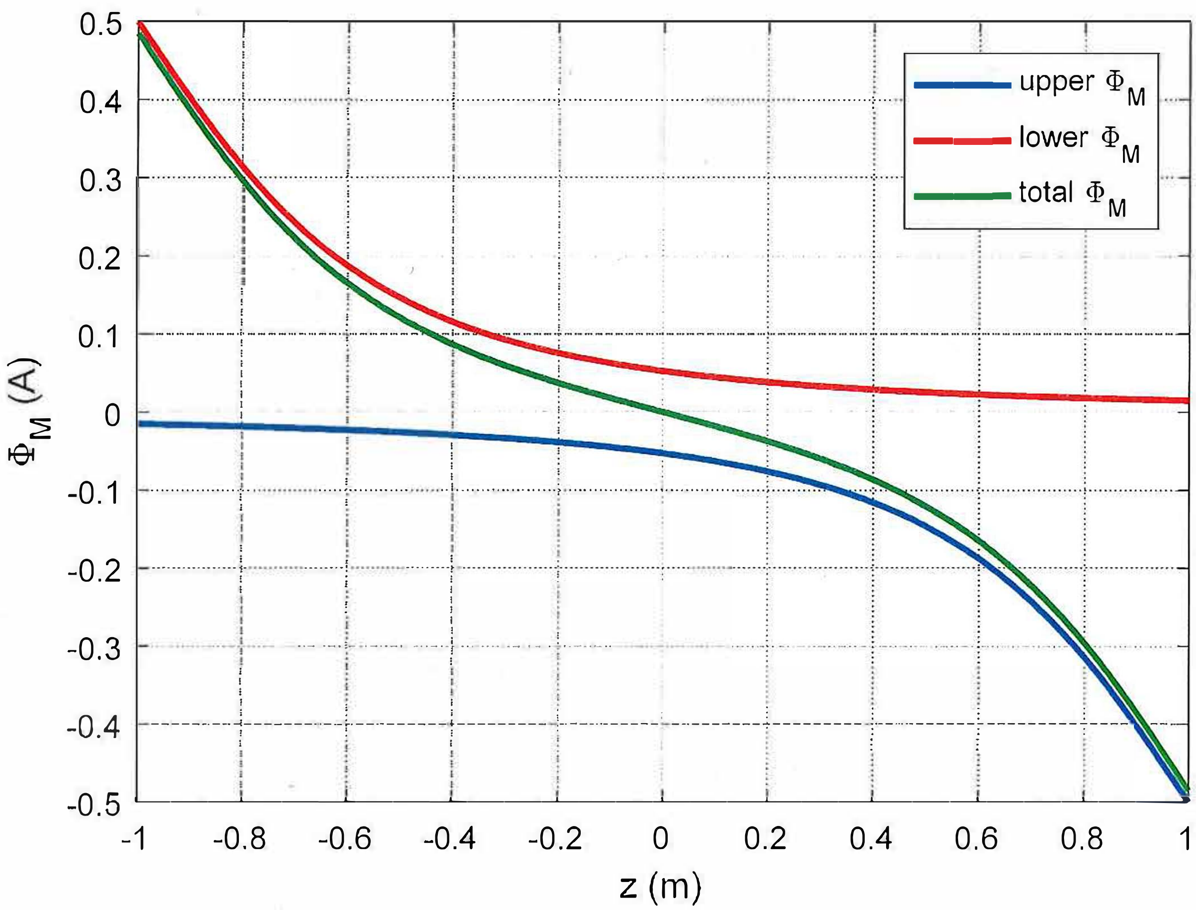

Figure 8. Magnetic scalar potential in the region between two coils

In order to use Fanselau's approach for optimizing a circular coil system's magnetic field uniformity, we must put Equation (23) into a form that resembles Equation (13). Therefore, we need to take the closed-form expression of Equation (23) and transform it into an infinite series. The generating function for Legendre polynomials is often useful in problems of this type. It is given by:[4]

$$g(x,h) = \frac{1}{\sqrt{1-2xh+h^2}} = \sum_{l=0}^{\infty} h^l P_l(x) \; ; \qquad |h| < 1 \; . \tag{24}$$

With the form of this generating function in mind, we will proceed by carrying out the squares under the square root terms in Equation (23):

$$\Phi_M(z) = -\frac{I}{2}\left[\frac{z-d}{\sqrt{z^2+(a^2+d^2)-2zd}} + \frac{z+d}{\sqrt{z^2+(a^2+d^2)+2zd}}\right] . \tag{25}$$

Referring back to Figure 7, it is evident that

$$R^2 = a^2 + d^2 \tag{26}$$

and

$$d = R\cos\theta_1 \; . \tag{27}$$

Substituting Equations (26) and (27) into Equation (25), we get:

$$\Phi_M(z) = -\frac{I}{2}\left(\frac{z-R\cos\theta_1}{\sqrt{z^2+R^2-2zR\cos\theta_1}} + \frac{z+R\cos\theta_1}{\sqrt{z^2+R^2+2zR\cos\theta_1}}\right) . \tag{28}$$

Also from Figure 7, we see that:

$$R_- = \sqrt{a^2+(d-z)^2} = \sqrt{z^2+R^2-2zR\cos\theta_1} \tag{29}$$

and

$$R_+ = \sqrt{a^2+(d+z)^2} = \sqrt{z^2+R^2+2zR\cos\theta_1} \; . \tag{30}$$

Therefore, upon making these substitutions, Equation (28) becomes:

$$\Phi_M(z) = -\frac{I}{2}\left(\frac{z-R\cos\theta_1}{R_-} + \frac{z+R\cos\theta_1}{R_+}\right) . \tag{31}$$

If we let $x = \cos\theta_1$ and $h = z/R$, Equation (24) for the generating function gives

$$g(\cos\theta_1, z/R) = \frac{1}{\sqrt{1-2\left(\frac{z}{R}\right)\cos\theta_1+\left(\frac{z}{R}\right)^2}} = \sum_{l=0}^{\infty}\left(\frac{z}{R}\right)^l P_l(\cos\theta_1) \; . \tag{32}$$

Therefore, we can express the reciprocal of Equation (29) as

$$\frac{1}{R_-} = \frac{1}{R\sqrt{1-2\left(\frac{z}{R}\right)\cos\theta_1+\left(\frac{z}{R}\right)^2}} = \sum_{l=0}^{\infty}\frac{z^l}{R^{l+1}}P_l(\cos\theta_1) \ , \tag{33}$$

and similarly, from Equation (30) we have

$$\frac{1}{R_+} = \frac{1}{R\sqrt{1+2\left(\frac{z}{R}\right)\cos\theta_1+\left(\frac{z}{R}\right)^2}} = \sum_{l=0}^{\infty}\frac{z^l}{R^{l+1}}P_l(-\cos\theta_1) \ . \tag{34}$$

Substituting Equations (33) and (34) into Equation (31), the potential is

$$\Phi_M(z) = -\frac{I}{2}\sum_{l=0}^{\infty}\frac{z^l}{R^{l+1}}\left[(z-R\cos\theta_1)P_l(\cos\theta_1)+(z+R\cos\theta_1)P_l(-\cos\theta_1)\right] \ . \tag{35}$$

The Legendre polynomials with odd values of l are odd functions and those with even l are even (see Figure 4), therefore,

$$P_l(-\cos\theta_1) = (-1)^l P_l(\cos\theta_1) \ . \tag{36}$$

Using Equation (36) to separate the $P_l(-\cos\theta_1)$ terms of the summation of Equation (35) into even and odd summations leads to

$$\begin{aligned}\Phi_M(z) = -\frac{I}{2}\Bigg[&\sum_{l=0}^{\infty}\left(\frac{z}{R}\right)^{l+1}P_l(\cos\theta_1) - \sum_{l=0}^{\infty}\left(\frac{z}{R}\right)^{l}\cos\theta_1\,P_l(\cos\theta_1)\\ &-\sum_{l\ odd}^{\infty}\left(\frac{z}{R}\right)^{l+1}P_l(\cos\theta_1) + \sum_{l\ even}^{\infty}\left(\frac{z}{R}\right)^{l+1}P_l(\cos\theta_1)\\ &-\sum_{l\ odd}^{\infty}\left(\frac{z}{R}\right)^{l}\cos\theta_1\,P_l(\cos\theta_1) + \sum_{l\ even}^{\infty}\left(\frac{z}{R}\right)^{l}\cos\theta_1\,P_l(\cos\theta_1)\Bigg] \ ,\end{aligned} \tag{37}$$

which simplifies to

$$\Phi_M(z) = -I\sum_{l\ even}^{\infty}\left(\frac{z}{R}\right)^{l+1}P_l(\cos\theta_1) + I\sum_{l\ odd}^{\infty}\left(\frac{z}{R}\right)^{l}\cos\theta_1\,P_l(\cos\theta_1) \ . \tag{38}$$

The even summation of Equation (38) can be written as a summation over odd l if the index is shifted down by one ($l \to l-1$), so that

$$-I \sum_{l\ even}^{\infty} \left(\frac{z}{R}\right)^{l+1} P_l(\cos\theta_1) = -I \sum_{l\ odd}^{\infty} \left(\frac{z}{R}\right)^{l} P_{l-1}(\cos\theta_1) \quad . \tag{399}$$

Inserting this equation into Equation (38), the expression for the potential becomes

$$\Phi_M(z) = -I \sum_{l\ odd}^{\infty} \left(\frac{z}{R}\right)^{l} \left[P_{l-1}(\cos\theta_1) - \cos\theta_1\, P_l(\cos\theta_1)\right] \tag{40}$$

An established recursive relationship that holds true for the Legendre polynomials is[5]

$$(1-x^2)P_l'(x) = lP_{l-1}(x) - lxP_l(x) \tag{41}$$

where

$$P_l'(x) = \frac{d}{dx}P_l(x) \,. \tag{42}$$

Taking $x = \cos\theta_1$ and rearranging Equation (41), we see that

$$\begin{aligned} P_{l-1}(\cos\theta_1) - \cos\theta_1\, P_l(\cos\theta_1) &= \frac{1}{l}(1-\cos^2\theta_1)P_l'(\cos\theta_1) \\ &= \frac{\sin^2\theta_1}{l} P_l'(\cos\theta_1) \quad . \end{aligned} \tag{43}$$

Substituting Equation (43) into Equation (40) leaves us with

$$\Phi_M(z) = -I\sin^2\theta_1 \sum_{l\ odd}^{\infty} \frac{1}{l}\left(\frac{z}{R}\right)^{l} P_l'(\cos\theta_1) \quad . \tag{44}$$

We can generalize this expression that is restricted to positions along the z-axis between the coils, to include all r and θ in this region, by comparing Equation (44) with the form of the general solution given in terms of spherical coordinates by Equation (13); and so the complete solution valid for all positions between the coils must be

$$\Phi_M(r,\theta) = -I\sin^2\theta_1 \sum_{l\ odd}^{\infty} \frac{1}{l}\left(\frac{r}{R}\right)^{l} P_l(\cos\theta)\, P_l'(\cos\theta_1) \ , \tag{45}$$

where the general coefficient of Equation (13) is given by

$$A_l = -\frac{I\sin^2\theta_1}{lR^l} P_l'(\cos\theta_1) \quad . \tag{46}$$

Rewriting Equation (45) to separate the coil-geometry factors from the general factors, we get

$$\Phi_M(r,\theta) = -I\sum_{l\ odd}^{\infty}\frac{1}{l}r^l P_l(\cos\theta)\left[\frac{\sin^2\theta_1}{R^l}P_l'(\cos\theta_1)\right], \tag{47}$$

where the term in square brackets exclusively contains the information about the relative size and separation of the coils.

From the above derivation, it is clear on how to proceed to compute the total magnetic scalar potential if additional circular coil-pairs are added to the magnetic coil system, assuming the midpoint of each pair is centered at the origin carrying current I, and all pairs share a common axis. The complete potential for such a system consisting of K coil-pairs can be obtained by replacing the bracketed coil-geometry term in Equation (47) by a summation over the coil-geometry terms appropriate to each coil pair. The resulting potential is

$$\Phi_M(r,\theta) = -I\sum_{l\ odd}^{\infty}\frac{1}{l}r^l P_l(\cos\theta)\left[\sum_{k=1}^{K}\frac{\sin^2\theta_k}{R_k^l}P_l'(\cos\theta_k)\right]. \tag{48}$$

where R_k and $\theta_k$are the radial distance and polar angle, respectively, defining the geometry of coil-pair k, generalizing the nomenclature we originally used for a single coil-pair in Figure 7.

As stated earlier, any terms in the magnetic scalar potential of a coil system having $l > 1$ introduce nonuniformity to the corresponding magnetic field. Since the potential of the system given by Equation (48) contains all of the terms containing odd values of l out to $l = \infty$, it is impossible to generate a perfectly uniform magnetic field with a coil system of this type. However, field uniformity of such a coil system can be maximized by adjusting the coil-geometry parameters R_k and θ_k for each coil-pair so that as many of the largest terms having $l \geq 3$ vanish. Each term of Equation (48) in the summation over l gets smaller as l increases because of the $1/l$ and $(r/R_k)^l$ factors. Therefore, making as many of the lowest l-terms vanish as is possible, starting with $l = 3$ and progressing upward in l if the number of adjustable parameters allows, produces the greatest increase in field uniformity.

Using Equation (48) for $\Phi_M(r,\theta)$, we can now calculate coil parameters for systems composed of various numbers of coil-pairs. We will demonstrate Fanselau's method in the computation of the coil geometries for the case of a single coil-pair ($K = 1$), resulting in the familiar Helmholtz coil geometry; and for a system based on two pairs of coils ($K = 2$), the so-called Fanselau coil. Braunbeck's improved solution to the Fanselau coil geometry is then discussed.

Helmholtz Coil

For the case of one pair of coils, $K = 1$, and Equation (48) reduces to the expression for the potential given in Equation (47). The coils each have a radius a, and are separated by a distance of $2d$. A diagram illustrating the coil geometry parameters $R_1$and $\theta_1$is shown in Figure 9. In a practical coil design, in most cases, either the coil radius or separation will be specified based on requirements, and their ratio will be determined by the solution for θ_1. Therefore, this coil configuration has only the single adjustable parameter θ_1, but the coil dimensions can be scaled to arbitrary size based on this angle.

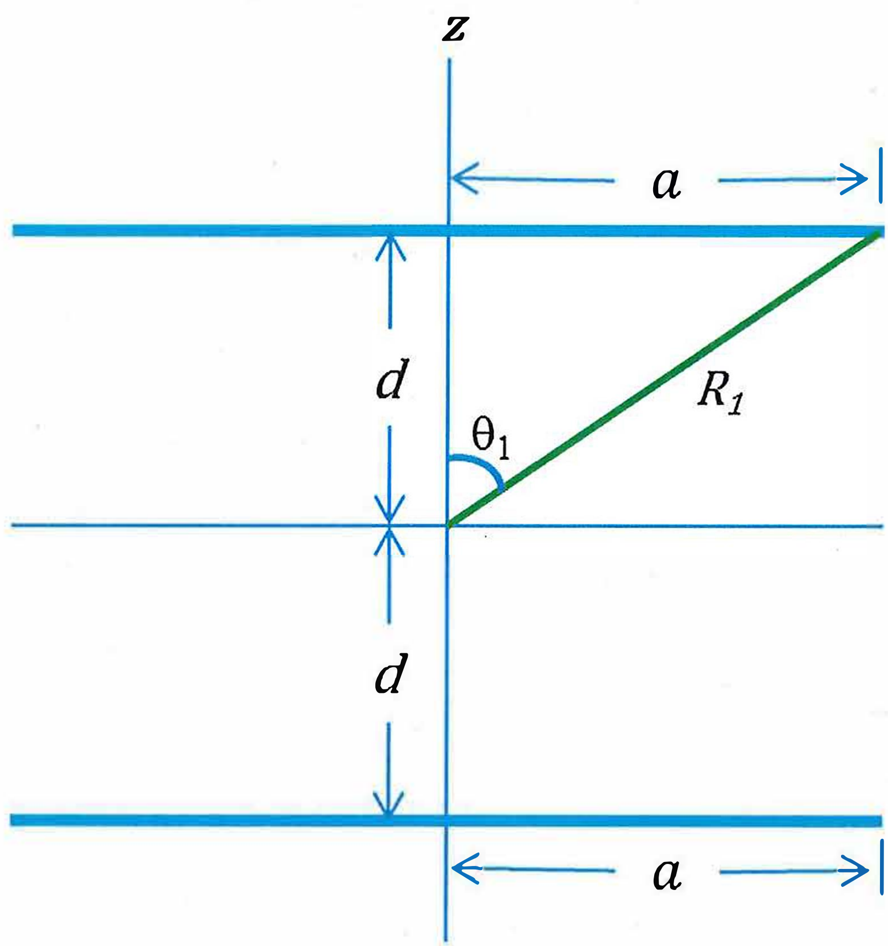

Figure 9. Geometrical parameters for the Helmholtz coil analysis

To maximize the magnetic field uniformity, the bracketed term of Equation (47) must vanish for the $l = 3$ term. Thus, this condition requires

$$\frac{\sin^2\theta_1}{R_1^3} P_3'(\cos\theta_1) = 0 \ , \tag{49}$$

which, after eliminating the nonphysical trivial solutions ($\theta_1 = 0$ or $R_1 = \infty$), becomes

$$P_3'(\cos\theta_1) = 0 \ . \tag{50}$$

From Table 1, the Legendre polynomial of degree 3 is

$$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta) \ , \tag{51}$$

and therefore, its first derivative with respect to $\cos\theta$ is given as

$$P_3'(\cos\theta) = \frac{3}{2}(5\cos^2\theta - 1) \ . \tag{52}$$

Substituting Equation (52) into Equation (50) and solving gives

$$\cos\theta_1 = \pm\frac{1}{\sqrt{5}} \ , \tag{53}$$

where the positive solution corresponds to $\theta_1 = 63.4°$ for the upper coil and the negative solution to $\theta_1 = 116.6°$ for the lower coil. From Figure 9 we see that

$$d = R_1\cos\theta_1 = \frac{R_1}{\sqrt{5}} \ , \tag{54}$$

and so it follows (using the Pythagorean theorem) that the coil radius is

$$a = R_1\sin\theta_1 = \frac{2R_1}{\sqrt{5}} \ . \tag{55}$$

From Equations (54) and (55), we conclude that $a = 2d$, indicating that the coil radius is equal to the coil separation, confirming the well-established design of a Helmholtz coil.

Fanselau Coil

We will now examine a system configuration containing two pairs of coils as illustrated in Figure 10. Fanselau's analysis of this coil system allows the relative relationships between the four parameters (a_1, a_2, d_1, d_2) to be determined.

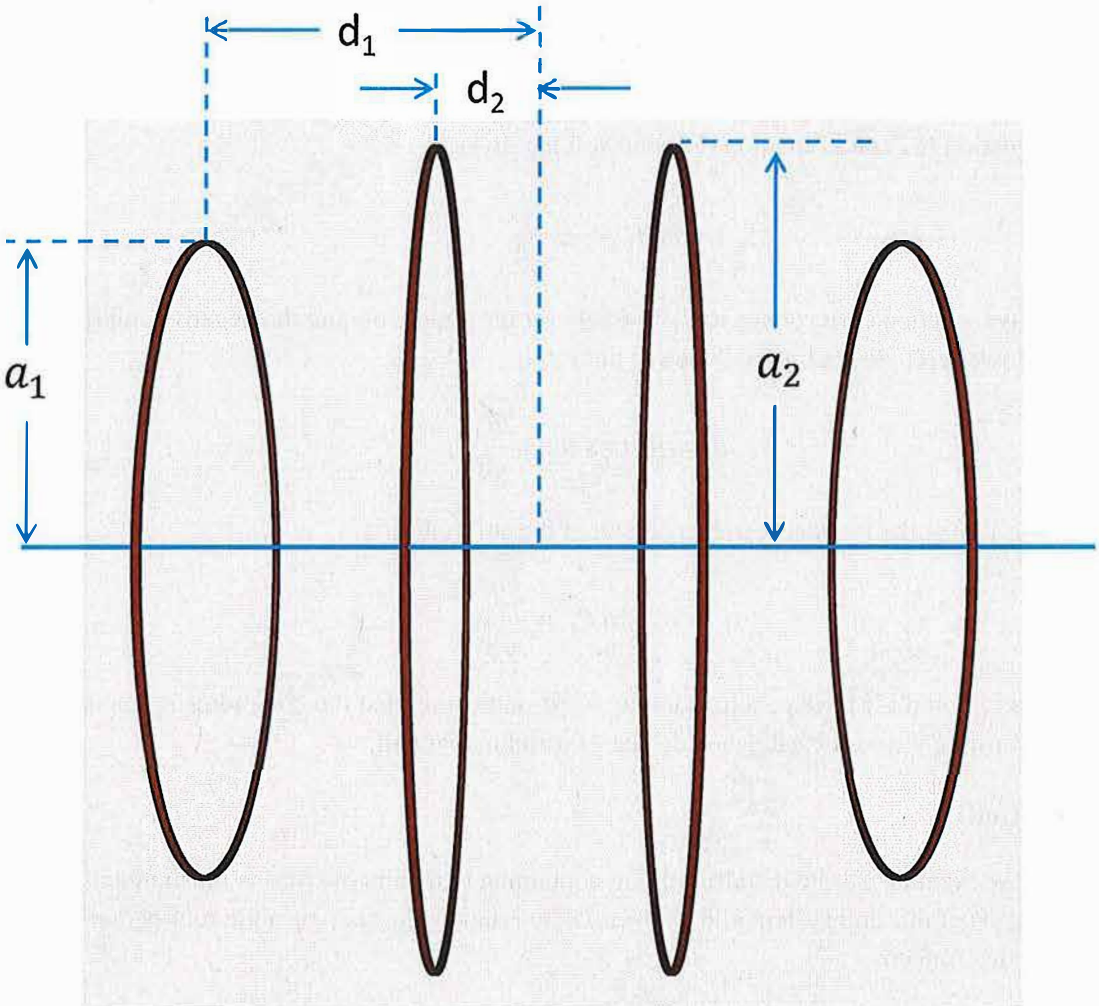

Figure 10. Fanselau coil configuration

Figure 11 defines the geometry of the four-coil system. The outer coils are shown in red and the inner coils in blue. This coil configuration has three adjustable parameters: the angles θ_1 and θ_2, and the ratio R_1/R_2, which relates the dimensions of one coil pair to the other. Fanselau's approach to optimize the magnetic field uniformity of this system is to find parameter values that make the $l = 3$ and $l = 5$ terms of the potential vanish. He begins his solution of the $K = 2$ case by setting the bracketed term of Equation (48) to zero for $l = 5$, giving:

$$\frac{\sin^2 \theta_1 P_5'(\cos \theta_1)}{R_1^5} + \frac{\sin^2 \theta_2 P_5'(\cos \theta_2)}{R_2^5} = 0 \ . \tag{56}$$

Defining the variable $\rho = R_1/R_2$, we can rewrite the above equation as

$$\sin^2 \theta_1 P_5'(\cos \theta_1) + \rho^5 \sin^2 \theta_2 P_5'(\cos \theta_2) = 0 \ . \tag{57}$$

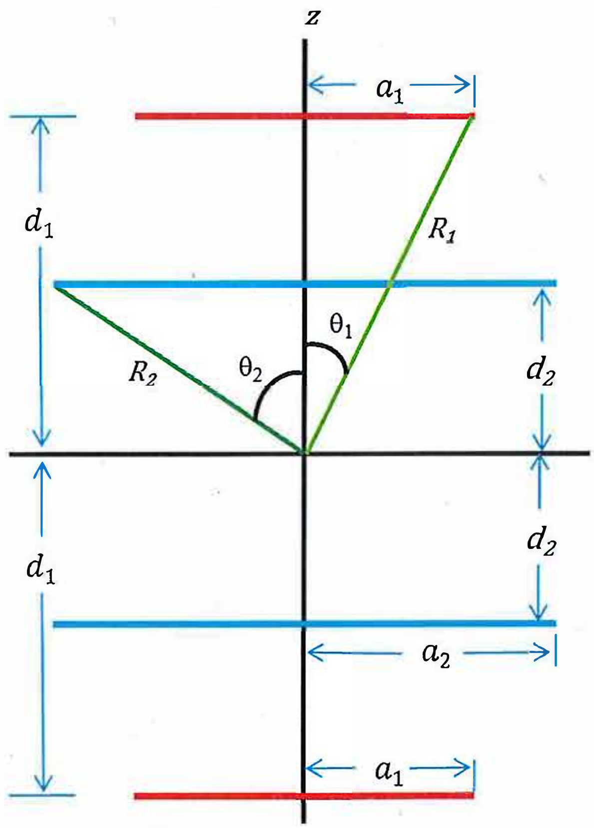

Figure 11. Fanselau coil geometry diagram

Though Equation (57) has three unknowns, Fanselau recognizes he can solve for the two angular parameters by setting the two individual terms of the sum equal to zero. This reduces the problem to independently choosing $\theta_1$and $\theta_2$so that $P_5'(\cos\theta_1)$ and $P_5'(\cos\theta_2)$ each vanish, analogous to the manner in which he solved the Helmholtz coil geometry. This approach allows for an analytical solution. The Legendre polynomial with $l = 5$ (see Table 1) is

$$P_5(\cos\theta) = \frac{1}{8}(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta) \ . \tag{58}$$

Setting its derivative to zero, produces

$$P_5'(\cos\theta) = \frac{1}{8}(315\cos^4\theta - 210\cos^2\theta + 15) = 0 \ . \tag{59}$$

This equation reduces further and is then easily solved for $\cos^2\theta$ by using the quadratic formula. The final solutions are simply the square roots of the $\cos^2\theta$ solutions, which are

$$\cos\theta = \pm\left(\frac{1}{3} \pm \frac{2}{21}\sqrt{7}\right)^{1/2} \ . \tag{60}$$

Based on the angular designations given in Figure 11, the angle for the outer pair of coils corresponds to the cosine solution with the larger magnitude

$$\cos\theta_1 = \pm\left(\frac{1}{3}+\frac{2}{21}\sqrt{7}\right)^{1/2} = \pm 0.765055 \ , \tag{61}$$

and the inner coil solution is therefore equal to

$$\cos\theta_2 = \pm\left(\frac{1}{3}-\frac{2}{21}\sqrt{7}\right)^{1/2} = \pm 0.28523 \ . \tag{62}$$

These results give angular values of $\theta_1 = 40.088°$ and $\theta_2 = 73.427°$, for the outer and inner coils, respectively.

Next, Fanselau solves for ρ by making the $l = 3$ term of the potential vanish. Setting the corresponding square-bracketed term in Equation (48) equal to zero, gives us

$$\frac{\sin^2\theta_1 P_3'(\cos\theta_1)}{R_1^3}+\frac{\sin^2\theta_2 P_3'(\cos\theta_2)}{R_2^3} = 0 \ . \tag{63}$$

Upon rearranging and using the above solutions for the angular parameters $\theta_1$and θ_2, the solution for ρ is

$$\rho = \left[-\frac{\sin^2\theta_1\, P_3'(\cos\theta_1)}{\sin^2\theta_2\, P_3'(\cos\theta_2)}\right]^{1/3} = 1.1359 \ , \tag{64}$$

where Equation (52) was used for the computations of $P_3'(\cos\theta)$.

Since $\sin\theta_i = a_i/R_i$ and $\cos\theta_i = d_i/R_i$ for $i = 1$ or 2 (see Figure 11), the following dimensional ratios result from Fanselau's solution of the four coil problem:

$$\frac{a_2}{a_1} = \frac{\sin\theta_2}{\rho\sin\theta_1} = 1.3102 \tag{65}$$

$$\frac{d_1}{a_1} = \frac{\cos\theta_1}{\sin\theta_1} = 1.1880 \tag{66}$$

$$\frac{d_2}{a_1} = \frac{\cos\theta_2}{\rho\sin\theta_1} = 0.3899 \tag{67}$$

$$\frac{d_2}{a_2} = \frac{\cos\theta_2}{\sin\theta_2} = 0.2976 \ . \tag{68}$$

Braunbeck Coil

Five years after Fanselau's analysis of the two-coil-pair geometry, Braunbeck published a paper with an improved solution of the $K = 2$ case of Equation (48). With two coil pairs, as stated earlier, there are three parameters in the solution: θ_1, θ_2, and ρ. Fanselau was able to solve for these parameters analytically by setting the $l = 3$ and $l = 5$ terms of the magnetic scalar potential to zero. Due to the particular structure of Equation (57), he was able to break it into two independent equations, with each one depending on the angle corresponding to a different coil pair. Fanselau simply solved Equation (59) twice by finding two angles (θ_1 and θ_2) that made $P_5'(\cos\theta)$ vanish. He then used the equation making the $l = 3$ term of the potential vanish to solve for ρ.

Braunbeck realized that he could use the three adjustable parameters to not only zero-out the $l = 3$ and $l = 5$ terms of the potential, but also the $l = 7$ term. This modification results in an increase in field uniformity over the Fanselau solution because an additional nonuniform contribution to the field vanishes. He set the term that is in brackets in Equation (48) to zero for $l = 3$, 5, and 7, to produce the following set of equations:

$$\sin^2\theta_1 P_3'(\cos\theta_1) + \rho^3 \sin^2\theta_2 P_3'(\cos\theta_2) = 0 \tag{69}$$

$$\sin^2\theta_1 P_5'(\cos\theta_1) + \rho^5 \sin^2\theta_2 P_5'(\cos\theta_2) = 0 \tag{70}$$

$$\sin^2\theta_1 P_7'(\cos\theta_1) + \rho^7 \sin^2\theta_2 P_7'(\cos\theta_2) = 0 \ . \tag{71}$$

These nonlinear equations cannot be solved analytically. Braunbeck solved these three equations in three unknowns with a numerical technique, the details of which were left unspecified in his paper. The solutions he obtained for the parameters are:

$$\cos\theta_1 = \pm 0.74213 \tag{72}$$

$$\cos\theta_2 = \pm 0.26789 \tag{73}$$

$$\rho = 1.0980 \ . \tag{74}$$

Braunbeck claims there is some uncertainty in the last digit of his quoted results. To check Braunbeck's results, we solved the set of Equations (69), (70), and (71) using a brute-force approach on a desktop personal computer. The computer program used nested loops to step through values of the parameters $\cos\theta_1$ and $\cos\theta_2$ in very fine increments. Within the inner loop, ρ was computed using Equation (70). This value was used along with the stepped cosine values to compute the left sides of Equations (69) and (71). If the absolute values of both of these computed terms were lower than their previous minimum values, the current values of the parameters were stored as the solution. This process was repeated over the full range of programed cosine values. To increase the precision of the results, after the solution region of parameter space was narrowed down, the cosine step sizes and their range were progressively reduced until the desired number of significant figures was attained. Table 2 summarizes the results obtained for the two-coil-pair configuration by Fanselau, Braunbeck, and our computer solution.

Braunbeck's higher field-uniformity coil-geometry parameters were slightly different from those computed by Fanselau. His θ_1 and θ_2 angles were two degrees and one degree higher, respectively, than Fanselau calculated. His value for ρ was approximately 3.5 percent lower than Fanselau's result. Our computer results (carried out to nine decimal places) confirmed the accuracy of Braunbeck's 1934 numerical approach.

Table 2. Fanselau/Braunbeck Coil Parameters

Parameter	Fanselau	Braunbeck	Computer
$\cos\theta_1$	± 0.7651	± 0.74213	± 0.742070427
$\cos\theta_2$	± 0.2852	± 0.26789	± 0.267867793
ρ	1.1359	1.0980	1.097954859
θ_1	40.09°	42.087°	42.0919°
θ_2	73.43°	74.461°	74.4626°
a_2/a_1	1.310	1.309	1.30907
d_1/a_1	1.188	1.107	1.10704
d_2/a_1	0.3899	0.364	0.36396
d_2/a_2	0.2976	0.278	0.27803

SUMMARY

We have presented the details of the approach used by Fanselau in his 1929 paper to design circular magnet coil systems providing a high degree of field uniformity. He based his method around constructing an infinite series to represent the magnetic scalar potential of a coil system containing a particular number of coaxial coil pairs. He analytically solved for the system's dimensional parameters (i.e., radius of each coil pair and coil spacing) by setting as many of the largest terms of the series contributing to a nonuniform field to zero as the number of dimensional degrees of freedom allowed. We demonstrated Fanselau's analysis for the cases of both a single coil pair and a system of two pairs of coils, the latter of which is known as a "Fanselau Coil." Next, we presented Braunbeck's improved solution to the Fanselau coil geometry. Braunbeck increased field uniformity by removing an additional nonuniform term in the potential by numerically solving the equations for the coil parameters.

REFERENCES

1. Fanselau, G., The Generation of Largely Homogeneous Magnetic Fields by Circular Currents, "*Zeitschrift für Physik*," Vol. 54, Issue 3-4, March 1929, pp. 260-269.
2. Braunbeck, W., The Generation of Largely Homogeneous Magnetic Fields by Circular Currents, "*Zeitschrift für Physik*," Vol. 88, Issue 5-6, May 1934, pp. 399-402.
3. Jackson, J. D., *Classical Electrodynamics*, Second Edition, John Wiley & Sons, Inc., New York, NY, 1975, p. 90.
4. Boas, Mary L., *Mathematical Methods in the Physical Sciences*, John Wiley & Sons, Inc., New York, NY, 1966, p. 544.
5. Abramowitz, M., and Stegun, I., Editors, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Inc., New York, NY, 1965, p. 334.

DISTRIBUTION

	Copies
DEFENSE TECHNICAL INFORMATION CENTER ATTN DTIC-0 8725 JOHN J KINGMAN ROAD FORT BELVOIR VA 22060-6218	1
NAVAL SURFACE WARFARE CENTER, PANAMA CITY DIVISION ATTN MR. JOHN T. BONO (X23) 110 VERNON AVENUE PANAMA CITY, FL 32407-7001	1
NAVAL SURFACE WARFARE CENTER, PANAMA CITY DIVISION ATTN MR. JOHN W. PURPURA (X12) 110 VERNON AVENUE PANAMA CITY, FL 32407-7001	1

NAVSEA
WARFARE CENTERS
PANAMA CITY