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Final report on the NSSEFF grant "Mathematical Modeling in Random media - From homogenization to Stochasticity", PI: Lenya Ryzhik

The overarching goal of the grant was to understand the propagation of uncertainty from the a priori unknown coefficients in partial differential equations to the solutions of such equations. The underlying equations model phenomena as varied as the propagation of waves in heterogeneous media (e.g., radars), to the heat distribution or the electric current in a non-uniform medium. The unifying aspect of these problems is that the physical parameters in the system cannot be perfectly known because of the fluctuations and small-scale heterogeneities. Such phenomena are ubiquitous in nature and the basic problem that is both of the theoretical and practical importance is quantifying the effect of uncertainty at the microscopic scales on the macroscopic phenomena.

The main interest in studying problems of this type is to describe the effect of the small scale fluctuations on the solutions on the macroscopic scales, much longer than the scales of the heterogeneities, on the order of thousands or even tens of thousands of the microscopic lengths. Such problems remain numerically prohibitive even today, especially in three dimensions. Moreover, even if the numerical simulations were available, they would not be sufficient in themselves, precisely because the small fluctuations of the media can not be measured precisely, and thus can not be the basis of the numerical simulations.

Such understanding is, in particular, critical in several types of problems: first, a forward problem involves, generically, a signal transmission through a heterogeneous environment – one is interested in the shape of the transmitted signal, as affected by the medium heterogeneities. Mathematically, this problem corresponds to an initial or boundary value problem for a partial differential equation with random coefficients. Second, the parameter estimation problems – one is interested in recovering the macroscopic properties of a complex medium but not the fine scale oscillations, as well as in the inverse problems where the goal is to find via remote measurements an object hidden in a medium and obscured by the heterogeneities. Mathematically, these problems correspond to the inverse problems for partial differential equations with random coefficients. It is well-known that such inverse problems are extremely ill-posed, so the mathematical challenge is to discover the observables that are affected by the heterogeneities in a non-trivial way but are stable with respect to the small scale fluctuations, in the sense that they depend only on the overall statistics of the fluctuations, and not on a particular realization of the medium. Both the forward and the inverse problems then hinge on two fundamental questions: (i) find measurements that are as immune to effects of noise as possible; and (ii) understand how noise propagates up to the macroscopic scales, and affects the large scale phenomena. The work was done by the group of Guillaume Bal at Columbia University and the group of Lenya Ryzhik at Stanford University.

Stability of measurements and propagation of stochasticity

Wave propagation in heterogeneous media is one example of a problem modeled by a partial differential equation involving random and highly oscillatory coefficients. One central component of the grant was to understand and whenever possible rigorously justify how stochasticity propagates from the coefficients in the PDE to its solution. As an example of a problem of interest, in the setting of wave propagation in heterogeneous media, consider the reconstruction of an inclusion buried behind a blocker and probed by a radiation source. The medium between the source and the inclusion may be highly turbid and its precise reconstruction is both practically infeasible, and useless the fluctuations do not carry useful information, we are only interested in the object hidden behind the blocker. Such turbid media must then be modeled as random. The goal is to understand the effects of randomness on measurements on the detector array and recover the inclusion from the noisy measurements. As no direct signal comes from the inclusion to the array due to the blocker, it is only because of the heterogeneities that we can hope to recover the inclusion at all but for the same reason the recorded signal at the array is inherently noisy, making the problem highly challenging. The goal in this setting is to extract some information about the object behind the blocker from the noisy measurements.

One of the accomplishments of the grant was the analysis of the functionals of the measurements on the array of detectors that would be (i) sensitive to the presence of the inclusion, and (ii) as immune to the details of the unknown randomness as possible. Examples of such functionals are the appropriately defined wave energy densities, as well as the field-to-field correlations. These quantities are homogenized: they are affected by the randomness but the overall effect of the noise is averaging to a non-trivial but deterministic effect. However, when the noise becomes too strong, or the propagation distances are too long, the effect of the randomness even on these functionals is no longer deterministic, and they experience fluctuations of order one. This is what is mathematically referred to as the transition from homogenization to stochasticity. As we have mentioned, mathematically, measurements are functionals of the solutions to the PDE and the errors committed in the reconstructions of the parameters of interest are directly related to errors, or fluctuations, in the measurements. The limitations in the forward and inverse problems, or in the parameter estimations, are, therefore, to a large extent driven by how stochasticity propagates from noise to measurements.

The analysis of such models has a long history, both in the mathematical and engineering communities. However, most work is done in the so-called homogenization regime, where randomness in the coefficients affects the solutions of PDEs in a highly non-trivial way but nevertheless the solutions are nearly deterministic. This regime is valid when the random fluctuations oscillate on a much smaller scale than both the macroscopic size of the domain of interest, and the scale of the initial signal, or the boundary data. This regime is crucial as it provides us with a model on how measurements depend on the parameters of interest, such as the buried inclusion in the preceding example. However, it gives us no information about the size of the inevitable random fluctuations in our measurements that arise as soon as we are slightly away from the homogenization regime, as always happens in practice.

The core objective of the grant was to understand the fluctuations in the available measurements beyond the homogenization limit and the dependence of such fluctuations on the randomness in the coefficients of the PDE. Such fundamental questions turn out to be particularly challenging. Only in some PDE models do we have a complete understanding of this problem with major contributions from the work supported by the grant.

Random fluctuations with slowly decaying correlations

Homogenization regime arises when the medium properties have short range correlations, so that the random coefficients rapidly forget values at nearby points. In contrast, media with slowly decaying correlations are fractal and have structure on all scales. A major accomplishment of the grant is the discovery of the dichotomy between these regimes – solutions of partial differential equations exhibit drastically different behavior in these two types of media. One important class of partial differential equations for which this analysis has been carried out is the parabolic equations with random potentials. This class includes, for instance, the parabolic models of radars in turbid environments.

For such models, both when the random fluctuations are strong and weak, we have discovered that the propagation of stochasticity strongly depends on the decorrelation properties of the random potential. For partial dofferential equations with random coefficients that have short-range correlations, we have proved that the solution of the corresponding partial differential equation is well approximated by a solution of the deterministic homogenized problem, in accordance with the general philosophy of the homogenization theory. Moreover, the random fluctuations in the solutions can be asymptotically described by a stochastic partial differential equation with additive or multiplicative noise, depending on the precise problem, or with random initial data. This completely characterizes the propagation of stochasticity in this case.

In contrast, for the partial differential equations with random coefficients that have long-range correlations, we have shown that the solutions remained stochastic at the macroscopic level. Moreover, it was shown, in a range of problems, that even the weakly random fluctuations in the coefficients affect the solution in a non-trivial way on time scales much shorter than that for rapidly decorrelating random coefficients. In particular, inverse problems in random media with long range correlations can not be addressed in the same way as in random media with short range correlations. The multi-scale nature of the small scale fluctuations affects measurements in a quantifiable but much stronger way.

We have also discovered a completely new phenomenon that long range correlations introduce a continuum of temporal scales on which various observables become non-trivially affected: for instance, in the wave problems, the wave amplitude becomes randomized on a time scale much shorter than the phase. In particle dispersion problems, the positions of individual tracers become randomized much sooner than the multi-particle correlations: there exists a long intermediate time scale, on which a collection of tracers behaves as rigid body with a center of mass performing a stochastic motion. This phenomenon, in particular, dramatically affects our understanding of which wave functionals should be measured in turbid media.

Such results were obtained for other PDE models including those used in wave propagation waves in complex media. The salient feature of such results is a characterization of the effect of the correlation properties of a random medium on the stochasticity of the PDE solutions.

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