# Decomposing Berge Graphs Containing Proper Wheels work in progress

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#### Abstract

If a Berge graph contains certain wheels, then it contains a "good" skew partition.

### 1 Introduction

A graph G is *perfect* if, for all induced subgraphs of G, the size of a largest clique is equal to the chromatic number [1]. Lovász [8] showed that a graph G is perfect if and only if its complement  $\overline{G}$  is perfect. A graph is *minimally* 

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*imperfect* if it is not perfect but all its proper induced subgraphs are. The only known minimally imperfect graphs are the odd holes and their complements. Berge [1] conjectured that there are no other (Strong Perfect Graph Conjecture). A graph is called *Berge* if it contains no odd hole or its complement. Every perfect graph is Berge. The Strong Perfect Graph Conjecture states that every Berge graph is perfect.

A graph G has a skew partition if the nodes V(G) can be partitioned into nonempty sets A, B, C, D such that every node of A is adjacent to every node of B and there is no edge between C and D. Chvátal [4] conjectured that a minimally imperfect graph cannot have a skew partition. Chvátal [4] proved this when A or B has cardinality one (the star cutset lemma).

Hoàng [7] proved the conjecture for special types of skew partitions. A T-cutset is a skew partition with  $u \in C$  and  $v \in D$  such that every node of A is adjacent to both u and v.

**Theorem 1** (Hoàng [7]) No minimally imperfect graph has a T-cutset.

This work was generalized by Robertson, Seymour, Thomas [10]. A skew partition (A, B, C, D) is good if  $C \cup D$  contains a node u that is adjacent to every node of A or B.

**Theorem 2** (Robertson, Seymour, Thomas [10]) No minimally imperfect graph has a good skew partition.

Chvátal's skew partition conjecture was solved recently in its generality:

**Theorem 3** (Chudnovsky, Robertson, Seymour, Thomas [3]) No minimally imperfect graph has a skew partition.

In these notes, we show that, if a Berge graph contains certain types of induced subgraphs called wheels, then it has a good skew partition. This shows that no minimally imperfect graph can contain these types of wheels.

## 2 The Wonderful Lemma

Given a set  $X \subset V(G)$  and a node  $x \notin X$ , we say that x is universal for X if x is adjacent to every node of X. We say that an edge e = yz such that  $y, z \notin X$ , sees X if both y and z are universal for X.

Given a chordless path (or a hole) P in  $G \setminus S$ , we denote by  $E_S(P)$  the set of edges in P that see S. |P| denotes the length (number of edges) of P. int(P) denotes the set of internal nodes of P.

The following lemma, due to Roussel and Rubio [11], plays a fundamental role in this paper. This lemma was proved independently by Robertson, Seymour and Thomas [10], who named it *The Wonderful Lemma*.

**Lemma 4 (Roussel and Rubio [11])** Let G be a Berge graph where V(G) can be partitioned into a co-connected set S and an odd chordless path  $P = u, u', \ldots, v', v$  of length at least 3 such that u, v are both universal for S. Then one of the following holds:

- (i) An odd number of edges of P see S.
- (ii) |P| = 3 and  $S \cup \{u', v'\}$  contains an odd chordless anti-path between u' and v'.
- (iii)  $|P| \ge 5$  and there exist two nonadjacent nodes x, x' in S such that  $(V(P) \setminus \{u, v\}) \cup \{x, x'\}$  induces a chord less path.

*Proof:* The proof is by induction on |S| + |P|.

Note that, for every  $x \in S$ , there is an odd number of edges in E(P) that see x, otherwise  $V(P) \cup \{x\}$  contains an odd hole. We can therefore assume that  $|S| \geq 2$ .

Claim 1: Lemma 4 holds if |P| = 3.

If |P| = 3 and (i) does not hold, then S can be partitioned into 3 sets  $S_1$ ,  $S_2$  and  $S_3$  such that every node in  $S_1$  (resp.  $S_2$ ) is adjacent to u' (resp. v') but not to v' (resp. u'), every node in  $S_3$  is adjacent to u' and v', and both  $S_1$  and  $S_2$  are nonempty. Given two nodes  $x_1 \in S_1$  and  $x_2 \in S_2$  with minimum distance in  $\overline{G}[S]$ , let P' be a shortest  $x_1, x_2$ -anti-path in S, then  $(x_1, P', x_2, u', v, u, v', x_1)$  is an anti-hole that is even if and only if P' has odd length. But then  $v', x_1, P', x_2, u'$  is a chordless odd anti-path in  $S \cup \{u', v'\}$  and (ii) holds.

We may assume, then, that  $|P| \ge 5$  and  $|S| \ge 2$ .

**Claim 2:** Lemma 4 holds if S contains two nonadjacent nodes x, x' such that  $V(P) \setminus \{u, v\} \cup \{x, x'\}$  contains an odd chordless path P' between x and x'.

Assume, by contradiction, that such nodes a path P' between two nodes x and x' in S exists. If (iii) holds then we are done. Therefore x or x' must have a neighbor in the interior of P distinct from u' and v', so u or v has no neighbors in the interior of P', say, w.l.o.g., u. But then (u, x, P', x', u) is an odd hole, a contradiction.

**Claim 3:** The interior of P does not contain two adjacent nodes y, y' such that  $S \cup \{y, y'\}$  contains a chordless odd anti-path P' between y and y'.

Assume not. Then, since  $|P| \ge 5$ , either u or v is adjacent to neither y nor y', say, w.l.o.g., u. But then (u, y, P', y', u) is an odd anti-hole, a contradiction.

**Claim 4:** For every co-connected nonempty subset S' of S, and for every odd subpath P' = z, ..., z' of P such that z, z' are universal for S' and  $G[S' \cup V(P')]$  is a proper subgraph of G, we may assume that  $E_{S'}(P_{zz'})$  has odd cardinality.

Assume not. Then, by induction, either S' contains two nonadjacent nodes x, x' such that  $V(P_{zz'}) \setminus \{z, z'\} \cup \{x, x'\}$  contains an odd path between x and x', and we are done by Claim 2, or the interior of  $P_{zz'}$  contains two adjacent nodes y, y' such that  $S' \cup \{y, y'\}$  contains a chordless odd anti-path between y and y', contradicting Claim 3.

Claim 5: No node in int(P) is universal for S.

Assume not. Then P can be partitioned into proper subpaths  $P_1, ..., P_k$  such that, for every  $1 \leq i \leq k$ ,  $P_i = u_i, ..., u_{i+1}$ ,  $u_i$  is universal for S for every  $1 \leq i \leq k+1, u_1 = u, u_{k+1} = v$  and no intermediate node of  $P_i$  is universal for S. Since P is an odd path, there is an odd number of paths  $P_i, 1 \leq i \leq k$  of odd length and, since (i) does not hold,  $E_S(P)$  has even cardinality. Therefore there exists  $j, 1 \leq j \leq k$ , such that  $P_j$  is an odd path of length at least 3, but  $E_S(P_i) = 0$ , contradicting Claim 4.

Let  $s_1$ ,  $s_2$  be two nodes with maximum distance in G[S], and let P' be a shortest anti-path between  $s_1$  and  $s_2$  contained in S. Let  $S_1 = S \setminus s_1$ ,  $S_2 = S \setminus s_2$  and  $S' = S_1 \cap S_2$ . By our choice of  $s_1$  and  $s_2$ ,  $S_1$ ,  $S_2$  and S' are all co-connected.

Claim 6: P' has odd length.

By Claim 4,  $E_{S_i}(P)$  has odd cardinality, i = 1, 2, and, by Claim 5, no node universal for  $S_1$  is also universal for  $S_2$ . Therefore, since  $|P| \ge 5$ , there exist two nonadjacent nodes  $z_1$  and  $z_2$  in the interior of P such that  $z_1$  (resp.  $z_2$ ) is universal for  $S_1$  (resp.  $S_2$ ) but not for  $S_2$  (resp.  $S_1$ ). Since both  $z_1$  and  $z_2$  are universal for S', then, if P' has even length,  $(z_1, s_1, P', s_2, z_2, z_1)$  is an odd anti-hole, a contradiction.

Since  $E_{S_i}(P) \neq \emptyset$ , i = 1, 2, then P can be partitioned into proper subpaths  $P_1, \dots, P_k$  where, for every  $1 \leq i \leq k$ ,  $P_i = u_i, \dots, u_{i+1}, u_i$  is universal for  $S_1$  or  $S_2$  for every  $1 \leq i \leq k+1$ ,  $u_1 = u$ ,  $u_{k+1} = v$  and no node in  $int(P_i)$  is universal for  $S_1$  or  $S_2$ .

**Claim 7:** There exists  $j, 1 \leq j \leq k$ , such that  $P_j$  is an odd path of length at least 3,  $u_j$  is universal for  $S_1$  and  $u_{j+1}$  is universal for  $S_2$ .

We first show that for any  $i, 1 \leq i \leq k$ , if  $P_i$  has length 1 then  $u_i u_{i+1} \in E_{S_1}(P) \cup E_{S_2}(P)$ . Suppose otherwise. W.l.o.g.  $s_1$  is adjacent to  $u_i$  but not  $u_{i+1}$  and  $s_2$  is adjacent to  $u_{i+1}$  but not  $u_i$ . Since  $|P| \geq 5$ , then either u or v is adjacent to neither  $u_i$  nor  $u_{i+1}$ , say, w.l.o.g., u. But then, by Claim 6,  $(u, u_{i+1}, s_1, P', s_2, u_i, u)$  is an odd anti-hole, a contradiction. Since P is an odd path, then there is an odd number of paths  $P_i, 1 \leq i \leq k$  of odd length. By Claim 4,  $E_{S_i}(P)$  has odd cardinality for i = 1, 2. By Claim 5,  $E_{S_1}(P) \cap E_{S_2}(P) = \emptyset$ , so  $E_{S_1}(P) \cup E_{S_2}(P)$  has even cardinality. Therefore there exists  $j, 1 \leq j \leq k$ , such that  $P_j$  is an odd path of length at least 3. If both  $u_j$  and  $u_{j+1}$  are universal for  $S_1$  (resp.  $S_2$ ), then by Claim 4,  $E_{S_1}(P_j)$  (resp.  $E_{S_1}(P_j)$ ) has odd cardinality so, since  $|P_j| \geq 3$ , there is a node in the interior of  $P_j$  that is universal for  $S_1$  (resp.  $S_2$ ), a contradiction. Hence  $P_j$  satisfies Claim 8.

Claim 8: Lemma 4 holds if |S| = 2.

If |S| = 2 then, in the odd path  $P_j$  of Claim 7,  $u_j$  is adjacent to  $s_2$ , and  $u_{j+1}$  is adjacent to  $s_1$ , and no node in  $int(P_j)$  is adjacent to  $s_1$  or  $s_2$ . Since G has no odd hole,  $s_1$  is not adjacent to  $u_j$  and  $s_2$  is not adjacent to  $u_{j+1}$ . But then  $s_2, u_j, P_j, u_{j+1}, s_1$  is an odd path and we are done by Claim 2.

Claim 9: S is a stable set.

Consider the odd path  $P_j$  of Claim 7. Since  $S' \neq \emptyset$ , then by Claim 4, there is an odd number of edges in  $P_j$  that see S'. Hence, since  $|P_j| \ge 3$ , there exists a node z in the interior of  $P_j$  that is universal for S'. If S is not a stable set, P' is an odd anti-path of length at least 3, therefore  $(z, s_1, P', s_2, z)$  is an odd anti-hole, a contradiction.

Let  $s_1, s_2, s_3 \in S$  and let  $S_i = S \setminus s_i, i = 1, 2, 3$ .

By Claim 4,  $E_{S_i}(P)$  is odd, for i = 1, 2, 3, and, by Claim 5, given  $e \in E_{S_i}(P)$ ,  $e' \in E_{S_i}(P)$ , for  $1 \le i < j \le 3$ , e and e' have no endnode in common,

hence there must be some  $k \in \{1, 2, 3\}$  and an edge in  $yy' \in E_{S_k}(P)$  such that  $\{y, y'\} \cap \{u', v'\} = \emptyset$ .

Assume y is closer to u in P than y'. Let z be the neighbor of  $s_k$  in  $P_{uy}$ closest to y and z' be the neighbor of  $s_k$  in  $P_{y'v}$  closest to y'. By Claim 5,  $y \neq z$  and  $y' \neq z'$ .  $P_{zz'}$  is even, otherwise  $(s_k, z, P_{zz'}, z', s_k)$  would be an odd hole, therefore either  $P_{zy}$  and  $P_{yz'}$  are both odd paths, or  $P_{zy'}$  and  $P_{y'z'}$ are both odd paths. Let  $w \in \{y, y'\}$  be such that  $P_{zw}$  and  $P_{wz'}$  are both odd paths. Since P is an odd path, then either  $P_{uw}$  or  $P_{wv}$  has even length. Assume, w.l.o.g., that  $P_{uw}$  is an even path. Let G' be the graph induced by S, together with v and the nodes of  $P_{uw}$ , plus a new edge wv.

Claim 10: G' is a Berge graph.

Assume not. Then G' contains either an odd hole or an odd anti-hole. If G' contains an odd hole H, then H must contain wv (otherwise H would be an odd hole in G). Since v is universal for S, H must contain exactly one node in S, and such node must be  $s_k$ , since any other node in S is adjacent to both w and v. The only hole in G' containing  $s_k$ , w and v is  $(z, P_{zw}, w, v, s_k, z)$ , which, by construction, is even. If G' contains an odd anti-hole H, then H contains, at most, two nodes in S, since S is a stable set, and at most four nodes in P, since every set of nodes of P with at least five elements contains a stable set of size 3. But then H is a 5-anti-hole, therefore H is also a 5-hole.

By construction, since  $P_{uw}$  and  $P_{wv}$  have both length at least 2, G' has a number of nodes strictly smaller than G, while  $P' = u, P_{uw}, w, v$  is an odd chordless path of length at least 3. Then, by induction, Lemma 4 holds for G'. Since, by Claim 5, there is no node in int(P') universal for S, then either there exist two nodes x and x' in S such that  $x, u', P_{u'w}, w, x'$  is a path, and we are done by Claim 2, or there exist two adjacent nodes t and t' in int(P')such that  $S \cup \{t, t'\}$  contains an odd anti-path, contradicting Claim 3.

The following is an easy consequence of Lemma 4.

**Lemma 5** Assume G is a Berge graph containing a co-connected set S and an odd chordless path P = u, u', ..., v', v disjoint from S of length at least 3 such that u, v are both universal for the set S. Furthermore, assume that  $G \setminus (S \cup V(P))$  contains a node w universal for S such that no intermediate node of P is adjacent to w. Then an odd number of edges of P see S.

*Proof:* Assume not. Then, by Lemma 4, either |P| = 3 and  $S \cup \{u', v'\}$  contains an odd anti-path Q between u' and v', or  $|P| \ge 5$  and there exist

two nonadjacent nodes x, x' in S such that  $x, u', P_{u'v'}, v', x', w$  is a chordless path. In the first case, w, u', Q, v', w is an odd anti-hole, and in the other case  $w, x, u', P_{u'v'}, v', x', w$  is an odd hole, a contradiction.

## 3 Definitions

A wheel, denoted by (H, v), is a graph induced by a hole H and a node  $v \notin V(H)$  having at least three neighbors in H. A wheel is *odd* if it contains an odd number of triangles. A wheel (H, v) is a *twin wheel* if v has exactly three neighbors in H and (H, v) contains exactly two triangles; the neighbor of v in H that is adjacent to all the other neighbors of v in H is said the *twin of* v *in* H. A wheel (H, v) is a *line wheel* if v has exactly four neighbors in H and (H, v) contains exactly two triangles and these two triangles have only the center v in common. A universal wheel is a wheel (H, v) where the center v is adjacent to all the nodes of H. A *triangle-free wheel* is a wheel containing no triangle. These four types of wheels are depicted in Figure 1, where solid lines represent edges and dotted lines represent paths. A *proper wheel* is a wheel that is not any of the above four types.

A  $3PC(x_1x_2x_3, y)$  is a graph induced by three chordless paths  $P^1 = x_1, \ldots, y, P^2 = x_2, \ldots, y$  and  $P^3 = x_3, \ldots, y$ , having no common nodes other than y and such that the only adjacencies between nodes of  $P^i \setminus y$  and  $P^j \setminus y$ , for  $i, j \in \{1, 2, 3\}$  distinct, are the edges of the clique of size three induced by  $\{x_1, x_2, x_3\}$ . Also, at most one of the paths  $P^1, P^2, P^3$  is an edge. We say that a graph G contains a  $3PC(\Delta, .)$  if it contains a  $3PC(x_1x_2x_3, y)$ for some  $x_1, x_2, x_3, y \in V(G)$ .

**Remark 6** Since both odd wheels and  $3PC(\Delta, \cdot)$ 's contain an odd hole, they are never contained in a Berge graph as an induced subgraph.

The following graphs will play an important role in this paper.

**Definition 7** A  $3PC(x_1x_2x_3, y_1y_2y_3)$  is a graph induced by three chordless paths  $P^1 = x_1, \ldots, y_1, P^2 = x_2, \ldots, y_2$  and  $P^3 = x_3, \ldots, y_3$ , having no common nodes and such that, for  $i, j \in \{1, 2, 3\}$  distinct,  $x_i$  is not adjacent to  $y_j$ and the only adjacencies between nodes of  $V(P^i) \setminus \{y_i\}$  and  $V(P^j) \setminus \{y_j\}$  are the edges of the clique of size three induced by  $\{x_1, x_2, x_3\}$  and the only adjacencies between nodes of  $V(P^i) \setminus \{x_i\}$  and  $V(P^j) \setminus \{x_j\}$ , for  $i, j \in \{1, 2, 3\}$ distinct, are the edges of the clique of size three induced by  $\{y_1, y_2, y_3\}$ . We



Figure 2: Connected diamonds

say that a graph G contains a  $3PC(\Delta, \Delta)$  if it contains a  $3PC(x_1x_2x_3, y_1y_2y_3)$ for some  $x_1, x_2, x_3, y_1, y_2, y_3 \in V(G)$ . We say that a  $3PC(x_1x_2x_3, y_1y_2y_3)$  is long if  $P_1$ ,  $P_2$  and  $P_3$  are not all of length 1.

**Definition 8** Connected diamonds consist of two node disjoint sets  $\{a_1, \ldots, a_4\}$ and  $\{b_1, \ldots, b_4\}$  each of which induces a diamond (the graph on four nodes with five edges) such that  $a_1a_4$  and  $b_1b_4$  are not edges, together with four chordless paths  $P^1, \ldots, P^4$  such that for  $i = 1, \ldots, 4$ ,  $P^i$  is a path between  $a_i$ and  $b_i$ . Paths  $P^1, \ldots, P^4$  are node disjoint and the only adjacencies between them are the edges of the two diamonds.

Let H be a hole and let  $x_1, x_2, x_3, y_1, y_2, y_3$  be distinct nodes of H such that  $x_2$  is adjacent to  $x_1$  and  $x_3$ , and  $y_2$  is adjacent to  $y_1$  and  $y_3$ . We say that (H, x, y) is a *double beetle* if x and y are not adjacent, x is adjacent to  $x_1, x_2, x_3$  and  $y_2$ , and y is adjacent to  $y_1, y_2, y_3$  and  $x_2$ . Note that a double beetle is a special case of connected diamonds.

**Definition 9** Given a graph G and  $e = uv \in E(G)$ , the graph G' obtained by subdividing e is the graph obtained from G by deleting the edge e and adding one node w adjacent only to u and v. Given two graphs G and G', G'is a subdivision of G if G' can be obtained from G by iteratively subdividing edges of G. We say that G' is a bipartite subdivision of G if G' is a bipartite graph that is a subdivision of G.

A class of graphs that will play an important role in this paper is the class of line graphs of bipartite subdivisions of  $K_4$  (the clique on four nodes). An example is depicted in Figure 3.

## 4 Hubs

Let H be a hole and  $N \subseteq V(H)$ . We say that two nodes of N are *consecutive* if at least one of the two subpaths of H joining them contains no node of N in its interior.

**Theorem 10** Let G be a Berge graph, H a hole of length at least 6, and S a co-connected set of nodes in  $G \setminus V(H)$ . One of the following holds:

(1) an even number of edges of H see S, or



Figure 3: Bipartite subdivision of a  $K_4$  and its line graph.

- (2) S contains nonadjacent nodes x, y such that (H, x) and (H, y) are twin wheels and exactly one edge of H sees both x and y or
- (3) S contains a node x with exactly 2 neighbors u and v in H, where u and v are adjacent.

*Proof:* The proof is by induction on |S| + |H|. When |S| = 1, the theorem is immediate, since we already observed that G cannot contain an odd wheel. We can therefore assume that S has at least 2 nodes. Also, by inductive hypothesis, for every co-connected set  $S' \subset S$ ,  $E_{S'}(H)$  is even, else (2) or (3) holds.

If  $|E_S(H)|$  is even, then we are done. Hence, assume that  $|E_S(H)|$  is odd and let  $uv \in E_S(H)$ .

**Claim 1**  $E_S(H) = \{uv\}$  and no other node in H is universal for S. Assume not, then there exists an odd chordless subpath  $P = x_1, \ldots, x_n$  of H such that  $|P| \ge 3$ ,  $x_1$  and  $x_n$  are both universal for S and no intermediate node of P is universal for S. Since P does not contain both u and v, let  $w \in \{u, v\} \setminus V(P)$ . Then the choice of S, P and w contradicts Lemma 5.

Let  $s_1$  and  $s_2$  be two nodes at maximum distance in  $\overline{G}[S]$ , and let P' be a shortest anti-path between  $s_1$  and  $s_2$  in S. Let  $S_1 = S \setminus s_1$ ,  $S_2 = S \setminus s_2$  and  $S' = S_1 \cap S_2$ . By our choice of  $s_1$  and  $s_2$ ,  $S_1$ ,  $S_2$  and S' are all co-connected. **Claim 2** P' has odd length.

Since  $E_{S_i}(H) \setminus \{uv\} \neq \emptyset$ , for i = 1, 2, and no node universal for  $S_1$  in  $V(H) \setminus \{u, v\}$  is also universal for  $S_2$ , then, since  $|H| \ge 6$ , there exist two

nonadjacent nodes  $z_1$  and  $z_2$  in  $V(H) \setminus \{u, v\}$  such that  $z_1$  (resp.  $z_2$ ) is universal for  $S_1$  (resp.  $S_2$ ) but not for  $S_2$  (resp.  $S_1$ ). Therefore, if P' has even length, then  $(z_1, s_1, P', s_2, z_2, z_1)$  is an odd anti-hole, a contradiction.

Let  $u_1, \ldots, u_{k+1}$  be all the nodes of H that are universal for  $S_1$  or  $S_2$  in the order they appear going from u to v in  $H \setminus uv$ . By definition,  $u_1 = u$ ,  $u_{k+1} = v$ . For every  $i, 1 \leq i \leq k$ , let  $P_i$  be the path from  $u_i$  to  $u_{i+1}$  in  $H \setminus uv$ . Obviously, for every  $i, 1 \leq i \leq k$ , no node in the interior of  $P_i$  is universal for  $S_1$  or  $S_2$ . Since  $E_{S_i}(H) \setminus \{uv\} \neq \emptyset$ , i = 1, 2, then  $k \geq 2$ .

**Claim 3** There exists  $j, 1 \leq j \leq k$ , such that  $P_j$  is an odd path of length at least 3,  $u_j$  is universal for  $S_1$  but not for  $S_2$  and  $u_{j+1}$  is universal for  $S_2$  but not for  $S_1$ .

For any  $i, 1 \leq i \leq k$ , if  $P_i$  has length 1 then  $u_i u_{i+1} \in (E_{S_1}(H) \cup E_{S_2}(H)) \setminus \{uv\}$ , otherwise we may assume, w.l.o.g., that  $s_1$  is adjacent to  $u_i$  but not  $u_{i+1}$  and  $s_2$  is adjacent to  $u_{i+1}$  but not  $u_i$ . Since  $|H| \geq 6$ , then either u or v is not adjacent to  $u_i$  and  $u_{i+1}$ , say, w.l.o.g., u. But then, by Claim 2,  $(u, u_{i+1}, s_1, P', s_2, u_i, u)$  is an odd anti-hole, a contradiction.

Since  $H \setminus uv$  is an odd chordless path, then there is an odd number of paths  $P_i$ ,  $1 \leq i \leq k$  of odd length. By Claim 1,  $E_{S_1}(H) \cap E_{S_2}(H) = \{uv\}$ , so  $E_{S_1}(H) \cup E_{S_2}(H) \setminus \{uv\}$  has even cardinality, therefore there exists j,  $1 \leq j \leq k$ , such that  $P_j$  is an odd path of length at least 3. If both  $u_j$  and  $u_{j+1}$  are universal for  $S_1$  (resp.  $S_2$ ), then by Lemma 5 applied to  $S_1$  (resp.  $S_2$ ),  $P_j$  and either node u or node v (since one of the two has no neighbor in the interior of  $P_j$ ),  $P_j$  has an odd number of edges that see  $S_1$  (resp.  $S_2$ ), so there is a node in the interior of  $P_j$  that is universal for  $S_1$  (resp.  $S_2$ ), a contradiction. Hence  $P_j$  satisfies Claim 3.

Let u', v' be, respectively, the neighbors of u and v in  $H \setminus uv$ .

Claim 4 Theorem 10 holds if |S| = 2.

Assume |S| = 2. Let  $P_j$  be the path defined in Claim 3. If  $u_j = u'$  and  $u_{j+1} = v'$ , then Theorem 10 (2) holds. Hence we may assume, w.l.o.g.,  $u' \neq u_j$ , but then  $(u, s_2, u_j, P_j, u_{j+1}, s_1, u)$  is an odd hole, a contradiction.

By Claim 4, we may assume  $|S| \ge 3$ 

Claim 5 S is a stable set.

Since  $S' \neq \emptyset$ , then by Lemma 5 applied to S',  $P_j$  and u, there is an odd number of edges in  $P_j$  that see S'. Hence there exists a node z in the interior of  $P_j$  that is universal for S'. If S is not a stable set, P' is an odd anti-path of length at least 3, therefore  $(z, s_1, P', s_2, z)$  is an odd anti-hole, a contradiction. Let  $s_1, s_2, s_3 \in S$  and let  $S_i = S \setminus s_i, i = 1, 2, 3$ .

Since  $E_{S_i}(H) \setminus \{uv\}$  has odd cardinality, i = 1, 2, 3, then, given  $e \in$  $E_{S_i}(H) \setminus \{uv\}, e' \in E_{S_i}(H) \setminus \{uv\}, \text{ for } 1 \leq i < j \leq 3, \text{ by Claim 1 } e \text{ and}$ e' have no endnode in common, hence there exists  $k \in \{1, 2, 3\}$  and an edge  $yy' \in E_{S_k}(P)$  such that  $\{y, y'\} \cap \{u', v'\} = \emptyset$ . For every pair s, t of nodes of H, let us denote by  $H_{st}$  the path between s and t in  $H \setminus uv$ . Assume y is closer to u in  $H \setminus uv$  than y'. Let z be the neighbor of  $s_k$  closest to y in  $H_{uy}$  and z' be the neighbor of  $s_k$  closest to y' in  $H_{y'v}$ . By Claim 1,  $y \neq z$ and  $y' \neq z'$ .  $H_{zz'}$  is even, otherwise  $(s_k, z, H_{zz'}, z', s_k)$  would be an odd hole, therefore either  $H_{zy}$  and  $H_{yz'}$  are both odd paths, or  $H_{zy'}$  and  $H_{y'z'}$  are both odd paths. Let  $w \in \{y, y'\}$  be such that  $H_{zw}$  and  $H_{wz'}$  are both odd paths. Since H is an even hole, then either  $H_{uw}$  or  $H_{wv}$  has even length. Assume, w.l.o.g., that  $H_{uw}$  is an even path. Let G' be the graph induced by S together with v and  $H_{uw}$ , plus a new edge wv. Let  $H' = (u, H_{uw}, w, v, u)$ ; H' is an even hole in G'. In particular, H' must have length at least 6, otherwise zis adjacent to u, w is adjacent to z and, given any node s in  $S_k$  that is not adjacent to z,  $(s, w, z, s_k, v, s)$  is a 5-hole in G.

#### Claim 6 G' is a Berge graph.

Assume not. Then G' contains either an odd hole or an odd anti-hole. If G' contains an odd hole Q, then Q must contain wv, otherwise Q would be an odd hole in G. Also, Q must contain a node in S, otherwise Q = H' that is an even hole. Since every node in  $S_k$  is adjacent to both w and v, Q must contain exactly one node in S, namely  $s_k$ . The only hole in G' containing  $s_k$ , w and v is  $(z, P_{zy}, w, v, s_k, z)$ , which, by construction, is even. If G' contains an odd anti-hole Q, then Q contains, at most, two nodes in S, since S is a stable set, and at most four nodes in H', since every subset of nodes of H' with at least five elements contains a stable set of size 3. But then Q is a 5-anti-hole, therefore Q is also a 5-hole, a contradiction.

Since, by construction,  $H_{uw}$  and  $H_{wv}$  have both length at least 2, H' has length strictly smaller than H. Therefore, by induction, Theorem 10 holds in G' for H' and S. Since  $E_S(H') = \{uv\}$  and every node of S has at least three neighbors in H', then the only possibility is that z is adjacent to u and there exists a node s in  $S_k$  whose only neighbors in H' are u, v and w. But then, in G,  $(z, H_{zw}, w, s, v, s_k, z)$  is an odd hole, a contradiction  $\Box$ 

Note that an edge set C of H of even cardinality induces a bicoloring of the nodes of H: two nodes of H are colored with distinct colors if and only if the subpaths of H connecting them contain an odd number of edges in C.

**Definition 11** Given a Berge graph G, a hub of G is a pair (H, S) where H is a hole of G of length at least 6 and S is a co-connected set in  $G \setminus V(H)$  that sees a positive even number of edges of H. A sector of a hub (H, S) is a maximal subpath of H containing no edge of  $E_S(H)$ .

**Remark 12** Let G be a Berge graph and (H, S) a hub of G. Then the endnodes of a sector are endnodes of edges of  $E_S(H)$  and every sector of (H, S) has even length.

*Proof:* By maximality in the definition of sector, every endnode of a sector must be an endnode of an edge in  $E_S(H)$ . Assume there exists a sector  $P = x_1, \ldots, x_n$  of (H, S) of odd length. Let w be the endnode of some edge in  $E_S(H)$  distinct from  $x_1$  and  $x_n$ . Since both  $x_1$  and  $x_n$  are universal for S and P has length at least 3, then by Lemma 5 applied to S, P and w, there is an odd number of edges of P that sees S, a contradiction.

**Corollary 13** Let G be a Berge graph and (H, S) be a hub of G. Let  $y \in V(G) \setminus (V(H) \cup S)$  be a node that sees an odd number of edges in a sector of (H, S). Assume  $S \cup y$  is co-connected. Then

- (i) y has exactly two neighbors in H and they are adjacent or
- (ii) There exists  $x \in S$  not adjacent to y such that (H, x) and (H, y) are twin wheels and exactly one edge of H sees both x and y or
- (iii) S contains a node x not adjacent to y such that (H, y) and (H, x) are both line wheels and no edge of H sees both x and y or
- (iv) |H| = 6, (H, y) is a line wheel and  $S \cup y$  contains an odd chordless anti-path Q of length at least 3 between y and a node x such that (H, x)is a line wheel, no edge of H sees both x and y and every intermediate node of Q is adjacent to every node in H.

*Proof:* If y has exactly two neighbors in H then conclusion (i) holds. Assume then that y has at least 3 neighbors in H. If  $E_{S\cup y}(H)$  has odd cardinality, then, by Theorem 10, conclusion (ii) holds. So  $E_{S\cup y}(H)$  has even cardinality. Since there is an even number of edges of H that sees y and y sees an odd number of edges in some sector of (H, S), then there are at least 2 sectors  $P = x_1, \ldots, x_h$  and  $P' = x'_1, \ldots, x'_k$  of (H, S) such that an odd number of edges of P and P', respectively, sees y. Let  $y_1, y_2$ , (resp.  $y'_1, y'_2$ ) be the neighbors of y in P (resp. P') closest to  $x_1$  and  $x_h$  (resp.  $x'_1$  and  $x'_k$ ) respectively.

Since an odd number of edges of P sees y, then  $P_{x_1y_1}$  and  $P_{y_2x_h}$  have length of distinct parity. We can therefore assume that  $P_{x_1y_1}$  has odd length and  $P_{y_2x_h}$  has even length. Analogously, assume that  $P'_{x'_1y'_1}$  has odd length and  $P_{y'_2x'_h}$  has even length.

If  $y_1$  and  $y_2$  are nonadjacent, then  $F = x_1, P_{x_1y_1}, y_1, y, y_2, P_{y_2x_h}, x_h$  is an odd path so, by Lemma 5 applied to S, F and  $x'_1$ , F has an odd number of edges that see S, contradicting either the definition of sector or the assumption that  $S \cup y$  is co-connected. Hence  $y_1y_2$  is an edge and, analogously,  $y'_1y'_2$  is an edge. Let now  $F = x_1, P_{x_1y_1}, y_1, y, y'_2, P_{y'_2x'_k}$ . If F is a chordless path then F is odd and by Lemma 5 applied to S, F and  $x'_1$ , F has an odd number of edges that see S, a contradiction. Therefore F is not a chordless path, but then  $x_1$  must be adjacent to  $x'_k$ . Analogously, by repeating the previous argument for  $F' = x'_1, P_{x'_1y'_1}, y'_1, y, y_2, P_{y_2x_h}, x_h$  must be adjacent to  $x'_1$ . Therefore (H, y)is an L-wheel.

#### **Case 1:** |H| > 6

Then, w.l.o.g.,  $H' = (x'_1, P_{x'_1y'_1}, y'_1, y, y_2, P_{y_2x_h}, x_h, x'_1)$  is a hole of length at least 6. Since  $E_S(H') = \{x'_1x_h\}$ , Theorem 10 applies.

**Case 1.1:** Conclusion (3) of Theorem 10 holds.

Then there exists a node x in S such that the only neighbors of x in H' are  $x_h$  and  $x'_1$ . Since x sees an odd number of edges in a sector of (H, y), then, by the previous argument, (H, x) is an L-wheel and (iii) holds.

**Case 1.2:** Conclusion (2) of Theorem 10 holds.

Then there exists two nodes x and x' in S such that (H', x) and (H', x')are both twin wheels. Let w, w' be, respectively, the neighbors of x and x' in  $V(H') \setminus \{x_h, x'_1\}$  and let F be the path between w and w' induced by  $V(H') \setminus \{x_h, x'_1\}$ . Since F has odd length,  $(x_1, x, w, F, w', x', x_1)$  is an odd hole, a contradiction.

#### **Case 2:** |H| = 6

Then  $y_2 = x_h$  and  $y'_2 = x'_k$ . Since  $y_1$  and  $y'_1$  are not universal for S and  $S \cup y$  is co-connected, let Q be a shortest anti-path in  $S \cup y$  from y to a node x that is not adjacent to both  $y_1$  and  $y'_1$ . Assume, w.l.o.g., that x is not adjacent to  $y_1$ , then  $(y, Q, x, y_1, x'_1, y)$  is an anti-hole, therefore Q must be an odd anti-path. If x is adjacent to  $y'_1$ , then  $(y, Q, x, y_1, x'_1, y)$  is a line wheel. If Q has length 1 then (iii) holds, else (iv) holds.

## 5 Connections from blue to red sectors of a hub

Let P be a connected subgraph of  $G \setminus (H \cup S)$ . The *attachments* of P to H are the nodes of H adjacent to at least one node of P.

**Theorem 14** Let (H, S) be a hub of a Berge graph G. Let  $P = x_1, \ldots, x_n$ be a minimal chordless path in  $G \setminus (V(H) \cup S)$  containing no node that is universal for S, such that  $x_1$  has a blue neighbor in H and  $x_n$  has a red neighbor w.r.t. the bicoloring induced by  $E_S(H)$  (n = 1 is allowed). If there exist consecutive attachments of P with distinct colors that are not adjacent, then one of the following holds.

- (a) There exists  $y \in S$  such that  $V(H) \cup V(P) \cup \{y\}$  induces the line graph of a bipartite subdivision of  $K_4$ .
- (b) n = 1, |H| = 6,  $(H, x_1)$  is a line wheel and  $S \cup x_1$  contains a chordless odd anti-path Q of length at least 3 between  $x_1$  and a node  $y \in S$  such that (H, y) is a line wheel, no edge of H sees both  $x_1$  and y and every intermediate node of Q is adjacent to every node in H.
- (c) There exists  $y \in S$  such that  $V(H) \cup V(P) \cup \{y\}$  induces connected diamonds.
- (d) n = 1 and there exists  $y \in S$  nonadjacent to  $x_1$  such that  $(H, x_1)$  and (H, y) are twin wheels and exactly one edge of H sees both  $x_1$  and y.
- (e) There exists  $y \in S$  such that (H, y) is a twin wheel, no node of P is a neighbor of y,  $x_1$  is adjacent to the twin of y in H and no other node in H while  $x_n$  is not adjacent to both the other neighbors of y in H.
- (f) n = 1, H contains a subpath u, z, w, z', u' such that  $E_S(H) = \{wz, wz'\}$ ,  $x_1$  is adjacent to u, w and u' but not z and  $z', S \cup x_1$  contains a chordless odd anti-path Q of length at least 3 between  $x_1$  and a node  $y \in S$  such that y is nonadjacent to u and u' and every intermediate node of Q is adjacent to both u and u'.
- (g) n = 1, H contains a subpath w, z, u, z', w' such that wz and w'z' are edges of  $E_S(H)$ ,  $x_1$  is adjacent to u, w and w' but not z and  $z', S \cup x_1$ contains an even anti-path Q between  $x_1$  and a node  $y \in S$  such that

y is nonadjacent to u and every intermediate node of Q is adjacent to u. Furthermore, every node in  $V(H) \setminus \{z, z'\}$  that is universal for S is adjacent to  $x_1$ .

- (h) n > 1, II contains a subpath w, z, u, z', w' such that wz and w'z' are edges of  $E_S(H)$ ,  $x_1$  is adjacent w and w' but not u, z and z', while  $x_n$  is adjacent to u but not w, z, w' and z'. Furthermore S contains two nonadjacent nodes y and y' such that the only neighbors of y in  $V(P) \cup \{w, z, u, z', w'\}$  are u, z, z', w, w' while the only neighbors of y' in  $V(P) \cup \{w, z, u, z', w'\}$  are  $x_1, z, z', w, w'$ .
- (k) n > 1, H = (v, w, z, u, z', w', v),  $E_S(H) = \{wz, w'z'\}$ ,  $x_1$  is adjacent only to v in H and  $x_n$  is adjacent only to u in H. Furthermore, Scontains two nonadjacent nodes y and y' such that y and y' are adjacent to every node in H except v and u, respectively, and no node in P is adjacent to y or y'.

*Proof:* Note that, by the minimality assumption on P, no intermediate node of P has a neighbor in H.

**Case 1:**  $x_1$  or  $x_n$  sees an odd number of edges in some sector of (H, S).

Assume, w.l.o.g., that  $x_1$  sees an odd number of edges in some sector of (H, S): then conclusion (i), (ii), (iii) or (iv) of Corollary 13 holds. If conclusion (ii) of Corollary 13 holds, then (d) holds. If conclusion (iii) of Corollary 13 holds, n = 1 and there exists y in S non adjacent to  $x_1$  such that  $(H, x_1)$  and (H, y) are line wheels and no edge in H sees both  $x_1$  and y, but then one can verify that  $V(H) \cup \{x_1, y\}$  is the line graph of a bipartite subdivision of  $K_4$ , so (a) holds. If conclusion (iv) of Corollary 13 holds, then (b) holds. Therefore we can assume that conclusion (i) of Corollary 13 holds and  $x_1$  has exactly two neighbors u, u' in H, u and u' are adjacent and they are both blue. If  $x_n$  has exactly one neighbor t in H, then there is a  $3PC(x_1uu', t)$ . If  $x_n$  has two neighbors in H that are not adjacent, then there is a  $3PC(x_1uu', x_n)$ . Hence  $x_n$  has exactly two neighbors v and v' in H and they are adjacent and both red. Assume that u and v are consecutive attachments of P and u', v' are consecutive attachments of P. W.l.o.g., u and v are non adjacent. Let  $H_{uv}$  and  $H_{u'v'}$  be, respectively, the paths between u and v and between u' and v' in H such that no intermediate node of  $H_{uv}$  or  $H_{u'v'}$  is an attachment of P. Since u and v are nonadjacent, then  $H' = (u, H_{uv}, v, x_n, P, x_1)$  is a hole of length at least 6 and, since u and v

have distinct colors and no node in P is universal for S, an odd number of edges of H' see S. Also  $H'' = (u', H_{u'v'}, v', x_n, P, x_1)$  is a hole (possibly of length 4) and an odd number of edges of H'' sees S. By Theorem 10, exactly one edge wz of H' and one edge of w'z' of H'' sees S and one of the following cases holds.

**Case 1.1:** There exists a node  $y \in S$  such that y has only two neighbors in H'.

But then y sees an odd number of edges in  $H_{u'v'}$ , so y must see exactly one edge in  $H_{u'v'}$ , otherwise  $V(H_{u'v'}) \cup V(P) \cup \{y\}$  would induce an odd wheel. But then (H, y) is a line wheel and one can verify that  $V(H) \cup V(P) \cup \{y\}$ induces the line graph of a bipartite subdivision of  $K_4$ , hence (a) holds.

**Case 1.2:** There exist non adjacent nodes  $y, y' \in S$  such that (H', y) and (H', y') are twin wheels.

Let t and t' be the neighbors of y and y', respectively, in  $V(H') \setminus \{w, z\}$ . If u' and v' are nonadjacent, then at least one node among w' and z' has no neighbor in P, say w', but then  $(V(H') \cup \{w', y, y'\}) \setminus \{w, z\}$  induces an odd hole, a contradiction. In particular, w.l.o.g. t = u and t' = v, else (H, y) or (H, y') is an odd wheel. Since H' is even, P must be odd, therefore  $(y, u, x_1, P, x_n, v', y)$  is an odd hole, a contradiction.

**Case 2:** Both  $x_1$  and  $x_n$  see an even number of edges in every sector of (H, S).

Let u and v be two consecutive, nonadjacent attachments of P with distinct colors in the bicoloring of H induced by  $E_S(H)$ . Assume, w.l.o.g., vis adjacent to  $x_1$  and u to  $x_n$ . Let  $H_{uv}$  be a subpath of H between u and v containing no attachments of P except u and v. Since u and v have distinct colors,  $H_{uv}$  contains an odd number of edges of  $E_S(H)$ , therefore the hole  $H' = (x_1, P, x_n, u, H_{uv}, v, x_1)$  has an odd number of edges that see S, otherwise P would contain some node universal for S. By Theorem 10, H'must contain a unique edge of  $E_S(H)$ , say edge zw, and no node universal for S except z and w. Assume, w.l.o.g., that z is one endnode of the sector Z containing u, and let z' be the other endnode of Z. Let w' be the neighbor of z' in  $V(H) \setminus V(Z)$ ; hence  $z'w' \in E_S(H)$ . Since H' is an even hole,  $H_{uv}$ has length of the same parity as P. Since u and v are nonadjacent, we may assume, w.l.o.g, that u and z are distinct. Let  $H_{uz}$  be the path between uand z in  $H_{uv}$  and  $H_{wv}$  be the path between w and v in  $H_{uv}$ .

Case 2.1: w = w'.

Then w = w' = v and  $E_S(H) = \{wz, wz'\}.$ 

**Case 2.1.1:** There exists a node  $y \in S$  whose only neighbors in H' are w and z.

If (H, y) is a twin wheel, then case (e) applies. If (H, y) is not a twin wheel, y has at least a neighbor in  $V(H) \setminus \{w, z, z'\}$ . If u is the only neighbor of  $x_n$  in Z, then G contains a 3PC(zwy, u), hence  $x_n$  has a neighbor in Zdistinct from u. Furthermore, since  $x_n$  sees an even number of edges in Z,  $x_n$  has a neighbor in Z that is not adjacent to u. If y has a neighbor in Zthat is not adjacent to u, then there is a  $3PC(zwy, x_n)$ , hence y has a unique neighbor t in Z and t is adjacent to u. Furthermore, t is adjacent to  $x_n$ , else there is a 3PC(zwy, u). Let u' be the neighbor of  $x_n$  in Z closest to z', then  $u' \neq t$ . If u' is not adjacent to t, then there is a  $3PC(x_ntu, y)$ . So u' is adjacent to t and hence  $V(H) \cup V(P) \cup \{y\}$  induces connected diamonds, so conclusion (c) holds.

**Case 2.1.2:** Every node in S has at least 3 neighbors in H'.

If  $|H'| \ge 6$  then, by Theorem 10, S contains two nonadjacent nodes yand y' such that (H', y) and (H', y') are twin wheels and wz is the only edge of H' that sees both y and y'. But then  $(V(H') \cup \{y, y'\}) \setminus \{w, z\}$  induces an odd path R between y and y' and (z', y, R, y', z') is an odd hole unless z' is adjacent to  $x_n$ . But then, since  $x_n$  sees an even number of edges in Z,  $H_{zu}$ must have even length. W.l.o.g. assume that y is not adjacent to  $x_1$ , then  $(V(H_{uz}) \cup \{y, z', x_n\}) \setminus \{z\}$  induces an odd hole, a contradiction.

Hence |H'| = 4, so u is adjacent to z and n = 1. Let u' be the neighbor of  $x_1$  in Z closest to z'. Then, since  $x_1$  sees an even number of edges in Zand u is adjacent to z, u' and z' have odd distance in H. By repeating the previous argument on the hole H'' containing w, u' and  $x_1$  in  $V(Z) \cup \{x_1, w\}$ instead of H', we argue that u' and z' must be adjacent. Since u and u' are not universal for S, let Q be a shortest possible anti-path in  $S \cup x_1$  between  $x_1$  and a node y not adjacent to both u and u'. Assume, w.l.o.g, that y is not adjacent to u. Q must have odd length, or else  $(x_1, Q, y, u, z', x_1)$  is an odd anti-hole. Moreover, since every node in S has at least 3 neighbors in H', Qhas length at least 3. Finally, if u' is adjacent to y, then  $(x_1, Q, y, u, u', z, x_1)$ is an odd anti-hole, a contradiction. Hence conclusion (f) holds.

Case 2.2:  $w \neq w'$ .

Note that, since w' is universal for S and distinct from w and z, then w' is not in  $H_{uv}$ . Let s be the neighbor of  $x_n$  in Z closest to z' and let  $H_{sz'}$  be the path between s and z' in Z. Since  $x_n$  sees an even number of edges in Z and

 $H_{zu}$  has length of the same parity as  $H_{sz'}$ . Let  $F = w, H_{wv}, v, x_1, P, s, H_{sz'}, z'$ . Since H' is an even hole and  $H_{zu}$  has the same length as  $H_{sz'}$ , F is an odd path between w and z'. If z is not adjacent to s then, by Lemma 5 applied to S, F and z, an odd number of edges of F see S, a contradiction. Hence u is the unique neighbor of  $x_n$  in Z and it is adjacent to z. Also, given any node t in  $V(H) \setminus \{z, z', w\}$  universal for S, if t is not an attachment of Pthen, by Lemma 5 applied to S, F and t, an odd number of edges of F see S, a contradiction. In particular, w' must be adjacent to  $x_1$  or to v.

If w' is adjacent to v then  $F' = w', v, x_1, P, x_n, u, z$  is an odd path, therefore, by a similar argument, z' is adjacent to u and w is also adjacent to v(since  $x_1$  sees an even number of edges in every sector, hence w cannot be adjacent to  $x_1$ ). Therefore |H| = 6 and, since F' must have length at least 5, by Lemma 4 there exists two nonadjacent nodes nodes y and and y' in Ssuch that y is adjacent to every node in H except v, y' is adjacent to every node in H except u and neither y nor y' has a neighbor in P, hence (k) holds.

If w' is adjacent to  $x_1$  then  $F' = w', x_1, P, x_n, u, z$  is an odd path, therefore, by the usual argument, z' is adjacent to u and w is adjacent to  $x_1$ . If |F'| = 3, then n = 1 and, by Lemma 4, there exists an odd anti-path  $x_1, Q, y, u$  between  $x_1$  and u in  $S \cup \{u, x_1\}$ , hence case (g) holds. If  $|F'| \ge 5$ , then by Lemma 4 S contains two nonadjacent nodes y and y' such that y is adjacent to  $x_1, z,$ z', w, w' an no other node in  $V(P) \cup \{w, z, u, z', w'\}$  while y' is adjacent to u,z, z', w, w' an no other node in  $V(P) \cup \{w, z, u, z', w'\}$ , hence case (h) holds.

Given a hub (H, S) and an edge  $ab \in E_S(H)$ , an *ear on ab* (with respect to (H, S)) is a chordless path  $P = x_1, \ldots, x_n$  in  $G \setminus (V(H) \cup S)$  such that  $x_1$  is adjacent to  $a, x_n$  is adjacent to b, no node in  $V(H) \setminus \{a, b\}$  has a neighbor in P, no node of P is universal for S, and P is minimal with these properties.

**Theorem 15** Let (H, S) be a hub of a Berge graph G where S is maximal with the property that (H, S) is a hub. Let  $P = x_1, \ldots, x_n$  be a minimal chordless path in  $G \setminus (H \cup S)$  containing no node universal for S such that  $x_1$  has a blue neighbor in H and  $x_n$  has a red neighbor (n = 1 is allowed). If every pair of consecutive attachments of P with distinct colors are adjacent, then one of the following holds.

- (a) P is an ear on some edge of  $E_S(H)$ .
- (b) n > 1, there exist two adjacent edges ab, bc of  $E_S(H)$  such that b is the only neighbor of  $x_1$  in H and  $x_n$  is adjacent to a, c and not to b.

Moreover, if  $E_S(H) \supseteq \{ab, bc\}$ , then no node of P has a neighbor in  $V(H) \setminus \{a, b, c\}$ .

(c) n > 1,  $E_S(H)$  contains at least two nonadjacent edges,  $x_1$  is adjacent to all the blue endnodes of the edges of II that see S (and possibly to other blue nodes of H),  $x_n$  is adjacent to all the red endnodes of the edges of H that see S (and possibly to other red nodes of H). If n > 2, then there exist nonadjacent  $y, z \in S$  such that y is adjacent to  $x_1$  and to no other node of P, and z is adjacent to  $x_n$  and to no other node of P. If n = 2, then  $S \cup \{x_1, x_2\}$  contains an odd anti-path between  $x_1$ and  $x_2$ .

*Proof:* Note that, by the minimality assumption on P, no intermediate node of P has a neighbor in H. Let a and b be two consecutive attachments of P with distinct colors. Then, by assumption, a and b are adjacent and  $ab \in E_S(H)$ . Assume, w.l.o.g., that a is adjacent to  $x_n$  and b is adjacent to  $x_1$ . Let c be the neighbor of b in  $V(H) \setminus \{a\}$ . If P has no neighbor in  $V(H) \setminus \{a, b\}$ , then P is an ear of ab and (a) occurs. Therefore we may assume, w.l.o.g., that  $x_n$  has a neighbor in  $V(H) \setminus \{a, b\}$ . Note that n > 1, otherwise either  $S \cup x_1$ sees a positive even number of edges of H, contradicting the maximality of S, or ab is the only edge of H that sees  $S \cup x_1$ , and by Theorem 10 there exists  $y \in S$  nonadjacent to  $x_1$  such that  $(H, x_1)$  and (H, y) are twin wheels and exactly one edge of H sees both  $x_1$  and y, thus contradicting the assumption that every two consecutive attachments of P with distinct colors are adjacent. Therefore  $x_1$  has only blue neighbors and  $x_n$  has only red neighbors. If  $x_n$ sees an odd number of edges in some sector of (H, S) then, by Corollary 13, the only neighbors of  $x_n$  in H are a and the neighbor d of a in  $V(H) \setminus \{b\}$ . If  $x_1$  has no neighbor in  $V(H) \setminus \{b\}$ , then G contains a  $3PC(x_nad, b)$ . If  $x_1$  has two nonadjacent neighbors in H, then G contains a  $3PC(x_nad, x_1)$ . Therefore  $x_1$  is adjacent to b, c and no other node in H. But then c and d are consecutive, non adjacent attachments of P with distinct colors in the bicoloring of H induced by  $E_S(H)$ , a contradiction. Therefore  $x_n$  sees an even number of edges in every sector of (H, S) and, by the same argument, also  $x_1$  sees an even number of edges in every sector of (H, S).

We may assume that  $x_n$  has at least as many neighbors in H as  $x_1$  does. If  $E_S(H) = \{ab, bc\}$  then (b) holds. Next we show that if  $x_n$  has no neighbor in  $H \setminus \{a, c\}$ , then (b) holds. Suppose that  $x_n$  has no neighbor in  $H \setminus \{a, c\}$ . Then  $x_n$  is adjacent to c. If  $x_1$  has no neighbors in  $H \setminus b$  then (b) holds.

Otherwise,  $x_1$  has exactly two neighbors in H, b and say d. Since all pairs of consecutive attachments of P having distinct colors are adjacent, then a, d and c, d are adjacent, hence |H| = 4, contradicting the assumption that (H, S) is a hub. Now we may assume that (b) does not hold, hence there exists a red sector  $Z = z_1, \ldots, z_k$  of (H, S) such that  $\{a, c\} \neq \{z_1, z_k\}$  and such that  $x_n$  has a neighbor in  $V(Z) \setminus \{a, c\}$ . Assume, w.l.o.g, that  $z_1 \notin \{a, c\}$ and  $x_n$  has a neighbor in  $V(Z) \setminus \{z_k\}$ . Let  $z_i$  be the neighbor of  $x_n$  of lowest index in Z, and let  $H_{z_1z_i}$  be the subpath between  $z_1$  and  $z_i$  in Z. Note that i < k. Since  $x_n$  sees an even number of edges in every sector of (H, S) and  $x_n$ has only red neighbors in H, then  $H_{z_1z_i}$  has even length (since  $x_n$  is adjacent to a) and also  $z_k$  and  $z_i$  have even distance in Z, hence they are not adjacent. Moreover,  $H' = (a, b, x_1, P, x_n, a)$  is an even hole, therefore P is an odd path. But then  $F = b, x_1, P, x_n, z_i, H_{z_1 z_i}, z_1$  is an odd chordless path. If there exists a node w universal for S in  $V(H) \setminus \{a, b, z_1\}$  that has no neighbor in the interior of F, then Lemma 5 applied to S, F and w implies that there exists an odd number of edges in F that see S, a contradiction. Therefore every node universal for S in  $V(H) \setminus \{a, b, z_1\}$  is adjacent either to  $x_1$  or to  $x_n$ . Let t be the unique blue neighbor of  $z_1$  in H. Note that t is adjacent to  $x_1$ . Since t and  $z_i$  are consecutive attachments of P, they must be adjacent. So  $x_n$  is adjacent to  $z_1$ . Hence every node of H that is universal for S must be adjacent to  $x_1$  or  $x_n$ . In particular,  $x_1$  is adjacent to all the blue endnodes of the edges of H that see S,  $x_n$  is adjacent to all the red endnodes of the edges of H that see S. If n > 2, then F has length at least 5 and by Lemma 4 there exist nonadjacent  $y, z \in S$  such that y is adjacent to  $x_1$  and to no other node of P, and z is adjacent to  $x_n$  and to no other node of P. If n = 1, then |F| = 3 and, by Lemma 4,  $S \cup \{x_1, x_2\}$  contains an odd anti-path between  $x_1$  and  $x_2$ . So conclusion (c) holds. 

In the bicoloring of H induced by  $E_S(H)$ , we say that a node u of H is an *inner* blue (resp. red) node if both neighbors of u in H are blue (resp. red).

**Theorem 16** Let (H, S) be the hub of a Berge graph G. Assume that S is a maximal set such that (H, S) is a hub with the further property that S does not contain any center of a twin wheel w.r.t. H. Let  $P = x_1, \ldots, x_n$  be a minimal chordless path in  $G \setminus (V(H) \cup S)$  containing no node universal for S such that  $x_1$  has a red neighbor, no other node of P has a red neighbor and

 $x_n$  has a blue neighbor b in H so that neither of the neighbors of b in H is a red neighbor of  $x_1$ . Then one of the following holds:

- (a) P has two consecutive attachments of different colors that are nonadjacent, and P is of one of the types in Theorem 14 (a)-(c) or (f)-(k).
- (b) There exist two adjacent edges  $ab_1$ ,  $ab_2$  of  $E_S(H)$  such that a is the only red neighbor of  $x_1$  in H and at least one node of P is adjacent to both  $b_1$  and  $b_2$ . If  $E_S(H) \supseteq \{ab_1, ab_2\}$  or if S contains a node s with no neighbors in P, then the path  $Q = a, x_1, \ldots, x_n$  contains an odd number of edges that see both  $b_1$  and  $b_2$ .
- (c) n > 1,  $E_S(H)$  contains at least two nonadjacent edges,  $x_1$  is adjacent to all the red endnodes of the edges of H that see S and the node  $x_j$ of lowest index adjacent to some blue node is adjacent to all the blue endnodes of the edges of H that see S. If j > 2, then S contains two nonadjacent nodes y and z such that y is adjacent to  $x_1$  and to no other node of  $P_{x_1x_j}$ , and z is adjacent to  $x_j$  and to no other node of  $P_{x_1x_j}$ . If j = 2, then  $S \cup \{x_1, x_2\}$  contains an odd chordless anti-path between  $x_1$  and  $x_2$ .

Note that every path  $P = x_1, \ldots, x_n$  such that  $x_1$  has a red neighbor and  $x_n$  has an inner blue neighbor contains a subpath as in the hypothesis of Theorem 16.

*Proof:* Let  $x_j$  be the node of P of lowest index having a blue neighbor. If the path  $P_{x_1x_j}$  has consecutive attachments of distinct colors that are not adjacent, then  $P_{x_1x_j}$  satisfies the hypothesis of Theorem 14, hence one the cases (a)-(c) or (f)-(k) of Theorem 14 apply (cases (d) and (e) cannot occur since S does not contain any center of a twin wheel w.r.t. H). Since in any of these cases  $x_j$  has a blue neighbor that is not adjacent to any red neighbor of  $x_1$  in H, then j = n and case (a) holds.

Hence we may assume that every pair of consecutive attachments with distinct colors of  $P_{x_1x_j}$  are adjacent, so case (a)-(c) of Theorem 15 occur. If case (c) occurs, then case (c) of Theorem 16 holds and we are done. Hence we may assume that case (a) or (b) of Theorem 15 holds. In particular,  $x_1$  has a unique red neighbor, say a and, given  $b_1$  and  $b_2$  the two neighbors of a in H,  $ab_1$  sees S and  $x_j$  is adjacent to  $b_1$ . Since  $x_n$  has a blue neighbor in H neither of whose neighbors in H is a red neighbor of  $x_1$ , n > 1.

Claim 1  $ab_2$  sees S and  $b_2$  has a neighbor in P.

Let t be the attachment of P in  $V(H) \setminus \{a, b_1\}$  that is closest to a in the path induced by  $V(H) \setminus \{b_1\}$ . Since a is the unique red attachment of P, then t is blue. If  $t = b_2$  then  $ab_2$  sees S and we are done. Assume then that  $t \neq b_2$ , hence no neighbor of t in H is a red neighbor of  $x_1$  so t is adjacent to  $x_n$  and no other node in P. Let  $H_{b_2t}$  be the path between  $b_2$  and t in the graph induced by  $V(H) \setminus \{b_1\}$ , and let  $H' = (a, x_1, P, x_n, t, H_{b_2t}, b_2, a)$ . Then H' is an hole of length at least 6 and, since a and t have distinct colors in the bicoloring of H induced by  $E_S(H)$  and no node in P is universal for S, an odd number of edges of H' sees S, therefore, by Theorem 10, exactly one edge of H' sees S and no node of H' is universal for S except the endnodes of such edge. Since a is universal for S, then the unique edge in H' that sees S must be  $ab_2$ . Also, by Theorem 10, we have two possibilities.

**Case 1:** There exists a node  $y \in S$  such that the only neighbors of y in H' are a and  $b_2$ .

Then t is not adjacent to  $b_1$ , otherwise (H, y) would be a twin wheel. Let  $Z = z_1, \ldots, z_k$  be the path induced by  $V(H) \setminus (V(H_{b_2t}) \cup \{a, b_1\})$ , where  $z_1$  is adjacent to t and  $z_k$  is adjacent to  $b_1$ . Since (H, y) is not a twin wheel, then y has a neighbor in Z. If  $x_n$  does not have a neighbor in Z, then there is a  $3PC(yab_2, t)$ . If both y and  $x_n$  have a neighbor in Z distinct from  $z_1$ , then there is a  $3PC(yab_2, x_n)$ . Note that  $b_1$  has a neighbor in  $V(P) \setminus \{x_1\}$ , otherwise  $(y, b_1, x_1, P, x_n, t, H_{b_2t}, b_2, y)$  is an odd hole.

If  $x_n$  has no neighbor in Z except  $z_1$ , then t and  $z_1$  are the only neighbors of  $x_n$  in H, otherwise  $(H, x_n)$  is an odd wheel. Since  $b_1$  has a neighbor in  $V(P) \setminus \{x_1\}$ , then there is a  $3PC(x_ntz_1, b_1)$ .

Hence  $x_n$  has a neighbor in  $V(Z) \setminus \{z_1\}$ , therefore the only neighbor of y in Z is  $z_1$ . Also  $x_n$  is adjacent to  $z_1$  otherwise there is a  $3PC(yab_2, t)$ . Consider now the hole  $H'' = (z_1, y, a, x_1, P, x_n, z_1)$ . Since  $b_1$  sees at least one edge in H'' and  $b_1$  has at least one neighbor in  $V(P) \setminus \{x_1\}$ , then either  $(H', b_1)$  or  $(H'', b_1)$  is an odd wheel since  $b_1$  sees in H'' exactly one edge more than in H'.

**Case 2:** S contains two nonadjacent nodes y and z such that the only neighbors of y in H' are a,  $b_2$  and  $x_1$  and the only neighbors of z in H' are a,  $b_2$  and  $x_1$  and the only neighbors of z in H' are a,  $b_2$  and the node  $c \neq a$  adjacent to  $b_2$  in H'.

Then t is not adjacent to  $b_1$ , otherwise (H, y) would be a twin wheel. Let  $Z = z_1, \ldots, z_k$  be the path induced by  $V(H) \setminus (V(H_{b_2t}) \cup \{a, b_1\})$ , where  $z_1$  is adjacent to t and  $z_k$  is adjacent to  $b_1$ . Since (H, y) is not a twin wheel, then

y has a neighbor in Z. Also, since (H, z) is not an odd wheel, also z has a neighbor in Z. Let p and q be two neighbors in Z of y and z respectively with minimum distance in Z. Let  $Z_{pq}$  be the path between p and q in Z.  $Z_{pq}$ is an even path, otherwise  $(a, y, p, Z_{pq}, q, z, a)$  would be an odd hole. If  $b_1$  has a neighbor in  $P \setminus x_1$ , then  $(P \setminus x_1) \cup H_{b_2t} \cup \{y, z, b_1\}$  contains a  $3PC(b_2zc, b_1)$ . So  $x_1$  is the unique neighbor of  $b_1$  in P. If  $x_n$  has no neighbors in Z, then  $H \cup P$  induces a  $3PC(x_1ab_1, t)$ . If  $z_1$  is not the unique neighbor of  $x_n$  in Z, then  $H \cup P$  contains a  $3PC(x_1ab_1, x_n)$ . So  $z_1$  is the unique neighbor of  $x_n$  in Z. If  $Z_{pq}$  contains  $z_1$ , then  $V(Z_{pq}) \cup V(P) \cup \{y, z, a\}$  induces a  $3PC(x_1ay, z_1)$ . Otherwise,  $V(P) \cup (V(H_{b_2t}) \setminus b_2) \cup V(Z_{pq}) \cup \{y, z\}$  induces an odd hole. This concludes the proof of Claim 1.

**Claim 2** There exists a node in P that is adjacent to both  $b_1$  and  $b_2$ .

Assume not. Let  $x_k$  be the node of P of lowest index that is adjacent to  $b_2$ . Since we assumed that the node  $x_j$  of lowest index in P adjacent to some blue node is adjacent to  $b_1$ , then k > j.

**Case 1:**  $x_1$  is the unique neighbor of  $b_1$  in  $P_{x_1x_k}$ .

Then  $x_k$  must be adjacent to the neighbor c of  $b_2$  in  $V(H) \setminus \{a\}$  and to no other node in  $V(H) \setminus \{b_2, c\}$ , or else there is either a  $3PC(ab_1x_1, b_2)$  or a  $3PC(ab_1x_1, x_k)$ . Let  $F = b_1, x_1, P_{x_1x_k}, x_k, b_2$ . F is an odd path and  $b_1$  and  $b_2$ are universal for S. Since P does not contain any node universal for S, then conclusion (ii) or (iii) of Lemma 4 holds.

If conclusion (ii) holds, then F has length 3 and  $S \cup \{x_1, x_2\}$  contains an odd anti-path Q between  $x_1$  and  $x_2$ . Since no node of  $V(H) \setminus \{a, b_1, b_2, c\}$  is adjacent to  $x_1$  or  $x_2$  and a is universal for all intermediate nodes of Q, then we can apply Lemma 5 in  $\overline{G}$  to the set  $V(H) \setminus \{a, b_1, b_2, c\}$ , the path Q and the node a. Therefore there must exists an intermediate node y of Q with no neighbors in  $V(H) \setminus \{a, b_1, b_2, c\}$ . But then the only neighbors of y in H are  $a, b_1$  and  $b_2$  and (H, y) is a twin wheel, a contradiction.

If conclusion (iii) holds, then S contains two nonadjacent nodes y and z such that y is adjacent to  $x_1$  and no other node of  $P_{x_1x_k}$  while z is adjacent to  $x_k$  and no other node of  $P_{x_1x_k}$ . Since S does not contain any center of twin wheels w.r.t. H, then y and z must have neighbors in  $V(H) \setminus \{a, b_1, b_2, c\}$ . Let p and q be two neighbors of y and z, respectively, that are closest possible in  $V(H) \setminus \{a, b_1, b_2, c\}$  and let Z be the path between p and q in the graph induced by  $V(H) \setminus \{a, b_1, b_2, c\}$ . Z must have even length otherwise (a, y, p, Z, q, z, a) is an odd hole, but then  $(y, x_1, P_{x_1x_k}, x_k, z, q, Z, p, y)$  is an odd hole, a contradiction. **Case 2:**  $b_1$  has a neighbor in  $P_{x_2x_k}$ .

Then k > 2 and  $H' = (a, x_1, P_{x_1x_k}, x_k, b_2, a)$  is a hole of length at least 6. The only edge of H' that sees S is  $ab_2$  hence conclusion (2) or (3) of Theorem 10 holds.

If conclusion (2) holds, then S contains two nonadjacent nodes y and z such that y is adjacent to  $x_1$  and no other node of  $P_{x_1x_k}$  while z is adjacent to  $x_k$  and no other node of  $P_{x_1x_k}$ , but then there exists a  $3PC(zb_2x_k, b_1)$ .

If conclusion (3) holds, then S contains a node y whose only neighbors in H' are a and  $b_2$ . Let P' be the shortest path between  $x_1$  and y in the graph induced by  $(V(P) \cup V(H) \cup \{y\}) \setminus \{a, b_1, b_2\}$ . Then  $H'' = (a, x_1, P', y, a)$  is a hole. Both  $b_1$  and  $b_2$  see the edge ay of H'', both  $b_1$  and  $b_2$  have a neighbor in  $P_{x_1x_j}$  and y is not adjacent to  $x_k$ , therefore by Theorem 10  $b_1$  and  $b_2$  see an even number of edges in H'', but then there exists a node of P that is adjacent to both  $b_1$  and  $b_2$ .

This concludes the proof of Claim 2.

**Claim 3** If  $E_S(H) \stackrel{\supset}{\neq} \{ab_1, ab_2\}$  then the path  $Q = a, x_1, \ldots, x_n$  contains an odd number of edges that see both  $b_1$  and  $b_2$ .

Assume that  $E_S(H) \supseteq \{ab_1, ab_2\}$ . Suppose it is not the case that an odd number of edges of Q see both  $b_1$  and  $b_2$ . Let  $x_l$  be the node of highest index that is adjacent to both  $b_1$  and  $b_2$ . Then l > 1. Suppose l is odd. Then  $F = a, x_1, P_{x_1x_l}, x_l$  is an odd path and hence by Lemma 4 applied to F and set  $\{b_1, b_2\}, b_1$  is adjacent to  $x_1, x_l$  and no other node in  $P_{x_1x_l}$  while  $b_2$  is adjacent to  $x_{l-1}$ ,  $x_l$  and no other node in  $P_{x_1x_l}$ . But then  $(V(H) \cup V(P_{x_1x_{l-1}})) \setminus \{a\}$ induces an odd hole, a contradiction. Therefore l is even. Let  $x_h$  and  $x_k$ be the nodes of highest index adjacent to, respectively,  $b_1$  and  $b_2$ . W.l.o.g.,  $h \leq k$ . We want to show that  $P_{x_l x_h}$  has even length. Assume not, then l < h, therefore, by definition of l, h and k, h < k. Since  $P_{x_l x_h}$  has odd length, then  $b_1$  must see an odd number of edges of  $P_{x_l x_h}$ . Let  $l = k_1 \leq \ldots \leq k_m = k$  be all the indexes between l and k such that  $b_2$  is adjacent to  $x_{k_i}$ . Then there exists  $i, 1 \leq i \leq m-1$  such that  $b_1$  sees an odd number of edges in  $P_{x_{k_i}x_{k_{i+1}}}$ . But then  $P_{x_{k_i}x_{k_{i+1}}}$  has length at least 2 and  $C = (b_2, x_{k_i}, P_{x_{k_i}x_{k_{i+1}}}, x_{k_{i+1}}, b_2)$  is an hole, therefore  $b_1$  sees exactly one edge uv in C, and  $V(C) \cup \{a, b_1\}$  induces a  $3PC(b_1uv, b_2)$ , a contradiction. Hence we have proven that  $a, x_1, P_{x_1x_h}, x_h$ has even length.

**Case 1:**  $x_n$  sees an odd number of edges in some sector of (H, S).

Since  $x_n$  has only blue neighbors in H, by Corollary 13,  $x_n$  has exactly two neighbors u and v in H and they are adjacent. Suppose  $x_n$  is not adjacent to  $b_2$ . If h < k then there is a  $3PC(x_nuv, b_2)$ . If h = k then there is a  $3PC(x_nuv, x_k)$ . So  $x_n$  is adjacent to  $b_2$ .  $P_{x_hx_n}$  has odd length, else  $(V(H) \cup V(P_{x_hx_n})) \setminus \{a, b_2\}$  induces an odd hole. Let c be the neighbor of  $b_2$  in  $H \setminus a$ . Then c is adjacent to  $x_n$ . Let z be the endnode distinct from  $b_2$  of the sector Z of (H, S) containing c, and let F be the path between c and z in Z. Since  $E_S(H) \supseteq \{ab_1, ab_2\}$ , then  $z \neq b_1$ . Moreover F has odd length, therefore  $R = a, x_1, P, x_n, c, F, z$  has odd length. Let w be the neighbor of z in  $V(H) \setminus V(Z)$ , then  $zw \in E_S(H)$  and, by Lemma 5 applied to S, R and w, there is an odd number of edges of R that sees S, a contradiction.

**Case 2:**  $x_n$  sees an even number of edges in every sector of (H, S).

Let u be the neighbor of  $x_n$  closest to  $b_1$  in the graph induced by  $V(H) \setminus \{a, b_2\}$  and  $H_{ub_1}$  be the path between u and  $b_1$  in the graph induced by  $V(H) \setminus \{a, b_2\}$ . We want to show that  $P_{x_hx_n}$  has length of the same parity as the length of  $H_{ub_1}$ . If not then  $u \neq b_1$  and  $x_h \neq x_n$ , but then  $(b_1, x_h, P_{x_hx_n}, x_n, u, H_{ub_1}, b_1)$  is an odd hole. Let z be the endnode distinct from  $b_1$  and  $b_2$  of the sector Z of (H, S) containing u (the existence of such a node is guaranteed by the hypothesis  $E_S(H) \supseteq \{ab_1, ab_2\}$ ). Let u' be the neighbor of  $x_n$  closest to z in Z and let F be the path between u' and z in Z. Since  $x_n$  sees an even number of edges in Z, then  $H_{ub_1}$  and F have lengths of the same parity, therefore  $R = a, x_1, P, x_n, u', F, z$  has odd length. Let w be the neighbor of z in  $V(H) \setminus V(Z)$ , then  $zw \in E_S(H)$  and, by Lemma 5 applied to S, R and w, there is an odd number of edges of R that sees S, a contradiction.

This concludes the proof of Claim 3.

**Claim 4** If S contains a node s with no neighbors in P, then the path  $Q = a, x_1, \ldots, x_n$  contains an odd number of edges that see both  $b_1$  and  $b_2$ .

Let F be the shortest path between  $x_1$  and s in the graph induced by  $(V(H) \cup V(P) \cup \{s\}) \setminus \{a, b_1, b_2\}$ . Then  $H' = (s, a, x_1, F, s)$  is a hole. Since as sees both  $b_1$  and  $b_2$  and there exists a further node in P that is adjacent to both  $b_1$  and  $b_2$  then, by Theorem 10, H' contains an even number of edges that see both  $b_1$  and  $b_2$ , but then  $Q = a, x_1, P, x_n$  has an odd number of edges that see both  $b_1$  and  $b_2$ . This concludes the proof of Claim 4.

### 6 Ears on isolated edges of a hub

Given an hub (H, S) in a Berge graph G, an edge uv in  $E_S(H)$  is *isolated* if no other edge in  $E_S(H)$  is adjacent to uv. **Lemma 17** Let (H, S) be a hub of a Berge graph G such that H contains an edge uv in  $E_S(H)$  that is isolated. Assume that S is maximal with such a property. Let  $P = x_1, ..., x_n$  be an ear on uv. Let  $Q = y_1, ..., y_m$  be a minimal path in  $G \setminus (V(H) \cup V(P) \cup S)$  such that  $y_1$  has a neighbor in P and  $y_m$  has a neighbor in the interior of some sector of (H, S). Then Q contains a node that is universal for S.

Proof: By contradiction, let  $Q = y_1, ..., y_m$  be a minimal path in  $G \setminus (V(H) \cup V(P) \cup S)$  such that  $y_1$  has a neighbor in P,  $y_m$  has a neighbor in the interior of some sector of (H, S) and no node in Q is universal for S. Note that we only need to prove the statement in the case in which Q does not contain any node whose only neighbors in H are u and v. In fact, if Q contains such a node and  $y_i$  is the node of highest index whose only neighbors are u and v, then  $P' = y_i$  is an ear on uv and  $Q' = y_{i+1}, Q_{y_{i+1}y_m}, y_m$  is a path such that  $y_{i+1}$  has a neighbor in P' and  $y_m$  has a neighbor in the interior of some sector of (H, S) but no node of Q' is adjacent to u, v and no other node of H. Let us assume, then, that Q does not contain any node whose only neighbors in H are u and v.

Claim 1: No node in Q is adjacent to both u and v.

Assume there exists  $i, 1 \leq i \leq m$ , such that  $y_i$  is adjacent to u and v. Since  $y_i$  is not universal for S, then  $S \cup y_i$  is co-connected. By the maximality of S,  $(H, S \cup y_i)$  is not a hub, hence uv is the only edge of H that sees  $S \cup y_i$ . Since uv is isolated, S does not contain any center of a twin wheel w.r.t. H, hence, by Theorem 10,  $y_i$  is adjacent only to u and v in H, a contradiction.

**Claim 2:** Let  $y_i$  be a node with a neighbor in H distinct from v (resp. u). Let s be the neighbor of  $y_i$  closest to u (resp. v) in  $V(H) \setminus \{v\}$  (resp.  $V(H) \setminus \{u\}$ ) and assume that no node in  $Q_{y_1y_{i-1}}$  has a neighbor closer to u (resp. v) in  $V(H) \setminus \{v\}$  (resp.  $V(H) \setminus \{u\}$ ) than s. Then s and u (resp. v) have the same color.

Assume, w.l.o.g., that  $y_i$  has a neighbor in H distinct from v. By contradiction, assume s and u have distinct colors, then  $s \neq u$ . Let w and w' be the endnodes of the sector Z of (H, S) containing s and assume w is closer to uin  $V(H) \setminus \{v\}$  than w'. Since uv is isolated, then w is not adjacent to u. Let F be the shortest path between w and u in  $V(Z) \cup V(Q_{y_1y_i}) \cup V(P) \cup \{u\}$  and F' be the path between u and w in  $V(H) \setminus \{v\}$ . Since H' = (u, F', w, F, u)is a hole, then F and F' have length of the same parity. Since w and u have distinct colors in the bicoloring of H induced by  $E_S(H)$ , then F' has odd length, therefore F is an odd chordless path. Since F' is odd and uv is isolated, F' contains a node t, distinct from u and v, that is universal for S. Lemma 5 applied to S, F and t implies that F contain an odd number of edges that see S, a contradiction.

Let  $y_j$  be the node of Q of lowest index such that  $y_j$  has a neighbor in H distinct from u and v. Let s be the neighbor of  $y_j$  closest to v in  $V(H) \setminus \{u\}$  and t be the neighbor of  $y_j$  closest to u in  $V(H) \setminus \{v\}$ .

**Claim 3**: st is an edge of H that sees S and  $st \neq uv$ . Furthermore,  $P = x_1$  and no node in  $Q_{y_1y_{j-1}}$  has a neighbor in H.

By Claim 2 applied to  $y_j$ , s has the same color of v and t has the same color of u in the bicoloring induced on H by  $E_S(H)$ . By Claim 1, either  $s \neq v$  or  $t \neq u$ . Assume, w.l.o.g., that  $u \neq t$ . Assume s and t are nonadjacent. Then s and t are consecutive neighbors of  $y_j$  with distinct colors in H that are nonadjacent, therefore we can apply Theorem 14 to the path consisting of  $y_j$ . Since  $E_S(H)$  contains an isolated edge, then conclusion (a), (b) or (g) of Theorem 14 holds.

**Case 1:** Case (a) or (b) of Theorem 14 holds.

Then  $E_S(H)$  consists of two nonadjacent edges uv and u'v' while  $(H, y_j)$ is a line wheel. Assume v and v' have the same color. By symmetry, we may assume that  $u \neq t$  and v' is not adjacent to  $y_j$ . Let F be the shortest path between u and  $y_j$  in  $V(P) \cup V(Q_{y_1y_j}) \cup \{u\}$  and let F' be the path between uand t in  $V(H) \setminus \{v\}$ . Since  $u \neq t$ ,  $H' = (u, F', t, y_j, F, u)$  is a hole, hence F'has distinct parity from F. But then, since  $y_j$  sees an odd number of edges in the sector of (H, S) with endnodes u and u', the shortest path F'' from u to u' in  $(V(H) \cup V(P) \cup V(F)) \setminus \{v, v', t\}$  has odd length. By Lemma 5 applied to S, F'' and v', an odd number of edges of F'' see S, a contradiction.

**Case 2:** Case (g) of Theorem 14 holds.

Then s = v, u and t are adjacent and H contains a path v, u, t, u', v'where u'v' sees S and  $y_j$  is adjacent to v, t, v' but not to u or u'. Let Fbe the shortest path between u and  $y_j$  in  $V(P) \cup V(Q_{y_1y_j}) \cup \{u\}$ . Since  $H' = (u, t, y_j, F, u)$  is a hole, F has even parity, but then  $u, F, y_j, v'$  is an odd chordless path and Lemma 5 applied to  $S, u, F, y_j, v'$  and u', implies that an odd number of edges of F see S, a contradiction.

Therefore s and t must be adjacent and, since they have distinct colors, st sees S. To conclude the proof of Claim 3, let  $F = v_1, ..., v_k$  be a shortest path

in  $V(Q_{y_1y_j}) \cup V(P)$  such that  $v_k = y_j$  and  $v_1$  is adjacent to u or v. If  $v_1$  is not adjacent to both u and v, say  $v_1$  is not adjacent to v, then  $V(H) \cup V(F)$ induces a  $3PC(sty_j, u)$ , a contradiction. Therefore  $P = x_1, v_1 = x_1$  and no node in  $Q_{y_1y_{j-1}}$  has a neighbor in H. This concludes the proof of Claim 3.

Let  $H_{ut}$  be the path in  $V(H) \setminus \{v\}$  between u and t and  $H_{vs}$  be the path in  $V(H) \setminus \{u\}$  between v and s. Note that  $H_{ut}$  and  $H_{vs}$  have both even length. Let  $y_k$  be the node of lowest index in Q such that k > j and  $y_k$  has a neighbor in  $V(H) \setminus \{s, t\}$ .

**Claim 4:**  $y_k$  has a neighbor both in  $V(H_{ut}) \setminus \{t\}$  and in  $V(H_{vs}) \setminus \{s\}$ .

Assume, w.l.o.g, that  $y_k$  has a neighbor in  $H_{ut}$  distinct from t and let p be the neighbor of  $y_k$  closest to u in  $H_{ut}$  (possibly u = p). By Claim 2, p and u must have the same color. Let F be the shortest path between p and s in  $V(Q_{y_jy_k}) \cup \{p, s\}$  and let F' be the path between u and p in  $H_{ut}$ . If  $y_k$  has no neighbors in  $V(H_{vs}) \setminus \{s\}$ , then  $H' = (u, F', p, F, s, H_{vs}, v, u)$  is a hole, then R = u, F', p, F, s is an odd path so, by Lemma 5 applied to S, R and v, R contains an odd number of edges that see S. Since u and p have the same color, then S sees an even number of edges of F', therefore S must see an odd number of edges of F, a contradiction.

Let p be the neighbor of  $y_k$  closest to u in  $H_{ut}$  and let q be the neighbor of  $y_k$  closest to v in  $H_{vs}$ . By Claim 1 and Claim 4, p and q are nonadjacent and, by Claim 2, p has the same color of u and q has the same color of v. We can also assume, w.l.o.g., that  $u \neq p$ .

Then p and q are consecutive neighbors of  $y_k$  with distinct colors in H that are nonadjacent, therefore we can apply Theorem 14 to the path consisting of  $y_k$ . Since  $E_S(H)$  contains an isolated edge, then conclusion (a), (b) or (g) of Theorem 14 holds.

**Case 1:** Case (a) or (b) of Theorem 14 holds.

Then  $E_S(H)$  consists only of uv and st. Note that st is an isolated edge of  $E_S(H)$ ,  $P' = y_j$  is an ear of st and S is maximal with this property. Moreover  $Q' = Q_{y_{j+1}y_k}$  is a path in  $G \setminus (V(H) \cup V(P') \cup S)$  such that  $y_{i+1}$  has a neighbor in P' and  $y_k$  has a neighbor in the interior of a sector of (H, S). But now P' and Q' contradict Claim 3.

Case 2: Case (g) of Theorem 14 holds.

Then q = v, u and p are adjacent and H contains a path v, u, p, u', v'where u'v' sees S and  $y_i$  is adjacent to v, p, v' but not to u or u'.

We have two cases:

Case 2.1:  $u'v' \neq st$ .

Then u'v' is not adjacent to st, since v' is in  $H_{ut}$  and v' and t have distinct colors. Let F be the shortest path between u and  $y_k$  in  $V(P) \cup V(Q_{y_1y_k}) \cup \{u\}$ . Since  $H' = (u, p, y_k, F, u)$  is a hole, then F is even, but then  $u, F, y_k, v'$  is an odd chordless path and Lemma 5 applied to  $S, u, F, y_k, v'$  and u', implies that an odd number of edges of F see S, a contradiction.

Case 2.2: u'v' = st.

Then u' = t and v' = s. Let F be the shortest path between t and  $y_k$  in  $V(Q_{y_jy_k}) \cup \{t\}$ . Since  $H' = (t, p, y_k, F, t)$  is a hole, then F is even, but then  $t, F, y_k, v$  is an odd chordless path and Lemma 5 applied to  $S, t, F, y_k, v$  and u, implies that an odd number of edges of F see S, a contradiction.

**Theorem 18** Let (H, S) be a hub of a Berge graph. If G contains an ear P on an isolated edge uv of  $E_S(H)$ , then G has a skew partition.

Proof: Let A be a maximal set containing S such that (H, A) is a hub and uv sees A. Assume that u is colored red in the bicoloring of (H, A) induced by  $E_A(H)$ . Let B be the set containing all the endnodes of the edges of  $E_A(H)$  and all the nodes in  $G \setminus (V(H) \cup A)$  that are universal for A. If  $G \setminus (A \cup B)$  is not connected, then G contains a skew-partition. Assume that  $G \setminus (A \cup B)$  is connected, then there exists a minimal path  $Q = y_1, ..., y_m$  in  $G \setminus (V(H) \cup V(P) \cup A \cup B)$  such that  $y_1$  has a neighbor in P and  $y_m$  has a neighbor in the interior of some sector of (H, A), but such a path would contradict Lemma 17.

## 7 Hubs in graphs containing no "large" line graphs

**ASSUMPTION:** Throughout this section, we will assume that G is a Berge graph such that G and  $\overline{G}$  contain no long  $3PC(\Delta, \Delta)$  and no line graph of a bipartite subdivision of  $K_4$ .

**Lemma 19** Let (H, S) be a hub of a Berge graph G such that G and  $\overline{G}$  contain no long  $3PC(\Delta, \Delta)$  and no line graph of a bipartite subdivision of  $K_4$ . Let  $P = x_1, \ldots, x_n$  be a minimal chordless path in  $G \setminus (V(H) \cup S)$  containing no node that is universal for S, such that  $x_1$  has a blue neighbor

in H and  $x_n$  has a red neighbor (n = 1 is allowed). If there exist consecutive attachments of P with distinct colors that are not adjacent, then one of the following holds.

- (a) |H| = 6, n = 1 and there exists  $y \in S$  such that  $V(H) \cup \{x_1, y\}$  induces a double beetle.
- (b) n = 1 and there exists  $y \in S$  nonadjacent to  $x_1$  such that  $(H, x_1)$  and (H, y) are twin wheels and exactly one edge of H sees both  $x_1$  and y.
- (c) There exists  $y \in S$  such that (H, y) is a twin wheel, no node of P is a neighbor of y,  $x_1$  is adjacent to the twin of y in H and no other node in H while  $x_n$  is not adjacent to both the other neighbors of y in H.

*Proof:* Assume not, then P is of one of the types (a)-(c) or (f)-(k) of Theorem 14. If P is of type (c), then  $V(H) \cup V(P) \cup \{y\}$  contains a long  $3PC(\Delta, \Delta)$ unless n = 1 and |H| = 6, so case (a) of Lemma 19 holds. P cannot be of type (a) by assumption. If P is of type (b), then n = 1, |H| = 6,  $(H, x_1)$ is a line wheel and  $S \cup x_1$  contains an odd chordless anti-path Q of length at least 3 between  $x_1$  and a node  $y \in S$  such that (H, y) is a line wheel, no edge of H sees both  $x_1$  and y and every intermediate node of Q is adjacent to every node in H. One can verify that  $G[V(H) \cup V(Q)]$  is the line graph of a bipartite subdivision of  $K_4$ . If P is of type (f), then n = 1, H contains a subpath u, z, w, z', u' such that  $E_S(H) = \{wz, wz'\}, x_1$  is adjacent to u, w and u' but not z and z',  $S \cup x_1$  contains an odd chordless anti-path Q of length at least 3 between  $x_1$  and a node  $y \in S$  such that y is nonadjacent to u and u' and every intermediate node of Q is adjacent to both u and u'. One can verify that  $G[V(Q) \cup \{u, z, z', u'\}]$  is a  $3PC(uu'y, z'zx_1)$ , and such  $3PC(\Delta, \Delta)$  is long since Q has length at least 3. If P is of type (g), then n = 1, H contains a subpath w, z, u, z', w' such that wz and w'z' are edges of  $E_S(H)$ ,  $x_1$  is adjacent to u, w and w' but not z and  $z', S \cup x_1$  contains an even chordless anti-path Q between  $x_1$  and a node  $y \in S$  such that y is nonadjacent to u and every intermediate node of Q is adjacent to u. One can verify that  $\overline{G}[V(Q) \cup \{w, z, u, z', w'\}]$  is a  $3PC(ww'u, z'zx_1)$ , which is long since Q has positive even length. If P is of type (h), then n > 1, H contains a subpath w, z, u, z', w' such that wz and w'z' are edges of  $E_S(H)$ ,  $x_1$  is adjacent w and w' but not u, z and z', while  $x_n$  is adjacent to u but not w, z, w' and z'. Furthermore S contains two nodes y and y' such that the only neighbors of y in  $V(P) \cup \{w, z, u, z', w'\}$  are u, z, z', w, w' while the only neighbors of y' in  $V(P) \cup \{w, z, u, z', w'\}$  are  $x_1, z, z', w, w'$ . One can verify that  $G[V(P) \cup \{y, y', u, z, w'\}]$  is a long  $3PC(uyz, x_1w'y')$ . If P is of type (k), then H = (v, w, z, u, z', w', v),  $E_S(H) = \{wz, w'z'\}$ ,  $x_1$  is adjacent only to v in H and  $x_n$  is adjacent only to u in H. Furthermore, S contains two nonadjacent nodes y and y' such that y and y' are adjacent to every node in H except v and u, respectively, and no node in P is adjacent to y or y'. One can verify that  $G[V(P) \cup \{y, y', u, v, z, w'\}]$  is a long 3PC(uyz, vw'y').  $\Box$ 

**Lemma 20** Let (H, S) be the hub of a Berge graph G such that G and G contain no long  $3PC(\Delta, \Delta)$  and no line graph of a bipartite subdivision of  $K_4$ . Assume that S is a maximal set such that (H, S) is a hub with the further property that S does not contain any center of a twin wheel w.r.t. H. Let  $P = x_1, \ldots, x_n$  be a minimal chordless path in  $G \setminus (V(H) \cup S)$  containing no node universal for S such that  $x_1$  has a red neighbor, no other node of P has a red neighbor and  $x_n$  has a blue neighbor whose neighbors in H are not red neighbors of  $x_1$ . Then one of the following holds:

- (1) There exist two adjacent edges  $ab_1$ ,  $ab_2$  of  $E_S(H)$  such that a is the only red neighbor of  $x_1$  in H and at least one node of P is adjacent to both  $b_1$  and  $b_2$ . If  $E_S(H) \supseteq \{ab_1, ab_2\}$  or if S contains a node s with no neighbors in P, then the path  $Q = a, x_1, \ldots, x_n$  contains an odd number of edges that see both  $b_1$  and  $b_2$ .
- (2) |H| = 6, n = 1 and there exists  $y \in S$  such that  $V(H) \cup \{x_1, y\}$  induces a double beetle.

*Proof:* Obviously, one of the conclusions of Theorem 16 must occur. If conclusion (a) of Theorem 16 holds, then by Lemma 19 conclusion (2) holds (since S does not contain any center of a twin wheel) and we are done. If conclusion (b) holds, then conclusion (1) holds and we are done.

So we may assume that conclusion (c) of Theorem 16 holds. Then n > 1,  $E_S(H)$  contains at least two nonadjacent edges,  $x_1$  is adjacent to all the red endnodes of the edges of H that see S and the node  $x_j$  of lowest index adjacent to some blue node is adjacent to all the blue endnodes of the edges of H that see S. If j > 2, then S contains two nonadjacent nodes y and z such that y is adjacent to  $x_1$  and to no other node of  $P_{x_1x_j}$ , and z is adjacent to  $x_j$  and to no other node of  $P_{x_1x_j}$ . If j = 2, then  $S \cup \{x_1, x_2\}$  contains an odd chordless anti-path between  $x_1$  and  $x_2$ .

Let uv and u'v' be two nonadjacent edges of  $E_S(H)$  and assume, w.l.o.g., that  $x_1$  is adjacent to u and u' and  $x_j$  is adjacent to v and v'. If j > 2then  $G[V(P_{x_1x_j}) \cup \{y, z, u, v'\}]$  is a long  $3PC(x_1yu, x_jv'z)$ . If j = 2 then  $\overline{G}[V(Q) \cup \{u, u', v, v'\}]$  is a long  $3PC(x_1vv', x_2u'u)$ .  $\Box$ 

#### 7.1 Good hubs

We say that a hub (H, S) is good if H has an inner blue node and an inner red node w.r.t. the bicoloring induced on H by  $E_S(H)$ . Equivalently, given the maximal paths  $P^1, \ldots, P^k$  induced by the endnodes of the edges of  $E_S(H)$ , (H, S) is a good hub if and only if there exists  $i, 1 \leq i \leq k$ , such that  $P^i$  has odd length.

**Lemma 21** Let (H, S) be a good hub of a Berge graph G such that G and  $\overline{G}$  contain no long  $3PC(\Delta, \Delta)$  and no line graph of a bipartite subdivision of  $K_4$ . Let  $y \in G \setminus (V(H) \cup S)$  be a node such that  $(H, S \cup y)$  is a hub. Then either  $(H, S \cup y)$  is a good hub or  $V(H) \cup y$  contains a hole H' such that (H', S) is a good hub with  $E_S(H') \subsetneq E_S(H)$ .

Proof: Since (H, S) is a good hub, by Lemma 19 every pair of consecutive neighbors of y in H with distinct colors are adjacent. Assume  $(H, S \cup y)$ is not a good hub. Let  $P^1, \ldots, P^k$  be the maximal paths induced by the endnodes of the edges of  $E_S(H)$  and assume, w.l.o.g, that  $P^1 = y_1, \ldots, y_m$ has odd length. If y has no neighbor in  $P^1$ , then  $P^1$  is contained in a sector  $Q = s, \ldots, t$  of (H, y), therefore, given H' = (y, s, Q, t, y), (H', S) is a good hub and  $E_S(H') \subsetneqq E_S(H)$ . Therefore we may assume that y has a neighbor in  $P^1$ . Let r be the neighbor of y closest to  $y_1$  in  $P^1$  and s be the neighbor of y closest to  $y_m$  in  $P^1$  (possibly r = s). Since  $(H, S \cup y)$  is not a good hub, then y sees an even number of edges of  $P^1$ , therefore  $P^1_{rs}$  has even length. Since  $P^1$  has odd length, we can assume, w.l.o.g., that  $P^1_{sy_m}$  has odd length. Let  $Q = s, \ldots, t$  be the sector of (H, y) containing  $P^1_{sy_m}$ , then, given H' = (y, s, Q, t, y), (H', S) is a good hub and  $E_S(H') \subsetneqq E_S(H)$  (since  $(H, S \cup y)$  is a hub).

**Theorem 22** Let G be a Berge graph such that G and  $\overline{G}$  contain no long  $3PC(\Delta, \Delta)$  and no line graph of a bipartite subdivision of  $K_4$ . If G contains a good hub (H, S), then G has a good skew partition.

*Proof:* Assume that, among all the good hubs contained in G, (H, S) is chosen so that  $E_S(H)$  is minimal (i.e. there is no good hub (H', S') such that  $E_{S'}(H') \subseteq E_S(H)$ . Let A be a maximal set containing S such that (H, A) is a hub. Then, by Lemma 21 and by the minimality assumption on  $E_S(H)$ , (H, A) is a good hub and  $E_A(H) = E_S(H)$ . Let B be the set containing all the nodes that are universal for A in  $G \setminus (V(H) \cup A)$  and all the blue endnodes of the edges in  $E_S(H)$ . If in  $G \setminus (A \cup B)$  the red nodes of H are in distinct connected components than the blue nodes of H, then G has a skew partition. Otherwise there exists a chordless path  $P = x_1, ..., x_n$  in  $G \setminus (V(H) \cup A)$  containing no node universal for S such that  $x_1$  is adjacent to a red node of H, no other node of P has a red node of H and  $x_n$  is adjacent to an inner blue node of H. Let j be the node of P with lowest index that is adjacent to a blue node b in H so that neither of the neighbors of b in H is a red neighbor of  $x_1$ . Then either conclusion (1) or (2) of Lemma 20 holds for  $P_{x_1x_j}$ . Conclusion (2) cannot hold since (H, A) is a good hub. Hence conclusion (1) holds, so there exist two adjacent edges  $ab_1$ ,  $ab_2$  of  $E_A(H)$  such that a is the only red neighbor of  $x_1$  in H and at least one node of  $P_{x_1x_j}$  is adjacent to both  $b_1$  and  $b_2$ . Since (H, A) is a good hub,  $E_A(H) \supseteq \{ab_1, ab_2\}$  so by Lemma 20 the path  $Q = a, x_1, \ldots, x_j$ contains an odd number of edges that see both  $b_1$  and  $b_2$ . If j = 1, then  $(H, A \cup x_1)$  is a hub, contradicting the maximality of A. Therefore j > 1 and there exists a node  $x_i$ , i < j, adjacent to  $b_1$  and  $b_2$  and to no other node in  $V(H) \setminus \{a, b_1, b_2\}$ . Thus  $(V(H) \cup \{x_i\}) \setminus \{a\}$  induces a hole H' and (H', A) is a good hub with  $E_A(H') \subseteq E_S(H)$ , contradicting the minimality of  $E_S(H)$ .

Hence G contains a skew partition (A, B, C, D) where C contains all the red nodes of H and D contains all the inner blue nodes of H (w.r.t. the bicoloring induced on H by  $E_A(H)$ ). Let u be any red endpoint of some edge in  $E_A(H)$ , then  $u \in C$  and u is universal for A, hence (A, B, C, D) is a good skew partition.

Recently, Chudnovsky, Robertson, Seymour and Thomas [3] showed that a minimally imperfect graph cannot contain a long  $3PC(\Delta, \Delta)$  or the line graph of a bipartite subdivision of  $K_4$ . This result, together with Theorems 2 and 22, implies the following.

**Theorem 23** No minimally imperfect graph contains a good hub.

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