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ON COMBINATORIAL METHODS IN THE THEORY OF STOCHASTIC PROCESSES

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1. Introduction

The main object of this paper is to prove a simple theorem of combinatorial nature and to show its usefulness in the theory of stochastic processes. The theorem mentioned is as follows.

THEOREM 1. *Let $\varphi(u)$, $0 \leq u < \infty$, be a nondecreasing step function satisfying the conditions $\varphi(0) = 0$ and $\varphi(t + u) = \varphi(t) + \varphi(u)$ for $u \geq 0$ where t is a finite positive number. Define*

$$\delta(u) = \begin{cases} 1 & \text{if} \quad v - \varphi(v) \geq u - \varphi(u) \quad \text{for} \quad v \geq u, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then

$$\int_0^t \delta(u)\, du = \begin{cases} t - \varphi(t) & \text{if} \quad 0 \leq \varphi(t) \leq t, \\ 0 & \text{if} \quad \varphi(t) \geq t. \end{cases} \tag{2}$$

PROOF. If $\varphi(t) > t$, then $\delta(u) = 0$ for every u, and thus the theorem is obviously true.

Now consider the case $0 \leq \varphi(t) \leq t$. For $u \geq 0$ define $\psi(u) = \inf \{v - \varphi(v)$ for $v \geq u\}$. We have $\psi(u) \leq u - \varphi(u)$, and $\psi(u) = u - \varphi(u)$ if and only if $\delta(u) = 1$ (compare figures 1, 2, 3).

It is clear that $\psi(u + t) = \psi(u) + t - \varphi(t)$ for $u \geq 0$ and that $0 \leq \psi(v) - \psi(u) \leq v - u$ for $0 \leq u \leq v$. Thus $\psi'(u)$ exists for almost all u, $0 \leq \psi'(u) \leq 1$, and

$$\int_0^t \psi'(u)\, du = \psi(t) - \psi(0) = t - \varphi(t) \tag{3}$$

because $\psi(u)$ is a monotone and absolutely continuous function of u. We also note that $\varphi(u + 0) = \varphi(u)$ and $\varphi'(u) = 0$ for almost all u.

First we prove that

$$\psi'(u) \leq \delta(u) \qquad \text{for almost all } u. \tag{4}$$

If $\psi'(u)$ exists and if $\psi'(u) = 0$, then (4) evidently holds. Now we shall prove

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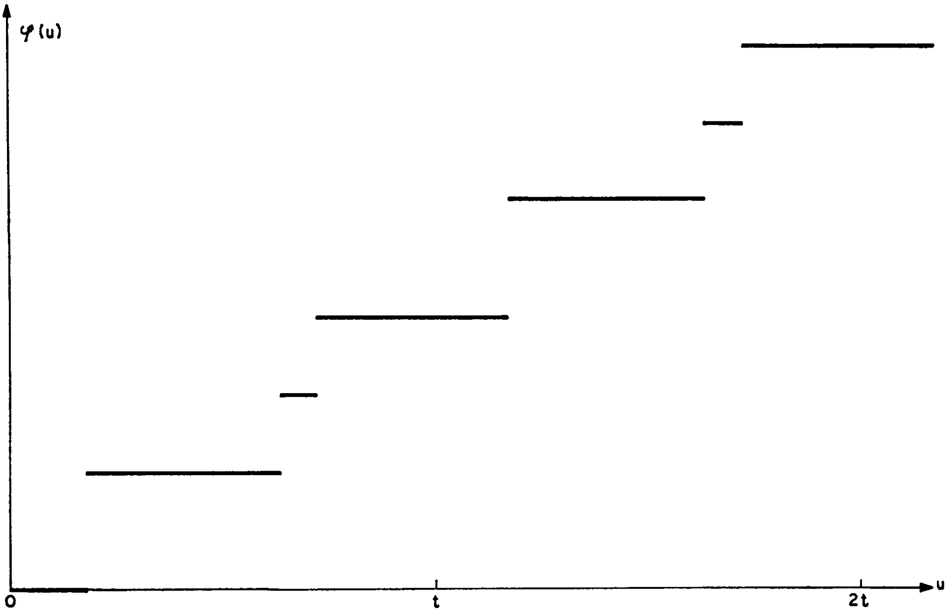

FIGURE 1

that if $\psi'(u)$ exists, if $\psi'(u) > 0$ and if $\varphi(u + 0) = \varphi(u)$, then $\delta(u) = 1$. If $\psi'(u) > 0$, then $\psi(v) > \psi(u)$ for $v > u$, and therefore $\psi(u) = \inf_{u \leq s < v} [s - \varphi(s)]$ for $v > u$. Thus $u - \varphi(v) \leq \psi(u) \leq u - \varphi(u)$ for $v > u$, and consequently, $u - \varphi(u + 0) \leq \psi(u) \leq u - \varphi(u)$. If $\varphi(u + 0) = \varphi(u)$, then $\psi(u) = u - \varphi(u)$ which implies that $\delta(u) = 1$. Since $\psi'(u) \leq 1$ always holds, (4) follows.

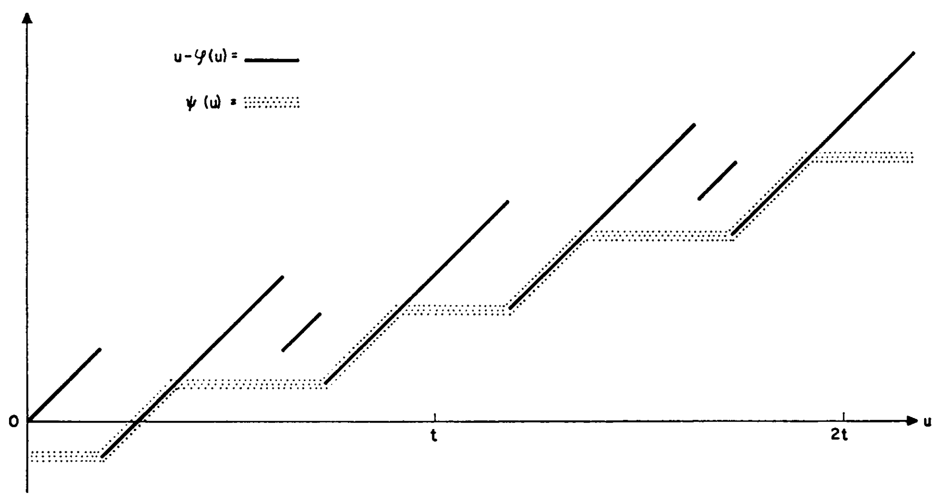

FIGURE 2

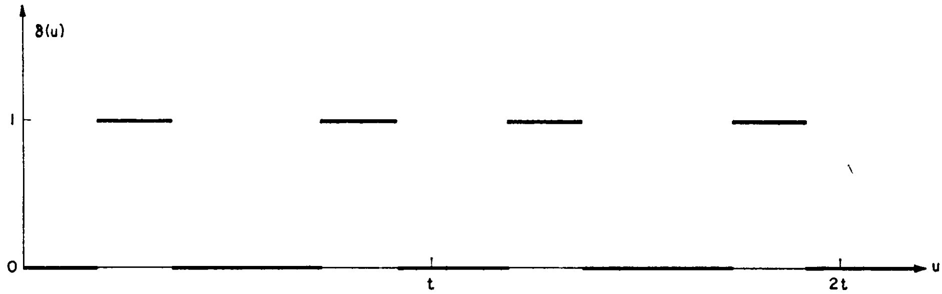

FIGURE 3

Second, we prove that

$$\delta(u) \leq \psi'(u) \qquad \text{for almost all } u. \tag{5}$$

If $\delta(u) = 0$ and $\psi'(u)$ exists, then (5) evidently holds. Now we shall prove that if $\delta(u) = 1$, if $\psi'(u)$ exists, if $\varphi'(u) = 0$, and if u is an accumulation point of the set $D = \{u: \delta(u) = 1, 0 \leq u < \infty\}$, then $\psi'(u) = 1$. Suppose that $u \in D$ and $u = \lim_{n\to\infty} u_n$ where $u_n \in D$ and $u_n \neq u$. Then $\psi(u) = u - \varphi(u)$ and $\psi(u_n) = u_n - \varphi(u_n)$. Accordingly, if $\psi'(u)$ exists and if $\varphi'(u) = 0$, we have

$$\psi'(u) = \lim_{n\to\infty} \frac{\psi(u) - \psi(u_n)}{u - u_n} = 1 - \lim_{n\to\infty} \frac{\varphi(u) - \varphi(u_n)}{u - u_n} = 1 - \varphi'(u) = 1. \tag{6}$$

Since the isolated points of the set D form a countable set (possibly empty), (5) follows.

If we compare (4) and (5), then we obtain that $\psi'(u) = \delta(u)$ for almost all u. Hence, by (3) we get (2) for $\varphi(t) \leq t$. This completes the proof of the theorem.

We note that if we alter the definition of $\delta(u)$ such that $\delta(u) = 1$ when $v - \varphi(v) > u - \varphi(u)$ for all $v > u$, and $\delta(u) = 0$ otherwise, then (2) remains unchanged.

Furthermore, if u is a discontinuity point of $\varphi(u)$, then $\varphi(u)$ may take any value in the interval $[\varphi(u - 0), \varphi(u + 0)]$.

By using theorem 1, we shall formulate a theorem for stochastic processes which will play a fundamental role in our considerations. By this theorem we shall find the distribution of the supremum for certain types of stochastic processes. The results obtained will be applied in the theory of order statistics, in the theory of queues, in the theory of dams, and in the theory of mathematical risk. We shall also prove some results for a random walk process.

REMARK 1. In a similar way we can prove the following discrete version of theorem 1.

THEOREM 2. *Let $\varphi(u)$, $u = 0, 1, 2, \cdots$, be a nondecreasing function of u satisfying the conditions $\varphi(0) = 0$ and $\varphi(t + u) = \varphi(t) + \varphi(u)$ for $u = 0, 1, 2, \cdots$ where t is a positive integer. Define*

$$\delta(u) = \begin{cases} 1 & \text{if} \quad v - \varphi(v) > u - \varphi(u) \quad \text{for} \quad v > u, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Then

$$\sum_{u=1}^{t} \delta(u) = \begin{cases} t - \varphi(t) & \text{if} \quad 0 \le \varphi(t) \le t, \\ 0 \quad \text{if} & \varphi(t) \ge t. \end{cases} \tag{8}$$

PROOF. The case of $\varphi(t) > t$ is trivial. Suppose that $0 \le \varphi(t) \le t$. Let $\psi(u) = \inf \{v - \varphi(v) \text{ for } v \ge u\}$ for $u = 0, 1, 2, \cdots$. Evidently $\psi(u + t) = \psi(u) + t - \varphi(t)$ and $\delta(u) = \psi(u + 1) - \psi(u)$ for $u \ge 0$. Therefore

$$\sum_{u=1}^{t} \delta(u) = \psi(t + 1) - \psi(1) = t - \varphi(t), \tag{9}$$

which proves the statement.

Among others, (8) yields an immediate proof for a generalization of the classical ballot theorem. (Cf. Takács [12].)

If we would deal with stochastic sequences instead of stochastic processes, then by using theorem 2 instead of theorem 1 we could replace each theorem proved for stochastic processes by an analogous theorem for stochastic sequences.

2. Stochastic processes with cyclically interchangeable increments

Let $\{\chi(u), 0 \le u \le t\}$ be a stochastic process where t is a finite positive number. We associate a stochastic process $\{\chi^*(u), 0 \le u \le \infty\}$ with $\{\chi(u), 0 \le u \le t\}$ such that $\chi^*(u) = \chi(u)$ for $0 \le u \le t$ and $\chi^*(t + u) = \chi^*(t) + \chi^*(u)$ for $u > 0$. If the finite dimensional distributions of $\{\chi^*(v + u) - \chi^*(v), 0 \le u \le t\}$ are independent of v for $v \ge 0$, then the process $\{\chi(u), 0 \le u \le t\}$ is said to have *cyclically interchangeable increments.*

First we shall give a simple example for such a process. For $0 \le u \le t$, define

$$\chi(u) = \sum_{0 \le \tau_r \le u} \chi_r \tag{10}$$

where $\tau_1, \tau_2, \cdots, \tau_n$ are mutually independent random variables having a uniform distribution over the interval $(0, t)$ and $\chi_1, \chi_2, \cdots, \chi_n$ are cyclically interchangeable random variables; that is, all the n cyclic permutations of $\chi_1, \chi_2, \cdots, \chi_n$ have a common joint distribution. If $\{\tau_r\}$ and $\{\chi_r\}$ are independent sequences, then $\{\chi(u), 0 \le u \le t\}$ is a stochastic process with cyclically interchangeable increments.

Now we shall prove our fundamental theorem.

THEOREM 3. *If $\{\chi(u), 0 \le u \le t\}$ is a separable stochastic process with cyclically interchangeable increments and if almost all sample functions are non-decreasing step functions which vanish at $u = 0$, then*

$$P\{\chi(u) \le u \quad \text{for} \quad 0 \le u \le t | \chi(t)\} = \begin{cases} \left(1 - \dfrac{\chi(t)}{t}\right) & \text{if} \quad 0 \le \chi(t) \le t, \\ 0 \quad \text{if} \quad \chi(t) \ge t, \end{cases} \tag{11}$$

with probability 1.

PROOF. Let $\chi^*(u) = \chi(u)$ for $0 \le u \le t$ and $\chi^*(t + u) = \chi^*(t) + \chi^*(u)$ for $u \ge 0$. Define

$$(12)\qquad \delta^*(u) = \begin{cases} 1 & \text{if}\quad \chi^*(v) - \chi^*(u) \le v - u \quad \text{for}\quad v \ge u, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\delta^*(u)$ is a random variable which has the same distribution for all $u \ge 0$. Now we have

$$(13)\qquad \begin{aligned} P\{\chi(u) \le u \text{ for } 0 \le u \le t|\chi(t)\} &= E\{\delta^*(0)|\chi(t)\} \\ &= \frac{1}{t}\int_0^t E\{\delta^*(u)|\chi(t)\}\,du = E\left\{\frac{1}{t}\int_0^t \delta^*(u)\,du|\chi(t)\right\} \\ &= \begin{cases} \left(1 - \dfrac{\chi(t)}{t}\right) & \text{if}\quad 0 \le \chi(t) \le t, \\ 0 & \text{if}\quad \chi(t) \ge t, \end{cases} \end{aligned}$$

with probability 1, because by (2),

$$(14)\qquad \int_0^t \delta^*(u)\,du = \begin{cases} t - \chi(t) & \text{if}\quad 0 \le \chi(t) \le t, \\ 0 & \text{if}\quad \chi(t) \ge t \end{cases}$$

holds for almost all sample functions. This completes the proof of the theorem.

Finally, we note that from (11) it follows that

$$(15)\qquad P\{\chi(u) \le u \text{ for } 0 \le u \le t\} = E\left\{\left[1 - \frac{\chi(t)}{t}\right]^+\right\}$$

where $[x]^+$ denotes the positive part of x.

3. Stochastic processes with interchangeable increments and stochastic processes with stationary independent increments

A stochastic process $\{\chi(u), 0 \le u \le T\}$ is said to have *interchangeable increments* if for all $n = 2, 3, \cdots$ and for all $t \in (0, T]$

$$(16)\qquad \chi\left(\frac{rt}{n}\right) - \chi\left(\frac{rt - t}{n}\right), \qquad (r = 1, 2, \cdots, n),$$

are interchangeable random variables; that is, all the $n!$ permutations of the random variables (16) have a common joint distribution.

If, in particular, for all $n = 2, 3, \cdots$ and for all $t \in (0, T]$ the random variables (16) are mutually independent, identically distributed random variables, then the stochastic process $\{\chi(u), 0 \le u \le T\}$ is said to have *stationary, independent increments.*

If $P\{\chi(0) = 0\} = 1$, then in both cases the stochastic process $\{\chi(u), 0 \le u \le t\}$ has cyclically interchangeable increments for all finite $t \in (0, T]$.

In all subsequent considerations we are concerned with stochastic processes $\{\chi(u), 0 \le u \le T\}$ having either interchangeable increments or stationary, independent increments and for which almost all sample functions are nondecreasing step functions vanishing at $u = 0$. The parameter range $[0, T]$ may be either finite or infinite.

First of all, I should like to mention a few basic properties of the processes $\{\chi(u), 0 \leq u \leq T\}$. Many theorems valid for stochastic processes with stationary, independent increments can be carried over to stochastic processes with interchangeable increments, because interchangeability is equivalent to conditional independence with common distribution (Cf. M. Loève [6], p. 365 and H. Bühlmann [2].)

For both types of processes,

$$E\{\chi(t)\} = \rho t \tag{17}$$

if $0 \leq t \leq T$ where ρ is a nonnegative number (possibly $\rho = \infty$). If $\rho = 0$, then $P\{\chi(t) = 0\} = 1$ for all $t \in [0, T]$.

If $\{\chi(u), 0 \leq u < \infty\}$ has stationary independent increments and $\rho < \infty$, then for $\{\chi(u), 0 \leq u < \infty\}$ both the weak law and strong law of large numbers hold. Namely for any $\epsilon > 0$,

$$\lim_{t\to\infty} P\left\{\left|\frac{\chi(t)}{t} - \rho\right| < \epsilon\right\} = 1 \tag{18}$$

and

$$P\left\{\lim_{t\to\infty} \frac{\chi(t)}{t} = \rho\right\} = 1. \tag{19}$$

(Cf. J. L. Doob [4], p. 364.)

If $\{\chi(u), 0 \leq u < \infty\}$ is a separable stochastic process, then (15) holds for all $t > 0$. If $t \to \infty$ in (15), then by using (18) we obtain

$$P\{\chi(u) \leq u \text{ for } 0 \leq u < \infty\} = \begin{cases} 1 - \rho & \text{if } \rho < 1, \\ 0 & \text{if } \rho \geq 1. \end{cases} \tag{20}$$

The left-hand side follows from the continuity theorem for probabilities, and the right-hand side from the fact that $[1 - (\chi(t)/t)]^+$ is bounded and converges in probability to $[1 - \rho]^+$.

We also mention that

$$P\{\sup_{0 \leq u < \infty} [\chi(u) - u] < \infty\} = \begin{cases} 1 & \text{if } \rho < 1, \\ 0 & \text{if } \rho \geq 1. \end{cases} \tag{21}$$

For $\rho \neq 1$ this follows from (19), and for $\rho = 1$ by a theorem of K. L. Chung and W. H. J. Fuchs [3].

If $\{\chi(u), 0 \leq u < \infty\}$ has nonnegative, stationary, independent increments, then for $\Re(s) \geq 0$,

$$E\{e^{-s\chi(u)}\} = e^{-u\Phi(s)} \tag{22}$$

with an appropriate $\Phi(s)$ and $\rho = \lim_{s\to 0} \Phi(s)/s$.

4. The distribution of the supremum for stochastic processes with interchangeable increments

In the theory and applications of stochastic processes there frequently arises the problem of finding the distribution of $\sup_{0 \leq u \leq t} \xi(u)$ where $\{\xi(u), 0 \leq u \leq T\}$

is a separable stochastic process. For stochastic processes with stationary independent increments G. Baxter and M. D. Donsker [1] solved this problem in principle. They determined the double Laplace-Stieltjes transform of $P\{\sup_{0\le u\le t}\xi(u)\le x\}$ for such processes. However, even in simple cases, it seems too complicated to invert the transforms.

In this section we shall show that for a wide class of stochastic processes the distribution of $\sup_{0\le u\le t}\xi(u)$ can be found in a simple way by making use of theorem 3.

In this section we suppose that $\{\chi(u), 0\le u\le T\}$ is a separable stochastic process with interchangeable increments and that almost all sample functions are nondecreasing step functions which vanish at $u = 0$.

We shall consider the following two processes with interchangeable increments: $\xi_1(u) = \chi(u) - u$ and $\xi_2(u) = u - \chi(u)$ for $0\le u\le T$, and we shall find the distribution of $\sup_{0\le u\le t}\xi_1(u)$ and that of $\sup_{0\le u\le t}\xi_2(u)$ for $0\le t\le T$.

In what follows we shall use the notation $d_xP\{\chi(u)\le x\} = P\{x\le\chi(u)\le x + dx\}$ regardless of whether u depends on x or not.

THEOREM 4. *If $\{\chi(u), 0\le u\le T\}$ is a separable stochastic process with interchangeable increments and if almost all sample functions are nondecreasing step functions which vanish at $u = 0$, then*

$$(23)\qquad P\{\sup_{0\le u\le t}[\chi(u) - u]\le x\} = P\{\chi(t)\le t + x\} - \iint_{0\le y\le z\le t}\left(\frac{t-z}{t-y}\right)d_yd_zP\{\chi(y)\le y + x,\ \chi(t)\le z + x\}$$

for all x and for all finite $t\in(0, T]$.

PROOF. We shall prove that the subtrahend on the right-hand side of (23) is the probability that $\chi(t)\le t + x$ and $\chi(u) > u + x$ for some $u\in[0, t]$ (compare figure 4). Suppose that $\chi(t) = z + x$ where $0\le z\le t$ and that the last passage of $\chi(u)$ through $u + x$ occurs at $u = y$. Then $\chi(y) = y + x$ and $\chi(u) - \chi(y)\le u - y$ for $y\le u\le t$. Given that $\chi(y) = y + x$ and $\chi(t) = z + x$, by theorem 3, the event $\{\chi(u) - \chi(y)\le u - y$ for $y\le u\le t\}$ has probability $(t - z)/(t - y)$ if $0\le y\le z\le t$. If we integrate $(t - z)/(t - y)$ with respect to $P\{y + x\le\chi(y)\le y + x + dy, z + x\le\chi(t)\le z + x + dz\}$ over the domain $0\le y\le z\le t$, then we get the subtrahend on the right-hand side of (23).

If, in particular, $x = 0$ in (23), then by (15),

$$(24)\qquad P\{\sup_{0\le u\le t}[\chi(u) - u]\le 0\} = \int_0^t\left(1 - \frac{y}{t}\right)d_yP\{\chi(t)\le y\}.$$

THEOREM 5. *If $\{\chi(u), 0\le u\le T\}$ is a separable stochastic process with interchangeable increments and if almost all sample functions are nondecreasing step functions which vanish at $u = 0$, then*

$$(25)\qquad P\{\sup_{0\le u\le t}[u - \chi(u)]\le x\} = 1 - \int_x^t\frac{x}{y}\,d_yP\{\chi(y)\le y - x\}$$

for $0 < x\le t\le T$.

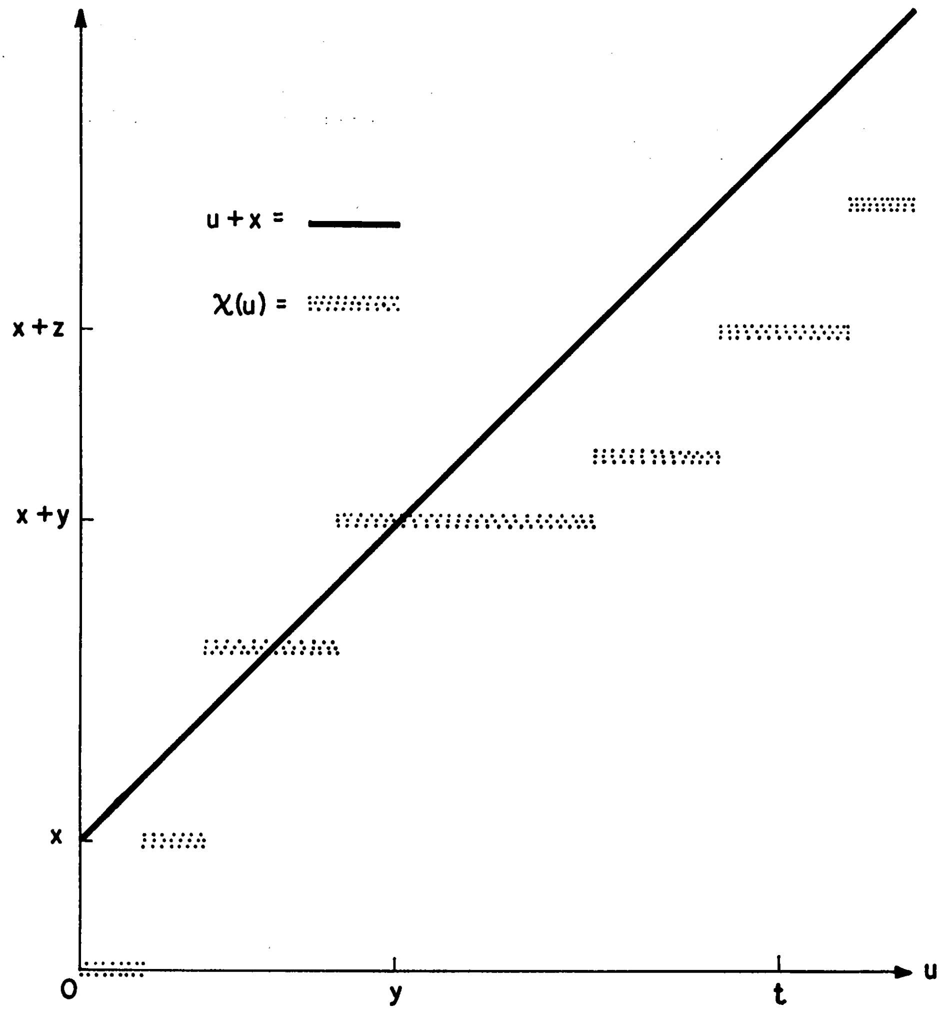

FIGURE 4

PROOF. We shall prove that the subtrahend on the right-hand side of (25) is the probability that $\chi(u) < u - x$ for some $u \in (0, t]$ (compare figure 5). Suppose that the first passage of $\chi(u)$ through $u - x$ occurs at $u = y$ where $x \le y \le t$. Then $\chi(y) = y - x$ and $\chi(u) \ge u - x$ for $0 \le u \le y$, or equivalently, $\chi(y) - \chi(u) \le y - u$ for $0 \le u \le y$. Given that $\chi(y) = y - x$, by theorem 3, the event $\{\chi(y) - \chi(u) \le y - u$ for $0 \le u \le y\}$ has probability x/y for $0 < x \le y$. If we integrate x/y with respect to $P\{y - x \le \chi(y) \le y - x + dy\}$ from x to t, then we get the subtrahend on the right-hand side of (25).

EXAMPLE 1. *Theory of order statistics* (cf. L. Takács [15] and [16]). Let $\xi_1, \xi_2, \cdots, \xi_n$ be mutually independent random variables having a common

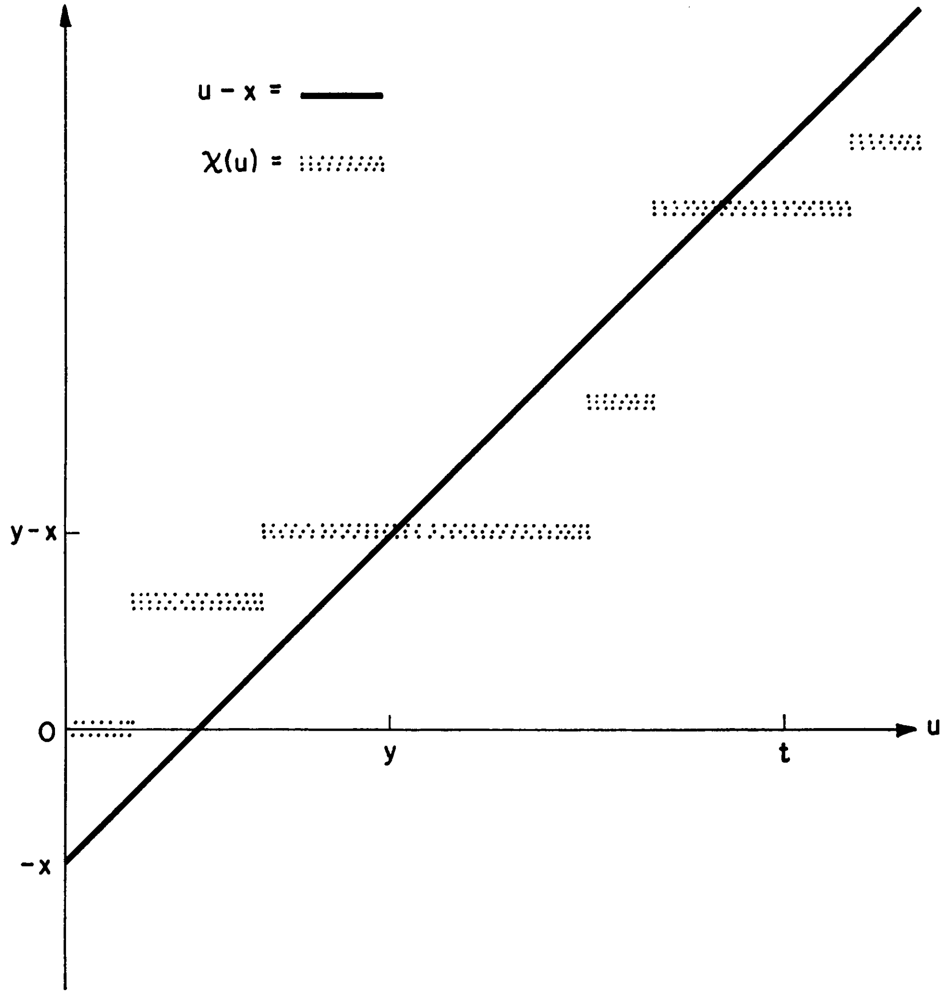

FIGURE 5

continuous distribution function $F(u)$. Denote by $F_n(u)$ the empirical distribution function of the sample $(\xi_1, \xi_2, \cdots, \xi_n)$. For $0 \leq \alpha < \beta \leq 1$, define

$$\delta_n^+(\alpha, \beta) = \sup_{\alpha \leq F(u) \leq \beta} [F_n(u) - F(u)]. \tag{26}$$

It can easily be seen that $\delta_n^+(\alpha, \beta)$ is a distribution-free statistic. To find the distribution of $\delta_n^+(\alpha, \beta)$ we may suppose that $F(u) = u$ for $0 \leq u \leq 1$. Then $F_n(u) = \chi(u)$ for $0 \leq u \leq 1$ where the process $\{\chi(u), 0 \leq u \leq 1\}$ is defined as follows. We choose n points independently in the interval $(0, 1)$ such that each point has a uniform distribution over $(0, 1)$. Denote by $\chi(u)$ the ratio of the number of points in the interval $(0, u]$ to n. Then the process $\{\chi(u), 0 \leq u \leq 1\}$ has interchangeable increments and satisfies the assumptions of theorem 4. Now

$$P\{\delta_n^+(\alpha, \beta) \le x\} = P\{\sup_{\alpha \le u \le \beta} [\chi(u) - u] \le x\} \tag{27}$$

and by a slight modification of (23) we get

$$P\{\sup_{\alpha \le u \le \beta} [\chi(u) - u] \le x\} = P\{\chi(\beta) \le \beta + x\} - \sum_{\alpha \le y \le z \le \beta}\sum \left(\frac{\beta - z}{\beta - y}\right) P\{\chi(y) = y + x, \chi(\beta) = z + x\} \tag{28}$$

for all x. In (28) $P\{\chi(y) = y + x, \chi(\beta) = z + x\} = 0$, except if $y = (j - nx)/n$ and $z = (k - nx)/n$ where $0 \le j \le k \le n$. Thus for $x \ge 0$,

$$P\{\delta_n^+(\alpha, \beta) \le x\} = \sum_{k \le n(\beta + x)} P\left\{\chi(\beta) = \frac{k}{n}\right\} - \sum_{n(x+\alpha) \le j \le k \le n(x+\beta)}\sum \left[\frac{n(x + \beta) - k}{n(x + \alpha) - j}\right] P\left\{\chi\left(\frac{j - nx}{n}\right) = \frac{j}{n}, \chi(\beta) = \frac{k}{n}\right\}, \tag{29}$$

and here

$$P\left\{\chi(u) = \frac{j}{n}\right\} = \binom{n}{j} u^j (1 - u)^{n-j} \tag{30}$$

for $0 \le j \le n$ and $0 \le u \le 1$, and

$$P\left\{\chi(u) = \frac{j}{n}, \chi(v) = \frac{k}{n}\right\} = \frac{n!}{j!(k - j)!(n - k)!} u^j (v - u)^{k-j} (1 - v)^{n-k} \tag{31}$$

for $0 \le j \le k \le n$ and $0 \le u \le v \le 1$.

5. The distribution of the supremum for stochastic processes with stationary independent increments

If $\{\chi(u), 0 \le u \le T\}$ has stationary independent increments, then (23) becomes

$$P\{\sup_{0 \le u \le t} [\chi(u) - u] \le x\} = P\{\chi(t) \le t + x\} - \iint\limits_{0 \le y \le z \le t} \left(\frac{t - z}{t - y}\right) d_y P\{\chi(y) \le y + x\} d_z P\{\chi(t - y) \le z - y\}, \tag{32}$$

which is valid for all x and for all finite $t \in (0, T]$. For, in this case, $\chi(y)$ and $\chi(t) - \chi(y)$ are independent variables and $\chi(t) - \chi(y)$ has the same distribution as $\chi(t - y)$. If we introduce the notation

$$W(t, x) = P\{\sup_{0 \le u \le t} [\chi(u) - u] \le x\} \tag{33}$$

and $W(x) = W(\infty, x)$, then by (24) we can write down (32) in the following form:

$$W(t, x) = P\{\chi(t) \le t + x\} - \int_0^t W(t - y, 0) d_y P\{\chi(y) \le y + x\}. \tag{34}$$

If $T = \infty$ and $t = \infty$, then (32) or (34) cannot be used to find $W(x) = P\{\sup_{0 \le u < \infty} [\chi(u) - u] \le x\}$; however, the following theorem is applicable.

THEOREM 6. *If $\{\chi(u), 0 \leq u < \infty\}$ is a separable stochastic process with stationary independent increments, if almost all sample functions are nondecreasing step functions which vanish at $u = 0$, and if $E\{\chi(u)\} = \rho u$, then for every x,*

$$(35) \qquad P\{\sup_{0 \leq u < \infty} [\chi(u) - u] \leq x\} = 1 - (1 - \rho) \int_0^\infty d_y P\{\chi(y) \leq y + x\}$$

whenever $0 \leq \rho < 1$. If $\rho \geq 1$, then the left-hand side of (35) is 0.

PROOF. By the continuity theorem for probabilities we have

$$W(x) = \lim_{t \to \infty} W(t, x).$$

First, let $0 \leq \rho < 1$. Then, by (20), we find that $W(0) = \lim_{t\to\infty} W(t, 0) = 1 - \rho$. If we let $t \to \infty$ in (34), then we get (35). It follows from (21) that $W(\infty) = 1$, that is, $W(x)$ is a proper distribution function. Evidently, $W(x) = 0$ if $x < 0$. If $\rho \geq 1$, then $W(x) = 0$ for every x, which follows from (21). If $\{\chi(u), 0 \leq u \leq T\}$ has stationary independent increments, then theorem 5 is applicable for every t (finite or infinite), and thus, in the case of $T = \infty$,

$$(36) \qquad P\{\sup_{0 \leq u < \infty} [u - \chi(u)] \leq x\} = 1 - \int_x^\infty \frac{x}{y} d_y P\{\chi(y) \leq y - x\}$$

for $x > 0$.

REMARK 2. If $\{\chi(u), 0 \leq u < \infty\}$ has stationary independent increments and $\Phi(s)$ is given by (22), then we can prove easily that the distributions (35) and (36) can also be obtained in the following way.

If $0 \leq \rho < 1$, then for the distribution function $W(x)$ defined by (35) we have

$$(37) \qquad \int_0^\infty e^{-sx} \, dW(x) = \frac{1 - \rho}{1 - \dfrac{\Phi(s)}{s}}$$

whenever $\Re(s) \geq 0$.

Further, for $x \geq 0$ we have

$$(38) \qquad P\{\sup_{0 \leq u < \infty} [u - \chi(u)] \leq x\} = 1 - e^{-\omega x}$$

where ω is the largest real root of the equation $\Phi(\omega) = \omega$. If $0 \leq \rho \leq 1$, then $\omega = 0$, and if $\rho > 1$, then $\omega > 0$. (Cf. L. Takács [17].)

EXAMPLE 2. *Theory of queues* (cf. L. Takács [9], [11], and [13]). Suppose that in the time interval $(0, \infty)$ customers arrive at a counter in accordance with a random process. The customers are served by a single server in the order of arrival. The server is idle if and only if there is no customer in the system. Denote by $\chi(u)$ the total service time of all those customers who arrive in the interval $(0, u]$. We suppose that $\{\chi(u), 0 \leq u < \infty\}$ is a separate stochastic process with nonnegative, stationary, independent increments and that almost all sample functions are nondecreasing step functions vanishing at $u = 0$. Denote by $\eta(t)$ the virtual waiting time at time t, that is, the time that a customer would have to wait if he arrived at time t. Let $\alpha(t)$ denote the total idle time of the server in the interval $(0, t)$. If $\eta(0) = 0$, then it can easily be seen that

$$P\{\eta(t) \leq x\} = P\{\sup_{0 \leq u \leq t} [\chi(u) - u] \leq x\} \tag{39}$$

and

$$P\{\alpha(t) \leq x\} = P\{\sup_{0 \leq u \leq t} [u - \chi(u)] \leq x\}. \tag{40}$$

If, in particular, customers arrive at the counter in accordance with a Poisson process of density λ and the service times are mutually independent, identically distributed random variables with distribution function $H(x)$ and independent of the arrival times, then $\{\chi(u), 0 \leq u < \infty\}$ has nonnegative stationary independent increments and

$$P\{\chi(u) \leq x\} = \sum_{n=0}^{\infty} e^{-\lambda u} \frac{(\lambda u)^n}{n!} H_n(x) \tag{41}$$

where $H_n(x)$ denotes the n-th iterated convolution of $H(x)$ with itself; $H_0(x) = 1$ if $x \geq 0$ and $H_0(x) = 0$ if $x < 0$. If a denotes the average service time and $\psi(s)$, the Laplace-Stieltjes transform of $H(x)$, then $\rho = \lambda a$ and $\Phi(s) = \lambda[1 - \psi(s)]$.

In this case the distributions and the limiting distributions of $\eta(t)$ and $\alpha(t)$ are given by (32), (35), (25), (36), (37), and (38).

EXAMPLE 3. *Theory of dams* (cf. D. G. Kendall [5], P. A. P. Moran [7], and L. Takács [14]). Consider a dam (reservoir) with infinite capacity and suppose that water is flowing into the dam in accordance with a random process. Denote by $\chi(u)$ the total quantity of water flowing into the dam in the interval $(0, u]$. Suppose that $\{\chi(u), 0 \leq u < \infty\}$ is a separable stochastic process with stationary independent increments and that almost all sample functions are nondecreasing step functions which vanish at $u = 0$. Suppose that the release is continuous at a constant unit rate when the dam is not empty. Denote by $\eta(t)$ the content of the dam at time t, and by $\alpha(t)$ the total time in the interval $(0, t)$ during which the dam is empty. If $\eta(0) = 0$, then the distributions of $\eta(t)$ and $\alpha(t)$ are given by (39) and (40) respectively.

Now I should like to mention two examples for input processes of this type.

(i) For $x \geq 0$,

$$P\{\chi(u) \leq x\} = \frac{1}{\Gamma(u)} \int_0^{\mu x} e^{-y} y^{u-1}\, dy \tag{42}$$

where μ is a positive constant. Then $\rho = 1/\mu$ and $\Phi(s) = \log(1 + (s/\mu))$.

(ii) For $x \geq 0$,

$$P\{\chi(u) \leq x\} = \frac{1}{\sqrt{4\pi}} \int_0^{x/u^2} e^{-1/4y} y^{-3/2}\, dy. \tag{43}$$

Then $\rho = \infty$ and $\Phi(s) = s^{1/2}$.

EXAMPLE 4. *Theory of mathematical risk* (cf. C. O. Segerdahl [8]). Suppose that a company deals with insurance and in the time interval $(0, u]$ receives the gross risk premium u, and the total claim in the time interval $(0, u]$ is $\chi(u)$ where $\{\chi(u), 0 \leq u < \infty\}$ is a separable stochastic process with stationary, independent increments almost all of whose sample functions are nondecreasing step functions which vanish at $u = 0$. Denote by $\gamma(u)$ the risk reserve at time u. If $\gamma(0) = x$, then $\gamma(u) = x + u - \chi(u)$. For $x \geq 0$, denote by θ_x the time when $\gamma(u)$ becomes 0 for the first time, that is, θ_x is the time when the risk reserve becomes depleted. The distribution of θ_x is determined by

$$P\{\theta_x > t\} = P\{\sup_{0 \leq u \leq t} [\chi(u) - u] \leq x\}, \tag{44}$$

and the right-hand side of (44) is given by (32). If $E\{\chi(u)\} = \rho u$ and $\rho \geq 1$, then θ_x is finite with probability 1, whereas if $0 \leq \rho < 1$, then there is a positive probability that $\theta_x = \infty$, that is, that the risk reserve will never be depleted.

If the insurance company deals with whole-life annuities, then the risk reserve can be expressed as $\gamma(u) = x + \chi(u) - u$ where x is the risk reserve at time $u = 0$ and the process $\{\chi(u), 0 \leq u < \infty\}$ has similar properties to the above one. Now if θ_x denotes the time when $\gamma(u)$ becomes 0 for the first time, then for $0 < x \leq t$ we have

$$P\{\theta_x > t\} = P\{\sup_{0 \leq u \leq t} [u - \chi(u)] \leq x\}, \tag{45}$$

and the right-hand side of (45) is given by (25), or also by (38), for $t = \infty$.

6. A random walk process

This section is independent of the preceding ones and illustrates that often very simple combinatorial arguments yield useful results in the theory of stochastic processes.

Suppose that a particle performs a random walk on the x-axis. Starting at $x = 0$ in each step the particle moves a unit distance to the right or a unit distance to the left with probabilities p and q respectively ($p + q = 1$, $0 < p < 1$). Suppose that the successive displacements are independent of each other. Denote by η_n the position of the particle after the n-th step; $\eta_0 = 0$. We have

$$P\{\eta_n = x\} = \binom{n}{\frac{n+x}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} \tag{46}$$

for $x = n, n - 2, \cdots, -n + 2, -n$.

If we suppose that the displacements of the particle occur at random times in the time interval $(0, \infty)$, and $\nu(u)$ denotes the number of steps taken in the time interval $(0, u]$, then $\chi(u) = \eta_{\nu(u)}$ is the position of the particle at time u.

We are interested in the investigation of the stochastic process $\{\chi(u), 0 \leq u < \infty\}$ in the case when $\{\eta_n\}$ and $\{\nu(u)\}$ are independent and with probability 1, $\nu(u)$, $0 \leq u < \infty$, increases only in jumps of magnitude 1.

In the particular case when $\{\nu(u), 0 \leq u < \infty\}$ is a Poisson process of constant density and $p = q = \frac{1}{2}$ by using analytical methods, G. Baxter and M. D. Donsker [1] found that

$$P\{\sup_{0 \leq u \leq t} \chi(u) < a\} = 1 - \int_0^t \frac{a}{u} P\{\chi(u) = a\}\, du \tag{47}$$

whenever a is a positive integer.

In this section we shall find in an elementary way the distribution of $\sup_{0 \le u \le t} \chi(u)$ and the joint distribution of $\sup_{0 \le u \le t} \chi(u)$ and $\inf_{0 \le u \le t} \chi(u)$ for the general process.

The following theorem is a generalization of (47) for an arbitrary process $\{\nu(u), 0 \le u < \infty\}$.

THEOREM 7. *If $p = q = \frac{1}{2}$ and a is a positive integer, then*

$$P\{\sup_{0 \le u \le t} \chi(u) < a\} = P\{-a \le \chi(t) < a\}. \tag{48}$$

PROOF. First,

$$P\{\sup_{0 \le u \le t} \chi(u) \ge a \text{ and } \chi(t) \ge a\} = P\{\chi(t) \ge a\} \tag{49}$$

evidently holds. Second, we have

$$\begin{aligned} P\{\sup_{0 \le u \le t} \chi(u) \ge a \text{ and } \chi(t) < a\} &= P\{\sup_{0 \le u \le t} \chi(u) \ge a \text{ and } \chi(t) > a\} \\ &= P\{\chi(t) > a\}. \end{aligned} \tag{50}$$

In proving (50) let τ be the first value of u for which $\chi(u) = a$. If we reflect the sample curve for $u > \tau$ in the line $x = a$, then we shall not change the probabilities because the changes in $\chi(u)$ after τ are equally likely to be positive or negative and are independent of the changes before τ. This implies (50). If we add (49) and (50), we get $P\{\sup_{0 \le u \le t} \chi(u) \ge a\}$, whence (48) follows.

The following theorem generalizes (48) further for arbitrary p.

THEOREM 8. *If a is a positive integer, then*

$$P\{\sup_{0 \le u \le t} \chi(u) < a\} = P\{\chi(t) < a\} - \left(\frac{p}{q}\right)^a P\{\chi(t) < -a\}. \tag{51}$$

PROOF. First we shall prove that for $x \le a$,

$$\begin{aligned} &P\{\eta_r < a \text{ for } r = 0, 1, \cdots, n \text{ and } \eta_n = x\} \\ &\qquad = P\{\eta_n = x\} - \left(\frac{p}{q}\right)^a P\{\eta_n = x - 2a\} \end{aligned} \tag{52}$$

where the distribution of η_n is given by (46).

Formula (52) can be proved as follows. If $p = q$, then $\{\eta_n, n = 0, 1, 2, \cdots\}$ describes the path of a symmetric random walk. By applying the reflection principle, we get

$$\begin{aligned} &P\{\eta_r < a \text{ for } r = 0, 1, \cdots, n \text{ and } \eta_n = x\} \\ &\qquad = P\{\eta_n = x\} - P\{\eta_n = x - 2a\} \end{aligned} \tag{53}$$

where now the probabilities on the right-hand side are given by (46) with $p = q = \frac{1}{2}$. In the symmetric random walk each favorable path $\{\eta_0, \eta_1, \cdots, \eta_n\}$ has probability $1/2^n$, and in the general case, each such path has probability $p^{\frac{1}{2}(n+x)}q^{\frac{1}{2}(n-x)}$. Accordingly, if we multiply (53) by $2^n p^{\frac{1}{2}(n+x)} q^{\frac{1}{2}(n-x)}$ and use the general notation (46), we get (52).

Summing (52) over $x < a$, we obtain

(54) $$P\{\eta_r < a \text{ for } r = 0, 1, \cdots, n\} = P\{\eta_n < a\} - \left(\frac{p}{q}\right)^a P\{\eta_n < -a\}.$$

If we multiply (54) by $P\{\nu(t) = n\}$ and add for $n = 0, 1, 2, \cdots$, then we get (51) which was to be proved.

REMARK 3. In the particular case where $\{\nu(u), 0 \le u < \infty\}$ is a Poisson process of density λ, we have

(55) $$P\{\chi(u) = k\} = e^{-\lambda u}\left(\frac{p}{q}\right)^{k/2} I_k(2\lambda p^{1/2}q^{1/2}u)$$

for $k = 0, \pm 1, \pm 2, \cdots$, where $I_k(x)$ is the modified Bessel function of order k defined by

(56) $$I_k(x) = \sum_{j=0}^{\infty} \frac{(x/2)^{2j+k}}{j!(j+k)!}$$

for $k \ge 0$ and $I_{-k}(x) = I_k(x)$.

If $\{\nu(u), 0 \le u < \infty\}$ is a Poisson process of density λ and we use $I_{k-1}(u) - I_{k+1}(u) = 2kI_k(u)/u$, then we can prove easily that (51) can be written in the form (47).

Finally we shall find the joint distribution of $\sup_{0 \le u \le t} \chi(u)$ and $\inf_{0 \le u \le t} \chi(u)$.

THEOREM 9. *If a and b are positive integers, then*

(57) $$\begin{aligned} &P\{-b < \chi(u) < a \text{ for } 0 \le u \le t\} \\ &= \sum_{k=-\infty}^{\infty} \left(\frac{p}{q}\right)^{-(a+b)k} P\{2(a+b)k - b < \chi(t) < 2(a+b)k + a\} \\ &- \sum_{k=-\infty}^{\infty} \left(\frac{p}{q}\right)^{(a+b)k+a} P\{-2(a+b)(k+1) + b < \chi(t) < -2(a+b)k - a\}. \end{aligned}$$

PROOF. In the particular case where $p = q$ we have for $-b < x < a$,

(58) $$\begin{aligned} &P\{-b < \eta_r < a \text{ for } r = 0, 1, \cdots, n \text{ and } \eta_n = x\} \\ &\qquad = \sum_{k=-\infty}^{\infty} [P\{\eta_n = x + 2(a+b)k\} - P\{\eta_n = x - 2(a+b)k - 2a\}] \end{aligned}$$

where the distribution of η_n is given by (46) with $p = q = \frac{1}{2}$. This follows from the theory of random walks. (Cf., for example, L. Takács [10], theorem 5.) In the particular case where $p = q = \frac{1}{2}$, each favorable path $\{\eta_0, \eta_1, \cdots, \eta_n\}$ has probability $1/2^n$, and in the general case each such path has probability $p^{\frac{1}{2}(n+x)}q^{\frac{1}{2}(n-x)}$. Accordingly, if we multiply (58) by $2^n p^{\frac{1}{2}(n+x)}q^{\frac{1}{2}(n-x)}$, and if we use the general notation (46), then we obtain in the general case

(59) $$\begin{aligned} &P\{-b < \eta_r < a \text{ for } r = 0, 1, \cdots, n \text{ and } \eta_n = x\} \\ &\qquad = \sum_{k=-\infty}^{\infty} \left(\frac{p}{q}\right)^{-(a+b)k} P\{\eta_n = x + 2(a+b)k\} \\ &\qquad - \sum_{k=-\infty}^{\infty} \left(\frac{p}{q}\right)^{(a+b)k} P\{\eta_n = x - 2(a+b)k - 2a\}. \end{aligned}$$

Hence,

$$P\{-b < \eta_r < a \text{ for } r = 0, 1, \cdots, n\} \tag{60}$$
$$= \sum_{k=-\infty}^{\infty} \left(\frac{p}{q}\right)^{-(a+b)k} P\{2(a+b)k - b < \eta_n < 2(a+b)k + a\}$$
$$- \sum_{k=-\infty}^{\infty} \left(\frac{p}{q}\right)^{(a+b)k+a} P\{-2(a+b)(k+1) + b < \eta_n < -2(a+b)k - a\}.$$

If we multiply (60) by $P\{\nu(t) = n\}$ and add for $n = 0, 1, 2, \cdots$, then we get (57) which was to be proved.

REMARK 4. In the particular case where $p = q = \frac{1}{2}$, (57) reduces to

$$P\{-b < \chi(u) < a \text{ for } 0 \le u \le t\} \tag{61}$$
$$= \sum_{k=-\infty}^{\infty} (-1)^k P\{(a+b)k - b < \chi(t) < (a+b)k + a\}.$$

We also note that if instead of (58) we use the following equivalent formula (cf. L. Takács [10], theorem 5),

$$P\{-b < \eta_r < a \text{ for } r = 0, 1, \cdots, n \text{ and } \eta_n = x\} \tag{62}$$
$$= \frac{2}{(a+b)} \sum_{k=0}^{a+b} \left(\cos \frac{k\pi}{a+b}\right)^n \sin \frac{k\pi a}{a+b} \sin \frac{k\pi(a-x)}{a+b},$$

then (61) can also be written in the following form:

$$P\{-b < \chi(u) < a \text{ for } 0 \le u \le t\} \tag{63}$$
$$= \frac{1}{(a+b)} \sum_{k=1}^{a+b-1} [1 - (-1)^k] G_t\left(\cos \frac{k\pi}{a+b}\right) \frac{\sin \frac{k\pi a}{a+b} \cos \frac{k\pi}{a+b}}{\sin \frac{k\pi}{a+b}}$$

where $G_t(z) = E\{z^{\nu(t)}\}$ is the generating function of the random variable $\nu(t)$. If $\{\nu(u), 0 \le u < \infty\}$ is a Poisson process of density λ, then $G_t(z) = e^{-\lambda t(1-z)}$.

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