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HEATING AND COOLING OF SIMPLE GEOMETRIC BODIES

by Dr. Eng. Heinrich Groeber, Berlin-Wilmersdorf

Communication from the Heat Propagation Committee of the Association of German Engineers. (This essay will shortly be published in an expanded version as research report.)

Zeit. Vereines Deutsch. Ingenieure 69(21): 705-11, 1925.

This article features graphs and curves on the heating and cooling time of spherical bodies, cylinders, and plates so that these developments can be easily traced. In looking at these results, we must keep in mind that many tasks arising in furnace and heating technology, metal-working and metal improvement can be traced back to the heating and cooling of these simple bodies. In some cases, the technical literature on the subject does not offer any data on the heat transfer [propagation] numbers. Here we can use the graphs in order to find the characteristic quantities and, from this, the heat transfer number. A supplementary note mentions a work by Williamson and L. H. Adams.

The technical theory of heat transfer has in the past been concerned mostly with heat transfer [conduction] processes in the settled state, although the processes of heating and cooling of solid bodies play a major role everywhere in the bast area of technology.

Mathematical physics has been exploring these processes for the simplest bodies quite accurately for a long time but the results do not show up in the technical literature on the subject. The reason for this is to be found in the unwieldiness and confusion of the final formulas, most of which contain Fourier or other similar endless series. But the practical operator in most instances does not have the necessary background, nor does he have the time; he cannot calculate and compute for days on end, just to get a single numerical value.

This being the situation, it might be a good idea to have one man tackle the computing and calculation job and to work out several important problems for the most frequently encountered numerical ranges. This would then give the general public and the specific user an easily handled body of numerical data. The management group of the Committee of Technical and Economic Experts of the Reich Coal Council for Fuel Consumption and of the Scientific Advisory Council of the Association of German Engineers immediately approved a suggestion along these lines and together prevailed upon their boards to make the necessary funds available. Mathematics research student Heinz Winterfeldt then made all of the computations on which I would like to report below.

Cooling the Sphere

A sphere with a radius of R, which was first heated in its entire mass to a temperature of t_1°C, is suddenly placed in the vicinity of lower temperature t_2°C. The sphere will then give off its warmth to the surrounding area; the outer layers will cool off quickly and strongly while the inner layers will follow slowly until, at the end — theoretically after an infinite period of time — the entire sphere has been cooled to t_2°C.

This entire process is so very simple that anyone without the slightest physics background can understand it; this is why we might be inclined to think that it should not be too difficult to follow the whole process mathematically. The following pages will show that this is not the case.

When we say that we want to follow the process mathematically, we mean that we want the mathematical answer to the following three questions:

1. What is the time curve of the surface temperature?

2. What is the time curve of the temperature at the center?

3. What is the time curve for the heat loss?

We can simplify the way we write all of these equations from the very outset by figuring the temperature not in degrees Centigrade or in degrees of absolute division but rather by setting the environmental temperature at zero. All temperatures of the sphere are then considered to be over-temperatures and can be designated with ϑ. (When we heat a cold sphere in a hot environment, we automatically get ϑ values with a negative sign.)

Similarly, we are not going to figure with the heat content of the sphere but rather only with the heat surplus above the environmental temperature.

Let us use the following symbols here:

R the radius of the sphere.......................... in m,
t the time... in hrs,

θ_0 the initial temperature of the sphere) in degrees of the 100-
θ_o the surface temperature of the sphere) part scale, measured as
θ_m temperature in the center of the sphere) overtemperatures,

$(WU)_0$	thermal conductivity of the sphere in kcal at time 0,	
λ	thermal conductivity of sphere material in	$\frac{kcal}{m\,°C\,h}$,
γ	density of sphere material	$\frac{kg}{m^3}$,
c	specific heat of sphere material in	$\frac{kcal}{kg\,°C}$,
a	temperature conductivity $= \frac{\lambda}{c\gamma}$ of sphere material in	$\frac{m^2}{h}$,
α	heat transfer number	in $\frac{kcal}{m^2\,°C\,h}$,
h	relative heat transfer number $= \frac{\alpha}{\lambda}$	in $\frac{1}{m}$

The computation for the time curve of the surface temperature then give us the expression (*)

$$\theta_o = \theta_0 \sum_{k=1}^{k=\infty} 2\,\frac{\sin\nu_k - \nu_k\cos\nu_k}{\nu_k - \sin\nu_k\cos\nu_k}\, e^{-\nu_k^2 \frac{at}{R^2}}\, \frac{\sin\nu_k}{\nu_k}\,. \tag{1a}$$

(*) Derivation of this and subsequent formulas: e.g., Groeber, Die Grundgesetze der Waermeleitung und des Waermeueberganges (The Basic Laws of Thermal Conductivity and Heat Transfer), 1921, pages 44, 51 and 54, Julius Springer Publishers.

Here of course ν_k would be the system of the infinitely large number of roots of the transcendental equation

$$\nu\cos\nu = (1 - hR)\sin\nu$$

There is only one thing we can easily tell from this equation: the value θ_o is proportional to the value θ_0; as for the rest, the expression is a rather confused function of the four independent variables t, R, a, and h. Anyone can plainly see that the mathematical evaluation of this equation is a rather timeconsuming process and that, no matter how good our mathematical training, it is impossible to get the course [curve] of the function from the fomula. Besides, because of the large number of independent variables it is impossible to show the numerical values of the function in a single numerical table or on a single sheet of curves or graphs. In short, the result of the mathematical compution in the form of equation (1a) has all of the properties that make it unsuitable for further practical use.

Now, there is one way out: we can say that $\frac{at}{R^2}$ is the only variable [a single, unique variable]; and at the same time we would of course note that the ν_k values, for their part, depend only of the single variable hR. In this way we can conceive of the infinite series as a function with only two variables and equation (1a) will then look like this:

$$\Theta_o = \Theta_e \Phi_o\left(\frac{a\,t}{R^2},\ h\,R\right). \qquad (1b)$$

The values $\frac{a\ t}{R^2}$ and h R

we now call the characteristic quantities; these are always pure numerical values, i.e., magnitudes without dimension. The possibility if grouping the four variables into groups of characteristic quantities can be explained with the help of the principle of similarity [similitude] or the theory of dimensions.

The numerical evaluation of the function Φ_o in equation (1b), with its two variables, still requires a rather big computation effort; but once we have done this job, we can bring out the results in a single numerical table or in a single graph.

In a similar manner we can also convert the equation for the temperature of the center. Here we have:

$$\Theta_m = \Theta_e \sum_{k=1}^{k=\infty} 2\frac{\sin\nu_k - \nu_k\cos\nu_k}{\nu_k - \sin\nu_k\cos\nu_k}\, e^{-\nu_k^2\frac{a\,t}{R^2}} \quad \ldots \ (2a)$$

$$= \Theta_e \Phi_m\left(\frac{a\,t}{R^2},\ h\,R\right) \quad \ldots\ldots\ldots\ldots \ (2b)$$

Numerical Table 1
Surface Temperature of Sphere When

$$\Theta_e = 1^0: \quad \Theta_0 = \Theta_e \Phi_0\left(\frac{a\,t}{R^2},\ h\,R\right).$$

$hR =$	0,0001	0,001	0,01	0,1	0,5	1	4	10	20	[illegible]	∞
$\frac{a\,t}{R^2} =$ 0,01	1,00	—	1,00	0,99	0,94	0,89	0,64	0,38	0,22	0,10	0,09
0,05	—	1,00	0,90	0,97	0,86	0,75	0,39	0,17	0,09	0,06	0,00
0,10	—	1,00	0,99	0,95	0,79	0,64	0,26	0,10	0,05	0,02	0,00
0,25	—	1,00	0,99	0,93	0,64	0,44	0,10	0,03	0,01	0,00	—
0,5	—	1,00	0,98	0,85	0,46	0,24	0,02	0,00	—	—	—
1,0	—	1,00	0,97	0,73	0,23	0,07	0,00	—	—	—	—
2,5	1,00	0,99	0,93	0,52	0,03	0,00	—	—	—	—	—
5,0	1,00	0,99	0,86	0,23	0,00	—	—	—	—	—	—
10,0	1,00	0,97	0,76	0,06	0,00	—	—	—	—	—	—
25,0	1,00	0,94	0,48	0,00	—	—	—	—	—	—	—

Numerical Table 2

Temperature in Center of Sphere When

$\Theta_0 = 1^0$: $\Theta_m = \Theta_0 \Phi_m \left(\frac{a t}{R^2}, h R\right)$.

$hR =$	0,0001	0,001	0,01	0,1	0,5	1	4	10	20	50	∞
$\frac{at}{R^2} = 0{,}01$	—	—	—	—	—	—	1,00	1,00	1,00	1,00	1,00
0,05	—	—	—	1,00	1,00	1,00	0,99	0,98	0,98	0,97	0,97
0,10	—	—	1,00	0,99	0,97	0,95	0,87	0,80	0,76	0,73	0,71
0,25	—	1,00	0,99	0,97	0,81	0,69	0,88	0,25	0,22	0,19	0,17
0,5	—	1,00	0,99	0,69	0,58	0,37	0,09	0,08	0,02	0,02	0,02
1,0	—	1,00	0,97	0,77	0,29	0,11	0,00	0,00	0,00	0,00	—
2,5	1,00	0,99	0,93	[illegible]	0,04	0,00	—	—	—	—	—
5,0	1,00	0,99	0,87	0,24	0,00	—	—	—	—	—	—
10,0	1,00	0,98	0,75	0,08	—	—	—	—	—	—	—
25,0	1,00	0,94	0,49	0,00	—	—	—	—	—	—	—

Numerical Table 3

Heat Loss of Sphere When $\Theta_0 = 1^0$.

$Q = (W\,U)_0 \Psi \left(\frac{a t}{R^2}, h R\right)$.

$hR =$	0,0001	0,001	0,01	0,1	0,5	1,0	4	10	20	50	∞
$\frac{at}{R^2} = 0{,}01$	—	—	—	0,00	0,02	0,03	0,09	0,16	0,22	0,27	0,31
0,05	—	—	0,00	0,02	0,07	0,12	0,32	0,46	0,53	0,57	0,61
0,10	—	—	0,00	0,03	0,13	0,23	0,51	0,66	0,71	0,75	0,77
0,25	—	0,00	0,01	0,07	0,29	0,47	0,80	0,90	0,92	0,94	0,95
0,5	—	0,00	0,02	0,14	0,49	0,71	0,96	0,99	0,99	0,99	1,00
1,0	—	0,00	0,03	0,26	0,74	0,92	1,00	1,00	1,00	1,00	—
2,5	—	0,00	0,08	0,54	0,97	1,00	—	—	—	—	—
5,0	0,00	0,01	0,16	0,77	1,00	—	—	—	—	—	—
10,0	0,00	0,02	0,27	0,96	1,00	—	—	—	—	—	—
25,0	0,00	0,05	0,56	0,99	1,00	—	—	—	—	—	—

Numerical Table 4

Surface Temperature of Cylinder When

$\Theta_0 = 1^0$: $\Theta_0 = \Theta_0 \Phi \left(\frac{a t}{R^2}, h R\right)$.

$hR =$	0,0001	0,001	0,01	0,1	0,5	1,0	4,0	10	20	50	∞
$\frac{at}{R^2} = 0{,}01$	—	—	1,00	0,99	0,95	0,89	0,66	0,42	0,24	0,11	0,00
0,05	—	1,00	0,99	0,97	0,87	0,77	0,42	0,21	0,10	0,05	0,00
0,10	—	1,00	0,99	0,96	0,82	0,69	0,31	0,14	0,06	0,03	0,00
0,25	—	1,00	0,99	0,93	0,71	0,53	0,17	0,06	0,03	0,01	0,00
0,50	—	1,00	0,99	0,88	0,57	0,35	0,07	0,02	0,01	0,00	—
1,0	—	1,00	0,98	0,80	0,36	0,16	0,01	0,00	0,00	—	—
2,5	—	1,00	0,95	0,60	0,10	0,02	0,00	—	—	—	—
5,0	1,00	0,99	0,90	0,87	0,01	0,00	—	—	—	—	—
10,0	1,00	0,98	0,82	0,14	0,00	—	—	—	—	—	—
25,0	1,00	0,95	0,61	0,01	—	—	—	—	—	—	—

Numerical Table 5
Temperature of Axis of Cylinder When

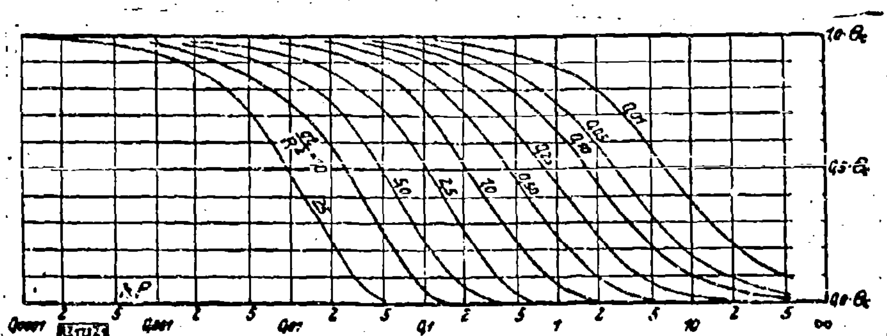
$\Theta_e = 1^0: \quad \Theta_m = \Theta_e \Phi_m \left(\frac{at}{R^2}, hR\right).$

$hR =$	0,0001	0,001	0,01	0,1	0,5	1,0	4,0	10	20	50	∞
$\frac{at}{R^2} =$ 0,01	—	—	—	—	—	—	—	1,00	1,00	1,00	1,00
0,05	—	—	—	—	1,00	1,00	1,00	0,99	0,99	0,99	[illegible]
0,10	—	—	—	1,00	0,99	0,97	0,94	0,90	0,88	0,87	[illegible]
0,25	—	—	1,00	0,98	0,89	0,81	0,59	0,48	0,43	0,40	[illegible]
0,50	—	—	0,99	0,93	0,72	0,55	0,24	0,15	0,12	0,10	[illegible]
1,0	—	—	0,98	0,84	0,46	0,26	0,04	0,01	0,01	0,01	0,01
2,5	—	1,00	0,96	0,63	0,12	0,02	0,00	0,00	0,00	0,00	0,00
5,0	1,00	0,99	0,91	0,38	0,01	0,00	—	—	—	—	—
10,0	1,00	0,98	[illegible]	0,14	0,00	—	—	—	—	—	—
25,0	1,00	0,96	0,61	0,01	—	—	—	—	—	—	—

Numerical Table 6
Heat Loss of Cylinder When

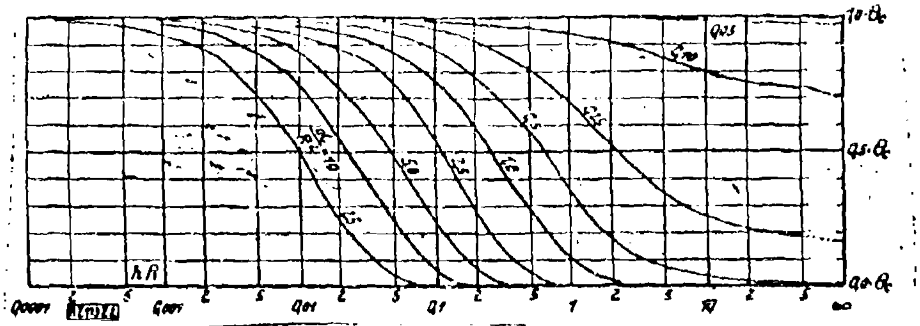
$\Theta_e = 1^0: \quad Q = (WU)_0 \Psi \left(\frac{at}{R^2}, hR\right).$

$hR =$	0,0001	0,001	0,01	0,1	0,5	1,0	4,0	10	20	50	∞
$\frac{at}{R^2} =$ 0,01	—	—	—	0,00	0,01	0,02	0,05	0,11	0,15	0,18	[illegible]
0,05	—	—	—	0,01	0,05	0,08	0,21	0,32	0,38	0,42	[illegible]
0,10	—	—	—	0,02	0,09	0,15	0,37	0,48	0,54	0,58	[illegible]
0,25	—	—	0,00	0,05	0,20	0,33	0,62	0,75	0,79	0,81	[illegible]
0,50	—	—	0,01	0,09	0,36	0,55	0,85	0,92	0,94	0,95	[illegible]
1,0	—	—	0,02	0,18	0,59	0,79	0,98	0,99	1,00	1,00	1,00
2,5	—	0,00	0,05	0,39	0,89	0,98	1,00	1,00	—	—	—
5,0	0,00	0,01	0,09	0,62	0,99	1,00	—	—	—	—	—
10,0	0,00	0,02	0,18	0,86	1,00	—	—	—	—	—	—
25,0	0,00	0,04	0,39	0,99	—	—	—	—	—	—	—



$\Theta_e = 1^0: \quad \Theta_0 = \Theta_e \Phi_0 \left(\frac{at}{R^2}, hR\right).$

Figure 1. Surface temperature of sphere when



$\Theta_0 = 1°$: $\Theta_m = \Theta_0 \Phi_m\left(\frac{at}{R^2}, hR\right) \ldots$

Figure 2. Temperature of center of sphere when

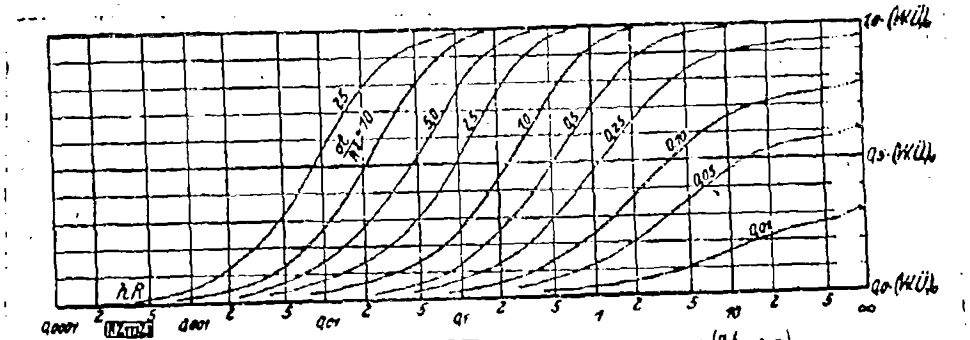


Figure 3. Heat loss of sphere when $\Theta_0 = 1°$: $Q = (WU)_0 \Psi\left(\frac{at}{R^2}, hR\right)$.

The third question we asked above -- the time curve for the heat loss -- can be answered with the following equation. The heat volume, which the sphere has given off until time t, is

$$Q = \frac{4}{3} R^3 \pi c \gamma \Theta_0 \sum_{k=1}^{k=\infty} 6 \frac{1}{\nu_k^3} \frac{(\sin \nu_k - \nu_k \cos \nu_k)^2}{\nu_k - \sin \nu_k \cos \nu_k} \left(1 - e^{-\nu_k^2 \frac{at}{R^2}}\right) \tag{3a}$$

Since Θ_c is the overtemperature of the sphere over [above] the environment, we have

$$\frac{4}{3} R^3 \pi c \gamma \Theta_0 = (WU)_0$$

which is equal to the heat surplus of the sphere at time t = 0.

In keeping with what we said earlier, we can now write:

$$Q = (WU)_0 \Psi\left(\frac{at}{R^2}, hR\right) \ldots \tag{3b}$$

Numerical tables 1, 2, and 3 and the curves in figures 1, 2, and 3 illustrate the values of the three functions for the sphere. Here the abscissa in the illustrations is plotted on the logarithmic scale so that the entire range of variables h R from 0.0001 to 50 can be shown.

Cooling the Cylinder

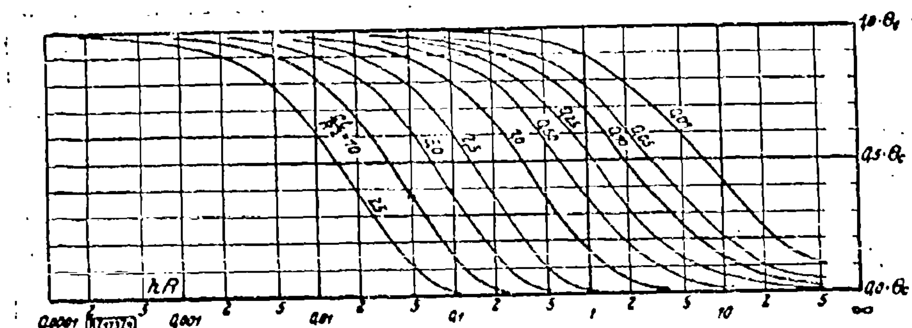
A very long cylinder with a radius of R is now suddenly brought from a high temperature to a low environmental temperature. We want to compute the time curve of the surface temperature θ_o, the temperature θ_m in the axis, and the heat loss for a section with a length L of this cylinder.

Disregarding the cooling of the terminal surfaces because of the great length of the cylinder, we have a computation here which is entirely similar to that of the sphere. The fact is that we have the following:

$$\Theta_o = \Theta_e \sum_{k=1}^{k=\infty} 2 \frac{1}{\mu_k} \frac{J_1(\mu_k)}{J_0^2(\mu_k) + J_1^2(\mu_k)} e^{-\mu_k^2 \frac{at}{R^2}} J_0(\mu_k) \qquad (4a)$$

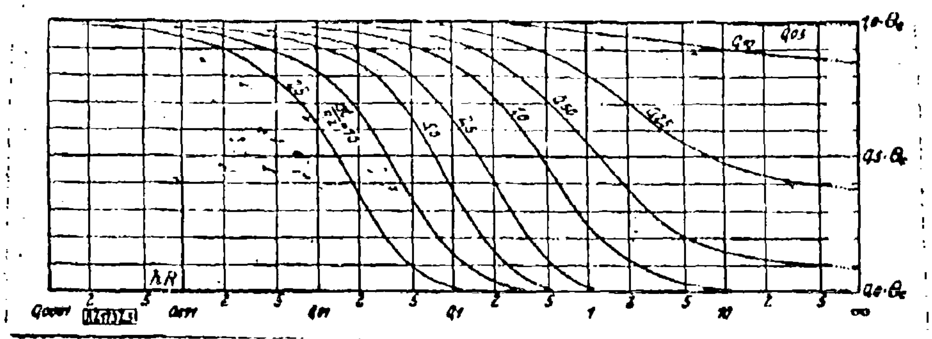
where μ_k represents the system of infinitely many roots of the transcendental equation μJ_1 (μ) hRJ$_0$ (μ) and J_0 as well as J_1 are two Bessel functions.

$$\Theta_m = \Theta_e \sum_{k=1}^{k=\infty} 2 \frac{1}{\mu_k} \frac{J_1(\mu_k)}{J_0^2(\mu_k) + J_1^2(\mu_k)} e^{-\mu_k^2 \frac{at}{R^2}} \qquad (5a)$$



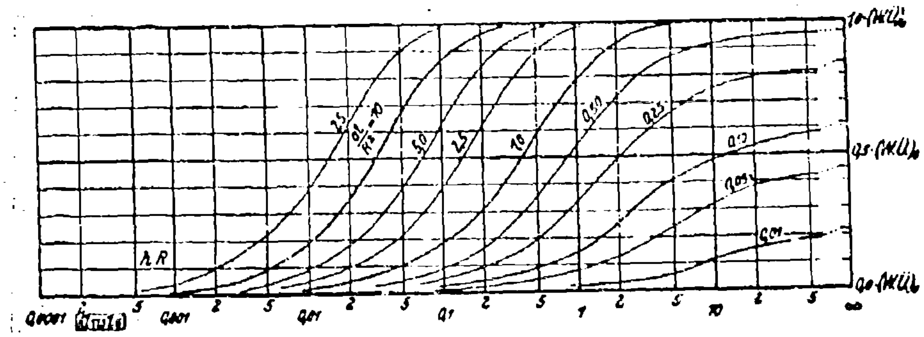
$\Theta_e = 1^0$: $\Theta_0 = \Theta_e \Phi_0 \left(\frac{at}{R^2}, hR\right)$.

Figure 1 Surface temperature of cylinder when



$$\Theta_o = 1^0: \quad \Theta_m = \Theta_o \Phi_m \left(\frac{a t}{R^2}, h R\right).$$

Figure 5. Temperature of axis of cylinder when



$$\Theta_o = 1^0: \quad Q = (W U)_0 \Psi \left(\frac{a t}{R^2}, h R\right).$$

Figure 6. Heat loss of cylinder when

$$Q = R^2 \pi L c \gamma \Theta_c \sum_{k=1}^{k=\infty} 4 \frac{1}{\mu_k^2} \frac{J_1^2(\mu_k)}{J_0^2(\mu_k) + J_1^2(\mu_k)} \left(1 - e^{-\mu_k^2 \frac{a t}{R^2}}\right) \qquad (6a)$$

The numerical evaluation of the three equations

$$\Theta_o = \Theta_o \Phi_0 \left(h R, \frac{a t}{R^2}\right) \ldots \qquad (4b)$$

$$\Theta_m = \Theta_o \Phi_m \left(h R, \frac{a t}{R^2}\right) \ldots \qquad (5b)$$

$$Q = (W U)_0 \Psi \left(h R, \frac{a t}{R^2}\right) \ldots \qquad (6b)$$

is shown in numerical tables 4, 5, and 6 and in figures 4, 5, and 6.

Unilateral Cooling of Plate

Let us take a very large plate with thickness X and let us insulate it completely on one side while the other three sides are free. If this plate is suddenly brought from a high temperature to a colder environment, it will gradually give off its entire heat content to the environment through the noninsulated surface. Now we are going to have to determine the time curve of temperature Θ_0 of the free surface, the temperature Θ_m of the insulated surface, and the heat loss Q for a section F of the plate!

The computation gives us the following

$$\Theta_0 = \Theta_c \sum_{k=1}^{k=\infty} 2 \frac{\sin \delta_k}{\delta_k + \sin \delta_k \cos \delta_k} e^{-\delta_k^2 \frac{a t}{X^2}} \cos \delta_k \qquad (7a)$$

where δ_K represents the infinitely many roots of the equation $\delta \sin \delta = (h X) \cos \delta$

$$\Theta_m = \Theta_c \sum_{k=1}^{k=\infty} 2 \frac{\sin \delta_k}{\delta_k + \sin \delta_k \cos \delta_k} e^{-\delta_k^2 \frac{a t}{X^2}} \ldots \qquad (8a)$$

$$Q = F X c \gamma \Theta_c \sum_{k=1}^{k=\infty} 2 \frac{\sin^2 \delta_k}{\delta_k^2 + \delta_k \sin \delta_k \cos \delta_k} \left(1 - e^{-\delta_k^2 \frac{a t}{X^2}}\right) \qquad (9a)$$

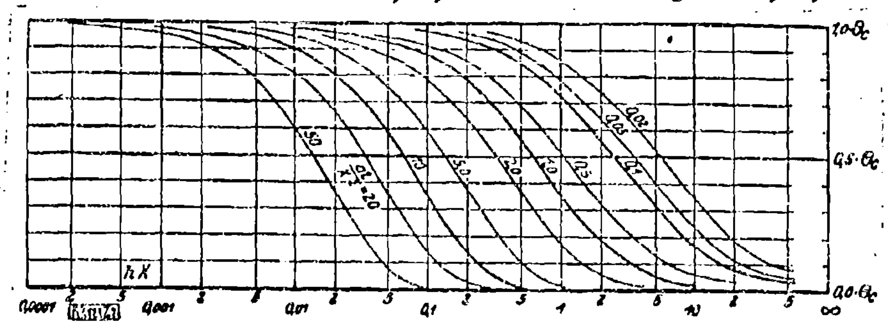
The numerical values from these three equations

$$\Theta_0 = \Theta_c \Phi_0 \left(h X, \frac{a t}{X^2}\right) \ldots \qquad (7b)$$

$$\Theta_m = \Theta_c \Phi_m \left(h X, \frac{a t}{X^2}\right) \ldots \qquad (8b)$$

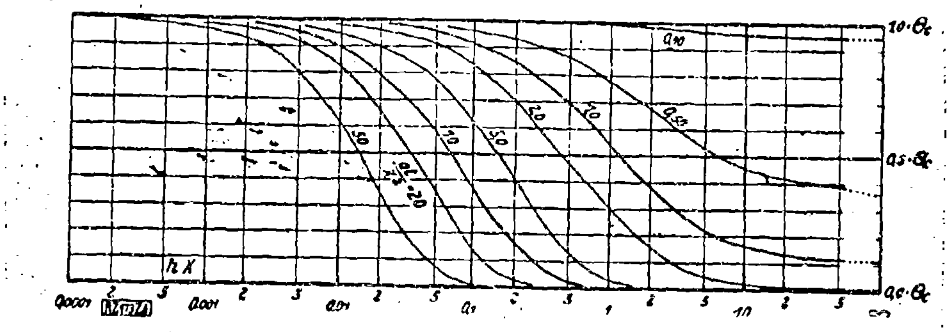
$$Q = (W U)_0 \Psi \left(h X, \frac{a t}{X^2}\right) \ldots \qquad (9b)$$

are shown in numerical tables 7, 8, and 9 and in figures 7, 8, and 9.



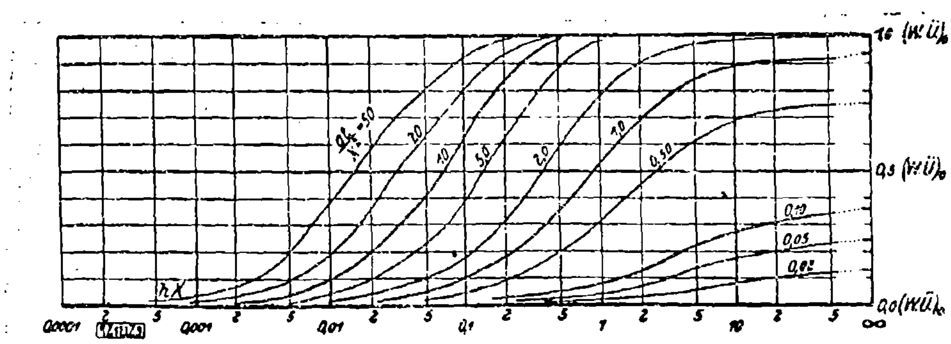
$\Theta_c = 1^0: \ \Theta_0 = \Theta_c \Phi_0 \left(\frac{a t}{X^2}, h X\right)$.

Figure 7. Temperature of free plate surface when



$\Theta_c = 1^0 : \ \Theta_m = \Theta_c \Phi_m \left(\frac{at}{X^2}, hX \right).$

Figure 8. Temperature of insulated plate surface when



$\Theta_c = 1^0 : \ Q = (W\ddot{U}) \Psi \left(\frac{at}{X^2}, hX \right).$

Figure 9. Heat loss of plate cooling on one side when

Bilateral Cooling of Plate

The last problem dealing with the onesided cooling also readily gives us the solution for an equal bilateral cooling of the plate.

If the cooling conditions of the plate on either side are exactly the same, then -- for reasons of symmetry -- the temperature gradient will have to be zero in the middle of the plate. But it so happens that in our last problem the condition /requirement/ for complete insulation meant that we found prescribed that the temperature gradient at that point likewise had to be zero. When the plate is cooled on both sides, we thus see that each half of the plate behaves as in the case of a unilaterally cooling plate.

If the plate has thickness D, then we use an auxiliary magnitude

$$X = \frac{D}{2}$$

and we can get the values for the functions Φ_c, Φ_m and ψ from figures 7, 8, and 9, if we use this magnitude. Here we must keep in mind that we have:

$$(W U)_0 = 2 X F c \gamma \Theta_e$$

How to Use the Figures /Illustrations/

Let us take a numerical example to show you how fast we can compute the cooling processes with the help of these illustrations.

A steel sphere with a diameter of 20 cm is heated to 280° C with its entire mass /weight/ and is then quickly dipped in an oil bath with a temperature of 30° C. Now, what is the temperature distribution in the sphere after 36 s, 3 min and 12 min, when we take the value $\alpha = 500$ /kcal/m h° C/ as the heat transfer number from the sphere to the oil bath?

Preparatory computation. The radius of the sphere must be given in meters; i.e.:

$$R = 0.01 \ [m].$$

The time is put in terms of hours:

$$\frac{36}{3600} = 0{,}01 \ [h],$$

$$t_2 = \frac{3}{60} = 0{,}05 \ [h],$$

$$t_3 = \frac{12}{60} = 0{,}20 \ [h].$$

Numerical Table 7
Temperature of Free Plate Surface When

$$\Theta_c = 1^0: \quad \Theta_0 = \Theta_e \Phi_0 \left(\frac{a t}{X^2}, h X\right).$$

$hX =$	0,0001	0,001	0,01	0,1	0,5	1	4	10	20	60	∞
$\frac{at}{X^2} =$ 0,02	—	1,00	0,99	0,98	0,93	0,86	0,59	0,34	0,19	0,08	0,00
0,06	—	1,00	0,99	0,98	0,89	0,79	0,46	0,23	0,12	0,05	0,00
0,1	—	1,00	0,99	0,97	0,85	0,73	0,37	0,17	0,08	0,04	0,00
0,5	—	1,00	0,99	0,92	0,69	0,51	0,17	0,06	0,03	0,01	0,00
1,0	—	1,00	0,98	0,88	0,66	0,35	0,08	0,02	0,01	0,00	—
2,0	—	1,00	0,97	0,79	0,37	0,17	0,02	0,00	0,00	—	—
5,0	—	1,00	0,96	0,59	0,10	0,02	0,00	—	—	—	—
10,0	1,00	0,99	0,90	0,36	0,01	0,00	—	—	—	—	—
20,0	1,00	0,98	0,81	0,13	0,00	—	—	—	—	—	—
50,0	1,00	0,96	0,60	0,01	—	—	—	—	—	—	—

Numerical Table 8
Temperature of Insulated Plate Surface When

$\Theta_o = 1^0; \quad \Theta_m = \Theta_o \Phi_m \left(\frac{a\,t}{X^2}, h\,X\right).$

$h\,X =$	0,0001	0,091	0,01	0,1	0,5	1	4	10	20	50	∞
$\frac{a\,t}{X^2} = 0{,}02$	—	—	—	—	—	—	—	—	—	—	—
0,06	—	—	—	—	—	—	1,00	1,00	1,00	1,00	1,00
0,1	—	—	1,00	1,00	1,00	1,00	0,98	0,97	0,96	0,95	0,95
0,5	—	1,00	0,99	0,97	0,87	0,78	0,56	0,46	0,41	0,39	0,37
1,0	—	1,00	0,99	0,92	0,70	0,54	0,25	0,16	0,13	0,12	0,11
2,0	—	1,00	0,98	0,83	6,46	0,31	0,06	0,02	0,01	0,01	0,01
5,0	—	1,00	0,95	0,62	0,18	0,03	0,00	0,00	0,00	0,00	0,00
10,0	1,00	0,99	0,90	0,37	0,02	0,00	—	—	—	—	—
20,0	1,00	0,98	0,82	0,14	0,00	—	—	—	—	—	—
50,0	1,00	0,96	0,61	0,01	—	—	—	—	—	—	—

Numerical Table 9
Heat Loss of Unilaterally Cooling Plate When

$\Theta_o = 1^0;$

$Q = (W\,U)\,\Psi\left(\frac{a\,t}{X^2}, h\,X\right).$

$h\,X =$	0,0001	0,001	0,01	0,1	0,2	0,5	1	4	10	50	∞
$\frac{a\,t}{X^2} = 0{,}02$	—	—	—	0,00	0,00	0,01	0,02	0,05	0,09	0,12	0,13
0,06	—	—	—	0,01	0,01	0,02	0,04	0,12	0,18	0,23	0,25
0,1	—	—	—	0,02	0,03	0,05	0,08	0,20	0,27	0,34	0,36
0,5	—	—	0,00	0,05	0,09	0,20	0,32	0,58	0,69	0,75	0,76
1,0	—	—	0,01	0,10	0,17	0,35	0,53	0,81	0,89	0,92	0,93
2,0	—	—	0,02	0,17	0,31	0,59	0,78	0,96	0,98	0,99	0,99
5,0	—	0,00	0,05	0,39	0,63	0,88	0,98	1,00	1,00	1,00	1,00
10,0	—	0,01	0,10	0,62	0,84	0,99	1,00	—	—	—	—
20,0	0,00	0,02	0,18	0,81	0,93	1,00	—	—	—	—	—
50,0	0,01	0,04	0,39	0,92	0,99	—	—	—	—	—	—

From the physical numerical tables we get the following figures for steel:

$$\lambda = 50\ [\text{kcal/m h }^0\text{C}],$$
$$\gamma = 7700\ [\text{kg/m}^3],$$
$$c = 0{,}13\ [\text{kcal/kg }^0\text{C}];$$

from which we can calculate:

$$a = \frac{\lambda}{c\,\gamma} = \frac{50}{0{,}13 \cdot 7700} = 0{,}05\ [\text{m}^2/\text{h}].$$

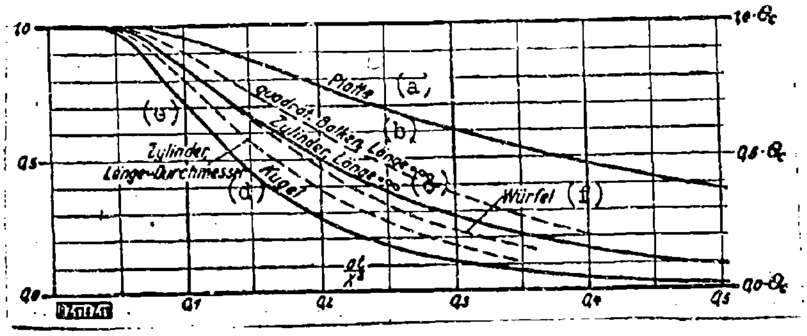


Figure 11. Cooling speed for center or axis of various bodies according to Williamson and Adams. Legend: a - plate; b - square beam, length infinite; c - cylinder, length infinite; d - sphere; e - cylinder, length equals diameter; f - cube.

Computation of characteristic quantities from these values:

$$h\,R = \frac{\alpha}{\lambda}\,R = \frac{500}{50}\cdot 0{,}1 = 1{,}0,$$
$$\left(\frac{a\,t}{R^2}\right)_1 = \frac{0{,}05}{0{,}01}\cdot 0{,}01 = 0{,}05,$$
$$\left(\frac{a\,t}{R^2}\right)_2 = \frac{0{,}05}{0{,}01}\cdot 0{,}05 = 0{,}25,$$
$$\left(\frac{a\,t}{R^2}\right)_3 = \frac{0{,}05}{0{,}01}\cdot 0{,}20 = 1{,}00.$$

We can now read off directly and get the following:

Numerical Table 10

(a) Werte zum Zwecke der Berechnung		$t_1 = 36$ s	$t_2 = 3$ min	$t_3 = 12$ min
1. der Oberflächentemperatur	$\frac{\Theta_o}{\Theta_e} =$	0,75	0,44	0,07
2. der Temperatur der Mitte	$\frac{\Theta_m}{\Theta_e} =$	1,00	0,69	0,11
3. der abgegebenen Wärme	$\frac{Q}{(W\,U_0)} =$	0,12	0,47	0,92
4. der zurückgebliebenen Wärme . .	$1 - \frac{Q}{(W\,U)_0} =$	0,88	0,53	0,08

Legend: a - values for computation purposes; 1. of surface temperature; 2. of temperature in the middle; 3. of heat given off; 4. of remaining heat.

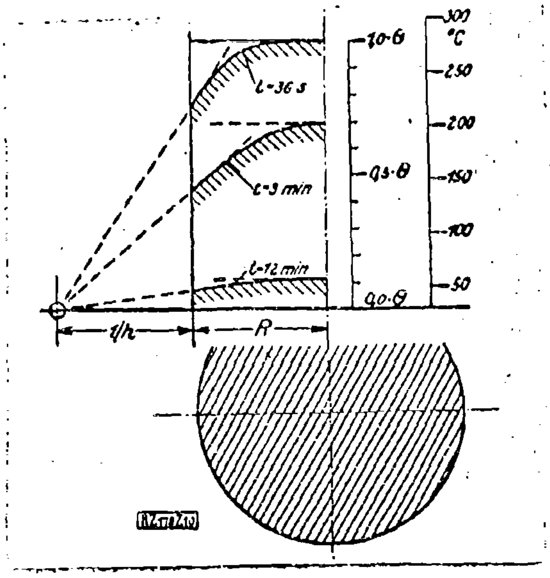


Figure 10. How to Illustrate the Determination of the Temperatures Along the Radius of a Sphere.

Plotting the temperature distribution. We plot the radius as the abscissa; as the ordinates, we plot, above the terminal points of the radius, on the one hand, the three values Θ_o / Θ_c and, on the other hand, the three values Θ_m / Θ_c . These six values alone give us a pretty good idea of the attenuation of the temperatures.

By drawing the illustration in Figure 10 we can get more detailed information on the temperature curve along the radius. In the center, this curve must have a horizontal tangent for symmetry reasons. Along the surface, however, the tangents of all curves must go through the same point which lies on the abscissa axis at a distance of 1/h [m] outside the surface (this applies equally to the cylinder add the plate; for explanation, see Groeber, loc. cit., page 14). With the help of these tangents, we can then draw the temperature curves with the kind of accuracy that will suffice for technical [engineering] purposes.

Computation of heat loss of sphere. The heat surplus of the sphere [cooled] from 280° C via the oil bath temperature of 30° C is

$$(W U)_0 = \frac{4}{3} R^3 \pi c \gamma \Theta_0$$
$$= \frac{4}{3} \cdot \frac{1}{1000} \pi \cdot 0{,}13 \cdot 7700 \cdot 250 = 1045 \text{ [kcal]}.$$

Of this amount, the fraction given in line 3 of Numerical Table 10 was transferred to the oil at times t_1, t_2, and t_3; the rest remained in the sphere.

Supplementary Comment

While this article was at the printer's, Prof. M. Jakob, Berlin, told me about an American work by Williamson and Adams who were after a similar objective (cf., Williamson and L. H. Adams, "Temperature Distribution in Solids during Heating or Cooling," Physical Review, Vol XIV Series II, 1919, page 99). Unfortunately, the authors always assume that it is not the environmental temperature but directly the surface temperature which is given, respectively, changed. This is the same thing as if in this work here, only the perpendicular series were given for h R and $h X = \infty$. In the first part of the work the authors furthermore assume that, after the settled state has set in, the surface temperature rises constantly from time t = 0 on in accordance with the law Θ_o = const. t. Since this case is of no consequence from the technical [engineering] viewpoint, we need not discuss it any further here.

In the second part the authors assume that, after the settled state has set in, the surface temperature suddenly jumps by the value Θ_c and that the heating or cooling process now takes place. They accurately computed the example of the sphere and give the values Θ / Θ_{c1} for varying distance r from the center of the sphere; see Numerical table 11. The values for r/R = 0 of course correspond to the values Θ_m / Θ_c for $h R = \infty$ in Numerical Table 2. On the other hand, the values of the other perpendicular series constitute an important supplement here.

But I think that Numerical Table 12 is particularly noteworthy here; this table is likewise taken from that work. It seems that the authors have computed and compiled the cooling speed of the center and the axis for the various bodies listed at the top of the table. See Figure 11.

Numerical Table 11
Temperature in the Interior of the Sphere When

$$\Theta_c = 1^0: \ \Theta_r = \Theta_c \Phi_r \left(\frac{a t}{R^2}\right).$$

$\frac{at}{R^2}$	r/R: 0	1/20	1/4	1/3	1/2	2/3	3/4	9/10	1
0,000	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00
0,004	1,00	1,00	1,00	1,00	1,00	1,00	0,99	0,39	0,00
0,016	1,00	1,00	1,00	1,00	0,99	0,91	0,79	0,18	0,00
0,036	0,99	0,99	0,98	0,96	0,88	0,68	0,53	0,10	0,00
0,064	0,91	0,91	0,86	0,81	0,68	0,47	0,35	0,06	0,00
0,100	0,71	0,70	0,65	0,60	0,47	0,32	0,23	0,04	0,00
0,196	0,29	0,29	0,26	0,24	0,18	0,12	0,09	0,02	0,00
0,256	0,16	0,16	0,14	0,13	0,10	0,07	0,06	0,01	0.00
0,400	0,04	0,04	0,03	0,03	0,02	0,02	0,01	0,00	0,00
∞	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0.00

Numerical Table 12

Temperatures at Center or Axis of Various Bodies When

$\vartheta_e = 1°$;

$$\Theta_m = \Theta_e \, \Phi_m \left(\frac{a\,t}{X^2} \right).$$

$\frac{a\,t}{X^2}$	Platte (a)	(b) Quadrat. Balken Länge = ∞	Würfel (c)	(d) Zylinder Länge = ∞	(e) Zylinder Länge = Dmr.	(f) Kugel
0,032	1,00	1,00	1,00	1,00	1,00	1,00
0,080	0,98	0,95	0,93	0,92	0,69	0,83
0,100	0,95	0,90	0,86	0,85	0,81	0,71
0,160	0,85	0,72	0,61	0,63	0,53	0,41
0,240	0,70	0,49	0,35	0,40	0,28	0,19
0,320	0,58	0,33	0,19	0,26	0,16	0,09
0,800	0,18	0,03	0,01	0,02	0,00	0,00
1,600	0,02	0,00	0,00	0,00	—	—
3,200	0,00	—	—	—	—	—

Legend: a - plate; b - square beam, length infinite; c - cube; d - cylinder, length infinite; e - cylinder, length equals diameter; f - sphere.

A comparison between the sphere and the cube or between the cylinder of infinite length and the cylinder whose length equals the diameter would, I think, be rather interesting here. Although these relations apply only to an infinitely large heat transfer number, i.e., only for $h\,R = \infty$, they nevertheless do permit a transfer /conversion/ to processes with a finite heat transfer number, if we are careful enough.