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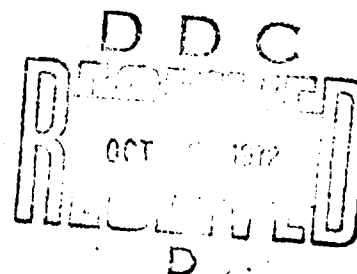
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A NOTE ON HOMOGENEOUS PROCESSES WITH INDEPENDENT INCREMENTS

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13. ABSTRACT

A class of stochastic processes, $X(t)$ is characterized based in the property that the conditional mean and variance of $X(t)$, given $X(t_0) = y$, for some $0 < t < t_1$, are linear functions of y . Two particular cases resulting in Wiener and Poisson processes are discussed.

KEY WORDS

Characterization
Stochastic process
Wiener Process
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1. INTRODUCTION.

Let $X(t)$, $t \geq 0$, be a homogeneous stochastic process with independent increments. Fix $t_1 > 0$. Then, given $X(t_1) = y$, the process $X(t)$, for $0 \leq t \leq t_1$, is called a tied-down process with end point equals y . Suppose $X(t)$ is a Poisson process, then the conditional distribution of $X(t)$ given $X(t_1) = y$, for all $0 \leq t \leq t_1$, is binomial with parameter $(y, t/t_1)$, and therefore the conditional expectation and variance of $X(t)$ given $X(t_1) = y$ are linear functions of y . Suppose $X(t)$ is a Wiener process, then the conditional distribution of $X(t)$ given $X(t_1) = y$, for all $0 \leq t \leq t_1$, is normal with parameter $((t/t_1)y, \sigma^2 t(1 - t/t_1))$, and hence the conditional expectation of $X(t)$ given $X(t_1) = y$ is a linear function of y and the conditional variance does not depend upon y .

In this note, we shall characterize a class of stochastic processes based on the property that the conditional mean and variance of $X(t)$, given $X(t_1) = y$, for some $0 < t < t_1$, are linear functions of y . It will be proved that if $E(X(t) \mid X(t_1) = y) = \alpha_0 + \alpha_1 y$, then
1) $\text{Var}(X(t) \mid X(t_1) = y) = \text{constant}$ a.e. if and only if $X(t) = W(t) + \mu t$, where $W(t)$ is a Wiener process and μ is a real constant,

1) $\text{Var}(X(t) \mid X(t_1) = y) = \beta_0 + \beta_1 y$ ($\beta_1 \neq 0$) if and only if $X(t) = cY(t) - vt$, where $Y(t)$ is a Poisson process and v and c are real constants. To avoid trivial cases, we shall assume that $X(t)$ is not a degenerate process. Also all stochastic processes $X(t)$, $t \geq 0$, considered in this note are assumed to be homogeneous, second-order and with independent increments.

For a recent survey of the results on characterizations of stochastic processes see the paper [1] by Lukacs.

2. THE RESULT.

We need the following two lemmas.

LEMMA 1. If $E(X(t) \mid X(t_1) = y) = \alpha_0 + \alpha_1 y$, then $\alpha_0 = 0$ and $\alpha_1 = t/t_1$, for all $0 \leq t \leq t_1 < \infty$.

PROOF. Since $X(t)$ is a homogeneous, second-order process with independent increments, it follows that $\mu(t) = E(X(t)) = \mu t$, $\sigma^2(t) = \text{Var}(X(t)) = \sigma^2 t$, for all $t \geq 0$, and $\rho(t_1, t_2) = \text{Corr. Coeff.}(X(t_1), X(t_2)) = \min(t_1, t_2)/\sqrt{t_1} \sqrt{t_2}$, for all $t_1 > 0$ and $t_2 > 0$, where $\mu = E(X(1))$ and $\sigma^2 = \text{Var}(X(1))$.

Therefore

$$\alpha_1 = \rho(t, t_1) \sigma(t)/\sigma(t_1) = t/t_1,$$

and

$$\alpha_0 = \mu(t) - \alpha_1 \mu(t_1) = 0.$$

LEMMA 2. Let $\kappa(s, t) = E(e^{isX(t)})$, be the characteristic function of $X(t)$ and $\chi(s) = \kappa(s, 1)$. Then

$$E(X(t_0)e^{isX(t_1)}) = -it_0 g^{t_1-1}(s) g'(s)$$

and

$$\begin{aligned} E(X^2(t_0)e^{isX(t_1)}) &= -t_0(t_0-1)g^{t_1-2}(s)(g'(s))^2 \\ &\quad - t_0 g^{t_1-1}(s) g''(s), \end{aligned}$$

for all $0 < t_0 < t_1$ and real s .

PROOF. Because $X(t)$ is a homogeneous, second-order process with independent increments, $g(s, t) = g^t(s)$ and the second partial derivative of $g(s, t)$ w.r.t. s exists for all t and s . It then follows that

$$E(X(t)e^{isX(t)}) = -itg^{t-1}(s) g'(s)$$

and

$$\begin{aligned} E(X^2(t)e^{isX(t)}) &= -t(t-1)g^{t-2}(s)(g'(s))^2 \\ &\quad - t g^{t-1}(s) g''(s), \end{aligned}$$

for all real s and $t > 0$.

Then Lemma 2 follows from the fact that

$$E(X^k(t_0)e^{isX(t_1)}) = E(X^k(t_0)e^{isX(t_0)}) g(s, t_1 - t_0)$$

for all k , $0 \leq t_0 \leq t_1$ and real s .

We now state and prove the main result of this note. For more detail of the proof, especially the last part, see the proof of theorem 2.1 in [2] by this author.

THEOREM. Let $0 < t_0 < t_1$. Then the necessary and sufficient condition that

$$(1) \quad E(X(t_0) \mid X(t_1) = y) = \alpha_0 + \alpha_1 y$$

and

$$(2) \quad \text{Var}(X(t_0) \mid X(t_1) = y) = \beta_0 + \beta_1 y \quad \text{a.e.,}$$

where α_0 , α_1 , β_0 and β_1 are constants w.r.t. y, is that

1) $X(t) = W(t) + \mu t$, if $\beta_1 = 0$, where $W(t)$ is a Wiener process
and μ is a real constant,

2) $X(t) = cY(t) - vt$, if $\beta_1 \neq 0$, where $Y(t)$ is a Poisson process
and v and c are real constants.

PROOF. The sufficient condition can be verified by a straight-forward calculation.

To prove the necessary condition. Suppose the conditions (1) and (2) hold. Then by lemma 1, the conditions (1) and (2) imply

$$(3) \quad E(X^2(t_0)e^{isX(t_1)}) - (t_0/t_1)^2 E(X^2(t_1)e^{isX(t_1)}) \\ = \beta_0 E(e^{isX(t_1)}) + \beta_1 E(X(t_1)e^{isX(t_1)})$$

for all real s .

By lemma 2, equation (3) is equivalent to

$$(4) \quad t_0(1 - t_0/t_1) \{ g^{t_1 - 2}(s) (g'(s))^2 - g^{t_1 - 1}(s) g''(s) \} \\ = \beta_0 g(s) - i\beta_1 t_1 g^{t_1 - 1}(s) g'(s),$$

for all real s .

Without loss of generality, we may assume that $g(s) \neq 0$ for all s .

And we rewrite equation (4) in the form

$$(5) \quad \frac{d}{ds} (g'(s)/g(s)) = -\beta_0 + i\beta_1 (g'(s)/g(s)),$$

where $B_0 = \beta_0 / (t_0(1 - t_0/t_1))$ and $B_1 = \beta_1 t_1 / (t_0(1 - t_0/t_1))$.

Because the second derivative of $g(s)$ exists and does not vanish for the s in a neighborhood N of the origin and $g'(s)/g(s)$ is independent of t_0 and t_1 , B_0 and B_1 are independent of t_0 and t_1 . In addition, it is easy to check that if $\beta_1 = 0$, then $B_0 > 0$. The solution of equation (4) is, if $\beta_1 = 0$,

$$g(s) = \exp \left\{ i\mu s - \frac{1}{2} \sigma^2 s^2 \right\},$$

where μ is a real constant and $\sigma^2 = B_0 > 0$, and if $\beta_1 \neq 0$,

$$g(s) = \exp \left\{ -i\nu s + \lambda(e^{ics} - 1) \right\},$$

where λ is a positive real constant independent of t and s , and

$$\nu = B_0/B_1, \quad c = B_1.$$

Therefore, the characteristic function of $X(t)$ is, if $\beta_1 = 0$,

$$g(s, t) = \exp \left\{ i\mu ts - \frac{1}{2} \sigma^2 ts^2 \right\},$$

and, if $\beta_1 \neq 0$,

$$g(s, t) = \exp \left\{ -i\nu ts + \lambda t(e^{ics} - 1) \right\}.$$

This completes our proof of the theorem.

The following two corollaries of the theorem are characterizations of the Wiener and the Poisson processes.

COROLLARY 1. If $E(X(t)) = 0$ for some $t > 0$. Then the necessary and sufficient condition that $X(t)$ is a Wiener process is that

$E(X(t_0) \mid X(t_1) = y)$ is a linear function of y and $\text{Var}(X(t_0) \mid X(t_1) = y)$ is constant a.e. for some $0 < t_0 < t_1$.

COROLLARY 2. The necessary and sufficient condition that $X(t)$ is a Poisson process is that $E(X(t_0) \mid X(t_1) = y)$ is a linear function of y and $\text{Var}(X(t_0) \mid X(t_1) = y) = (t_0/t_1)(1 - t_0/t_1)y$ a.e. for some $0 < t_0 < t_1$.

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