Odd Hole Recognition in Graphs of Bounded Clique Size

Michele Conforti
Dipartimento di Matematica Pura ed Applicata
Università di Padova
Via Belzoni 7, 35131 Padova, Italy
conforti@math.unipd.it

Gérard Cornuéjols [1]
Tepper School of Business
Carnegie Mellon University
Pittsburgh, PA 15213 and
LIF, Faculté des Sciences de Luminy
13288 Marseille, France
gc0v@andrew.cmu.edu

Xinming Liu
Tepper School of Business
Carnegie Mellon University
Pittsburgh, PA 15213
xinming.liu@gmail.com

Kristina Vušković [2]
School of Computing
University of Leeds
Leeds LS2 9JT, UK
vuskovi@comp.leeds.ac.uk

Giacomo Zambelli
Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
gzambell@math.uwaterloo.ca

July 19, 2004
Revised August 23, 2005

[1]Supported by NSF grant DMI-0352885 and ONR grant N00014-03-1-0133.
[2]Supported by EPSRC grant GR/R35629/01.

Abstract

In a graph G, an odd hole is an induced odd cycle of length at least five. A clique of G is a set of pairwise adjacent vertices. In this paper we consider the class $\mathcal{C}_k$ of graphs whose cliques have a size bounded by a constant k. Given a graph G in $\mathcal{C}_k$, we show how to recognize in polynomial time whether G contains an odd hole.

Keywords: odd hole, recognition algorithm, cleaning, decomposition

1 Introduction

A *hole* is a graph induced by a cycle of length at least four. A hole is *odd* if it contains an odd number of vertices. Otherwise, it is even. Graph G *contains* graph H if H is isomorphic to an induced subgraph of G. Chudnovsky, Cornuéjols, Liu, Seymour and Vušković recently proved that it is polynomial to test whether a graph contains an odd hole or its complement [2]. However, it is still an open problem to test whether a graph contains an odd hole. Bienstock [1] proved that it is NP-complete to test whether a graph contains an odd hole passing through a specific vertex. A *clique* is a set of pairwise adjacent vertices. The *clique number* of a graph is the size of its largest clique. In this paper, we show that it is polynomial to test whether a graph of bounded clique number contains an odd hole.

We use the same general strategy as in [2]. Let H be an odd hole in a graph G. We say that $u \in V(G) \setminus V(H)$ is *H-minor* if its neighbors in H lie in some 2-edge path of H. In particular, u is H-minor if u has no neighbor in H. A vertex $u \in V(G) \setminus V(H)$ is *H-major* if it is not H-minor. We say that H is *clean* if G contains no H-major vertex. A graph G is *clean* if either it is odd-hole-free or it contains a clean shortest odd hole. As in [2] our approach for testing whether a graph G of bounded clique number contains an odd hole consists of two steps:

(i) constructing in polynomial time a clean graph G' that contains an odd hole if and only if G does, or in some cases identifying an odd hole of G, and

(ii) checking whether the clean graph G' contains an odd hole.

For step (ii), we can use the polynomial algorithms in [2]. The main result of this paper is a polynomial algorithm for step (i). Step (i) is called *cleaning* the graph G.

1.1 Notation

For a graph G and a set B of vertices of G, we denote by $G(B)$ the subgraph of G induced by the vertex set B. For a vertex v, $N(v)$ denotes the set of vertices adjacent to v.

A *pyramid* $\Pi(xyz; u)$ is a graph induced by three paths $P_1 = x, \ldots, u$, $P_2 = y, \ldots, u$ and $P_3 = z, \ldots, u$ having no common or adjacent intermediate vertices, such that at most one of the paths is of length 1 and the vertex set $\{x, y, z\}$ induces a clique of size 3. Note that every two of the paths P_1, P_2, P_3 induce a hole. Since two of the three paths must have the same parity, one of these holes is odd. Therefore, every pyramid contains an odd hole.

A *wheel*, denoted by (H, x), is a graph induced by a hole H and a vertex $x \notin V(H)$ having at least three neighbors in H, say $x_1, \ldots, x_n$. Vertex x is the *center* of the wheel. A subpath of H connecting x_i and x_j is a *sector* if it contains no intermediate vertex x_l, $l \in \{1, \ldots, n\}$. A *short sector* is a sector of length 1, and a *long sector* is a sector of length at least 2. A wheel is *odd* if it contains an odd number of short sectors, and *even* otherwise. Each of the long sectors together with vertex x induces a hole. If each of these holes is even and the wheel (H, v) is odd then H is an odd hole, since the wheel (H, x) contains an odd number of short sectors. Therefore, every odd wheel contains an odd hole.

In a graph G, a *jewel* is a sequence $v_1, \cdots, v_5$, P such that $v_1, \cdots, v_5$ are distinct vertices, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1 are edges, v_1v_3, v_2v_4, v_1v_4 are nonedges, and P is a path of G between v_1

and v_4 such that v_2, v_3, v_5 have no neighbors in $V(P) \setminus \{v_1, v_4\}$. Clearly a jewel either contains an odd wheel or a 5-hole, so if there is a jewel in a graph G then there is an odd hole in G.

Chudnovsky and Seymour found an $O(|V(G)|^9)$ algorithm to test whether a graph G contains a pyramid and an $O(|V(G)|^6)$ algorithm to test whether a graph G contains a jewel (see [2]).

2 Cleaning

In this section, we show how to clean a graph G of bounded clique number. That is, we perform step (i) above. The cleaning algorithm produces a polynomial family of induced subgraphs of G such that if G contains a shortest odd hole H^*, then one of the graphs produced by the cleaning algorithm, say G', contains H^* and H^* is clean in G'.

Roughly speaking, this is accomplished by showing that there exists a set X of vertices of H^*, whose size depends only on the clique number, such that every major vertex for H^* has a neighbor in X. Since the set Y of vertices of H^* with neighbors in X has at most $2|X|$ elements, we may enumerate all possible choices for X and Y, and for each choice of X and Y add to the family the graph obtained by removing the vertices of $V(G) \setminus Y$ that have a neighbor in X.

2.1 Vertices with At Most Three Neighbors in H^*

Lemma 1 *Let H^* be a shortest odd hole in G. Suppose that G does not contain a pyramid. If a vertex $u \notin V(H^*)$ has a neighbor but no more than three neighbors in H^* then u is H^*-minor.*

Proof: If u has one neighbor in H^* then u is H^*-minor. Now suppose that u has two neighbors in H^*, say u_1 and u_2. Let P_1 and P_2 be the two u_1u_2-subpaths of H^*. Since H^* is odd, P_1 and P_2 have different parity, say P_1 is odd. If P_1 is of length 1 then u is H^*-minor. Otherwise, $V(P_1) \cup \{u\}$ induces an odd hole. Since this hole cannot be shorter than H^*, P_2 is of length 2, and hence u is H^*-minor.

Now assume that u has three neighbors in H^*, and let P_1, P_2 and P_3 be the three sectors of the wheel (H^*, u). If exactly one of the sectors is short then $V(H^*) \cup \{u\}$ induces a pyramid. If two of the sectors are short then u is H^*-minor. Finally suppose that all three sectors are long. Since H^* is odd, at least one of the sectors, say P_1, is odd. Then $V(P_1) \cup \{u\}$ induces an odd hole shorter than H^*, a contradiction. □

2.2 Vertices with More Than Three Neighbors in H^*

Let H^* be a shortest odd hole in G. Let $S(H^*)$ be the set of H^*-major vertices that have four or more neighbors in H^*. Note that, for any $u \in S(H^*)$, every long sector of the wheel (H^*, u) is of even length since H^* is a shortest odd hole of G; hence, (H^*, u) contains an odd number of short sectors.

Let $S \subseteq V(G)$. We say that vertex $x \in V(G) \setminus S$ is *S-complete* if x is adjacent to every vertex in S. We say that an edge xy is *S-complete* if both vertices x and y are S-complete.

Lemma 2 *Let H^* be a shortest odd hole in G. Suppose that G does not contain a jewel. If $u, v \in S(H^*)$ are not adjacent then an odd number of edges of H^* are $\{u, v\}$-complete.*

Proof: Let u and v be nonadjacent vertices of $S(H^*)$. Suppose that an even number of edges of H^* are $\{u,v\}$-complete. Then some long sector P of the wheel (H^*,u) contains an odd number of short sectors of (H^*,v). Let u_1 and u_2 be the endvertices of P. P has even length. Let P' be the subpath of H^* induced by $(V(H^*)\setminus V(P))\cup\{u_1,u_2\}$. P' has odd length. Note that P' must be of length at least four, since otherwise (H^*,u) is a jewel, a contradiction. If P contains three or more neighbors of v, then the vertex set $V(P)\cup\{u,v\}$ induces an odd wheel with center v, and hence contains an odd hole shorter than H^*, contradicting our choice of H^*. Otherwise, let v_1 and v_2 be the two neighbors of v in P. Vertex v cannot have exactly four neighbors in H^*, say v_1,v_2,v_3,v_4, such that both v_3u_1 and v_4u_2 are edges, because otherwise the vertex set $(V(H^*)\setminus V(P))\cup\{v\}$ induces a shorter odd hole than H^*, since P is even and P' is of length at least four. Therefore, there exist vertices $u_3,v_3\in V(H^*)\setminus V(P)$, the neighbors of u and v respectively, such that u and v have no other neighbors on u_3v_3-subpath of H^*, call it Q, and vertices u_3 and v_3 are not adjacent to u_1 or u_2. But now the vertex set $V(Q)\cup V(P)\cup\{u,v\}$ induces a pyramid $\Pi(v_1v_2v;u)$, and hence contains an odd hole shorter than H^*, contradicting our choice of H^*. □

The following, which is an easy consequence of Lemma 2, will be used in several places.

Lemma 3 *Let H^* be a shortest odd hole in G, P be a subpath of H^* such that $|V(H^*)\setminus V(P)|\geq 3$, and x,y be two nonadjacent vertices in $S(H^*)$. Assume that no ends of P are $\{x,y\}$-complete and there is no $\{x,y\}$-complete edge in P. Then there exists an $\{x,y\}$-complete vertex in H^* with no neighbor in P.*

Proof: By Lemma 2, there exists an $\{x,y\}$ complete edge e in H^*. One of the two endvertices of e has the desired property. □

Lemma 4 *Suppose that G does not contain a jewel. If $A\subseteq S(H^*)$ is a stable set, then an odd number of edges of H^* are A-complete.*

Proof: Let $A\subseteq S(H^*)$ be a stable set and suppose that an even number of edges of H^* are A-complete. Let A' be a smallest subset of A with the property that an even number of edges of H^* are A'-complete. Note that by Lemma 2, $|A'|\geq 3$. Let $s_1,\dots,s_m$ be the vertices of H^* adjacent to at least one vertex in A', encountered in that order when traversing H^* clockwise. For $i\in[m]$, let S_i be the s_is_{i+1}-subpath of H^* (indices taken modulo m), that does not contain any intermediate vertex s_j, $j\in[m]$.

Claim For every $i\in[m]$, S_i is either an edge whose endvertices are both adjacent to some vertex $x\in A$, or S_i has even length.

Proof of Claim: If there is a vertex $x\in A'$ adjacent to both s_i and s_{i+1}, then S_i is a sector of the wheel (H^*,x) and hence the result holds. Otherwise, let x_1 and x_2 be vertices of A' such that x_1 is adjacent to s_i and x_2 is adjacent to s_{i+1}. By Lemma 3 there exits an $\{x_1,x_2\}$-complete vertex u in H^* with no neighbor in S_i. Then the vertex set $V(S_i)\cup\{x_1,x_2,u\}$ induces a hole. Since both x_1 and x_2 have at least four neighbors in H^*, this hole is shorter than H^*, so it must be even, hence S_i is of even length. This completes the proof of the claim. ◇

For $C \subseteq A'$, let δ_C denote the number of edges of H^* that are C-complete. Let δ be the number of paths in $S_1, \ldots, S_m$ of length one. Then

$$\delta = \sum_{i=1}^{|A'|} (-1)^{i+1} \sum_{C \subseteq A', |C|=i} \delta_C$$

By the choice of A', for every $C \subseteq A'$ such that $C \neq A'$, δ_C is odd. Hence the parity of δ is equal to the parity of

$$\sum_{i=1}^{|A'|-1} \left(\begin{array}{c} |A'| \\ i \end{array} \right) + \delta_{A'}$$

which is itself equal to the parity of $\delta_{A'}$ since

$$\sum_{i=1}^{|A'|-1} \left(\begin{array}{c} |A'| \\ i \end{array} \right) = 2^{|A'|} - 2$$

By the Claim and because H^* is odd, δ is odd. Hence $\delta_{A'}$ must be odd as well, contradicting the choice of A'. □

Theorem 5 *Suppose that G does not contain a jewel. Let A be a stable set of $S(H^*)$ and let x_1x_2 be an edge of H^* such that every vertex of A is adjacent to both x_1 and x_2 (such an edge exists by Lemma 4). Let B be the set of vertices of $S(H^*)$ that have no neighbor in $\{x_1, x_2\}$, and have both a neighbor and a nonneighbor in A. Then there exists an edge y_1y_2 of H^* such that y_1 is A-complete and every vertex of B has a neighbor in $\{y_1, y_2\}$.*

Proof: If $B = \emptyset$ then the result is trivially true, so we may assume that $B \neq \emptyset$. Since every vertex of B is major, this implies that H^* is of length greater than 5.

Claim 1 For every $u \in B$, an edge of H^* is $(A \cup \{u\})$-complete.

Proof of Claim 1: Let A_1 be the neighbors of u in A and $A_2 = A \setminus A_1$. By Lemma 4, there is an edge u_1u_2 of H^* such that every vertex of $A_2 \cup \{u\}$ is adjacent to both u_1 and u_2. Since u has no neighbor in $\{x_1, x_2\}$, every vertex of A_1 must be adjacent to both u_1 and u_2, else there is a 5-hole. This completes the proof of Claim 1. ◇

Claim 2 If X is a stable set of B, then there exists an edge z_1z_2 of H^* such that z_1 is A-complete and every vertex of X has a neighbor in $\{z_1, z_2\}$.

Proof of Claim 2: We consider the following two cases.

Case 1 There is a vertex in A that is not adjacent to any vertex in X.

Let $A_1 \subseteq A$ be such that $A_1 \cup X$ is a maximal stable set. By Lemma 4, an edge of H^* is $(A_1 \cup X)$-complete, say u_1u_2. Let $w \in A \setminus A_1$. Note that w is adjacent to some $x \in X$. If w is not adjacent to u_1 or u_2, then there is a 5-hole in the graph induced by $\{x, y, w, u_1, u_2, x_1, x_2\}$, where $y \in A_1$. So every vertex of $A \setminus A_1$ is adjacent to both u_1 and u_2.

Case 2 Every vertex of A is adjacent to some vertex in X.

By Claim 1 and Case 1, we may assume w.l.o.g. that $|X| > 1$ and for every proper subset of X the result holds. Let $w \in A$ be such that $|N(w) \cap X|$ is minimum. Let $Z = N(w) \cap X$. Since every vertex of X has a non-neighbor in A and $|Z|$ is minimum, $|Z| < |X|$. By our assumption, there exists an edge y_1y_2 of H^* such that y_1 is A-complete and every vertex of $X \setminus Z$ has a neighbor in $\{y_1, y_2\}$. By Lemma 4 an edge of H^* is X-complete, say edge y_3y_4.

We may assume that vertices y_1, y_2, y_3, y_4 are all distinct and y_1y_3 and y_1y_4 are not edges, since otherwise the result trivially holds. Also w.l.o.g. y_2y_4 is not an edge.

Suppose that wy_4 is not an edge. We may assume that some $z \in Z$ is not adjacent to y_1, since otherwise the edge y_1y_2 satisfies the claim. If some $v \in X \setminus Z$ is adjacent to y_1, then $\{y_1, v, w, z, y_4\}$ induces a 5-hole. So for every $v \in X \setminus Z$, vy_1 is not an edge, and hence vy_2 is an edge. If w is adjacent to y_2, then $\{y_2, w, v, z, y_4\}$ induces a 5-hole. So w is not adjacent to y_2. By Lemma 3, there is a vertex u of H^* adjacent to both v and w, but with no neighbor in $\{y_1, y_2\}$. Then $\{y_1, y_2, u, v, w\}$ induces a 5-hole.

Therefore wy_4 is an edge. We now show that y_4 is A-complete. Let $w' \in A$ and assume $w'y_4$ is not an edge. By the choice of w and by the above argument, there is a vertex $v \in X \setminus Z$ adjacent to w'. But then the graph induced by $\{w, w', x_1, x_2, v, y_4\}$ contains a 5-hole. This completes the proof of Claim 2. $\diamond$

Claim 3 For every edge v_1v_2 in $G(B)$, there exists $v \in A$ that is adjacent to neither v_1 nor v_2.

Proof of Claim 3: Let A_1 be the set of neighbors of v_1 in A, and $A_2 = A \setminus A_1$. Suppose the claim does not hold. Then v_2 is universal for A_2. Let w_1 be a vertex of A_1 that v_2 is not adjacent to. Then $v_1, v_2, w_2, x_2, w_1, v_1$, where $w_2 \in A_2$, is a 5-hole. This completes the proof of Claim 3. $\diamond$

By Claim 1, we may assume that for every proper subset B' of B, the statement holds. By Claim 2 we may assume that B is not a stable set. Let v_1v_2 be an edge of $G(B)$. By Claim 3, let v be a vertex of A that is adjacent to neither v_1 nor v_2. Let y_1y_2 be an edge of H^* such that y_1 is A-complete and all vertices of $B \setminus v_2$ have a neighbor in $\{y_1, y_2\}$. Let y_3y_4 be an edge of H^* such that y_3 is A-complete and all vertices of $B \setminus v_1$ have a neighbor in $\{y_3, y_4\}$. Then the theorem follows from the following claim.

Claim 4 v_1 has a neighbor in $\{y_3, y_4\}$, or v_2 has a neighbor in $\{y_1, y_2\}$.

Proof of Claim 4: Suppose the claim does not hold. v_1 has no neighbor in $\{y_3, y_4\}$ and v_2 has no neighbor in $\{y_1, y_2\}$.

If a vertex of $\{y_1, y_2\}$ coincides with a vertex of $\{y_3, y_4\}$, then $\{y_1, y_2, y_3, y_4, v_1, v_2\}$ induces a 5-hole. Therefore, vertices y_1, y_2, y_3, y_4 are all distinct.

We now show that v and v_1 must have a common neighbor in $\{y_1, y_2\}$. Assume not. Then vy_1 and v_1y_2 are edges, and vy_2 and v_1y_1 are not. By Lemma 3, there is a vertex u of H^* that is $\{v, v_1\}$-complete but has no neighbor in $\{y_1, y_2\}$. Then $\{y_1, y_2, v, v_1, u\}$ induces a 5-hole. Therefore, v and v_1 have a common neighbor y in $\{y_1, y_2\}$, and similarly v and v_2 have a common neighbor y' in $\{y_3, y_4\}$. If yy' is not an edge, then $\{y, y', v, v_1, v_2\}$ induces a 5-hole. Therefore, yy' is an edge.

Let a, y, y', b be the subpath of H^* induced by $\{y_1, y_2, y_3, y_4\}$. Then vy, vy', v_1y, v_2y' are edges and v_2a, v_2y, v_1y', v_1b are not.

Let z_2 be the neighbor of v_2 in H^* that is closest to a in $H^* \setminus \{y, y'\}$. Note that $z_2 \neq b$ since v_2 is a major vertex. Let P_2 be the az_2-subpath of H^* that does not contain y.

Suppose v does not have a neighbor in P_2. By Lemma 3, some vertex u of H^* is $\{v, v_2\}$-complete and has no neighbor in P_2. Note that $u \neq b$ since b is not $\{v, v_2\}$-complete. But then $P_2 \cup \{y, y', v, v_2, u\}$ induces a pyramid $\Pi(vyy', v_2)$, and hence there is an odd hole shorter than H^*, a contradiction. Therefore v must have a neighbor in P_2.

We now show that a is the unique neighbor of v in P_2. Let v' be the neighbor of v in P_2 that is closest to z_2. Assume that $v' \neq a$. Let P' be the $v'z_2$-subpath of P_2. If v_1 has no neighbor in P', then the graph induced by $S = P' \cup \{y, y', v, v_1, v_2\}$ is a pyramid $\Pi(vyy', v_2)$ hence there is an odd hole shorter than H^*. If v_1 has a neighbor in $P' \setminus z_2$, then the graph induced by S contains a pyramid $\Pi(vyy', v_1)$ hence there is an odd hole shorter than H^*. So v_1 is adjacent to z_2. If the graph induced by $P_2 \cup \{y, y', v_1, v_2\}$ is an odd wheel with center v_1, there is an odd hole shorter than H^*. Hence v_1 must have a neighbor in $P_2 \setminus P'$. If v_1 has a neighbor z in P_2 that lies strictly between a and v', then there is a path Q from v to v_1 with interior in z, P_2, v'. But then $Q \cup \{y, y', v_2\}$ induces a pyramid $\Pi(vyy', v_1)$, which contains an odd hole shorter than H^*. Therefore a and z_2 are the only neighbors of v_1 in P_2. Then v is not adjacent to a for otherwise a, v, y', v_2, v_1, a is an odd hole. Let v'' be the neighbor of v closest to a in P_2. Note that $v'' \neq z_2$ since otherwise $P_2 \cup \{y, y', v_2, v\}$ induces an odd wheel with center v hence there is an odd hole shorter than H^*. Let P'' denote the av''-subpath of P_2. By Lemma 3, some vertex u of H^* is $\{v, v_1\}$-complete and has no neighbor in P''. But then the graph induced by $P'' \cup \{y, v, v_1, u\}$ is a pyramid $\Pi(ayv_1, v)$ hence there is an odd hole shorter than H^*. Therefore a is the unique neighbor of v in P_2.

Then v_1 is not adjacent to a for otherwise a, v, y', v_2, v_1, a is an odd hole. Suppose v_1 has a neighbor in P_2. By Lemma 3, there exists a vertex u of H^* adjacent to both v and v_1, but with no neighbor in P_2. Then the graph induced by $P_2 \cup \{y, v, v_1, u\}$ contains a pyramid $\Pi(ayv, v_1)$ hence there is an odd hole shorter than H^*. Therefore, v_1 has no neighbor in P_2.

Let z_1 be the neighbor of v_1 in H^* that is closest to b in $H^* \setminus \{y, y'\}$. Let P_1 be the bz_1-subpath of H^* that does not contain y. By symmetry, b is the unique neighbor of v in P_1 and v_2 has no neighbor in P_1. Since P_2, a, y, y' is a sector of wheel (H^*, v_2), P_2 must be even, and similarly P_1 is even. Note that z_1z_2 is not an edge since H^* and the path a, y, y', b have odd length and P_1, P_2 have even length. But then $P_1 \cup P_2 \cup \{v, v_1, v_2\}$ induces an odd hole shorter than H^*, a contradiction. □

2.3 Cleaning Algorithm

In this section, we present our cleaning algorithm for the class of graphs of bounded clique number. The running time depends on the clique number.

Input: A graph G of bounded clique number k.

Output: Either an odd hole or a family $\mathcal{F}$ of induced subgraphs of G that satisfies the following properties:

(1) G contains an odd hole if and only if some graph of $\mathcal{F}$ contains a clean shortest odd hole.

(2) $|\mathcal{F}|$ is $O(|V(G)|^{8k})$.

Step 1: Check whether G contains a jewel or a pyramid (by algorithms in [2]). If it does, output an odd hole and stop. Otherwise, set $\mathcal{F}_1 = \{G\}$ and $\mathcal{F}_2 = \emptyset$.

Step 2: Repeat the following k times. For each graph $F \in \mathcal{F}_1$ and every (P_1, P_2) where $P_1 = x_0, x_1, x_2, x_3$ and $P_2 = y_0, y_1, y_2, y_3$ are two induced paths of F, add to $\mathcal{F}_2$ the graph obtained from F by removing the vertex set $(N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)) \setminus (V(P_1) \cup V(P_2))$. Set $\mathcal{F}_1 = \mathcal{F}_2$ and $\mathcal{F}_2 = \emptyset$.

Step 3: Set $\mathcal{F} = \mathcal{F}_1$.

Theorem 6 *This algorithm produces the desired output, and its running time is* $O(|V(G)|^{8k})$.

Proof: Suppose that the algorithm does not output an odd hole. Suppose G contains a shortest odd hole H^*. By Step 1 G contains no jewel and no pyramid. Now we show how Step 2 generates a graph in $\mathcal{F}_1$ that contains H^* and H^* is clean in it.

By Lemma 1, $S(H^*)$ is the set of all H^*-major vertices. Let A be a maximal stable set of $S(H^*)$. We follow the notation in Theorem 5. Let $P_1 = x_0, x_1, x_2, x_3$ and $P_2 = y_0, y_1, y_2, y_3$ such that x_1x_2 and y_1y_2 satisfy the conditions stated in Theorem 5. Let $S'(H^*)$ denote the set of vertices of $S(H^*)$ that have no neighbor in $\{x_1, x_2\}$, and are A-complete. Let G' be the graph obtained from G by removing $(N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)) \setminus (V(P_1) \cup V(P_2))$. Then G' contains H^* and the set of major vertices for H^* in G' is contained in $S'(H^*)$. The clique number of the graph induced by $S'(H^*)$ is one less than the clique number of the graph induced by $S(H^*)$. Hence, by the fact that the clique number of G is bounded by k, Theorem 5 implies that, when the k iterations of Step 2 are completed, some graph $F \in \mathcal{F}_1$ contains H^* and H^* is clean in F. Hence (1) holds.

$O(|V(G)|^{8k})$ graphs are created in Step 2. Hence, (2) holds. The running time of Step 1 is $O(|V(G)|^9)$ as discussed in [2]. The running time of Steps 2 is $O(|V(G)|^{8k})$. Therefore, the overall running time is $O(|V(G)|^{8k})$. □

In [2] a polynomial time algorithm with following specification is obtained.

Input: A clean graph G.

Output: ODD-HOLE-FREE when G is odd-hole-free, and NOT ODD-HOLE-FREE otherwise.

The above two algorithms imply that it is polynomial to test whether a graph of bounded clique number contains an odd hole.

References

[1] D. Bienstock, On complexity of testing for odd holes and induced odd paths, *Discrete Mathematics 90* (1991) 85-92.

[2] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour and K. Vušković, Recognizing Berge graphs, *Combinatorica 25* (2005) 143-186.