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## SOME APPLICATIONS OF THE FAST FOURIER TRANSFORM

Feodor Theilheimer

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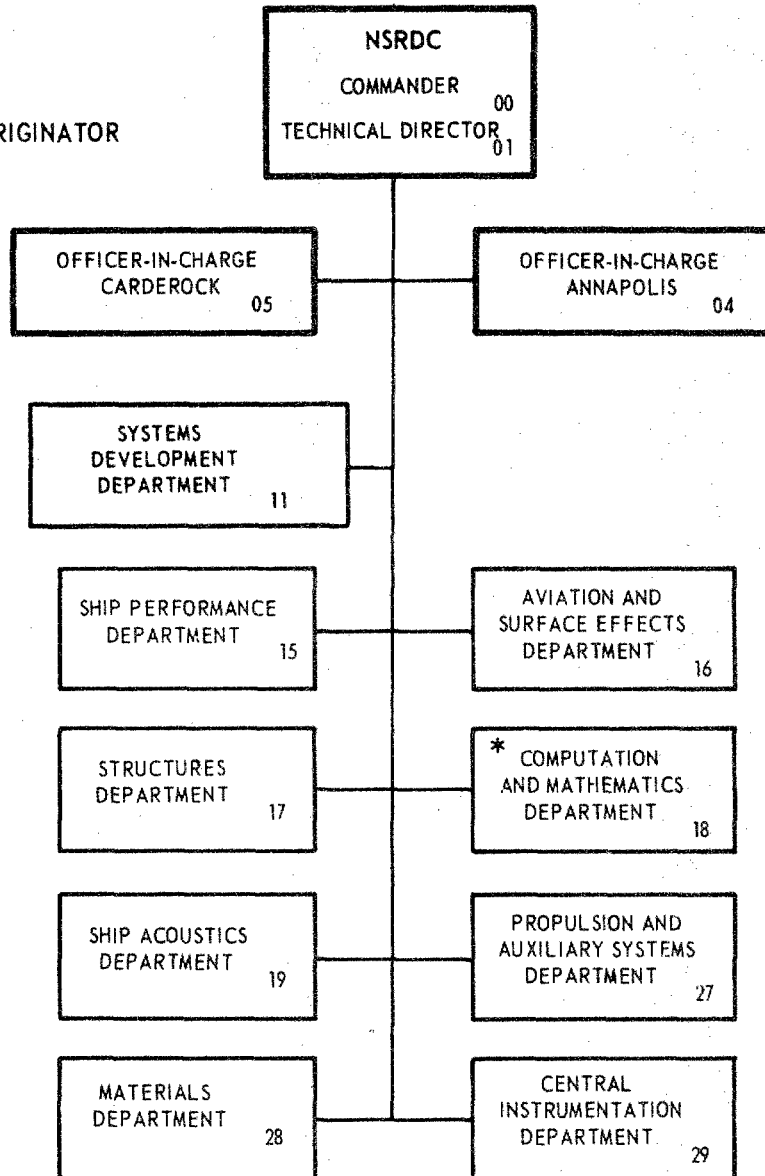## COMPUTATION AND MATHEMATICS DEPARTMENT

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SOME APPLICATIONS OF THE FAST

FOURIER TRANSFORM




by



Feodor Theilheimer

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## ABSTRACT

Fourier's approach to the problem of heat conduction served as the introduction to Fourier series and Fourier integral. This led later to the discrete Fourier transform and recently to the fast Fourier transform. The fast Fourier transform, serving as an expeditious means to pass from functions of time to functions of frequency, contributes to the analysis of time-invariant systems for deterministic as well as stochastic situations.

## ADMINISTRATIVE INFORMATION

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# HISTORICAL REMARKS ABOUT THE
# FOURIER TRANSFORM

The development of the Fourier transform goes back about 150 years to Jean Baptiste Joseph Fourier who was engaged at the time in trying to solve the problem of heat conduction. The partial differential equation of heat conduction is solved, for certain regions, by separation of variables. This leads among other things, to an ordinary differential equation which calls for a function proportional to its second derivative, the factor of proportionality being a negative number. Functions satisfying such differential equations are, of course, sinusoidal. To adjust solutions to the specific conditions of a particular heat problem, Fourier developed his famous theorem that any quite arbitrary and irregular function, either periodic or given only in a finite interval, can be expanded into an infinite sum of cosines and sines whose arguments are integer multiples of one basic argument. If a function $g(t)$ is periodic with period $T$, or is given only for the interval $0 < t < T$, then $g(t)$ can be written as

$$g(t) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{2\pi i}{T} nt}$$

where

$$a_n = \frac{1}{T} \int_0^T g(t) e^{-\frac{2\pi i}{T} nt} dt$$

This relationship is expressed here by means of exponentials, making use of the well-known relation $e^{ib} = \cos b + i \sin b$, rather than by means of sines and cosines. The question of precisely describing those functions which equal a convergent Fourier series has occupied mathematicians ever since the time of Fourier. Fourier's method of expanding periodic functions into series was extended to functions which are not periodic. This led to the Fourier integral, customarily written as

$$g(t) = \int_{-\infty}^{\infty} G(f) \, e^{2\pi i f t} \, dt$$

where

$$G(f) = \int_{-\infty}^{\infty} g(t) \, e^{-2\pi i f t} \, dt \, .$$

These formulas hold, of course, only if the improper integral $\int_{-\infty}^{\infty} |g(t)| \, dt$ is convergent. They associate a function of time, t, with a function of frequency, f. Until recently the term Fourier transform was reserved for these formulas.

If we are interested in numerical results we usually deal with a situation in which a function is given at N, usually equidistant, points: g(0), g(1), g(2), ..., g(N-2), g(N-1), and we want to find N quantities G(0), G(1), ..., G(N-1) such that

$$g(k) = \sum_{q=0}^{N-1} G(q) \, e^{\frac{2\pi i}{N} k q} \, .$$

Orthogonality relations permit this system of N equations in N unknowns to be readily solved for the G(q). Thus

$$G(q) = \frac{1}{N} \sum_{k=0}^{N-1} g(k) \, e^{-\frac{2\pi i}{N} k q}$$

The computation of the G(q), which we call the discrete Fourier transform, from given g(k) is a very straightforward process involving N multiplications and additions to find one G(q). To find all N values of G(q), $N^2$ operations are necessary; since $N^2$ for large N is a very large number, a shorter method was needed even for high-speed computers,

and the fast Fourier transform was developed.[1]


# THE FAST FOURIER TRANSFORM

A substantial saving of time can be achieved if the integer N has many factors. There are various ways of describing the time saving process of the fast Fourier transform (FFT), one of which follows: Let N be a power of 2, for example, $N = 64 = 2^6$. Then the original problem is equivalent to multiplying a 64 dimensional vector by a 64 by 64 matrix. None of the elements of the matrix are zero, since they are all of the form $e^{ib}$ which has an absolute value of 1. It can, however, be shown that this matrix can be factored into two matrices, one having 32 non-zeros per row and per column, and the other having two non-zeros per row and per column. We can then further factor the first matrix, the one with the 32 non-zeros in each row and each column, and continue the process until we obtain six matrices which have only two non-zero elements in each row and in each column. Thus, instead of needing $64^2 = 4096$ operations, we now need only 2 x 64 x 6 or 768 operations, a substantial reduction in the number of operations required. Note that reference to the matrices was made merely to illustrate how the FFT saves time. There is no intention to suggest that the FFT should be carried out by actually multiplying the matrices described.

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[1] Cooley, J.W. and J.W. Tuckey, "An algorithm for the machine calculation of complex Fourier Series," Math. Comp., vol. 10, pp. 297-301, April 1965.

# APPLICATIONS

The following applications have been chosen to illustrate the use of the Fourier transform in general and the fast Fourier transform in particular. No effort has been made to select the most important or the most recent applications. Note that the fast Fourier transform may be used to advantage in any large problem in which the Fourier transform is being used.

## LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

One of the most important applications of the Fourier transform, fast or otherwise, has to do with the solution of linear ordinary differential equations with constant coefficients. For instance we study

$$\sum_{m=0}^{M} b_m g^{(m)}(t) = p(t)$$

where $p(t)$ is a given function and $b_m$ are given coefficients, and where

$g^{(m)}(t) = \dfrac{d^m g(t)}{dt^m}$ . The differential equation is to be solved for the

unknown function $g(t)$. Applying the Fourier transform to both sides of the differential equation changes the left-hand side into a product of the transform $G(f)$ and a function of the frequency $f$, such that we may write $G(f) = H(f) \cdot P(f)$ where $P(f)$ is the transform of $p(t)$. From $G(f)$ we then go to $g(t)$ by taking the inverse Fourier transform. To use this method, it is, of course, important to be able to move easily from function of time to function of frequency, and vice versa, and it is this facility that the FFT provides.

# LINEAR TIME-INVARIANT SYSTEMS

Often we are in a worse predicament in which we are to solve the
equation

$$\Sigma \, b_m \, g^{(m)}(t) = p(t)$$

without actually being given the coefficients $b_m$. We are given only a
physical situation which takes input functions and furnishes output
functions. Such a physical system which is described by a linear differen-
tial equation with constant coefficients is called a linear time-invariant
system. The problem is to find $g(t)$ if $p(t)$ is given. We have to find
the function $H(f)$ without using the coefficients $b_m$. By assuming that
we have before us a physical system, we make an experiment with an
input which is sinusoidal with a particular frequency, say $f_0$. Then a
simple analysis shows that $H(f_0)$ becomes the ratio of output to input.
The disadvantage of this method is that it gives the function $H(f)$ for
only one particular f; so then for each f desired, another experiment
must be made.

A more expeditious way of finding $H(f)$ would be to choose $p(t)$ such
that $P(f) = 1$, so that the corresponding $G(f)$ will be the function $H(f)$
which governs the system. But it is shown in the standard books on
Fourier transforms that the inverse Fourier transform of the constant 1
is the Dirac function. If we find the response of our system to a Dirac
function, we will have a function $h(t)$, which we call the impulse response
whose Fourier transform is $H(f)$, the so-called transfer function. Now
if we want to find the $g(t)$ that goes with a given $p(t)$, we first find
$P(f)$, the Fourier transform of $p(t)$. Then $G(f) = H(f) \cdot P(f)$, and $g(t)$
is the inverse Fourier transform of $G(f)$.

A practical application of this method, described to me by a colleague, Theo Kooij,[2] concerns the prediction of the echo from an underwater object for various sonar signals. Assuming that the echo can be considered as output of a linear time-invariant system, we find the response to a Dirac function by finding the echo to the explosion of an explosive charge. This echo is the function h(t) associated with the system (or more exactly a constant multiple thereof). We digitize the hydrophone output and find the fast Fourier transform, which gives us H(f). For any sonar signal whose echo we want to find, we first find the Fourier transform, which can be done explicitly in many cases, and then multiply it by H(f) and take the fast inverse Fourier transform of the product. Thus the echo for various sonar signals can readily be found.

Another example of a time-invariant linear system is in connection with the response of a non-uniform beam to a transient forcing function. Such a situation is described by Francis Henderson in his report, "Transient Response Calculation in the Frequency Domain with General Bending Response Program."[3] In this case the unknown function is either the displacement or the moment, each a function of two variables, time t and longitudinal coordinate x. Here we have a differential equation which, after applying the Fourier transform with respect to the variable t, still contains derivatives with respect to the variable x. A difference equation scheme is used by dividing the x interval into, say, M sub-intervals. The differential equation then leads to a system of M linear equations which contain the frequency f as a parameter. This system must be solved for a large number of values of the frequency f. These

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[2] Personal Communication

[3] Henderson, F., "Transient Response Calculation in the Frequency Domain with General Bending Response Program," Naval Ship Research and Development Center Report 1613, February 1971.

solutions are obtained by utilizing the fact that the matrix in question has zeros in most positions except in the vicinity of the main diagonal. After having obtained the solution as functions of frequency, we then obtain the solution as a function of time by means of the inverse fast Fourier transform.

## STOCHASTIC PROCESSES

In the foregoing cases, a linear time-invariant device connected an input and an output function. There are, however, many physical situations in which it is more meaningful to consider the input and the output, not as deterministic functions of time, but as stochastic processes. For instance, we might be interested in studying the interrelation between the heights of the waves in the ocean and the response of a ship. One such study has been made by M. K. Ochi and W. E. Bolton,[4] in which they investigate the interrelation between the input stochastic process, given by the wave height at a chosen point in the ocean, and the output stochastic process, the response of the ship. The connection between input and output stochastic processes can again be described by means of the impulse response $h(t)$ or the transfer function $H(f)$, its Fourier transform, but the formulas look a bit different. Here, $S(f)$, the so-called power spectral density — often called simply the power spectrum — of the input stochastic process is related to $U(f)$, the power spectrum of the output stochastic process, by means of

$$U(f) = |H(f)|^2 S(f).$$

The power spectrum $S(f)$, and likewise $U(f)$, is a function of frequency characterized by the property that the integral over a frequency interval

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[4] Ochi, M. K. and W. E. Bolton, "Statistics for Prediction of Ship Performance in a Seaway," International Shipbuilding Progress, Vol. 20, No. 222 (1973).

$(f_1, f_2)$ namely $\int_{f_1}^{f_2} S(f) \, df$ represents the contribution to the variance from that frequency interval. (This definition by an integral explains why the longer name "power spectral density" is somewhat more precise than "power spectrum".) The power spectrum can be found as the Fourier transform of the autocorrelation which in turn is found from an observed particular realization of the stochastic process. An alternate way of computing the power spectrum, and here we refere to the discrete case for N given points, is to first find the Fourier transform of the given data, and to then find the squares of the absolute values. Before the introduction of the FFT this latter method was found to be much more time consuming than finding the autocorrelation for a certain number of lags and then its Fourier transform. However, the FFT now makes this alternate method, which does not use the autocorrelation, much faster than the method which uses autocorrelation.

Returning to our discussion of the interrelation of two stochastic processes (the wave height in the ocean and the response of a ship), let us consider a possible way of finding the function $H(f)$, or the square $|H(f)|^2$, known among naval engineers as the response amplitude operator. If we create waves of a certain frequency around a ship model, we can observe the response of the model, which may be a displacement, a force, a moment, whatever quantity we want to study. Then $H(f)$, the transfer function at a particular frequency, equals the ratio of output to input for a sinusoidal wave. Once the function $H(f)$ is determined, or more precisely, once the functions $H(f)$ with respect to each interesting response are determined, the corresponding power spectra of the responses are readily found for any given power spectrum of the waves of the ocean.

## SIGNAL ANALYSIS

We have thus far given examples of finding the relation between the input time function and the output time function by first finding the input as a function of frequency and then by a mere multiplication finding the output first as function of frequency and then, by inverse Fourier transform, as function of time. Sometimes, however, we have problems which do not require that all of these steps be taken. For instance, suppose we are listening to underwater sound in order to find out whether there are any interesting sound sources in the water which we would like to identify. We can consider the underwater sound as a stochastic process and use the time-series obtained by digitizing the output of a hydrophone to compute the power spectrum of the underwater sound. We need not subject the input stochastic process to a linear time-independent operation, but can be satisfied with studying the power spectrum of the underwater sound. We observe the characteristic features of the power spectrum, the location of its peaks and the like, and use this observation to compare it with previously obtained power spectra and store its essential characteristics for later comparisons. In this problem, we need only to perform a direct Fourier transform, the inverse Fourier transform is not needed.

A quite similar application has to do with listening to one of our own vessels. We again find the power spectrum and identify from it objectionable sources of sound on the vessel with the view of making the vessel more nearly silent. In this application, many of the computer programs developed for detection of objects by underwater sound are used.

## TURBULENCE STUDIES

One of the early applications of the power spectrum was in the study of turbulence. We are concerned with a fluid motion for which it is meaningful to consider the velocity as a stochastic process. To get better insight into the phenomenon we use the FFT to determine the power

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spectrum. In calling the fluid motion a stochastic process we need to give some further explanation. Thus far we have referred to stochastic processes which depend on time and on chance. A phenomenon depending on chance and on a coordinate x may just as well be called a stochastic process. In general, we may study stochastic processes that depend on chance, on time, and on the three coordinates x, y, z. In the case of a function of time we are led to a Fourier transform depending on frequency, where frequency has the dimension of time to the power minus one; frequency gives the number of waves per unit of time. A function of x, of length, yields a Fourier transform depending on inverse length, which may be called spatial frequency or wave number, giving the number of waves per unit length. In the study of turbulence we deal with stochastic processes depending on time as well as with those depending on space.

## PICTURE PROCESSING

Another application of Fourier analysis to functions of spatial coordinates is met in the field of picture processing. If, for instance, we want to remove what may be called optical noise from a picture, we can often do so by filtering, which in turn is done by first finding the Fourier transform, then removing components of undesirable spatial frequencies and finally taking the inverse Fourier transform.

## CONCLUDING REMARKS

In conclusion, attention should be called to the fact that the applications of the FFT are, of course, also applications of high-speed computers. It is no accident that the FFT was developed at a time when means were available to attack problems involving very many points, in situations where the FFT could show its vast superiority over other methods.

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13. ABSTRACT

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