THIS REPORT HAS BEEN DELIMITED
AND CLEARED FOR PUBLIC RELEASE
UNDER DOD DIRECTIVE 5200.20 AND
NO RESTRICTIONS ARE IMPOSED UPON
ITS USE AND DISCLOSURE.

DISTRIBUTION STATEMENT A

APPROVED FOR PUBLIC RELEASE;
DISTRIBUTION UNLIMITED.

AFATL-TR-74-41

AEROBALLISTIC RANGE DATA REDUCTION TECHNIQUE UTILIZING NUMERICAL INTEGRATION

ARMAMENT SYSTEMS DEPARTMENT
GENERAL ELECTRIC COMPANY

TECHNICAL REPORT AFATL-TR-74-41

FEBRUARY 1974

Distribution limited to U. S. Government agencies only; this report documents test and evaluation; distribution limitation applied February 1974. Other requests for this document must be referred to the Air Force Armament Laboratory (DLDL), Eglin Air Force Base, Florida 32542.

AIR FORCE ARMAMENT LABORATORY

AIR FORCE SYSTEMS COMMAND • UNITED STATES AIR FORCE

EGLIN AIR FORCE BASE, FLORIDA

Aeroballistic Range Data Reduction Technique Utilizing Numerical Integration

Robert H. Whyte
Wayne H. Hathaway

Distribution limited to U. S. Government agencies only; this report documents test and evaluation; distribution limitation applied February 1974. Other requests for this document must be referred to the Air Force Armament Laboratory (DLDL), Eglin Air Force Base, Florida 32542.

FOREWORD

This report documents work accomplished during the period from 26 June 1973 to 30 January 1974 by the Armament Systems Department, General Electric Company, Burlington, Vermont, under Contract F08635-73-C-0165 with the Air Force Armament Laboratory, Eglin Air Force Base, Florida. The program monitor for the Armament Laboratory was Mr. Gerald L. Winchenbach (DLDL).

The principal investigators for the contractor were Wayne H. Hathaway and Robert H. Whyte.

This technical report has been reviewed and is approved.

DALE M. DAVIS
Director, Guns and Rockets Division

ABSTRACT

The numerical integration technique to be utilized in the reduction and analysis of data gathered in the USAF Aeroballistic Research Facility located at Eglin Air Force Base is described. The method of Chapman and Kirk has been developed into a system of digital computer programs which may be easily and routinely used in the analysis of data derived from spark range firing. The equations of motion for a missile/projectile in free-flight are derived in this report utilizing the six degrees of freedom. The parametric equations, philosophy, and methods of implementation are also derived and discussed. Brief descriptions of the computer program involved are also given.

(The reverse of this page is blank.)

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

A	Reference area
CG	Center of gravity, % of length
C_{lp}	Spin deceleration coefficient
$C_{l\delta}$	Fin cant coefficient
C_m	Pitching moment coefficient
$C_{m\delta}$	Trim moment coefficient
C_{mq}	Damping in pitch coefficient
C_{np}	Magnus moment coefficient
C_D	Drag coefficient
C_N	Normal force coefficient
$C_{N\delta}$	Trim force coefficient
C_{Yp}	Magnus force coefficient
C_X	Axial force coefficient
I_x	Axial moment of inertia, slug-ft^2
I_y	Transverse moment of inertia, slug-ft^2
V	Total missile velocity, ft/sec

VREF, V_o	Reference velocity, ft/sec
u, v, w	Missile fixed velocities, ft/sec
d	Reference diameter
ℓ	Reference length, inches
m	Mass, slugs
p	Missile spin rate, rad/sec
p_R	Missile spin rate, rad/ft
$\overline{q}$	Dynamic pressure, lb/ft^2
θ, ψ, ϕ	Fixed plane angles, radians
g	Gravity
X_e, Y_e, Z_e	Earth fixed distances, ft
V_{Xe}, V_{Ye}, V_{Ze}	Earth fixed velocities, ft/sec
θ_m	Missile pitch angle, $\sin^{-1}\ [w/\sqrt{u^2 + w^2}]$
ψ_m	Missile yaw angle, $\sin^{-1}\ [-v/V]$
$\overline{\alpha}$	Missile total angle of attack, $\sin^{-1}\ [\sqrt{v^2 + w^2}/V]$
ε	Sin $\overline{\alpha}$
ρ	Density, lb sec^2/ft^4
δ^2	Mean squared yaw, degrees squared or radians squared

Superscripts

$\dot{}$	Derivative with respect to time
$\ddot{}$	Second derivative with respect to time

Subscript

α	Derivative with respect to sin α
α_3	Derivative with respect to $\sin^3 \alpha$
α_5	Derivative with respect to $\sin^5 \alpha$
α_2	Derivative with respect to $\sin^2 \alpha$
0	Initial condition
V	Derivative with respect to velocity

SECTION I

INTRODUCTION AND BACKGROUND

This report will discuss the rationale and mathematics involved in the implementation of a numerical integration data reduction technique to be utilized with the USAF Aeroballistic Research Facility[1], Air Force Armament Laboratory, Eglin Air Force Base, Florida.

This numerical technique was reported on by Chapman and Kirk[2] in 1969. Goodman[3] (1966) and Knadler[4] (1969) also reported on this general technique under separate efforts. Whyte and Beliveau of General Electric Company[5,6,7] (1969, 1970) applied this technique to ballistic range, yaw sonde, and wind tunnel data. Beginning in 1971 General Electric Company, under various in-house efforts and United States Government contracts, has developed and applied this technique to a wide variety of exterior ballistic and interior ballistic analysis problems.[8-15]

In October 1972 General Electric Company delivered to Arnold Engineering Development Center a series of numerical integration data reduction computer programs[13] for use with VKF Range G. These programs are now being utilized in the routine production of free-flight data from Range G.

The capability exists in these programs to analyze up to three experiments simultaneously and to identify Mach number non-linearities in addition to angle of attack non-linearities.

The original programs were obtained by AFATL from AEDC and have been modified to complement the large computational capability of ADTC. Most importantly, the programs have been tailored to provide maximum flexibility of use while decreasing the tasks required of the project engineer.

The inputs and outputs of the various computer programs will be discussed briefly in this report. A forthcoming document will report on the details, flow charts, input, output, and options.

SECTION II
AEROBALLISTIC RESEARCH FACILITY

The United States Air Force Aeroballistic Research Facility is part of the Air Force Armament Laboratory located at Eglin Air Force Base, Florida. This facility, now under construction, is described in detail in reference 1. The construction is expected to be complete and instrumentation installed by summer 1974.

This facility, essentially a ballistic range, is constructed of concrete and is approximately 780 feet in length. The basic instrumentation will be 50 spark shadowgraph stations. Each station will consist of two sparks and two cameras located in the wall and floor (pit).

The stations may be set up at 131 locations along the range thus allowing considerable spacing flexibility to enhance data acquisition for contemplated test programs.

Timing data will be provided at each of the 50 stations by electronic chronographs. They are operated with an IR detection system and the spark apparatus.

The estimated accuracies of measurements are as follows:

Time ±0.1 microsecond
Position ±0.0013 foot
Angle ±0.10 degree

This range will be continuously calibrated through the use of catenary wires suspended in front of the wall and ceiling reflective screens. Reference beads will be precisely positioned along the wires, with the location of all beads known to 0.0001 foot.

SECTION III

STATEMENT OF PROBLEM

The angular and translational motion of a projectile as it traverses a ballistic range may be observed, measured, and documented by means discussed in the previous section and reference 1. The following basic data are obtained as a function of time: X_e, Y_e, Z_e, θ, ψ, and ϕ. Knowing the physical properties of the projectile and the atmospheric properties of the range facility, the problem becomes one of identifying the aerodynamic coefficients and their magnitudes which are causing the observed motion.

The most prevalent method of analyzing ballistic range data is known as linear theory. Murphy[16-18], MacAllister[19,20], and others of BRL as well as Nicolaides[19,21,22] and Eikenberry[23] have developed this theory to the extent that certain types of non-linearities can also be analyzed. However, unless many cycles of data are present and/or multiple experiments conducted, the only non-linear coefficient which can be consistently identified is the cubic pitching moment.

In 1969 Chapman and Kirk of NASA Ames, in attempting to analyze free flight data, developed a technique which allowed the differential equations of motion to be used directly in the fitting process.

Linear theory, on the other hand, depends on using the closed form solution to the equations of motion in the fitting process. This technique is very effective providing the assumptions required to obtain the closed form solution, i.e., small velocity drop, nearly constant pd/2V, linear aerodynamics, and small angular motion are not drastically violated.

None of the above assumptions are required when using the differential equations of motion directly. The only drawbacks or restrictions are in the identification of the coefficients which are to be solved for and the efficient coding of a computer program such that a solution may be found at a reasonable cost.

The system in use at AEDC and Eglin AFB uses the linear theory programs in concert with the numerical integration programs such that the user has fewer problems in obtaining adequate reductions. Linear theory reduction and numerical integration reductions may be directly compared as they occur in the same computer run.

SECTION IV
THEORY AND EQUATIONS

1. EQUATIONS OF MOTION

The equations of motion are derived in a fixed plane coordinate system. The Euler angles, as well as the missile-fixed velocities and earth-fixed velocities, are integrated numerically (step-wise) to simulate the projectile motion.

This formulation was chosen for the following reasons:

1. Allows largest time step for integration.
2. Most accurate for spin-stabilized or fast spinning projectiles.
3. Consistent with data acquisition.
4. Allows slight aerodynamic asymmetries.

The following assumptions are contained in the derivations:

1. No earth rotation effect on angular motion
2. Flat earth
3. Rotational symmetry ($C_{m\alpha} = C_{n\beta}$, $I_y = I_z$, etc.)
4. Trim moments and forces are small.
5. Aerodynamic coefficients are expanded as polynomial functions of the sine of the total angle of attack.

These derivations are patterned after Barnett's of Picatinny Arsenal.[24,25] These references are interesting in that they describe the advantages of one type of coordinate system as opposed to another. Barnett discusses the problem associated with obtaining adequate (accurate) simulations of spin stabilized projectiles using a coordinate system which is attached to the projectile and rolling with it. His conclusions and those of the authors of this report are that adequate simulations of spin stabilized projectiles may most reliably be made with the fixed plane system. In addition a run time cost savings of over 25 to 1 are realized when using a fixed plane simulation as opposed to a rolling coordinate system simulation of questionable accuracy.

a. <u>Six Degrees of Freedom</u>

Force Equations of Motion:

$$\Sigma \overrightarrow{F} = m \frac{d^2{}_E \overrightarrow{R}}{d\,t^2} \quad \text{(Earth-fixed derivative)} \tag{1}$$

$\Sigma \overrightarrow{F}$ = Summation of forces acting on missile

$\overrightarrow{R}$ = Vector from earth center to current missile C.G.

$\overline{i}_H, \overline{j}_H, \overline{k}_H,$ = Unit vectors aligned with missile-fixed coordinate system (non-rotating)

$\overrightarrow{\omega}$ = Angular velocity of missile-fixed coordinate system relative to earth-fixed coordinate.

$$\overrightarrow{\omega} = \omega_{XH}\, \overline{i}_H + \omega_{YH}\, \overline{j}_H + \omega_{ZH}\, \overline{k}_H \tag{2}$$

in terms of missile-fixed coordinates

Equation (1) may be rewritten as:

$$\Sigma \overrightarrow{F} = m \frac{d^2{}_E \overrightarrow{R}}{d\,t^2} \quad m \frac{d_E \overrightarrow{V}}{d\,t} \tag{3}$$

where: $\overrightarrow{V}$ = time rate of change of $\overrightarrow{R}$, i.e., missile velocity

Equation (3) may be rewritten as follows:

$$\Sigma \overrightarrow{F} = m \frac{d_E \overrightarrow{V}}{d\,t} = m \left[\frac{d_H \overrightarrow{V}}{d\,t} + \overrightarrow{\omega} \times \overrightarrow{V} \right] \tag{4}$$

where: $\vec{V} \equiv u\,\bar{i}_H + v\,\bar{j}_H + w\,\bar{k}_H$ (5)

$$\frac{d_H \vec{V}}{d\,t} = \dot{u}\,\bar{i}_H + \dot{v}\,\bar{j}_H + \dot{w}\,\bar{k}_H \tag{6}$$

where all quantities in the right-hand side of Equation (4) are understood to be expressed in missile coordinates (H). Intuitively, the presence of the $\vec{\omega} \times \vec{V}$ term relates the motion of the missile coordinate system, to which forces and moments are referred, to earth-fixed coordinates.

Combining Equations (4), (5), and (6), and performing the indicated operations, one then obtains the three component force equations:

$$\begin{aligned}
\Sigma F_{XH} &= m\left[\dot{u} + \omega_{YH}\,w - \omega_{ZH}\,v\right]; \left(\underline{\text{in the } \bar{i}_H \text{ direction}}\right) \\
\Sigma F_{YH} &= m\left[\dot{v} + \omega_{ZH}\,u - \omega_{XH}\,w\right]; \left(\underline{\text{in the } \bar{j}_H \text{ direction}}\right) \\
\Sigma F_{ZH} &= m\left[\dot{w} + \omega_{XH}\,v - \omega_{YH}\,u\right]; \left(\underline{\text{in the } \bar{k}_H \text{ direction}}\right)
\end{aligned} \tag{7}$$

Moment Equations of Motion:

Treating the moment equation in a similar manner to the force equation:

$$\Sigma \vec{L} = \frac{d_E \vec{J}}{d\,t} = \frac{d_H \vec{J}}{d\,t} + \vec{\omega} \times \vec{J} \tag{8}$$

$\Sigma \vec{L}$ = Summation of moments acting on missile

$\vec{J}$ = Angular momentum of missile

$(\vec{\omega}_T)$ = Total angular velocity of the missile (rolling coordinates) relative to earth coordinates.

$$\overrightarrow{J} = I_x (\omega_T)_{XH} \bar{i}_H + I_y (\omega_T)_{YH} \bar{j}_H + I_y (\omega_T)_{ZH} \bar{k}_H \quad (9)$$

where I_x, I_y are the moments of inertia of the missile about the longitudinal and traverse axes, respectively. Note that the assumption of rotational symmetry ($I_z = I_y$) is implicitly made.

$$\frac{d_H \overrightarrow{J}}{d\,t} = I_x (\dot{\omega}_T)_{XH} \bar{i}_H + I_y (\dot{\omega}_T)_{YH} \bar{j}_H + I_y (\dot{\omega}_T)_{ZH} \bar{k}_H \quad (10)$$

Appropriately combining Equations (2), (8), (9), and (10) results in the component equations for the moments:

$$\begin{aligned}
\Sigma L_{XH} &= I_x (\dot{\omega}_T)_{XH} + \omega_{YH} I_y (\omega_T)_{ZH} - \omega_{ZH} I_y (\omega_T)_{YH} \\
\Sigma L_{YH} &= I_y (\dot{\omega}_T)_{YH} + \omega_{ZH} I_x (\omega_T)_{XH} - \omega_{XH} I_y (\omega_T)_{ZH} \qquad (11) \\
\Sigma L_{ZH} &= I_y (\dot{\omega}_T)_{ZH} + \omega_{XH} I_y (\omega_T)_{YH} - \omega_{YH} I_x (\omega_T)_{XH}
\end{aligned}$$

AERODYNAMIC FORCES

Three aerodynamic forces are considered:

Axial drag	-	Along missile axis
Normal force	-	Perpendicular to missile axis in plane of yaw
Magnus force	-	Perpendicular to plane of yaw

These forces are illustrated in Figure 1 (arrows indicate positive directions).

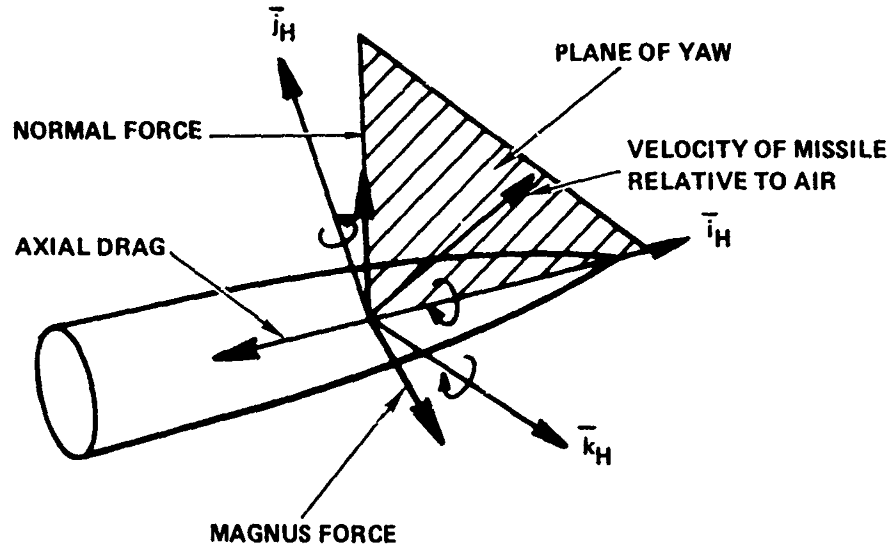


Figure 1. Force System

The axial drag acts along the negative $\bar{i}_H$ axis which, by definition, is directed from the CG towards the nose of the missile.

$$(\text{Axial Drag})_{XH} = - \frac{\rho\ d^2\ \pi\ (V)^2}{8}\ \bar{C}_X$$

$$= - \bar{q}\ A\ \bar{C}_X$$

Where $\bar{q}$ = dynamic pressure = $1/2\ \rho\ V^2$

A = cross sectional area

The normal force components act opposite to the directions of v and w. Using known geometry, one can deduce the components of the normal force:

$$(\text{Normal Force})_{YH} = - \bar{q}\ A\ \bar{C}_{N\alpha} \sin \bar{\alpha}\ \frac{v}{\sqrt{v^2 + w^2}}$$

where: $\bar{C}_{N\alpha}$ = The slope of the Normal Force coefficient plotted as a function of $\sin \bar{\alpha}$.

$$\sin \bar{\alpha} = \sqrt{v^2 + w^2} \;/V$$

$$\therefore \text{(Normal Force)}_{YH} = - \bar{q} \, A \, \bar{C}_{N\alpha} \, \frac{v}{V}$$

$$\text{(Normal Force)}_{ZH} = - \bar{q} \, A \, \bar{C}_{N\alpha} \sin \bar{\alpha} \, \frac{w}{\sqrt{v^2 + w^2}}$$

$$= - \bar{q} \, A \, \bar{C}_{N\alpha} \, \frac{w}{V}$$

The Magnus Force acts in a direction perpendicular to the plane of yaw. As in the case of the Normal Force, using known velocity geometry the Magnus Force components are:

$$\text{(Magnus Force)}_{YH} = + \frac{\bar{q} \, A \, d}{2 \, V} \, (\omega_T)_{XH} \, \bar{C}_{Yp\alpha} \sin \bar{\alpha} \, \frac{w}{\sqrt{v^2 + w^2}}$$

Where $\bar{C}_{Yp\alpha}$ = The slope of the Magnus Force coefficient plotted as a function of $\sin \bar{\alpha}$.

$$\text{(Magnus Force)}_{YH} = \frac{\bar{q} \, A \, d}{2 \, V} \, (\omega_T)_{XH} \, \bar{C}_{Yp\alpha} \, \frac{w}{V}$$

$$\text{(Magnus Force)}_{ZH} = - \frac{\bar{q} \, A \, d}{2 \, V} \, (\omega_T)_{XH} \, \bar{C}_{Yp\alpha} \, \frac{v}{V}$$

Summary of Forces: (excluding gravity)

$$\Sigma \, F_{XH} = - \bar{q} \, A \, \bar{C}_X$$

$$\Sigma \, F_{YH} = - \bar{q} \, A \, \bar{C}_{N\alpha} \, v/V + \frac{\bar{q} \, A \, d}{2 \, V} \, (\omega_T)_{XH} \, \bar{C}_{Yp\alpha} \, \frac{w}{V} \tag{12}$$

$$\Sigma \, F_{ZH} = - \bar{q} \, A \, \bar{C}_{N\alpha} \, \frac{w}{V} - \frac{\bar{q} \, A \, d}{2 \, V} \, (\omega_T)_{XH} \, \bar{C}_{Yp\alpha} \, \frac{v}{V}$$

AERODYNAMIC MOMENTS

The following moments are considered:

Pitching Moment	- Perpendicular to plane of yaw
Magnus Moment	- In plane of yaw
Damping Moment	- Perpendicular to missile axis
Spin Deceleration	- Along missile axis
Trim Moment	- Perpendicular to missile axis

Rotational sign convention is indicated in Figure 1.

Pitching Moment Components:

$$(\text{Pitching Moment})_{YH} = \bar{q}\ A\ d\ \bar{C}_{m\alpha} \sin \bar{\alpha} \quad \frac{w}{\sqrt{v^2 + w^2}}$$

Where: $\bar{C}_{m\alpha}$ = The slope of the Pitching Moment coefficient plotted as function of sin $\bar{\alpha}$.

$$(\text{Pitching Moment})_{YH} = \bar{q}\ A\ d\ \bar{C}_{m\alpha} \frac{w}{V}$$

$$(\text{Pitching Moment})_{ZH} = -\ \bar{q}\ A\ d\ \bar{C}_{m\alpha} \frac{v}{V}$$

Magnus Moment Components:

$$(\text{Magnus Moment})_{YH} = \frac{\bar{q}\ A\ d^2}{2\ V} (\omega_T)_{XH}\ \bar{C}_{np\alpha} \sin \bar{\alpha} \quad \frac{v}{\sqrt{v^2 + w^2}}$$

Where: $\bar{C}_{np\alpha}$ = The slope of the Magnus Moment coefficient plotted as a function of sin $\bar{\alpha}$.

$$(\text{Magnus Moment})_{YH} = \frac{\bar{q}\ A\ d^2}{2\ V} \ (\omega_T)_{XH}\ \bar{C}_{np\alpha} \frac{v}{V}$$

$$(\text{Magnus Moment})_{ZH} = \frac{q\ A\ d^2}{2\ V} \ (\omega_T)_{XH}\ \bar{C}_{np\alpha} \frac{w}{V}$$

Damping Moment Components:

$$(\text{Damping Moment})_{YH} = \frac{\bar{q}\,A\,d^2}{2\,V}\,(\omega_T)_{YH}\,\bar{C}_{mq}$$

Where: $\bar{C}_{mq}$ = Damping Moment coefficient.

$$(\text{Damping Moment})_{ZH} = \frac{\bar{q}\,A\,d^2}{2\,V}\,(\omega_T)_{ZH}\,\bar{C}_{mq}$$

Spin Deceleration Moment:

$$(\text{Spin Moment})_{XH} = \frac{\bar{q}\,A\,d^2}{2\,V}\,(\omega_T)_{XH}\,C_{1p}$$

C_{1p} = Spin deceleration coefficient

Trim Moment Components:

This moment is induced by missile asymmetries and is assumed negligible for spin-stabilized missiles.

$$(\text{Trim Moment})_{YH} = -\,\bar{q}\,A\,d\,[C_{m\delta}\,(\delta_B \sin\phi - \delta_A \cos\phi)]$$

$$(\text{Trim Moment})_{ZH} = \bar{q}\,A\,d\,[C_{m\delta}(\delta_B \cos\phi + \delta_A \sin\phi)]$$

Where: $C_{m\delta}$ = Trim Moment Coefficient

δ_A, δ_B = Components of the trim angle

ϕ = Roll angle

Summary of Moments:

$$\Sigma\, L_{XH} = \frac{\bar{q}\, A\, d^2}{2\, V} (\omega_T)_{XH}\, C_{1p}$$

$$\Sigma\, L_{YH} = \bar{q}\, A\, d\, \bar{C}_{m\alpha} \frac{w}{V} + \frac{\bar{q}\, A\, d^2}{2\, V} (\omega_T)_{XH}\, \bar{C}_{np\alpha} \frac{v}{V}$$

$$+ \frac{q\, A\, d^2}{2\, V} (\omega_T)_{YH}\, \bar{C}_{mq} - q\, A\, d\, [C_{m\delta}\, (\delta_B \sin\phi - \delta_A \cos\phi)] \tag{13}$$

$$\Sigma\, L_{ZH} = -\, \bar{q}\, A\, d\, \bar{C}_{m\alpha} \frac{v}{V} + \frac{q\, A\, d^2}{2\, V} (\omega_T)_{XH}\, \bar{C}_{np\alpha} \frac{w}{V}$$

$$+ \frac{\bar{q}\, A\, d^2}{2\, V} (\omega_T)_{ZH}\, \bar{C}_{mq} + \bar{q}\, A\, d\, [C_{m\delta}\, (\delta_B \cos\phi + \delta_A \sin\phi)]$$

Substituting Equations (12) into (7) and Equations (13) into (11), the six-degree-of-freedom equations describing the motion of a missile (excluding gravity) can be written as:

Forces:

$$-\, \bar{q}\, A\, \bar{C}_X = m\, [\dot{u} + \omega_{YH}\, w - \omega_{ZH}\, v]; \quad \left\{ \underline{\text{in } \bar{i}_H \text{ direction}} \right\} \tag{14}$$

$$-\, \bar{q}\, A\, \bar{C}_{N\alpha} \frac{v}{V} + \frac{\bar{q}\, A\, d}{2\, V} (\omega_T)_{XH}\, \bar{C}_{Yp\alpha} \frac{w}{V} = m\, [\dot{v} + \omega_{ZH}\, u - \omega_{XH}\, w]; \left\{ \underline{\bar{j}_H \text{ direction}} \right\} \tag{15}$$

$$-\, \bar{q}\, A\, \bar{C}_{n\alpha} \frac{w}{V} - \frac{\bar{q}\, A\, d}{2\, V} (\omega_T)_{XH}\, \bar{C}_{Yp\alpha} \frac{v}{V} = m\, [\dot{w} + \omega_{XH}\, v - \omega_{YH}\, u]; \left\{ \underline{\bar{k}_H \text{ direction}} \right\} \tag{16}$$

Moments:

$$\frac{\bar{q}\ A\ d^2}{2\ V}\ (\omega_T)_{XH}\ C_{lp} = I_x\ (\dot{\omega}_T)_{XH} + \omega_{YH}\ I_y\ (\omega_T)_{ZH} - \omega_{ZH}\ I_y\ (\omega_T)_{YH};\ \left\{\underline{\text{rotation } \bar{i}_H}\right\} \tag{17}$$

$$\bar{q}\ A\ d\ \bar{C}_{m\alpha}\ \frac{w}{V} + \frac{\bar{q}\ A\ d^2}{2\ V}\ (\omega_T)_{XH}\ \bar{C}_{np\alpha}\ \frac{v}{V} + \frac{\bar{q}\ A\ d^2}{2\ V}\ (\omega_T)_{YH}\ \bar{C}_{mq}$$

$$- \bar{q}\ A\ d\ [C_{m\delta}\ (\delta_B \sin\phi - \delta_A \cos\phi)] = I_y\ (\dot{\omega}_T)_{YH} + \omega_{ZH}\ I_x\ (\omega_T)_{XH}$$

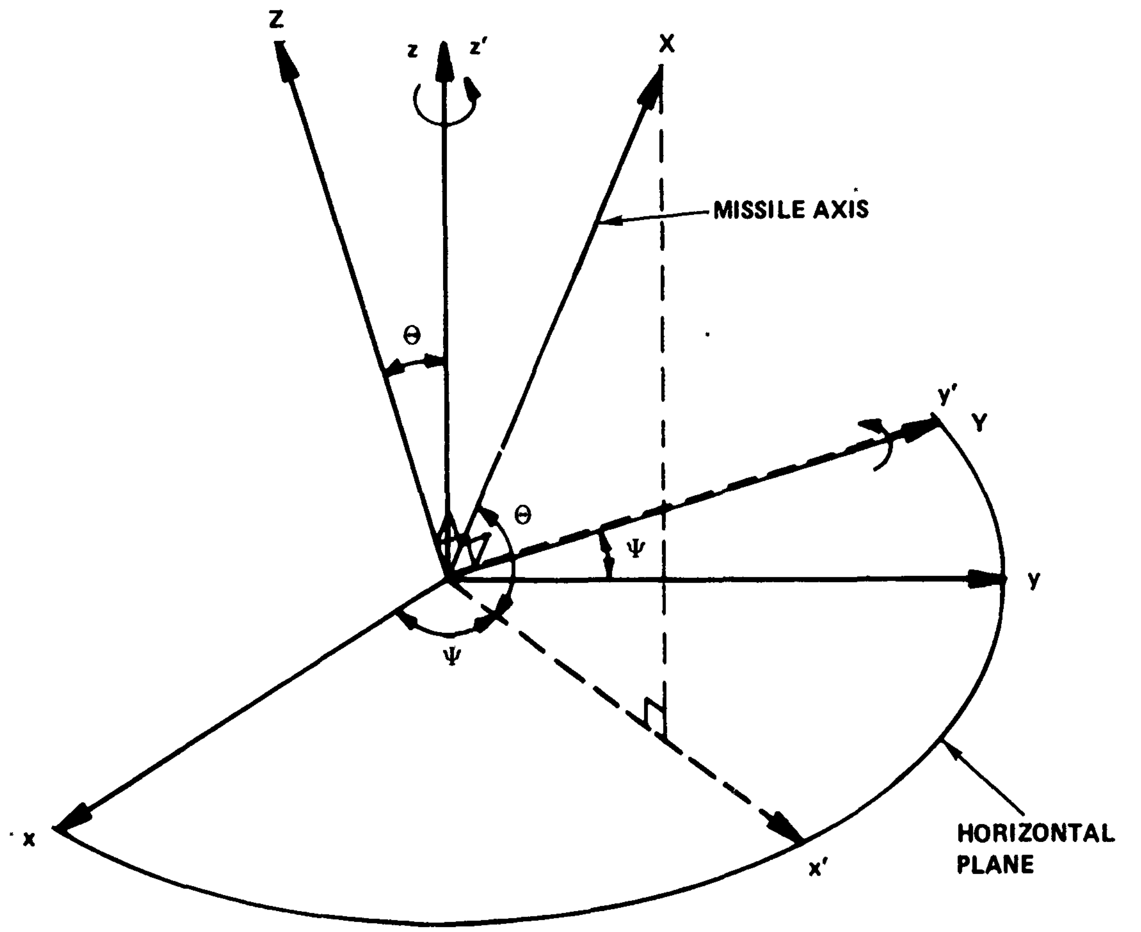
$$- \omega_{XH}\ I_y\ (\omega_T)_{ZH};\ \left\{\underline{\text{rotation } \bar{j}_H}\right\} \tag{18}$$

$$- \bar{q}\ A\ d\ \bar{C}_{m\alpha}\ \frac{v}{V} + \frac{\bar{q}\ A\ d^2}{2\ V}\ (\omega_T)_{XH}\ \bar{C}_{np\alpha}\ \frac{w}{V} + \frac{\bar{q}\ A\ d^2}{2\ V}\ (\omega_T)_{ZH}\ \bar{C}_{mq}$$

$$+ \bar{q}\ A\ d\ [C_{m\delta}\ (\delta_B \cos\phi + \delta_A \sin\phi)] = I_y\ (\dot{\omega}_T)_{ZH} + \omega_{XH}\ I_y\ (\omega_T)_{YH}$$

$$- \omega_{YH}\ I_x\ (\omega_T)_{XH};\ \left\{\underline{\text{rotation } \bar{k}_H}\right\} \tag{19}$$

Equations (14) through (19) are of limited use until one knows explicitly how the missile-fixed coordinate system (H) is moving with respect to earth-fixed coordinates (E). Further, one often has vector quantities expressed in earth coordinates (such as wind and gravitational attraction) which must be properly introduced into the (H) coordinates. Therefore, additional relationships between the (H) and (E) coordinates must be derived. In particular, one must know how the (H) coordinate system is oriented relative to the (E) coordinates at all times. The use of Euler angles seems to be the most straightforward approach. In it, one rotates a coordinate system, initially coincident with the (E) system, about selected axes so that after the rotations are performed in a specified sequence, this coordinate system will have the same orientation as the (H) system. These rotations are illustrated by Figure 2.



x, y, z — Earth Axis Coordinates

x', y', z' — Intermediate Axis Coordinates (after first rotation – through Ψ)

X, Y, Z — Missile Axis Coordinates

Θ and Ψ are the Euler angles

Figure 2. Euler Coordinate System

Figure 2 shows the (H) and (E) coordinate system. The missile-fixed rolling coordinate system (M) is defined as simply the (H) coordinate system rotating about the missile axis at an angular rate of $\dot{\phi}$ through the angle ϕ. Previously defined quantities in the equations of motion may now be related through the Euler angles to earth-fixed coordinates as follows:

$$\omega_{XH} = -\dot{\psi} \sin\theta$$

$$\omega_{YH} = \dot{\theta}$$

$$\omega_{ZH} = \dot{\psi} \cos\theta$$

$$\omega_{XH} = -\omega_{ZH} \tan\theta$$

$$(\omega_T)_{XH} = \dot{\phi} + \omega_{XH} = \dot{\phi} - \dot{\psi} \sin\theta \tag{20}$$

$$(\omega_T)_{YH} = \omega_{YH} = \dot{\theta}$$

$$(\omega_T)_{ZH} = \omega_{ZH} = \dot{\psi} \cos\theta$$

$$(\dot{\omega}_T)_{XH} = \ddot{\phi} - \dot{\omega}_{ZH} \tan\theta - \omega_{ZH}\, \dot{\theta} \sec^2\theta$$

$$= \ddot{\phi} - \tan\theta\, (\ddot{\psi} \cos\theta - \dot{\psi}\, \dot{\theta} \sin\theta) - \dot{\theta}\, \dot{\psi}/\cos\theta$$

$$(\dot{\omega}_T)_{YH} = \ddot{\theta}$$

$$(\dot{\omega}_T)_{ZH} = \ddot{\psi} \cos\theta - \dot{\psi}\, \dot{\theta} \sin\theta$$

Through appropriate substitution of Equations (20), the equations of motion may be rewritten as:

<u>Force Equations</u>:

$$-\overline{q}\, A\, \overline{C}_X = m\, [\dot{u} + \dot{\theta}\, w - \dot{\psi} \cos\theta\, v] \tag{21}$$

$$;\left\{\underline{\text{In } \overline{i}_H \text{ (X) direction}}\right\}$$

$$
\begin{aligned}
&- \bar{q} A \bar{C}_{N\alpha} \frac{v}{V} + \frac{\bar{q} A d}{2 V} \bar{C}_{Yp\alpha} \frac{w}{V} (\dot{\phi} - \dot{\psi} \sin \theta) = m [\dot{v} + \dot{\psi} u \cos \theta \\
&+ \dot{\psi} w \sin \theta] \qquad ; \{\text{In } \bar{j}_H \text{ (Y) direction}\}
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&- \bar{q} A \bar{C}_{N\alpha} \frac{w}{V} - \frac{\bar{q} A d}{2 V} \bar{C}_{Yp\alpha} \frac{v}{V} (\dot{\phi} - \dot{\psi} \sin \theta) = m [\dot{w} - \dot{\psi} v \sin \theta \\
&- \dot{\theta} u] \qquad ; \{\text{In } \bar{k}_H \text{ (Z) direction}\}
\end{aligned}
\tag{23}
$$

<u>Moment Equations:</u>

$$
\begin{aligned}
&\frac{\bar{q} A d^2}{2 V} C_{1p} (\dot{\phi} - \dot{\psi} \sin \theta) = I_x [\ddot{\phi} - \tan \theta (\ddot{\psi} \cos \theta - \dot{\psi} \dot{\theta} \sin \theta) \\
&- \dot{\psi} \dot{\theta}/\cos \theta] \qquad ; \{\text{Rotation about } \bar{i}_H \text{ (X)}\}
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&\bar{q} A d \bar{C}_{m\alpha} \frac{w}{V} + \frac{\bar{q} A d^2}{2 V} \bar{C}_{np\alpha} \frac{v}{V} (\dot{\phi} - \dot{\psi} \sin \theta) + \frac{\bar{q} A d^2}{2 V} \dot{\theta} \bar{C}_{mq} \\
&- \bar{q} A d [C_{m\delta} (\delta_B \sin \phi - \delta_A \cos \phi)] = I_y \ddot{\theta} + I_x \dot{\psi} \cos \theta (\dot{\phi} - \dot{\psi} \sin\theta) \\
&+ I_y \dot{\psi}^2 \cos \theta \sin \theta \qquad ; \{\text{Rotation about } \bar{j}_H \text{ (Y)}\}
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&- \bar{q} A d \bar{C}_{m\alpha} \frac{v}{V} + \frac{\bar{q} A d^2}{2 V} \bar{C}_{np\alpha} \frac{w}{V} (\dot{\phi} - \dot{\psi} \sin \theta) + \frac{\bar{q} A d^2}{2 V} \bar{C}_{mq} \dot{\psi} \cos \theta \\
&+ \bar{q} A d [C_{m\delta} (\delta_B \cos \phi + \delta_A \sin \phi)] = I_y (\ddot{\psi} \cos \theta - \dot{\psi} \dot{\theta} \sin \theta) \\
&+ I_y (- \dot{\psi} \sin \theta) \dot{\theta} - I_x \dot{\theta} (\dot{\phi} - \dot{\psi} \sin \theta) \qquad ; \{\text{Rotation about } \bar{k}_H \text{ (Z)}\}
\end{aligned}
\tag{26}
$$

Equations (21) through (26) are the 6-DOF equations of motion expressed in terms of Euler angles without gravitational effects.

Gravitational Force:

Assumption: Flat Earth

$$(\text{Gravity}) = -mg\ \bar{k}_E$$

where $\bar{k}_E$ is unit vector in the $+Z_E$ direction

Replacing $\bar{k}_E$ by its representation in (H) coordinates, one obtains:

$$-mg\ \bar{k}_E = mg \sin\theta\ \bar{i}_H - mg \cos\theta\ \bar{k}_H$$

The gravity force can now be substituted directly into Equations (21) and (23).

Spin is defined as the total angular velocity of the missile about its own axis.

$$\text{Spin} = p = (\omega_T)_{XH} = \dot{\phi} - \dot{\psi} \sin\theta$$

The 6-DOF equations of motion, upon substitution of the gravitational force and spin identity and rearranging terms, are written as follows:

Force Equations:

$$\dot{u} = -\bar{q}\frac{A}{m}\bar{C}_X + g \sin\theta - \dot{\theta}\, w + \dot{\psi} \cos\theta\, v \tag{27}$$

$$\dot{v} = -\bar{q}\frac{A}{m}\bar{C}_{N\alpha}\frac{v}{V} + \frac{\bar{q}\, A\, d\, p}{m\, 2\, V}\bar{C}_{Yp\alpha}\frac{w}{V} - \dot{\psi} \cos\theta\, [u + w \tan\theta] \tag{28}$$

$$\dot{w} = -\frac{\bar{q}\,A}{m}\,\bar{C}_{N\alpha}\,\frac{w}{V} - \frac{\bar{q}\,A}{m}\,\frac{pd}{2\,V}\,\bar{C}_{Yp\alpha}\,\frac{v}{V} - g\cos\theta + \dot{\psi}\,(\sin\theta)\,v + \dot{\theta}\,u \qquad (29)$$

Moment Equations:

$$\ddot{\phi} = \frac{\bar{q}\,A\,d^2\,p}{I_x\,2V}\,C_{1p} + \tan\theta[\ddot{\psi}\cos\theta - \dot{\psi}\,\dot{\theta}\sin\theta] + \dot{\theta}\,\dot{\psi}/\cos\theta \qquad (30)$$

$$\ddot{\theta} = \frac{\bar{q}\,A\,d}{I_y}\,\bar{C}_{m\alpha}\,\frac{w}{V} + \frac{\bar{q}\,A\,d^2}{2V\,I_y}\,\bar{C}_{mq}\,\dot{\theta} + \frac{\bar{q}\,A\,d^2\,p}{2V\,I_y}\,\bar{C}_{np\alpha}\,\frac{v}{V} \qquad (31)$$

$$- \frac{\bar{q}\,A\,d}{I_y}\,[C_{m\delta}(\delta_B\sin\phi - \delta_A\cos\phi)] - \dot{\psi}\cos\theta\,p\,\frac{I_x}{I_y}$$

$$- \dot{\psi}^2\cos\theta\sin\theta$$

$$\ddot{\psi} = [\,-\frac{\bar{q}\,A\,d}{I_y}\,\bar{C}_{m\alpha}\,\frac{v}{V} + \frac{\bar{q}\,A\,d^2}{2V\,I_y}\,\bar{C}_{mq}\,\dot{\psi}\cos\theta + \frac{\bar{q}\,A\,d^2\,p}{2V\,I_y}\,\bar{C}_{np\alpha}\,\frac{w}{V} \qquad (32)$$

$$+ \frac{\bar{q}\,A\,d}{I_y}\,[C_{m\delta}\,(\delta_B\cos\phi + \delta_A\sin\phi)] + 2\,\dot{\theta}\,\dot{\psi}\sin\theta$$

$$+ \dot{\theta}\,p\,I_x/I_y\,]\,/\cos\theta$$

For the data reduction technique being discussed, the aerodynamic coefficients $\bar{C}_{m\alpha}$, $\bar{C}_{np\alpha}$, and $\bar{C}_{mq}$ previously defined are assumed to be non-linear and are therefore expanded to include higher order dependencies on the angle of attack, $\bar{\alpha}$. These dependencies are assumed to exist as a function of sin $\bar{\alpha}$. The following non-linearities are assumed:

$$\overline{C}_{m\alpha} = C_{m\alpha} + C_{m\alpha_3} \sin^2 \overline{\alpha} + C_{m\alpha_5} \sin^4 \overline{\alpha} + C_{m\alpha_7} \sin^6 \overline{\alpha} + C_{m\alpha_V} (V_o - V)$$

$$\overline{C}_{mq} = C_{mq} + C_{mq_2} \sin^2 \overline{\alpha}$$

$$\overline{C}_{np\alpha} = C_{np\alpha} + C_{np\alpha_3} \sin^2 \overline{\alpha} + C_{np\alpha_5} \sin^4 \overline{\alpha}$$

The preceding moment coefficient expansions are utilized in the angular motion programs.

Similar expansions on the force coefficients are used in the translational equations of motion which will be developed next.

b. Translational Equations of Motion

In the derivation, the previously defined force equations of motion will be utilized with the addition of a trim force in the missile Y-Z plane, defined as components in the missile Y and Z directions. This trim force will account for any slight asymmetries present (intentional or otherwise) in the missile configuration.

$$\dot{u} = -\frac{\overline{q} A}{m} \overline{C}_X + g \sin\theta - \dot{\theta} w + \dot{\psi} \cos\theta \, v \tag{33}$$

$$\dot{v} = -\frac{\overline{q} A}{m} \overline{C}_{N\alpha} \frac{v}{V} + \frac{\overline{q} A}{m} \frac{p d}{2 V} \overline{C}_{Yp\alpha} \frac{w}{V} - \dot{\psi} \cos\theta \, [u + w \tan\theta]$$
$$- C_{N\delta_A} (\delta_B \cos\phi - \delta_A \sin\phi) \left(\frac{\overline{q} A}{m}\right) \tag{34}$$

$$\dot{w} = -\frac{\overline{q} A}{m} \overline{C}_{N\alpha} \frac{w}{V} - \frac{\overline{q} A}{m} \frac{p d}{2 V} \overline{C}_{Yp\alpha} \frac{v}{V} - g \cos\theta + \dot{\psi} \sin\theta \, v$$
$$+ \dot{\theta} u - C_{N\delta_A} (\delta_A \cos\phi + \delta_B \sin\phi) \left(\frac{\overline{q} A}{m}\right) \tag{35}$$

The following expressions relate the earth-fixed velocities to the missile velocities using the definitions of the Euler angles:

$$
\begin{aligned}
V_{Xe} &= u \cos\theta \cos\psi - v \sin\psi + w \sin\theta \cos\psi \\
V_{Ye} &= u \cos\theta \sin\psi + v \cos\psi + w \sin\theta \sin\psi \\
V_{Ze} &= - u \sin\theta + w \cos\theta
\end{aligned}
\tag{36}
$$

Taking the derivative of Equations (36) with respect to time, one obtains:

$$
\begin{aligned}
\dot{V}_{Xe} = {} & \dot{u} \cos\theta \cos\psi - \dot{v} \sin\psi + \dot{w} \sin\theta \cos\psi \\
& - u \sin\theta \cos\psi\, \dot{\theta} - u \cos\theta \sin\psi\, \dot{\psi} - v \cos\psi\, \dot{\psi} \\
& + w \cos\theta \cos\psi\, \dot{\theta} - w \sin\theta \sin\psi\, \dot{\psi}
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
\dot{V}_{Ye} = {} & \dot{u} \cos\theta \sin\psi + \dot{v} \cos\psi + \dot{w} \sin\theta \sin\psi \\
& - u \sin\theta \sin\psi\, \dot{\theta} + u \cos\theta \cos\psi\, \dot{\psi} - v \sin\psi\, \dot{\psi} \\
& + w \cos\theta \sin\psi\, \dot{\theta} + w \sin\theta \cos\psi\, \dot{\psi}
\end{aligned}
\tag{38}
$$

$$
\dot{V}_{Ze} = - \dot{u} \sin\theta + \dot{w} \cos\theta - u \cos\theta\, \dot{\theta} - w \sin\theta\, \dot{\theta}
\tag{39}
$$

Substitution of Equations (33), (34), and (35) into (37), (38), and (39) provides the translational equations of motion:

$$
\begin{aligned}
\dot{V}_{Xe} = {} & - \frac{\rho A}{2 m} V^2 \bar{C}_X \cos\theta \cos\psi + \frac{\rho A}{2 m} V^2 \bar{C}_{N\alpha} \frac{v}{V} \sin\psi \\
& - \frac{\rho A}{2 m} V^2 \bar{C}_{N\alpha} \frac{w}{V} \sin\theta \cos\psi - \frac{\rho A\, pd}{4 m} V \bar{C}_{Yp\alpha} \frac{w}{V} \sin\psi
\end{aligned}
\tag{40}
$$

$$- \frac{\rho A pd}{4 m} V \overline{C}_{Yp\alpha} \frac{v}{V} \sin\theta \cos\psi$$

$$+ \frac{\rho A}{2 m} V^2 [C_{N\delta_A} (\delta_B \cos\phi - \delta_A \sin\phi) \sin\psi$$

$$- C_{N\delta_A} (\delta_A \cos\phi + \delta_B \sin\phi) \sin\theta \cos\psi]$$

$$\dot{V}_{Ye} = - \frac{\rho A}{2 m} V^2 \overline{C}_X \cos\theta \sin\psi - \frac{\rho A}{2 m} V^2 \overline{C}_{N\alpha} \frac{v}{V} \cos\psi \tag{41}$$

$$- \frac{\rho A}{2 m} V^2 \overline{C}_{N\alpha} \frac{w}{V} \sin\theta \sin\psi + \frac{\rho A pd}{4 m} V \overline{C}_{Yp\alpha} \frac{w}{V} \cos\psi$$

$$- \frac{\rho A pd}{4 m} V \overline{C}_{Yp\alpha} \frac{v}{V} \sin\theta \sin\psi$$

$$- \frac{\rho A}{2 m} V^2 [C_{N\delta_A} (\delta_B \cos\phi - \delta_A \sin\phi) \cos\psi$$

$$+ C_{N\delta_A} (\delta_A \cos\phi + \delta_B \sin\phi) \sin\theta \sin\psi]$$

$$\dot{V}_{Ze} = \frac{\rho A}{2 m} V^2 \overline{C}_X \sin\theta - \frac{\rho A}{2 m} V^2 \overline{C}_{N\alpha} \frac{w}{V} \cos\theta \tag{42}$$

$$- \frac{\rho A pd}{4 m} V \overline{C}_{Yp\alpha} \frac{v}{V} \cos\theta$$

$$- C_{N\delta_A} (\delta_A \cos\phi + \delta_B \sin\phi) \cos\theta \left(\frac{A \rho}{2 m} V^2 \right) - g$$

The following non-linearities are assumed for the force coefficients:

$$\overline{C}_X = C_X + C_{X_2} \varepsilon^2 + C_{X_V} (V - V_o)$$

$$\overline{C}_{N\alpha} = C_{N\alpha} + C_{N\alpha_3} \varepsilon^2$$

$$\overline{C}_{Yp\alpha} = C_{Yp\alpha} + C_{Yp\alpha_3}\, \varepsilon^2$$

Where: $\varepsilon^2 = \sin^2 \overline{\alpha}$

V_o = Reference Velocity (constant)

c. Modified Six Degrees of Freedom

As ballistic ranges are level and the ballistic trajectories flat, the equations of motion from which the parametric equations are derived can be modified. This is a special case and cannot be applied to high angle fire problems ($\gamma_e > 30^o$) such as yaw sonde reductions. Reference 11 discusses the consequences of this assumption in the high angle fire mode.

The following assumptions are made:

$$\theta_m = \theta, \ \psi_m = \psi$$

and from linear theory (Ref. 17)

$$C_{mq_{(3D)}} = C_{mq} - C_{N\alpha}\ 2\ I_y/md^2$$

$$C_{np\alpha_{(3D)}} = C_{np\alpha} + C_{N\alpha}\ 2\ I_x/md^2$$

Essentially what is done is to approximate the 6-DOF equations with 3-DOF equations and allow correction for the effect on the damping of $C_{N\alpha}$.

The effect of trajectory curvature is accounted for by the angular rate g cos θ/V.

$$\dot{\theta}_{(3D)} = \dot{\theta}_{(6D)} + g \cos \theta/V$$

This term results in the generation of the yaw of repose.

It should be pointed out and understood that the modified equations are only used in deriving the parametric equations and are not used for computing theoretical motions. Equations (28), (29), (31), and (32) are utilized for motion generation.

Approximate 6-DOF Equations

Using the previous assumptions, Equations (31) and (32) reduce to:

$$
\begin{aligned}
\ddot{\theta} = {} & M_{m\alpha} \sin\theta \cos\psi + M_{mq} [\dot{\theta} + g \cos\theta/V] \\
& - M_{np\alpha} \sin\psi - M_{m\delta_{\theta}} - \dot{\psi} \cos\theta \; p \; I_x/I_y \\
& - \dot{\psi}^2 \cos\theta \sin\theta
\end{aligned}
$$

$$
\begin{aligned}
\ddot{\psi} = {} & [M_{m\alpha} \sin\psi + M_{mq} \dot{\psi} \cos\theta + M_{np\alpha} \sin\theta \cos\psi + M_{m\delta_{\psi}} \\
& + (\dot{\theta} + g \cos\theta/V) \; p \frac{I_x}{I_y} + 2 \sin\theta \, (\dot{\theta} + g \cos\theta/V) \, \dot{\psi}]/\cos\theta
\end{aligned}
$$

Where:

$$
M_{m\alpha} = \bar{q} \frac{A d}{I_y} [C_{m\alpha} + C_{m\alpha_3} \varepsilon^2 + C_{m\alpha_5} \varepsilon^4 + C_{m\alpha_v} (V_o - V)]
$$

$$
M_{mq} = \frac{\bar{q} A d^2}{2 V I_y} [C_{mq} + C_{mq_2} \varepsilon^2 - C_{N\alpha} \, 2 \, I_y/md^2]
$$

$$
M_{np\alpha} = \frac{\bar{q} A d^2}{2 V I_y} \; p \; [C_{np\alpha} + C_{np\alpha_3} \varepsilon^2 + C_{np\alpha_5} \varepsilon^4 + C_{N\alpha} \, 2 \, I_x/md^2]
$$

$$M_{m\delta_\theta} = \bar{q}\ \frac{A\ d}{I_y}\ [C_{m\delta}\ (\delta_B \sin\phi - \delta_A \cos\phi)]$$

$$M_{m\delta_\psi} = \bar{q}\ \frac{A\ d}{I_y}\ [C_{m\delta}\ (\delta_B \cos\phi + \delta_A \sin\phi)]$$

As the term $g \cos\theta/V$ is usually small and $1.0 > \cos\theta > 0.9$, this term is considered a constant when partial derivatives are taken.

d. Roll Equation

As previously derived and stated in Equation (30) the 6-DOF roll equation is:

$$\ddot{\phi} = \frac{\bar{q}\ A\ d^2}{2\ V\ I_x}\ C_{lp}\ (\dot{\phi} - \dot{\psi} \sin\theta) + \tan\theta\ (\ddot{\psi} \cos\theta - \dot{\psi}\ \dot{\theta} \sin\theta) + \frac{\dot{\psi}\ \dot{\theta}}{\cos\theta}$$

The preceding equation may be restated in the following equivalent form:

$$\dot{\phi} = p + \dot{\psi} \sin\theta \tag{30a}$$

$$\dot{p} = \frac{\bar{q}\ A\ d^2}{2\ V\ I_x}\ p\ C_{lp} \tag{30b}$$

The above two equations are integrated simultaneously to produce the roll angle, roll rate, and spin rate with the advantage of fewer higher order terms than equation (30).

$\dot{\psi}$ will be computed from the available ψ versus time history. It should be pointed out that the roll reduction is accomplished at the same time as the translational motion reduction.

A fin cant moment $(C_{l\delta}\ \delta\ \bar{q}\ \frac{A\ d}{I_x})$ and a velocity dependent $C_{lp} \left[C_{lpv}\ (V_o - V)\ \frac{\bar{q}\ A\ d^2\ p}{2\ V\ I_x} \right]$ are added to equation (30b) for expanded capability.

2. CHAPMAN-KIRK TECHNIQUE

The procedures utilized in the implementation of the Chapman-Kirk technique are as follows:

1. Formulate equations of motion which are adequate to simulate the experimental data.

   a. Select a broad general set of aerodynamic terms.

   b. Define possible non-linear terms; i.e., Mach number and angle of attack.

2. Equations of motion are partially differentiated with respect to each coefficient to form a set of parametric differential equations.

3. Integrate equations of motion numerically utilizing:

   a. Estimated Aerodynamic Coefficients
   b. Estimated Initial Conditions

4. Integrate parametric equations numerically to obtain values for the partial derivatives of the state variables with respect to each coefficient.

5. A differential corrections equation is set up from a Taylor expansion of the dependent variable of the equations of motion.

6. The experimental data are then compared to the dependent variables in a least squares sense, and corrections to the coefficients and initial conditions are obtained from the differential correction equations.

7. Step 3 is repeated with adjusted coefficients and initial conditions, and the process is repeated until convergence is achieved.

a. <u>Differential Corrections and Least Squares Theory</u>

The methods of least squares and differential corrections are employed by computer programs to obtain corrections to be applied to the coefficients so that the solutions to the equations of motion are a better fit to the test data. A brief description of this technique will be given considering two data sets. The primary goal of this description is to illustrate how two or more data sets may be handled simultaneously. Consider the following equations:

$$\ddot{\alpha}_1 + C_1\dot{\alpha}_1 + C_2\alpha_1 + C_3 = 0$$

$$\ddot{\alpha}_2 + C_1\dot{\alpha}_2 + C_2\alpha_2 + C_3 = 0$$

α_1 = Angles from data set one

α_2 = Angles from data set two

Although the above equations are linear, it is well known that aerodynamic equations of motion are often non-linear in nature. Regardless of the linearity or non-linearity of the equations of motion, the theory of least squares is not sufficient in itself to solve for aerodynamic properties, and differential corrections are employed. The above equations are presented only for simplicity and clarity. The real equations of motion cover all six degrees of freedom and are coupled.

Let: $C_4 = \alpha_{10}$ = initial condition on α_1

$C_5 = \dot{\alpha}_{10}$ = initial condition on $\dot{\alpha}_1$

$C_6 = \alpha_{20}$ = initial condition on α_2

$C_7 = \dot{\alpha}_{20}$ = initial condition on $\dot{\alpha}_2$

C_1, C_2, and C_3 are constant coefficients to be determined and are common to both data sets.

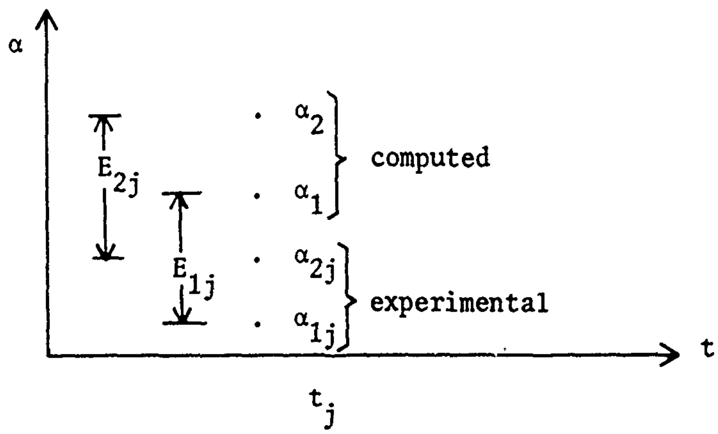
Let: t = The variable parameter (time)

$$\alpha_1 = f\ (C_1, C_2, C_3, C_4, C_5, t)$$

$$\alpha_2 = g\ (C_1, C_2, C_3, C_6, C_7, t)$$

$$E_{1j} = f\ (C_1, C_2, C_3, C_4, C_5, t_j) - \alpha_{1j}$$

$$E_{2j} = g\ (C_1, C_2, C_3, C_6, C_7, t_j) - \alpha_{2j}$$



With first approximations of the constants, a Taylor series expansion gives:

$$\alpha_{1j} + E_{1j} = f\ (C_1, C_2, C_3, C_4, C_5, t_j) = f\ (C_o, t_j) + \Delta C_1 f_{c_1}$$

$$+ \Delta C_2 f_{c_2} + \Delta C_3 f_{c_3} + \Delta C_4 f_{c_4}$$

$$+ \Delta C_5 f_{c_5} + \frac{[\Delta C_1]^2}{2!} f_{c_{1_2}} + \frac{[\Delta C_2]^2}{2!} f_{c_{2_2}}$$

$$+ \frac{[\Delta C_3]^2}{2!} f_{c_{3_2}} + \frac{[\Delta C_4]^2}{2!} f_{c_{4_2}} + \frac{[\Delta C_5]^2}{2!} f_{c_{5_2}} + \dots$$

$$\alpha_{2j} + E_{2j} = g\ (C_1, C_2, C_3, C_6, C_7, t_j) = g\ (C_o, t_j) + \Delta C_1 g_{c_1}$$

$$+ \Delta C_2 g_{c_2} + \Delta C_3 g_{c_3} + \Delta C_4 g_{c_4}$$

$$+ \Delta C_5 \, g_{c_5} + \frac{[\Delta C_1]^2}{2!} g_{c_{1_2}} + \frac{[\Delta C_2]^2}{2!} g_{c_{2_2}}$$

$$+ \frac{[\Delta C_3]^2}{2!} g_{c_{3_2}} + \frac{[\Delta C_4]^2}{2!} g_{c_{6_2}} + \frac{[\Delta C_5]^2}{2!} g_{c_{7_2}} + \ldots + \text{(higher order terms)}$$

Where:

α_{1j}, α_{2j} are experimental points of data sets one and two, respectively.

E_{1j}, E_{2j} are the errors between the data and the theoretical points of data sets one and two, respectively.

$f\,(C_o, t) = f$ [initial guesses at constants and initial conditions (α_1), t]

$g\,(C_o, t) = g$ [initial guesses at constants and initial conditions (α_2), t]

ΔC_i is the change in C_i between successive approximations.

f_{c_i}, g_{c_i} are the partials of f and g, respectively, with respect to C_i

$f_{c_{i_2}}$, $g_{c_{i_2}}$ are the second partials of f and g, respectively, with respect to C_i

Assuming that the Δ C's are small such that the second and higher order terms can be neglected, the following set of equations result:

$$\alpha_{1j} + E_{1j} = f\,(C_o, t_j) + \Delta C_1 f_{c_1} + \Delta C_2 f_{c_2} + \Delta C_3 f_{c_3} + \Delta C_4 f_{c_4} + \Delta C_5 f_{c_5}$$

$$\alpha_{2j} + E_{2j} = g\,(C_o, t_j) + \Delta C_1 g_{c_1} + \Delta C_2 g_{c_2} + \Delta C_3 g_{c_3} + \Delta C_6 g_{c_6} + \Delta C_7 g_{c_7}$$

Let the residuals be the error at each point that result from using the initial guesses of the constants and initial conditions.

$$R^2_{\alpha_{1j}} = [\alpha_{1j} - f\,(C_o,\, t_j)]^{\,2} = [\Delta\, C_1\, f_{c_1} + \Delta\, C_2\, f_{c_2} + \Delta\, C_3\, f_{c_3}$$

$$+ \Delta\, C_4\, f_{c_4} + \Delta\, C_5\, f_{c_5} - E_{1j}]^2$$

$$R^2_{\alpha_{2j}} = [\alpha_{2j} - g\,(C_o,\, t_j)]^{\,2} = [\Delta\, C_1\, g_{c_1} + \Delta\, C_2\, g_{c_2} + \Delta\, C_3\, g_{c_3}$$

$$+ \Delta\, C_6\, g_{c_6} + \Delta\, C_7\, g_{c_7} - E_{2j}]^2$$

Introducing least squares theory, the following results:

$$0 = \frac{\partial\, R^2_{\alpha_{1j}}}{\partial\, \Delta\, C_i} = 2\,[\Delta\, C_1\, f_{c_1} + \Delta\, C_2\, f_{c_2} + \Delta\, C_3\, f_{c_3} + \Delta\, C_4\, f_{c_4} + \Delta\, C_5\, f_{c_5} - E_{1j}]f_{c_i}$$

$$0 = \frac{\partial\, R^2_{\alpha_{2j}}}{\partial\, \Delta\, C_i} = 2\,[\Delta\, C_1\, g_{c_1} + \Delta\, C_2\, g_{c_2} + \Delta\, C_3\, g_{c_3} + \Delta\, C_6\, g_{c_6} + \Delta\, C_7\, g_{c_7} - E_{2j}]g_{c_i}$$

In matrix notation:

$$[A]\;[\Delta\; C] = [R]$$

Let $H_{ik} = f_{c_i}\, f_{c_k} + g_{c_i}\, g_{c_k}$

$$[A] = \begin{bmatrix} \Sigma H_{11} & \Sigma H_{12} & \Sigma H_{13} & \Sigma H_{14} & \Sigma H_{15} & \Sigma H_{16} & \Sigma H_{17} \\ \Sigma H_{21} & \Sigma H_{22} & \Sigma H_{23} & \Sigma H_{24} & \Sigma H_{25} & \Sigma H_{26} & \Sigma H_{27} \\ \Sigma H_{31} & \Sigma H_{32} & \Sigma H_{33} & \Sigma H_{34} & \Sigma H_{35} & \Sigma H_{36} & \Sigma H_{37} \\ \Sigma H_{41} & \Sigma H_{42} & \Sigma H_{43} & \Sigma H_{44} & \Sigma H_{45} & \Sigma H_{46} & \Sigma H_{47} \\ \Sigma H_{51} & \Sigma H_{52} & \Sigma H_{53} & \Sigma H_{54} & \Sigma H_{55} & \Sigma H_{56} & \Sigma H_{57} \\ \Sigma H_{61} & \Sigma H_{62} & \Sigma H_{63} & \Sigma H_{64} & \Sigma H_{65} & \Sigma H_{66} & \Sigma H_{67} \\ \Sigma H_{71} & \Sigma H_{72} & \Sigma H_{73} & \Sigma H_{74} & \Sigma H_{75} & \Sigma H_{76} & \Sigma H_{77} \end{bmatrix}$$

$$[R] = \begin{bmatrix} \Sigma\, E_1\, f_{c_1} + E_2\, g_{c_1} \\ \Sigma\, E_1\, f_{c_2} + E_2\, g_{c_2} \\ \Sigma\, E_1\, f_{c_3} + E_2\, g_{c_3} \\ \Sigma\, E_1\, f_{c_4} + E_2\, g_{c_4} \\ \Sigma\, E_1\, f_{c_5} + E_2\, g_{c_5} \\ \Sigma\, E_1\, f_{c_6} + E_2\, g_{c_6} \\ \Sigma\, E_1\, f_{c_7} + E_2\, g_{c_7} \end{bmatrix} \qquad [\Delta\, C] = \begin{bmatrix} \Delta\, C_1 \\ \Delta\, C_2 \\ \Delta\, C_3 \\ \Delta\, C_4 \\ \Delta\, C_5 \\ \Delta\, C_6 \\ \Delta\, C_7 \end{bmatrix}$$

The summation is from 1 to N, where N is the number of data points in each data set.

f_{c_6}, f_{c_7}, g_{c_4}, g_{c_5} do not appear in the equations and are equal to zero. The reason for this is that the initial conditions are unique to each data set, as opposed to the common set of coefficients used by all data sets.

b. Parametric Equations

(1) Derivation of Generalized Parametric Differential Equation for Translational Motion

Consider equation (40):

$$\dot{V}_{Xe} + \frac{\rho\, A}{2\, m} V^2 \left[C_X + C_{X2}\, \varepsilon^2 + C_{Xm}\,(V - V_o)\right] \cos\theta \cos\psi \tag{43}$$

$$- \frac{\rho\, A}{2\, m} V^2 \left[C_{N\alpha} + C_{N\alpha_3}\, \varepsilon^2\right] \frac{v}{V} \sin\psi$$

$$+ \frac{\rho\, A}{2\, m} V^2 \left[C_{N\alpha} + C_{N\alpha_3}\, \varepsilon^2\right] \frac{w}{V} \sin\theta \cos\psi$$

$$+ \frac{\rho\ A\ pd}{4\ m} \frac{V^2}{V} [C_{Yp\alpha} + C_{Yp\alpha_3}\ \varepsilon^2]\ \frac{w}{V} \sin\psi$$

$$+ \frac{\rho\ A\ pd}{4\ m} \frac{V^2}{V}\ [C_{Yp\alpha} + C_{Yp\alpha_3}\ \varepsilon^2]\ \frac{v}{V} \sin\theta\ \cos\psi$$

$$- \frac{\rho\ A}{2\ m} V^2\ C_{N\delta_A}\ (\delta_B \cos\phi - \delta_A \sin\phi) \sin\psi$$

$$+ \frac{\rho\ A}{2\ m} V^2\ C_{N\delta_A}\ (\delta_A \cos\phi + \delta_B \sin\phi) \sin\theta\ \cos\psi = 0$$

Let:

$C_1 = X_0$

$C_2 = \dot{X}_0$

$C_3 = Y_0$

$C_4 = \dot{Y}_0$

$C_5 = Z_0$

$C_6 = \dot{Z}_0$

$C_7 = \frac{\rho\ A}{2\ m} C_X$

$C_8 = \frac{\rho\ A}{2\ m} C_{N\alpha}$

$C_9 = \frac{\rho\ A}{4\ m} pd\ C_{Yp\alpha}$

$C_{10} = \frac{\rho\ A}{2\ m}\ C_{X2}$

$C_{11} = \frac{\rho\ A}{2\ m}\ C_{N\alpha_3}$

$C_{12} = \frac{\rho\ A\ pd}{4\ m}\ C_{Yp\alpha_3}$

$C_{13} = \frac{\rho\ A}{2\ m}\ C_{Xm}$

$C_{14} = C_{N\delta_A}\ \delta_A \left(\frac{\rho\ A}{2\ m}\right)$

$C_{15} = C_{N\delta_A}\ \delta_B \left(\frac{\rho\ A}{2\ m}\right)$

Rewriting Equation (43) with these definitions, one obtains:

$$\dot{V}_{Xe} + V^2 [C_7 + C_{10} \varepsilon^2 + C_{13} (V - V_o)] \cos \theta \cos \psi \tag{44}$$

$$- V^2 [C_8 + C_{11} \varepsilon^2] (\frac{v}{V} \sin \psi - \frac{w}{V} \sin \theta \cos \psi)$$

$$+ V [C_9 + C_{12} \varepsilon^2] (\frac{w}{V} \sin \psi + \frac{v}{V} \sin \theta \cos \psi)$$

$$- V^2 [(C_{15} \cos \phi - C_{14} \sin \phi) \sin \psi - (C_{14} \cos \phi$$

$$+ C_{15} \sin \phi) \sin \theta \cos \psi] = 0$$

To perform differential corrections the $\partial X/\partial C_j$ is required where C_j = the initial conditions on X_e, Y_e, Z_e, $\dot{X}_e$, $\dot{Y}_e$, $\dot{Z}_e$, and the aerodynamic coefficients (C_7 - C_{15}). The $\partial X/\partial C_j$ is obtained through integration of $\partial \dot{V}_{Xe}/\partial C_j$. For computational purposes $\partial \dot{V}_{Xe}/\partial C_j$ is expressed in generalized form such that it will be applicable to all C_j's.

Taking the generalized partial of Equation (44) with respect to C_j, the following is obtained:

$$\frac{\partial \dot{V}_{Xe}}{\partial C_j} + \Big\{ V (2 C_7 + 2 C_{10} \varepsilon^2 + C_{13} (3 V - 2 V_o)) \cos \theta \cos \psi \tag{45}$$

$$- 2 V (C_8 + C_{11} \varepsilon^2) (\frac{v}{V} \sin \psi - \frac{w}{V} \sin \theta \cos \psi)$$

$$+ (C_9 + C_{12} \varepsilon^2) (\frac{w}{V} \sin \psi + \frac{v}{V} \sin \theta \cos \psi)$$

$$- 2 V [(C_{15} \cos \phi - C_{14} \sin \phi) \sin \psi - (C_{14} \cos \phi$$

$$+ C_{15} \sin \phi) \sin \theta \cos \psi] \Big\} \frac{\partial V}{\partial C_j} = - K_j \quad \partial C_j/\partial C_j$$

where: K_j = the constant multiplier of the specific C_j from Equation (44), for example, if $C_j = C_8$, then $K_j = - V^2 (\frac{v}{V} \sin \psi - \frac{w}{V} \sin \theta \cos \psi)$

It should be noted that v/V and w/W are trigonometric functions of the missile angles and are assumed to be independent variables. θ and ψ are functions of θ_m, ψ_m, V_{Xe}, V_{Ye}, and V_{Ze}. However as θ_m and ψ_m are considered known and the angular components γ_e and δ_e small, θ and ψ have also been designated independent variables. It should be pointed out that new profiles of θ and ψ are computed for each iteration thus minimizing this assumption providing convergence is being achieved.

$\partial V/\partial C_j$ must be expressed in terms of $\partial V_{Xe}/\partial C_j$ and is done as follows:

$$V = [V_{Xe}^{\ 2} + V_{Ye}^{\ 2} + V_{Xe}^{\ 2}]^{\frac{1}{2}}$$

Taking the generalized partial derivative, one obtains:

$$\partial V/\partial C_j = \frac{1}{2}\ \frac{[2\ V_{Xe}\ \partial V_{Xe}/\partial C_j + 2\ V_{Ye}\ \partial V_{Ye}/\partial C_j + 2\ V_{Ze}\ \partial V_{Ze}/\partial C_j]}{[V_{Xe}^{\ 2} + V_{Ye}^{\ 2} + V_{Ze}^{\ 2}]^{\frac{1}{2}}}$$

$$\partial V/\partial C_j = [\ \frac{V_{Xe}}{V}\ \frac{\partial V_{Xe}}{\partial C_j} + \frac{V_{Ye}}{V}\ \frac{\partial V_{Ye}}{\partial C_j} + \frac{V_{Ze}}{V}\ \frac{\partial V_{Ze}}{\partial C_j}\ ] \tag{46}$$

Substituting Equation (46) into (45), the generalized partial derivative may be written as follows:

$$\frac{\partial \dot{V}_{Xe}}{\partial C_j} + A_1\ [\ \frac{V_{Xe}}{V}\ \frac{\partial V_{Xe}}{\partial C_j} + \frac{V_{Ye}}{V}\ \frac{\partial V_{Ye}}{\partial C_j} + \frac{V_{Ze}}{V}\ \frac{\partial V_{Ze}}{\partial C_j}\ ] = B_1 \tag{47}$$

where: $A_1 = V\ (2\ C_7 + 2\ C_{10}\ \varepsilon^2 + C_{13}\ (3\ V - 2\ V_o))\cos\theta\cos\psi$

$$- 2\ V\ (C_8 + C_{11}\ \varepsilon^2)\ (\frac{v}{V}\sin\psi - \frac{w}{V}\sin\theta\cos\psi)$$

$$+ (C_9 + C_{12}\ \varepsilon^2)\ (\frac{w}{V}\sin\psi + \frac{v}{V}\sin\theta\cos\psi)$$

$$- 2\ V\ [(C_{15}\cos\phi - C_{14}\sin\phi)\sin\psi$$

$$- (C_{14}\cos\phi + C_{15}\sin\phi)\sin\theta\cos\psi]$$

$$B_1 = -K_j$$

Consider the remaining two equations of motion:

$$\dot{V}_{Ye} + V^2 [C_7 + C_{10} \varepsilon^2 + C_{13} (V - V_o)] \cos\theta \sin\psi \tag{48}$$

$$+ V^2 [C_8 + C_{11} \varepsilon^2] (\frac{v}{V} \cos\psi + \frac{w}{V} \sin\theta \sin\psi)$$

$$- V [C_9 + C_{12} \varepsilon^2] (\frac{w}{V} \cos\psi - \frac{v}{V} \sin\theta \sin\psi)$$

$$+ V^2 [(C_{15} \cos\phi - C_{14} \sin\phi) \cos\psi + (C_{14} \cos\phi$$

$$+ C_{15} \sin\phi) \sin\theta \sin\psi] = 0$$

$$\dot{V}_{Ze} - V^2 [C_7 + C_{10} \varepsilon^2 + C_{13} (V - V_o)] \sin\theta + V^2 [C_8 + C_{11} \varepsilon^2] \frac{w}{V} \cos\theta \tag{49}$$

$$+ V [C_9 + C_{12} \varepsilon^2] \frac{v}{V} \cos\theta + V^2 (C_{14} \cos\phi + C_{15} \sin\phi) \cos\theta - g = 0$$

Similarly, as before, the generalized partial derivatives of Equations (48) and (49) may be expressed in the following form:

$$\frac{\partial \dot{V}_{Ye}}{\partial C_j} + A_2 \left[\frac{V_{Xe}}{V} \frac{\partial V_{Xe}}{\partial C_j} + \frac{V_{Ye}}{V} \frac{\partial V_{Ye}}{\partial C_j} + \frac{V_{Ze}}{V} \frac{\partial V_{Ze}}{\partial C_j} \right] = B_2 \tag{50}$$

$$\frac{\partial \dot{V}_{Ze}}{\partial C_j} + A_3 \left[\frac{V_{Xe}}{V} \frac{\partial V_{Xe}}{\partial C_j} + \frac{V_{Ye}}{V} \frac{\partial V_{Ye}}{\partial C_j} + \frac{V_{Ze}}{V} \frac{\partial V_{Ze}}{\partial C_j} \right] = B_3 \tag{51}$$

where: $A_2 = V (2 C_7 + 2 C_{10} \varepsilon^2 + C_{13} (3 V - 2 V_o)) \cos\theta \sin\psi$

$$+ 2 V (C_8 + C_{11} \varepsilon^2) (\frac{v}{V} \cos\psi + \frac{w}{V} \sin\theta \sin\psi)$$

$$- (C_9 + C_{12}\,\varepsilon^2)\,(\frac{w}{V}\cos\psi - \frac{v}{V}\sin\theta\,\sin\psi)$$

$$+ 2\,\dot{V}\,[(C_{15}\cos\phi - C_{14}\sin\phi)\cos\psi$$

$$+ (C_{14}\cos\phi + C_{15}\sin\phi)\sin\theta\,\sin\psi]$$

$$B_2 = -K_j \quad (\text{of } \dot{V}_{Ye} \text{ equation})$$

$$A_3 = -V\,(2\,C_7 + 2\,C_{10}\,\varepsilon^2 + C_{13}\,(3\,V - 2\,V_o))\sin\theta$$

$$+ 2\,V\,(C_8 + C_{11}\,\varepsilon^2)\,\frac{w}{V}\cos\theta$$

$$+ (C_9 + C_{12}\,\varepsilon^2)\,\frac{v}{V}\cos\theta$$

$$+ 2\,V\,(C_{14}\cos\phi + C_{15}\sin\phi)\cos\theta$$

$$B_3 = -K_j \quad (\text{of } \dot{V}_{Ze} \text{ equation})$$

(2) <u>Derivation of Parametric Differential Equations for Approximate Six-DOF Equations</u>

Consider equation (32):

$$\ddot{\psi}\cos\theta - M_{m\alpha}\sin\psi - M_{mq}\,\dot{\psi}\cos\theta - M_{np\alpha}\sin\theta\cos\psi \qquad (52)$$

$$- M_{m\delta\psi} - (\dot{\theta} + g\cos\theta/V)\,p\,\frac{I_x}{I_y}$$

$$- 2\sin\theta\,(\dot{\theta} + g\cos\theta/V)\,\dot{\psi} = 0$$

As previously stated, the term, $g \cos \theta/V$, is considered a constant in taking the partial derivative of Equation (52).

$$\ddot{\psi} \cos \theta - M_{m\alpha} \sin \psi - M_{mq} \dot{\psi} \cos \theta - M_{np\alpha} \sin \theta \cos \psi \tag{53}$$

$$- M_{m\delta\psi} - (\dot{\theta} + K_g) p \frac{I_x}{I_y} - 2 \sin \theta (\dot{\theta} + K_g) \dot{\psi} = 0$$

where: $$M_{m\alpha} = \frac{\bar{q} A d}{I_y} [C_{m\alpha} + C_{m\alpha_3} \varepsilon^2 + C_{m\alpha_5} \varepsilon^4 + C_{m\alpha_V} (VREF - V)]$$

$$M_{mq} = \frac{\bar{q} A d^2}{2 V I_y} [C_{mq} + C_{mq_2} \varepsilon^2 - C_{N\alpha} 2 I_y/md^2]$$

$$M_{np\alpha} = \frac{\bar{q} A d^2 p}{2 V I_y} [C_{np\alpha} + C_{np\alpha_3} \varepsilon^2 + C_{np\alpha_5} \varepsilon^4 + C_{N\alpha} 2 I_x/md^2]$$

$$M_{m\delta\theta} = \frac{\bar{q} A d}{I_y} [C_{m\delta} (\delta_B \sin \phi - \delta_A \cos \phi]$$

$$M_{m\delta\psi} = \frac{\bar{q} A d}{I_y} [C_{m\delta} (\delta_B \cos \phi + \delta_B \sin \phi]$$

$$\varepsilon^2 = \sin^2 \bar{\alpha} = \sin^2 \psi + \sin^2 \theta \cos^2 \psi$$

$$K_g = g \cos \theta/V$$

To perform differential corrections, the $\partial\psi/\partial C_j$ is required where C_j = the initial conditions on θ, ψ, $\dot{\theta}$, $\dot{\psi}$, and the aerodynamic moment coefficients. The $\partial\psi/\partial C_j$ is obtained through integration of $\partial\psi/\partial C_j$. For computational purposes $\partial\ddot{\psi}/\partial C_j$ is expressed in generalized form such that it will be applicable to all C_j's. The angular motion reduction follows the completion of the roll and translational motion reductions and thus the quantities V, $\bar{q}$ and p are considered known and can be designated independent variables.

Taking the generalized partial of Equation (53) with respect to C_j, the following is obtained:

$$\frac{\partial\ddot{\psi}}{\partial C_j}\cos\theta - \ddot{\psi}\sin\theta\frac{\partial\theta}{\partial C_j} - M_{m\alpha}\cos\psi\frac{\partial\psi}{\partial C_j} - \sin\psi\frac{\partial\varepsilon^2}{\partial C_j}\ \frac{\partial M_{m\alpha}}{\partial\varepsilon^2} \tag{54}$$

$$- M_{mq}\cos\theta\frac{\partial\dot{\psi}}{\partial C_j} + M_{mq}\dot{\psi}\sin\theta\frac{\partial\theta}{\partial C_j} - \dot{\psi}\cos\theta\frac{\partial M_{mq}}{\partial\varepsilon^2}\frac{\partial\varepsilon^2}{\partial C_j}$$

$$- M_{np\alpha}\cos\theta\cos\psi\frac{\partial\theta}{\partial C_j} + M_{np\alpha}\sin\theta\sin\psi\frac{\partial\psi}{\partial C_j} - \sin\theta\cos\psi\frac{\partial M_{np\alpha}}{\partial\varepsilon^2}\frac{\partial\varepsilon^2}{\partial C_j}$$

$$- p\frac{I_x}{I_y}\frac{\partial\dot{\theta}}{\partial C_j} - 2\sin\theta(\dot{\theta} + K_g)\frac{\partial\dot{\psi}}{\partial C_j} - 2\sin\theta\,\dot{\psi}\frac{\partial\dot{\theta}}{\partial C_j}$$

$$- 2\cos\theta(\dot{\theta} + K_g)\dot{\psi}\frac{\partial\theta}{\partial C_j} = - F_j\,\partial C_j/\partial C_j$$

where: F_j = the constant multiplier of the specific C_j from Equation (53); for example, if $C_j = C_{m\alpha_3}$, then $F_j = \frac{\bar{q}\,A\,d}{I_y}\varepsilon^2\sin\psi$.

From the definition of $\varepsilon^2 = \sin^2\bar{\alpha}$, the following generalized partial derivative is obtained:

$$\frac{\partial\varepsilon^2}{\partial C_j} = 2\sin\psi\cos\psi\frac{\partial\psi}{\partial C_j} + 2\cos^2\psi\sin\theta\cos\theta\frac{\partial\theta}{\partial C_j} \tag{55}$$

$$- 2\sin^2\theta\sin\psi\cos\psi\frac{\partial\psi}{\partial C_j}$$

$$= \cos^2\theta\sin 2\psi\frac{\partial\psi}{\partial C_j} + \cos^2\psi\sin 2\theta\frac{\partial\theta}{\partial C_j}$$

Substituting Equation (55) into (54), one obtains:

$$\frac{\partial\ddot{\psi}}{\partial C_j}\cos\theta - \ddot{\psi}\sin\theta\frac{\partial\theta}{\partial C_j} - M_{m\alpha}\cos\psi\frac{\partial}{\partial C_j} - M_{mq}\cos\theta\frac{\partial\dot{\psi}}{\partial C_j} \tag{56}$$

$$+ M_{mq}\dot{\psi}\sin\theta\frac{\partial\theta}{\partial C_j} - M_{np\alpha}\cos\theta\cos\psi\frac{\partial\theta}{\partial C_j} + M_{np\alpha}\sin\theta\sin\psi\frac{\partial\psi}{\partial C_j}$$

$$- p\frac{I_x}{I_y}\frac{\partial\dot{\theta}}{\partial C_j} - 2\sin\theta(\dot{\theta} + K_g)\frac{\partial\dot{\psi}}{\partial C_j} - 2\sin\theta\dot{\psi}\frac{\partial\dot{\theta}}{\partial C_j} - 2\cos\theta(\dot{\theta}$$

$$+ K_g)\dot{\psi}\frac{\partial\theta}{\partial C_j}$$

$$- [\sin\psi\frac{\partial M_{m\alpha}}{\partial\varepsilon^2} + \dot{\psi}\cos\theta\frac{\partial M_{mq}}{\partial\varepsilon^2} + \sin\theta\cos\psi\frac{\partial M_{np\alpha}}{\partial\varepsilon^2}]\;[\cos^2\theta\sin 2\psi\frac{\partial\psi}{\partial C_j}$$

$$+ \cos^2\psi\sin 2\theta\frac{\partial\theta}{\partial C_j}] = - F_j\;\partial C_j/\partial C_j$$

Equation (56) may be written in the following generalized form:

$$\frac{\partial\ddot{\psi}}{\partial C_j}\cos\theta + A_1\frac{\partial\theta}{\partial C_j} + B_1\frac{\partial\dot{\theta}}{\partial C_j} + C_1\frac{\partial\psi}{\partial C_j} + D_1\frac{\partial\dot{\psi}}{\partial C_j} = G_{1j}$$

where:

$$A_1 = -\ddot{\psi}\sin\theta + M_{mq}\dot{\psi}\sin\theta - M_{np\alpha}\cos\theta\cos\psi$$

$$- 2\cos\theta(\dot{\theta} + K_g)\dot{\psi} - [\sin\psi\frac{\partial M_{m\alpha}}{\partial\varepsilon^2} + \dot{\psi}\cos\theta\frac{\partial M_{mq}}{\partial\varepsilon^2}$$

$$+ \sin\theta\cos\psi\frac{\partial M_{np\alpha}}{\partial\varepsilon^2}]\;[\cos^2\psi\sin 2\theta]$$

$$B_1 = - p\frac{I_x}{I_y} - 2\sin\theta\dot{\psi}$$

$$C_1 = - M_{m\alpha} \cos \psi + M_{np\alpha} \sin \theta \sin \psi - [\sin \psi \frac{\partial M_{m\alpha}}{\partial \varepsilon^2}$$

$$+ \dot{\psi} \cos \theta \frac{\partial M_{mq}}{\partial \varepsilon^2} + \sin \theta \cos \psi \frac{\partial M_{np\alpha}}{\partial \varepsilon^2}] \sin 2 \psi \cos^2 \theta$$

$$D_1 = - M_{mq} \cos \theta - 2 \sin \theta (\dot{\theta} + K_g)$$

$$G_{1j} = - F_j$$

From the definitions of $M_{m\alpha}$, M_{mq}, $M_{np\alpha}$, the following partial derivatives are used in Equation (56):

$$\frac{\partial M_{m\alpha}}{\partial \varepsilon^2} = \frac{\bar{q} A d}{I_y} [C_{m\alpha_3} + 2 C_{m\alpha_5}]$$

$$\frac{\partial M_{mq}}{\partial \varepsilon^2} = \frac{\bar{q} A d^2}{2 V I_y} [C_{mq2}]$$

$$\frac{\partial M_{np\alpha}}{\partial \varepsilon^2} = \frac{\bar{q} A d^2 p}{2 V I_y} [C_{np\alpha_3} + 2 C_{np\alpha_5}]$$

Consider the remaining equation of motion used:

$$\ddot{\theta} - M_{m\alpha} \sin \theta \cos \psi - M_{mq} (\dot{\theta} + K_g) + M_{np\alpha} \sin \psi + M_{m\delta\theta} \tag{57}$$

$$+ \dot{\psi} \cos \theta \, p \frac{I_x}{I_y} + \dot{\psi}^2 \cos \theta \sin \theta = 0$$

Similarly, as before, Equation (57) may be expressed in the following generalized form:

$$\frac{\partial \ddot{\theta}}{\partial C_j} + A_2 \frac{\partial \theta}{\partial C_j} + B_2 \frac{\partial \dot{\theta}}{\partial C_j} + C_2 \frac{\partial \psi}{\partial C_j} + D_2 \frac{\partial \dot{\psi}}{\partial C_j} = G_{2j}$$

where: $$A_2 = - p \dot{\psi} \sin \theta \frac{I_x}{I_y} - M_{m\alpha} \cos \psi \cos \theta + \dot{\psi}^2 \cos 2 \theta$$

$$- [\cos \psi \sin \theta \frac{\partial M_{m\alpha}}{\partial \varepsilon^2} + \dot{\theta} \frac{\partial M_{mq}}{\partial \varepsilon^2} - \sin \psi \frac{\partial M_{np\alpha}}{\partial \varepsilon^2}] \cos^2 \psi \sin 2 \theta$$

$$B_2 = - M_{mq}$$

$$C_2 = M_{m\alpha} \sin \psi \sin \theta + M_{np\alpha} \cos \psi$$

$$- [\dot{\theta} \frac{\partial M_{mq}}{\partial \varepsilon^2} + \cos \psi \sin \theta \frac{\partial M_{m\alpha}}{\partial \varepsilon^2} - \sin \psi \frac{\partial M_{np\alpha}}{\partial \varepsilon^2}] \cos^2 \theta \sin 2 \psi$$

$$D_2 = p \cos \theta \frac{I_x}{I_y} + \sin 2 \theta \dot{\psi}$$

$$G_{2j} = - F_j$$

(3) Derivation of Partial Derivations for Approximate Roll Equation

Consider the equation:

$$\ddot{\phi} - \frac{\bar{q} A d^2}{2 V I_x} [C_{1p} + C_{1p_V} (V_o - V)] (\dot{\phi} - \dot{\psi} \sin \theta) - \frac{\bar{q} A d}{2 I_x} C_{1\delta} \delta$$
$$- \tan \theta (\ddot{\psi} \cos \theta - \dot{\psi} \dot{\theta} \sin \theta) + \frac{\dot{\psi} \dot{\theta}}{\cos \theta}$$

The experimental data is provided as the angle ϕ.

The initial conditions are $\dot{\phi}_0$ and ϕ_0.

$$\frac{\partial \ddot{\phi}}{\partial C_j} - \frac{\bar{q} A d^2}{2 V I_x} \frac{\partial \dot{\phi}}{\partial C_j} [C_{1p} + C_{1p_V} (V_o - V)] = - F_j \, \partial C_j / \partial C_j$$

Use is made of the equality $p = \dot{\phi} - \dot{\psi} \sin \theta$. θ, $\ddot{\psi}$, $\dot{\psi}$, ψ, V and $\overline{q}$ are considered known and therefore are designated independent variables. This reduces the complexity of this problem as most higher order terms $(\dot{\theta}, \ddot{\psi}, \dot{\psi})$ now vanish. The spin (p) was computed during the integration of the equations of motion, and as $(\dot{\phi} - \dot{\psi} \sin \theta)$ always appears as a pair, p is substituted.

SECTION V

TECHNIQUE APPLICATION

1. COMPUTER PROGRAMS

The equations derived in the previous section are applied through five computer programs interconnected through the use of overlays.

The five programs consist of:

1. PRENUT - A modified linear theory program used for initialization. This program provides sufficient initial guesses and program controls to properly execute the numerical integration data reduction process.

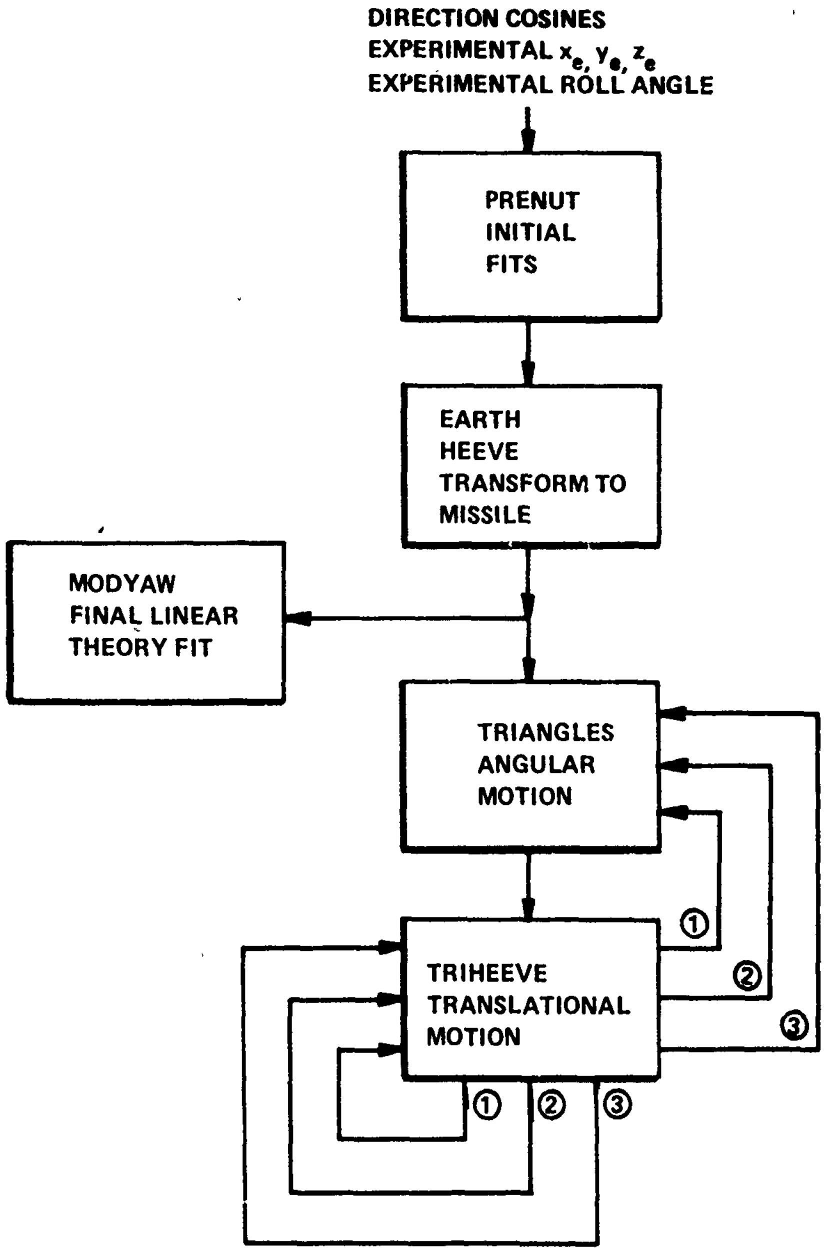
2. EARTH HEEVE - This program analyzes the translational and roll motion of the projectile, determining the earth-fixed velocity vectors and roll rate. It transforms the measured fixed plane angles to missile angles and determines the axial force, normal force and roll coefficients.

3. MODYAW - This program makes a final complete modified linear theory analysis of the rolling, translational, and angular motion of the projectile from the experimental data.

4. TRIANGLES - This numerical integration program is utilized to analyze the angular motion of one, two, or three experiments simultaneously. Several options are available to the user and can be utilized for obtaining non-linear aerodynamic coefficients.

5. TRIHEEVE - The translational motion of up to three experiments is fitted by this program. Options are available to assist the user in obtaining non-linear aerodynamic coefficients.

Each program's inputs and outputs except MODYAW are discussed in the following section. A diagram showing pictorially the flow of a typical reduction of three sets of data which are to be reduced simultaneously by TRIHEEVE and TRIANGLES is given as Figure 3.



After three designated experiments are reduced individually, a multiple TRIHEEVE reduction is made followed by a multiple TRIANGLES reduction.

Figure 3. Data Reduction System Flow Diagram

a. Program - PRENUT

Purpose - Perform a linear theory data reduction and provide estimates of initial conditions and aerodynamic coefficients for the EARTH HEEVE program and the TRIANGLES program.

Inputs - Earth fixed direction cosines $m_e,\ n_e,\ p_e$

Earth fixed coordinates $X_e,\ Y_e,\ Z_e$

Roll angle ϕ

$\left.\right\}$ versus t

Projectile physical properties $m,\ I_x,\ I_y,\ CG,\ d,\ \ell$

Estimates for $C_{N\alpha}$

Outputs - Experimental data $\theta,\ \psi,\ \phi,\ X_e,\ Y_e,\ Z_e,\ t$

Fitted values of $\theta_{m},\ \psi_m$ versus t

Initial conditions $X_{e_0},\ \dot{X}_{e_0},\ Y_{e_0},\ \dot{Y}_{e_0},\ Z_{e_0},\ \dot{Z}_{e_0}$ versus t

$\theta_{m_0},\ \psi_{m_0},\ \dot{\theta}_0,\ \dot{\psi}_0,\ \phi_0,\ p_0$

Aerodynamic coefficients $C_{X0},\ C_{m\alpha},\ C_{mq},\ C_{np\alpha},\ C_{N\alpha}$

Comments: The roll angle, ϕ, is measured from 0^o to 360^o degrees and must be correctly (exactly) unwound to be a continuous accumulative function of time (0^o to ∞). This is accomplished in a subroutine contained in PRENUT.

Equation of Motion - PRENUT

$$\bar{\alpha} = K_1 e^{(\lambda_1 + i\bar{\omega}_1)X_e} + K_2 e^{(\lambda_2 + i\bar{\omega}_2)X_e} + K_4$$

where:

	K_1	amplitude nutation vector
	K_2	amplitude precession vector
	λ_1	nutation damping factor
	λ_2	precession damping factor
	$\bar{\omega}_1$	nutation frequency $(\dot{\phi}_1 + \ddot{\phi}_1 X_e)$
	$\bar{\omega}_2$	precession frequency $(\dot{\phi}_2 + \ddot{\phi}_2 X_e)$
	K_4	yaw of repose vector
	Xe	distance traveled
	ϕ_1	nutation vector orientation
	ϕ_2	precession vector orientation

b. Program - EARTH HEEVE

Purpose - Perform a numerical integration reduction of the earth-fixed (translational) equations of motion (equations 40, 41 and 42) determining best fit to X_e, Y_e, Z_e.

Inputs - From PRENUT

Experimental data	θ, ψ, ϕ, X_e, Y_e, Z_e, versus t
Fitted data	θ_m, ψ_m versus t
Physical properties	m, I_x, I_y, CG, d, ℓ
Estimates for	C_{X0}, $C_{N\alpha}$, C_{X2}, $C_{Yp\alpha}$, $C_{m\alpha}$, $C_{np\alpha}$, C_{mq}, C_{1p}
Initial conditions	X_{e_0}, $\dot{X}_{e_0}$, Y_{e_0}, $\dot{Y}_{e_0}$, Z_{e_0}, $\dot{Z}_{e_0}$, versus t
	θ_{m_0}, ψ_{m_0}, $\dot{\theta}_0$, $\dot{\psi}_0$, ϕ_0, p_0

Outputs -

Experimental data	θ, ψ, ϕ, x_e, y_e, z_e versus t
Computed data	θ_m, ψ_m versus t
Computed initial conditions	X_{e_0}, $\dot{X}_{e_0}$, Y_{e_0}, $\dot{Y}_{e_0}$, Z_{e_0}, $\dot{Z}_{e_0}$
Computed parameters	V, $\overline{q}$, p, γ_e versus t (from best fit)
Computed coefficients	C_{X0}, $C_{N\alpha}$, $C_{Yp\alpha}$, C_{XV}, C_{X2} C_{1p}, $C_{1\delta}$

Comments: The final numerical integration roll fit is performed in this program with final values of C_{1p} and $C_{1\delta}$ determined. Exact transformation from measured fixed plane angles to missile angles are performed using the theoretical $\dot{X}_e$, $\dot{Y}_e$, and $\dot{Z}_e$ velocity profiles.

Equations of Motion - EARTH HEEVE

$$\dot{V}_{Xe} = -\bar{q}\frac{A}{m}[C_X + C_{X2}\,\varepsilon^2]\cos\theta\cos\psi$$

$$+\bar{q}\frac{A}{m}[C_{N\alpha} + C_{N\alpha_3}\,\varepsilon^2]\frac{v}{V}\sin\psi$$

$$-\bar{q}\frac{A}{m}[C_{N\alpha} + C_{N\alpha_3}\,\varepsilon^2]\frac{w}{V}\sin\theta\cos\psi$$

$$-\bar{q}\frac{A}{m}\frac{p\,d}{2\,V}[C_{Yp\alpha} + C_{Yp\alpha_3}\,\varepsilon^2]\frac{w}{V}\sin\psi$$

$$-\bar{q}\frac{A}{m}\frac{p\,d}{2\,V}[C_{Yp\alpha} + C_{Yp\alpha_3}\,\varepsilon^2]\frac{v}{V}\sin\theta\cos\psi$$

$$\dot{V}_{Ye} = -\bar{q}\frac{A}{m}[C_X + C_{X2}\,\varepsilon^2]\cos\theta\sin\psi$$

$$-\bar{q}\frac{A}{m}[C_{N\alpha} + C_{N\alpha_3}\,\varepsilon^2]\frac{v}{V}\cos\psi$$

$$-\bar{q}\frac{A}{m}[C_{N\alpha} + C_{N\alpha_3}\,\varepsilon^2]\frac{w}{V}\sin\theta\sin\psi$$

$$+\bar{q}\frac{A}{m}\frac{p\,d}{2\,V}[C_{Yp\alpha} + C_{Yp\alpha_3}\,\varepsilon^2]\frac{w}{V}\cos\psi$$

$$-\bar{q}\frac{A}{m}\frac{p\,d}{2\,V}[C_{Yp\alpha} + C_{Yp\alpha_3}\,\varepsilon^2]\frac{v}{V}\sin\theta\sin\psi$$

$$\dot{V}_{Ze} = \bar{q}\frac{A}{m}[C_X + C_{X2}\,\varepsilon^2]\sin\theta$$

$$-\bar{q}\frac{A}{m}[C_{N\alpha} + C_{N\alpha_3}\,\varepsilon^2]\frac{w}{V}\cos\theta$$

$$-\bar{q}\frac{A}{m}\frac{p\,d}{2\,V}[C_{Yp\alpha} + C_{Yp\alpha_3}\,\varepsilon^2]\frac{v}{V}\cos\theta - g$$

$$\dot{p} = \frac{\bar{q}\,A\,d^2\,p}{2\,I_x\,V}[C_{1p}] + \frac{\bar{q}\,A\,d}{I_x}[C_{1\delta}]$$

c. Program - TRIANGLES

Purpose - To provide a best fit to the angular motion experimental transformed data (θ_m, ψ_m) using the equations of motion (equations 30 and 31). To provide a multiple fit of more than one experimental case using common coefficients for all cases.

Inputs -	From EARTH HEEVE	
	Experimental data	$\theta, \psi, \phi, X_e, Y_e, Z_e$, versus t
	Transformed data	θ_m, ψ_m, versus t
	Computed data	$V, \bar{q}, p, \gamma_e$ versus t
	Computed coefficients	$C_{X0}, C_{N\alpha}, C_{Yp\alpha}, C_{XV}, C_{X2}$
	Estimated coefficients	$C_{m\alpha}, C_{mq}, C_{np\alpha}$
	Estimates for	$\theta_{m_0}, \psi_{m_0}, \dot{\theta}, \dot{\psi}$
	Physical properties	m, I_x, I_y, CG, d, ℓ
	Initial conditions	$X_{e_0}, \dot{X}_{e_0}, Y_{e_0}, \dot{Y}_{e_0}, Z_{e_0}, \dot{Z}_{e_0}$
Outputs -	Experimental data	$\theta, \psi, \phi, X_e, Y_e, Z_e$, versus t
	Best fit	θ_m, ψ_m versus t
	Computed data	p_R versus t, I_x/I_y
	Computed coefficients	$C_{X0}, C_{N\alpha}, C_{Yp\alpha}, C_{XV}, C_{X2}$
	Final computed coefficients	$C_{m\alpha}, C_{mq}, C_{np\alpha}$ plus non-linear terms
	Initial conditions	$X_{e_0}, \dot{X}_{e_0}, Y_{e_0}, \dot{Y}_{e_0}, Z_{e_0}, \dot{Z}_{e_0}, \theta_{m_0}, \dot{\theta}_0, \psi_{m_0}, \dot{\psi}_0$

Comments: The final aerodynamic moment coefficients $C_{m\alpha}$, C_{mq}, and $C_{np\alpha}$ (functions of angle of attack and Mach number) are derived from this program both for single and multiple fits.

Equations of Motions - TRIANGLES

$$\dot{u} = - \bar{q} \frac{A}{m} C_X + g \sin \theta - \dot{\theta} w + \dot{\psi} \cos \theta v \quad (1)$$

$$\dot{v} = - \bar{q} \frac{A}{m} C_{N\alpha} \frac{v}{V} + \bar{q} \frac{A}{m} \frac{p d}{2 V} C_{Yp\alpha} \frac{w}{V} - \dot{\psi} \cos \theta [u + w \tan \theta] \quad (2)$$

$$\dot{w} = - \bar{q} \frac{A}{m} C_{N\alpha} \frac{w}{V} - \bar{q} \frac{A}{m} \frac{p d}{2 V} C_{Yp\alpha} \frac{v}{V} - g \cos \theta + \dot{\psi} \sin \theta v + \dot{\theta} u \quad (3)$$

$$\ddot{\phi} = \bar{q} \frac{A d}{I_x} \frac{p d}{2 V} C_{1p} + \tan \theta [\cos \theta \ddot{\psi} - \dot{\psi} \dot{\theta} \sin \theta] + \dot{\theta} \dot{\psi}/\cos \theta \quad (4)$$

$$p = \dot{\phi} - \dot{\psi} \sin \theta$$

$$\ddot{\theta} = \bar{q} \frac{A d}{I_y} [C_{m\alpha} + C_{m\alpha_3} \varepsilon^2 + C_{m\alpha_5} \varepsilon^4 + C_{m\alpha_7} \varepsilon^6 + C_{m\alpha_V} (V_o - V)] \frac{w}{V} \quad (5)$$

$$+ \bar{q} \frac{A d^2}{2 V I_y} [C_{mq} + C_{mq_2} \varepsilon^2] \dot{\theta}$$

$$+ \bar{q} \frac{A d^2 p}{2 V I_y} [C_{np\alpha} + C_{np\alpha_3} \varepsilon^2 + C_{np\alpha_5} \varepsilon^4] \frac{v}{V}$$

$$- \bar{q} \frac{A d}{I_y} [C_{m\delta} (\delta_B \sin \phi - \delta_A \cos \phi)]$$

$$- \dot{\psi} \cos \theta p \frac{I_x}{I_y} - \dot{\psi}^2 \cos \theta \sin \theta$$

$$\ddot{\psi} = \Big[- \bar{q} \frac{A d}{I_y} [C_{m\alpha} + C_{m\alpha_3} \varepsilon^2 + C_{m\alpha_5} \varepsilon^4 + C_{m\alpha_7} \varepsilon^6 + C_{m\alpha_V} (V_o - V)] \frac{v}{V} \quad (6)$$

$$+ \bar{q} \frac{A d^2}{2 V I_y} [C_{mq} + C_{mq_2} \varepsilon^2] \dot{\psi} \cos \theta$$

$$+ \bar{q} \frac{A\ d^2\ p}{2\ V\ I_y} [C_{np\alpha} + C_{np\alpha_3}\ \varepsilon^2 + C_{np\alpha_5}\ \varepsilon^4] \frac{w}{V}$$

$$+ \bar{q} \frac{A\ d}{I_y} [C_{m\delta}\ (\delta_B \cos \phi + \delta_A \sin \phi)]$$

$$+ 2\ \dot{\theta}\ \dot{\psi} \sin \theta + \dot{\theta}\ p \frac{I_x}{I_y}\Big] / \cos \theta$$

d. Program - TRIHEEVE

Purpose - To provide a best fit of the equations of motion to the measured X, Y, and Z data. To provide a multiple fit of more than one experimental case using unique coefficients for all cases.

Inputs - From TRIANGLES

Experimental	θ, ψ, ϕ, X_e, Y_e, Z_e, versus t
Best fit	θ_m, ψ_m versus t
Computed coefficient (estimates)	C_{X0}, $C_{N\alpha}$, $C_{Yp\alpha}$, C_{X2}, C_{XV}
Computed initial conditions	X_{e_0}, $\dot{X}_{e_0}$, Y_{e_0}, $\dot{Y}_{e_0}$, Z_{e_0}, $\dot{Z}_{e_0}$
Computed	p_R vs t
Physical properties	m, I_x, I_y, CG, d, ℓ

Outputs -

Final best fits of	X_e, Y_e, Z_e versus t
Computed data	V, $\bar{q}$, p, γ_e versus t
Final computed coefficients	C_X, $C_{N\alpha}$, $C_{Yp\alpha}$ as function of Mach number and angle of attack

Transformed θ_m and ψ_m based on new velocity profile

Comments: TRIHEEVE computes the final best value of the axial force, normal force, and Magnus Force coefficients both for single and multiple fits.

$$\dot{V}_{Xe} = - \frac{\bar{q}\,A}{m} \cdot [C_X + C_{X2} \sin^2 \alpha + C_{XV} (V - VREF)] \cos \theta \cos \psi$$

$$+ \frac{\bar{q}\,A}{m} [C_{N\alpha} + C_{N\alpha_3} \sin^2 \alpha] [\frac{v}{V} \sin \psi - \frac{w}{V} \sin \theta \cos \psi]$$

$$- \frac{\bar{q}\,A\,p\,d}{m\,2\,V} [C_{Yp\alpha} + C_{Yp\alpha_3} \sin^2 \alpha] [\frac{w}{V} \sin \psi + \frac{v}{V} \cos \psi \sin \theta]$$

$$+ \frac{\bar{q}\,A}{m} [-C_{N\delta_A} (\sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi)$$

$$+ C_{N\delta_B} (\cos \phi \sin \psi - \sin \phi \sin \theta \cos \psi)]$$

$$\dot{V}_{Ye} = - \frac{\bar{q}\,A}{m} [C_X + C_{X2} \sin^2 \alpha + C_{XV} (V - VREF)] \cos \theta \sin \psi$$

$$- \frac{\bar{q}\,A}{m} [C_{N\alpha} + C_{N\alpha_3} \sin^2 \alpha] [\frac{v}{V} \cos \psi + \frac{w}{V} \sin \theta \sin \psi]$$

$$+ \frac{\bar{q}\,A\,p\,d}{m\,2\,V} [C_{Yp\alpha} + C_{Yp\alpha_3} \sin^2 \alpha] [\frac{w}{V} \cos \psi - \frac{v}{V} \sin \psi \sin \theta]$$

$$- \frac{\bar{q}\,A}{m} [- C_{N\delta_A} (\sin \phi \cos \psi - \cos \phi \sin \theta \sin \psi)$$

$$+ C_{N\delta_B} (\cos \phi \cos \psi + \sin \phi \sin \theta \sin \psi)]$$

$$\dot{V}_{Ze} = + \frac{\bar{q}\,A}{m} [C_X + C_{X2} \sin^2 \alpha + C_{XV} (V - VREF)] \sin \theta - \frac{\bar{q}\,A}{m} [C_{N\alpha}$$

$$+ C_{N\alpha_3} \sin^2] \frac{w}{V} \cos \theta$$

$$- \frac{\bar{q}\,A\,p\,d}{m\,2\,V} [C_{Yp\alpha} + C_{Yp\alpha_3} \sin^2 \alpha] \frac{v}{V} \cos \theta - \frac{\bar{q}\,A}{m} [C_{N\delta_A} \cos \phi$$

$$+ C_{N\delta_B} \sin \phi] \cos \theta - g$$

2. NUMERICAL ROUTINES

All numerical integration performed in the previously described programs utilize the fourth order Runge-Kutta method. It is acknowledged that there are extensive numerical integration techniques in existence, many of which are claimed to be much more efficient than the Runge-Kutta method. The two primary reasons for implementing this technique are that (1) Runge Kutta methods are self-starting, the interval between steps may be changed at will, and in general, they are particularly straightforward to apply on a digital computer, and (2) they are comparable in accuracy and often more accurate than corresponding order predictor-corrector methods except that the integration interval is a more significant parameter in terms of solution accuracy. Experience has demonstrated this technique to be reliable, stable, and of sufficient efficiency from a storage versus execution time viewpoint given the form of the differential equations solved for as previously described (equations of motion and partial differential correction equations). During development of these programs, test implementation of a modified Adams-Moulton predictor-corrector routine was made for purposes of comparison. The results indicated that both integration techniques achieved the same degree of accuracy and execution time. Comparable execution times resulted from the fact that the predictor-corrector method had difficulty in starting and time step adjustment.

The matrix inversion routine implemented in all the programs utilizes a Gaussian elimination method. This is a direct method (versus iterative) and eliminates truncation error. Over a period of years during which many sets of test data were reduced, no problems were encountered with either the numerical integration or the matrix inversion methods.

Additional details on these two numerical routines may be found in references 26 and 27.

SECTION VI
CONCLUSIONS AND FUTURE ACTION

An extensive set of data reduction computer programs for analyzing aeroballistic range data has been developed and sequenced to produce a sophisticated system. These programs are primarily geared to the reduction and analysis of spin-stabilized or spinning statically stable projectiles with small configuration asymmetries.

The need exists for a similar set of programs tailored toward the analysis of rolling or non-rolling projectiles with configurational and mass asymmetries. In this manner, the ballistic range could become a useful tool in the testing of winged re-entry bodies and non-symmetric nose cones.

APPENDIX I

COORDINATE SYSTEM IDENTITIES-TRANSFORMATIONS

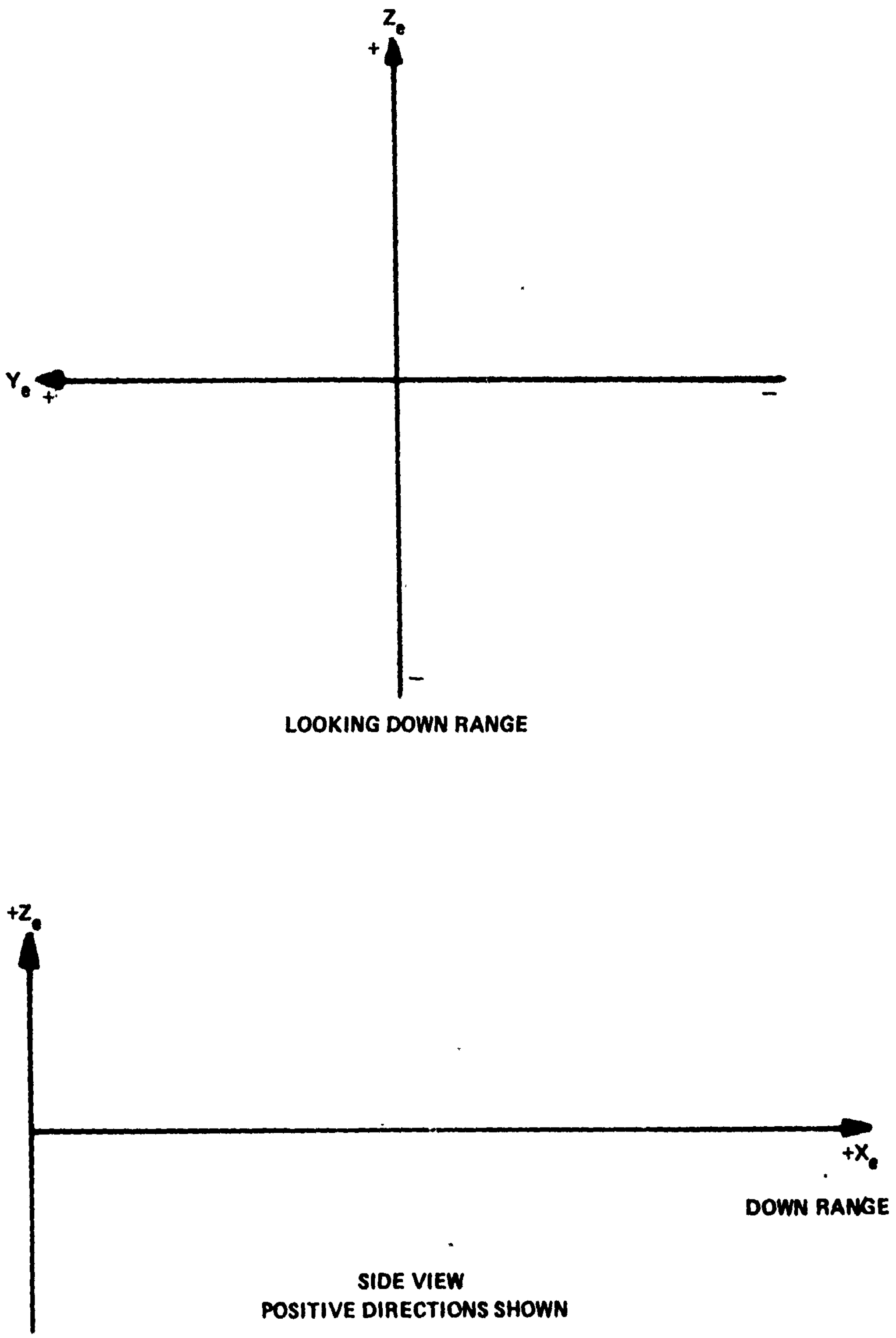


Figure I-1. Coordinate System Fixed Plane (Pcsitions)

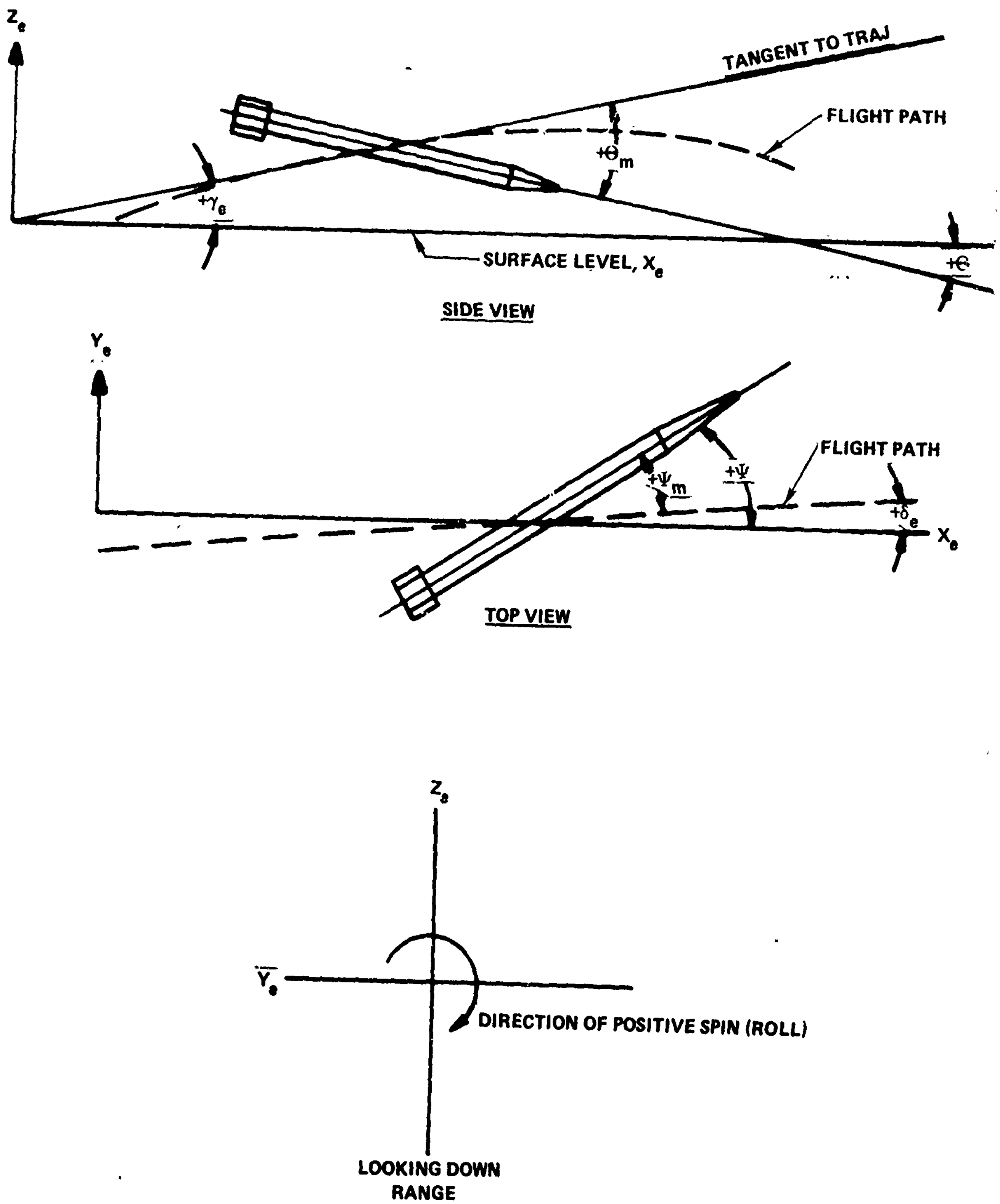


Figure I-2. Coordinate System Fixed Plane (Rotation)

Angular Identities
Missile Angles

$$\sin \bar{\alpha}_m = \sqrt{\sin^2 \psi_m + \cos^2 \psi_m \sin^2 \theta_m}$$

$$\sin \theta_m = \sin \chi_m / \cos \psi_m$$

$$\sin \theta_m = \sin \alpha_m$$

$$\sin \psi_m = - \sin \beta_m \cos \chi_m$$

$$\tan \beta_m = - \tan \psi_m / \cos \theta_m$$

Missile Angles - Earth Angles

$$\theta = \theta_m - \sin^{-1} [\sin \gamma_e / \cos \psi_m]$$

$$\psi = \sin^{-1} [\sin \psi_m / \cos \gamma_e] + \delta_e$$

Angular Identities
Earth Angles

$$\tan \theta = \tan \chi \cos \psi$$

Missile Velocities - Missile Angles

$$v = - V \sin \psi_m$$

$$w = V \cos \psi_m \sin \theta_m$$

$$u = V \cos \psi_m \cos \theta_m$$

Missile Angle - Velocity Relationship

ANGLE	SINE	COSINE	TANGENT
χ_m	$\frac{w}{\sqrt{u^2+v^2+w^2}}$	$\frac{\sqrt{u^2+v^2}}{\sqrt{u^2+v^2+w^2}}$	$\frac{w}{\sqrt{u^2+v^2}}$
ψ_m	$\frac{-v}{\sqrt{u^2+v^2+w^2}}$	$\frac{\sqrt{u^2+w^2}}{\sqrt{u^2+v^2+w^2}}$	$\frac{-v}{\sqrt{u^2+w^2}}$
θ_m	$\frac{w}{\sqrt{u^2+w^2}}$	$\frac{u}{\sqrt{u^2+w^2}}$	$\frac{w}{u}$
α_m	$\frac{w}{\sqrt{u^2+w^2}}$	$\frac{u}{\sqrt{u^2+w^2}}$	$\frac{w}{u}$
β_m	$\frac{v}{\sqrt{u^2+v^2}}$	$\frac{u}{\sqrt{u^2+v^2}}$	$\frac{v}{u}$
$\bar{\alpha}_m$	$\frac{\sqrt{v^2+w^2}}{\sqrt{u^2+v^2+w^2}}$	$\frac{u}{\sqrt{u^2+v^2+w^2}}$	$\frac{\sqrt{v^2+w^2}}{u}$

Trajectory Angle - Velocity Relationships

ANGLE	SINE	COSINE	TANGENT
γ_e	$\frac{V_{Ze}}{\sqrt{V_{Xe}^2+V_{Ye}^2+V_{Ze}^2}}$	$\frac{\sqrt{V_{Xe}^2+V_{Ye}^2}}{\sqrt{V_{Xe}^2+V_{Ye}^2+V_{Ze}^2}}$	$\frac{V_{Ze}}{\sqrt{V_{Xe}^2+V_{Ye}^2}}$
δ_e	$\frac{V_{Ye}}{\sqrt{V_{Xe}^2+V_{Ye}^2}}$	$\frac{V_{Xe}}{\sqrt{V_{Xe}^2+V_{Ye}^2}}$	$\frac{V_{Ye}}{V_{Xe}}$

Transformations
Missile to Earth

$$V_{Xe} = u \cos\theta \cos\psi - v \sin\psi + w \sin\theta \cos\psi$$

$$V_{Ye} = u \cos\theta \sin\psi + v \cos\psi + w \sin\theta \sin\psi$$

$$V_{Ze} = - u \sin\theta + w \cos\theta$$

Earth to Missile

$$u = V_{Xe} \cos\theta \cos\psi + V_{Ye} \cos\theta \sin\psi - V_{Ze} \sin\theta$$

$$v = - V_{Xe} \sin\psi + V_{Ye} \cos\psi$$

$$w = V_{Xe} \sin\theta \cos\psi + V_{Ye} \sin\theta \sin\psi + V_{Ze} \cos\theta$$

Direction Cosines to Fixed Plane Angles

$$\theta = - \sin^{-1} (n)$$

$$\psi = \sin^{-1}(m_e/\sqrt{m_e^2 + p_e^2})$$

where: n_e, m_e, p_e are the direction cosines of the missiles X axis relative to the range coordinate system (fixed plane).

REFERENCES

1. Daniel, D.C., Winchenbach, G.L., and Browell, E.V., "The USAF Aeroballistic Research Facility", Proceedings of the 24th meeting of the Aeroballistics Range Association, Tullahoma, Tenn., Sept. 1973.

2. Chapman, G.T. and Kirk, D.B., "A Method for Extracting Aerodynamic Coefficients from Free Flight Data," AIAA Journal, Vol. 8, No. 4, April 1970.

3. Goodman, T.R., "System Identification and Prediction--An Algorithm Using a Newtonian Iteration Procedure", Quarterly of Applied Mathematics, 24 No. 3, 1966, pp. 249-255.

4. Knadler, C.E., "A Method for Determining the Parameters of Ordinary Differential Equations," NOL TR 68-192, November 1968.

5. Whyte, R.H., and Beliveau, J.G., "An Investigation of the Chapman-Kirk Free Flight Data Reduction Technique", Advance Munitions Report, GE Company, Burlington, Vermont, August 1969.

6. Whyte, R.H. and Beliveau, J.G., "Nonlinear Free Flight Aerodynamic Reduction Using Angle of Attack or Angular Rate Data," General Electric Report TIS R69APB6, September 1969.

7. Whyte, R.H., and Beliveau, J.G., "Reduction of 4.2 Inch Mortar Free Flight Spark Range Data by Numerical Integration", Advance Munitions Report, GE Company, Burlington, Vermont, August 1970.

8. Whyte, R.H. and Hathaway, W.H., "Reduction of Yaw Sonde and Position Time Radar Data by Numerical Integration," General Electric Report 72APB506, January 1972.

9. Whyte, R.H., Jeung, A., and Bradley, J.W., "Chapman-Kirk Reduction of Free Flight Range Data to Obtain Nonlinear Aerodynamic Coefficients", BRL MR 2298, May 1973.

10. Whyte, R.H., and Mermagen, W.H., "A Method for Obtaining Aerodynamic Coefficients from Yawsonde and Radar Data", Journal of Spacecraft and Rockets, Vol. 10, No. 6, June 1973, pp. 384-388.

11. Whyte, R.H., and Houghton, R.C., "Reduction of Range and Yaw Sonde Data Using Numerical Integration Techniques", General Electric, Burlington, Vermont, Report No. 72APB539, September 1972.

12. Whyte, R.H., Houghton, R.C. and Hathaway, W.H., "Description of Yaw Sonde Numerical Integration Data Reduction Computer Programs", General Electric, Burlington, Vermont, Report No. 73APB514, May 1973.

13. Whyte, R.H., Houghton, R.C., "User Manual for AEDC Range G Free Flight Reduction Computer Programs", General Electric, Burlington, Vermont, Report No. 72APB550, October 1972.

14. Hathaway, W.H., and Whyte, R.H., "Interior Ballistic Data Reduction Technique", General Electric, Burlington, Vermont, Report No. 71APB561, December 1971.

15. Cochran, G. and Wolf, J., "Rotating Band/Rifling Interaction Study", General Electric, Burlington, Vermont, Report No. 72APB552, November 1972.

16. Murphy, C.H., "Data Reduction for the Free Flight Spark Ranges", BRL Report 900, February 1954.

17. Murphy, C.H., "Free Flight Motion of Symmetric Missiles", BRL Report 1216, July 1963.

18. Murphy, C.H., "The Measurement of Non-Linear Forces and Moments by Means of Free Flight Tests", BRL Report No. 974, Feb. 1956.

19. Nicolaides, J.D., and MacAllister, L.C., "A Review of Aeroballistic Range Research on Winged and/or Finned Missile" Ballistic TN No. 5, Bureau of Ordinance, Dept. of Navy, 1955.

20. Krial, K.S., and MacAllister, L.C., "Aerodynamic Properties of a Family of Shell of Similar Shape--105MM XM380E5, XM380E6, T388 and 155MM T387" BRL MR 2023, Feb. 1970.

21. Nicolaides, J.D., "Free Flight Dynamics", University of Notre Dame, South Bend, Indiana, 1967.

22. Nicolaides, J.D., "On the Free Flight Motion of Missiles Having Slight Configurational Asymmetries", BRL Report No. 858, June 1953.

23. Eikenberry, R.S., "Analysis of the Angular Motion of Missiles", Univ. of Notre Dame, Sandia Report SC-CR-70-6051, Feb. 1970.

24. Nielson, J.N., "Computer Program for Six Degrees of Freedom Missile Trajectories," Picatinny Arsenal Report TM1292, November 1963.

25. Barnett, B., "Trajectory Equations for a Six Degree of Freedom Missile Using a Fixed Plane Coordinate System," Picatinny Arsenal Report TR3391, June 1966.

26. Hildebrand, F.B., "Introduction to Numerical Analysis", McGraw Hill Book Company, Library of Congress Catalog Number 55-8284.

27. Carnahan, B., Luther, H.A., and Wilkes, J.O., "Applied Numerical Methods", John Wiley & Sons, Library of Congress, Catalog Number 67-27555.

(The reverse of this page is blank.)

INITIAL DISTRIBUTION

ASD (ENYS)	1
AUL (AUL-LES-70-239)	1
Frankford Arsenal (Library)	1
Harry Diamond Lab/AMXDO-TC	1
USN Wpns Ctr/Code 753	2
DDC	2
USNOL (Tech Library)	6
General Electric	5
Dept of the Army Lake City Ammunition Plant	3
Picatinny Ars (Tech Library)	3
US Army Munitions Comd	1
NASA Ames, Langley Res Ctr	2
LSU (Dept of Engr Res)	1
Univ of Notre Dame, Dept of Aero Space & Mech Engr	1
USNWL	5
AEDC	8
Sandia Corp	1
BRL (Technical Library)	7
Alpha Research Inc	1
NSRDC	2
AOMC	1
Honeywell, Inc	2
AFATL (DL)	1
AFATL (DLOSL)	2
AFATL (DLB)	1
AFATL (DLDL)	5
AFATL (DLYE)	1
AFATL (DLYD)	1
Hq USAF/SAMI	1
TRADOC/ADTC-DO	1

(The reverse of this page is blank)

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Armament Systems Department General Electric Company Burlington, Vermont	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

AEROBALLISTIC RANGE DATA REDUCTION TECHNIQUE UTILIZING NUMERICAL INTEGRATION

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Final Report - 26 June 1973 to 30 January 1974

5. AUTHOR(S) (First name, middle initial, last name)

Robert H. Whyte

Wayne H. Hathaway

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
February 1974	75	27

8a. CONTRACT OR GRANT NO. F08635-73-C-0165	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. 2547	
c. Task No. 03	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) AFATL-TR-74-41
d. Work Unit No. 010	

10. DISTRIBUTION STATEMENT

Distribution limited to U. S. Government agencies only; this report documents test and evaluation; distribution limitation applied February 1974 . Other requests for this document must be referred to the Air Force Armament Laboratory (DLDL), Eglin Air Force Base, Florida 32542.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
Available in DDC	Air Force Armament Laboratory Air Force Systems Command Eglin Air Force Base, Florida 32542

13. ABSTRACT

The numerical integration technique to be utilized in the reduction and analysis of data gathered in the USAF Aeroballistic Research Facility located at Eglin Air Force Base is described. The method of Chapman and Kirk has been developed into a system of digital computer programs which may be easily and routinely used in the analysis of data derived from spark range firing. The equations of motion for a missile/projectile in free-flight are derived in this report utilizing the six degrees of freedom. The parametric equations, philosophy, and methods of implementation are also derived and discussed. Brief descriptions of the computer program involved are also given.

UNCLASSIFIED

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
AEROBALLISTIC RANGE DATA REDUCTION TECHNIQUE						
NUMERICAL INTEGRATION						
SPARK RANGE FIRING						

UNCLASSIFIED

Security Classification