AD 744809

AD

REPORT NO. RL-TR-72-5

THE EMPLOYMENT OF GALERKIN'S VECTOR IN DEFINING CONCENTRATED LOADS AND MOMENTS IN LINEAR ISOTROPIC ELASTICITY

by

James J. Richardson

April 1972

Approved for public release; distribution unlimited.

U.S. ARMY MISSILE COMMAND

U.S. ARMY MISSILE COMMAND

Redstone Arsenal, Alabama

DISPOSITION INSTRUCTIONS

Destroy this report when it is no longer needed. Do not return it to the originator.

DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

TRADE NAMES

Use of trade names or manufacturers in this report does not constitute an official indorsement or approval of the use of such commercial hardware or software.

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Ground Equipment and Materials Directorate Directorate for Research, Development, Engineering and Missile Systems Laboratory U.S. Army Missile Command Redstone Arsenal, Alabama 35809	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP NA

3. REPORT TITLE

THE EMPLOYMENT OF GALERKIN'S VECTOR IN DEFINING CONCENTRATED LOADS AND MOMENTS IN LINEAR ISOTROPIC ELASTICITY

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (First name, middle initial, last name)

James J. Richardson

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
25 April 1972	36	9

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. (DA) 1T061101A91A	RL-TR-72-5
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	AD_______

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	Same as No. 1

13. ABSTRACT

This report contains a discussion of a formulation of Galerkin's vector in isotropic linear elasticity and an employment of this vector in describing concentrated loads and moments. Various forms of these concentrated phenomena are developed by the familiar process of superposition and then by a limit solution technique.

The report is primarily based on work contained in the author's doctoral dissertation written at the University of Illinois at Champaign-Urbana. In addition, methods and results obtained from several unpublished papers by Professor Marvin Stippes of the University of Illinois are incorporated.

I

DD FORM 1 NOV 65 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Galerkin's vector Linear isotropic elasticity Cauchy's equations Concentrated loads and moments						

25 April 1972 Report No. RL-TR-72-5

THE EMPLOYMENT OF GALERKIN'S VECTOR IN DEFINING CONCENTRATED LOADS AND MOMENTS IN LINEAR ISOTROPIC ELASTICITY

by

James J. Richardson

DA Project No. 1T061101A91A

Ground Equipment and Materials Directorate
Directorate for Research, Development, Engineering
and Missile Systems Laboratory
U.S. Army Missile Command
Redstone Arsenal, Alabama 35809

ABSTRACT

This report contains a discussion of a formulation of Galerkin's vector in isotropic linear elasticity and an employment of this vector in describing concentrated loads and moments. Various forms of these concentrated phenomena are developed by the familiar process of superposition and then by a limit solution technique.

The report is primarily based on work contained in the author's doctoral dissertation written at the University of Illinois at Champaign-Urbana. In addition, methods and results obtained from several unpublished papers by Professor Marvin Stippes of the University of Illinois are incorporated.

IV

CONTENTS

SYMBOLS

$C_{ik}(D)$	Cauchy's operator
D_i	Differential operator $\left(\frac{\partial}{\partial x_i}\right)$
D^2	$D_i D_i$ or ∇^2
$\tilde{E}$	Strain dyadic
F_i	Body force field
g_i	Galerkin's vector
$\tilde{I}$	Idemfactor
$\underline{L}$	Concentrated load
S	State of stress
$\tilde{S}$	Stress dyadic
T_i	Surface tractions
u_i	Displacement field
V	Region of elastic space
∂V	Surface of V
∇	Convention del operator $\left(\frac{\partial}{\partial x_i} \underline{i}_i\right)$
$\Delta(D)$	Determinant of $C_{ik}(D)$
δ_{ij}	Kroenecker delta
ϵ_{ijl}	Permutation symbol
μ, λ	Lamé's constants
ν	Poisson's ratio
ρ	Mass density

VI

1. Introduction

The equations of equilibrium of an isotropic linear elastic body V are called Cauchy's equations and may be written as

$$C_{ik}(D)\ U_k + \rho F_i = 0 \tag{1.1}$$

where $C_{ik}(D)$ is called Cauchy's operator and is defined below

$$C_{ik}(D) = \mu\ D^2\ \delta_{ik} + (\lambda + \mu)\ D_i D_k \quad . \tag{1.2}$$

The Cauchy equations are formulated with the assumption that V is a continuous medium. This ultimately leads to the restriction that u_i is at least piecewise continuous of class C^2 throughout V. A solution to Eq. (1.1) which satisfies this restriction is called a regular solution*. The form of a regular solution to Eq. (1.1) depends upon the shape of the body and the type of loading imposed.

Singular solutions exist, however, which fail to exhibit this continuity at a point, along an arc or over a surface of V. The purpose of this report is to discuss solutions which are singular in some deleted neighborhood of one or more points in V. Such a solution has one or more isolated singularities and is called an isolated singular solution.

As an isolated (nonremovable) singular point $\underline{x}$ in V is approached, the limits of u_i and their derivatives do not exist. In fact, the displacements and stresses at $\underline{x}$ are unbounded. The isolated singularity may be considered as the manifestation of some type of load or moment. In this report the physical significance which may be attached to various forms of isolated singularities will be discussed.

It will be readily seen that at least three useful functions are served by furthering one's understanding of isolated singularities of Eq. (1.1). There are numerous cases in which loading is closely approximated by a concentrated force internal to a body. Also, singular solutions may be used as influence functions to obtain regular solutions. This is done, for example, in Betti's [1] adaption of the method of singularities to isotropic elasticity. Finally, the investigation of point defects in metallurgy involves these solutions (e.g., see Simmons et al [2]).

---

*Additionally, if V is unbounded it is required that the $u_k = 0(1/r)$ and $u_{k,j} = 0(1/r^2)$ uniformly in the spherical coordinates θ, ϕ as r approaches infinity.

First, a solution of Eq. (1.1) must be obtained. Because the Cauchy equations are elliptic (assuming that $C_{ik}(D)$ is positive definite), techniques analogous to those found in potential theory may be used [3]. In Paragraph 2, Cauchy's equations are written in biharmonic form. This is possible through the employment of the Galerkin vector which is developed in Paragraph 2.

2. Galerkin Vector

A very concise derivation of the Galerkin vector was presented by Somigliana [4]. Choosing a transformation $K_{ij}(D)$ so that

$$u_i = K_{ik}(D)\, v_k \quad , \tag{2.1}$$

the Cauchy equations may be written as

$$C_{ik}(D)\, K_{km}(D)\, v_m + \rho F_i = 0 \quad . \tag{2.2}$$

Now it is specified that K_{km} is the matrix of cofactors of Cauchy's vector; that is

$$K_{ip}(D) = \frac{1}{2!}\, \epsilon_{ijk}\, \epsilon_{pmn}\, C_{jm}\, C_{kn} = \mu D^2 \left[(\lambda + 2\mu) D^2 \delta_{ip} - (\lambda + \mu) D_i D_p\right] \quad . \tag{2.3}$$

By definition

$$C_{ik}(D)\, K_{km}(D) = \delta_{im}\, \Delta(D) \tag{2.4}$$

where $\Delta(D)$ represents the determinant of $C_{ik}(D)$

$$\Delta(D) = \mu^2 (\lambda + 2\mu)\, D^6 \quad . \tag{2.5}$$

Thus, Cauchy's equations may be written in terms of the variable v_i, which will be called an auxillary vector function

$$\mu^2 (\lambda + 2\mu)\, D^6\, v_i + \rho F_i = 0 \quad . \tag{2.6}$$

Finally, a new auxilliary vector function is defined as

$$g_i = \nabla^2 v_i \tag{2.7}$$

so that

$$u_i = \mu(\lambda + 2\mu)\, D^2 g_i - (\lambda + \mu)\, D_m D_i g_m \quad . \tag{2.8}$$

A particular solution to Eq. (2.6) may be found by using the concept of influence functions which are analogous to Green's functions in potential theory. An insight into this approach can be gained from the one dimensional string problem. Consider a string L units in length which is fixed at both ends and which lies along the x_1 axis. If the string, under a tension t, is subjected to a transverse unit force applied at $x_1 = s$, the equation of equilibrium is

$$t x_{2,11} + \delta(x_1 - s) = 0 \quad . \tag{2.9}$$

$\delta(x_1 - s)$ is the Dirac delta function defined by Chen [5]

$$\delta(x_1 - s) = \begin{cases} 0, & \text{for } x_1 \neq s \\ \infty, & \text{for } x_1 = s \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x_1 - s)\, dx = 1 \quad .$$

A solution of Eq. (2.9) indicating the vertical displacement of the string is

$$x_2 = K(x_1, s) \tag{2.10}$$

where $K(x_1, s)$ is called an influence coefficient. Importantly, Eq. (2.10) can be used to solve problems concerning distributed transverse loadings $q(x_1)$ because it can be shown that the resulting equilibrium equation has the solution

$$x_2 = \int_0^L K(x_1, s)\, q(s)\, ds \quad . \tag{2.11}$$

A solution to Eq. (2.6) which corresponds to a body force ρF_i can be determined in an identical manner. To avoid complications introduced by finite regions, a body V^∞ of infinite proportions is chosen. Imagine a body force ρF_i which is only nonzero over a finite region V. Then, if $S(\underline{x}, \underline{\xi})$ is the solution of

$$\Delta(D)\ S(\underline{x}, \underline{\xi}) + \delta_3(\underline{x} - \underline{\xi}) = 0 \tag{2.12}$$

where $\delta_3(\underline{x} - \underline{\xi})$ is the volume Dirac delta function, then a solution to Eq. (2.6) at a point $\underline{x}$ is

$$q_i(\underline{x}) = \int S(\underline{x}, \underline{\xi})\ \rho F_i(\underline{\xi})\ dD(\underline{\xi}) \quad . \tag{2.13}$$

Fritz John [6] presents the general form of $S(\underline{x}, \underline{\xi})$. The particular form of $S(\underline{x}, \underline{\xi})$ for Cauchy's equations was found to be [7]

$$S(\underline{x}, \underline{\xi}) = \alpha R(\underline{x}, \underline{\xi}) \tag{2.14}$$

where

$$\alpha = \frac{1}{4\pi\mu^2(\lambda + 2\mu)} \ , \quad R(\underline{x}, \underline{\xi}) = \left[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2\right]^{\frac{1}{2}} \quad . \tag{2.15}$$

Thus, a particular solution to Eq. (2.6) is

$$q_i = \alpha \int \rho F_i(\underline{\xi})\ R(\underline{x}, \underline{\xi})\ dD(\underline{\xi}) \quad . \tag{2.16}$$

In this form the Galerkin vector q_i in Eq. (2.8) yields a solution u_i which represents the displacement within an infinite linearly elastic body V^∞ because of a body force field ρF_i acting only within a finite region V.

3. Limit Solution

The purpose of this section is to employ the limit solution technique of Sternberg and Eubanks [8] to develop and discuss various types of isolated singularities in linear isotropic elasticity.

Again, entire three dimensional space V^{∞} is considered. This time, imagine a sequence of regions $\{V^{(n)}\}$ each containing the origin and a corresponding sequence of body forces $\{\rho F_i^{(n)}\}$ such that $\rho F_i^{(1)}$ acts over region $V^{(1)}$, $\rho F_i^{(2)}$ over $V^{(2)}$, etc. It is required that each $\rho F_i^{(n)}$ be of class C^2 in V^{∞} and be zero at any point outside of $V^{(n)}$. It has been shown that, under these circumstances, a particular solution $u_i^{(n)}(\underline{x})$ exists for each $V^{(n)}$ and corresponding body force field $\rho F_i^{(n)}$. In addition, as n approaches infinity the length of the longest chord within $V^{(n)}$ must uniformly approach zero.

The transformation represented by Eq. (2.14) is now employed, yielding

$$u_k^{(n)}(\underline{x}) = K_{ki}\, g_i^{(n)}(\underline{x}) \quad . \tag{3.1}$$

Thus, if $\underline{\xi}$ is a typical point within $V^{(n)}$ and $\underline{x}$ is any other point within V^{∞},

$$g_i^{(n)}(\underline{x}) = \alpha \int R(\underline{x}, \underline{\xi})\, \rho F_i^{(n)}(\underline{\xi})\, dV^{(n)}(\underline{\xi}) \quad . \tag{3.2}$$

The limiting case of Eq. (3.2) is defined as

$$G_i(\underline{x}) = \lim_{n \to \infty} g_i^{(n)}(\underline{x}) = \lim_{n \to \infty} \alpha \int R(\underline{x}, \underline{\xi})\, \rho F_i^{(n)}(\underline{\xi})\, dV^{(n)}(\underline{\xi}) \quad . \tag{3.3}$$

Richardson [7] showed that, in general,

$$G_{i,jk}(\underline{x}) = \lim_{n \to \infty} g_{i,jk}^{(n)} \quad . \tag{3.4}$$

Therefore, U_k computed from G_k represents the limiting solution, a displacement field with an isolated singularity at the origin. The forms which this solution assumes will be discussed in later sections.

$R(\underline{x}, \underline{\xi})$ may be expanded in the Taylor's series for three variables shown below

$$R(\underline{x}, \underline{\xi}) = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}}{i!j!k!} D_1^i D_2^j D_3^k (r)\, \xi_1^i \xi_2^j \xi_3^k , \qquad r = r(\underline{x}) \quad . \tag{3.5}$$

Thus, Eq. (3.3) becomes

$$\underline{G}(\underline{x}) = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}}{i!j!k!} D_1^i D_2^j D_3^k (r) \lim_{n \to \infty} \int \rho \underline{F}^{(n)}(\underline{\xi})\, \xi_1^i \xi_2^j \xi_3^k \, dV^{(n)}(\underline{\xi}) \quad . \tag{3.6}$$

Note that the summation sign may be removed from under the integral sign because Taylor's series is uniformly convergent. Defining the moment integrals $\underline{M}^{ijk}$ which are of order i+j+k and operators ^{ijk} by

$$\underline{M}^{ijk} = \lim_{n \to \infty} \int \rho \underline{F}^{(n)}(\underline{\xi})\, \xi_1^i \xi_2^j \xi_3^k \, dV^{(n)}(\underline{\xi}) \tag{3.7}$$

$${}^{ijk} = \frac{(-1)^{i+j+k}}{i!j!k!} D_1^i D_2^j D_3^k \quad , \tag{3.8}$$

where i+j+k = p which is the order of the operator ^{ijk}, allows one to write the Galerkin vector as

$$G(\underline{x}) = \sum_{i,j,k=0} {}^{ijk}(r)\, \underline{M}^{ijk} \quad . \tag{3.9}$$

This is a series which represents the Galerkin vector for isolated singularities of all orders. Additionally, by choosing appropriate terms of the series, one can write the Galerkin vector corresponding to specific forms of isolated singularities.

It is of interest to investigate the nature of $\underline{M}^{ijk}$. Note that if the sequence of regions $V^{(n)}$ and the corresponding body force fields $\rho F_i^{(n)}$ are symmetric about all axes, then the integrals $\underline{M}^{ijk}$ with an odd i, j, or k are zero. The shape of the region $V^{(n)}$ is arbitrary so symmetry can be assumed; however, $\rho F_i^{(n)}$ are not generally symmetric. Later in this report some manifestations of symmetry of $\rho F_i^{(n)}$ will be discussed.

In Paragraphs 4, 5, and 6 it will be shown that Eq. (3.9) may be written in the form

$$G_i(\underline{x}) = \phi(r)m_i + \phi_j(r)m_{ij} + \phi_{jk}(r)m_{ijk} + \ldots \quad . \qquad (3.10)$$

4. Fundamental or First Order Singularity

The term in Eq. (3.10) corresponding to i=j=k=0 is

$$G_i^1(\underline{x}) = \alpha r \lim_{n\to\infty} \int \rho F_i^{(n)}(\underline{\xi})\, dV^{(n)}(\underline{\xi}) \quad . \qquad (4.1)$$

From Eq. (2.8) it is seen that the displacement vector is expressed in terms of the second derivatives of g_i. Therefore, if g_i is of $O(r)$ as $r \to 0$ then u_i will be of $O(1/r)$ as $r \to 0$. This case shall be called the first order singularity. Superscripts on G_i will indicate the order of singularity. Defining

$$\phi(r) = \alpha r,\ m_i = \lim_{n\to\infty} \int \rho F_i^{(n)}(\underline{\xi})\, dV^{(n)}(\underline{\xi}) \quad , \qquad (4.2)$$

produces the first term of Eq. (3.10)

$$G_i^1(\underline{x}) = \phi(r)m_i \quad . \qquad (4.3)$$

It is readily noted that Eq. (4.2) may be interpreted as the Galerkin vector leading to a solution which corresponds to a concentrated load acting at r = 0 if it is specified that

$$\lim_{n\to\infty} \int \rho F_i^{(n)}(\underline{\xi})\, dV^{(n)}(\underline{\xi}) = L_i \quad . \tag{4.4}$$

In the following, the fundamental displacements, displacement gradients, strains, and rotations are given. Throughout the remainder of this section, $i \neq j \neq k$ and i, j, k will not be summed.

The strain-displacement relations are given by

$$U_i^1 = \frac{1}{8\pi\mu(1-\nu)r^3}\Big[4(1-\nu)\, m_i x_i^2 + (3-4\nu)\, m_i x_j^2 + (3-4\nu)\, m_i x_k^2 + m_j x_i x_j + m_k x_i x_k\Big] \quad . \tag{4.5}$$

The displacement gradients are

$$U_{i,i}^1 = \frac{1}{8\pi\mu(1-\nu)r^5}\Big[4(\nu-1)\, m_i x_i^3 + m_j x_j^3 + m_k x_k^3 - 2m_j x_i^2 x_j - 2m_k x_i^2 x_k + (4\nu-1)\, m_i x_i x_j^2 + (4\nu-1)\, m_i x_i x_k^2 + m_j x_j x_k^2 + m_k x_j^2 x_k\Big] \quad . \tag{4.6}$$

However, for the off-diagonal terms,

$$U_{i,j}^1 = \frac{1}{8\pi\mu(1-\nu)r^5}\Big[m_j x_i^3 + (4\nu-3)\, m_i x_j^3 + (4\nu-6)\, m_i x_i^2 x_j - 2m_j x_i x_j^2 + m_j x_i x_k^2 + (4\nu-3)\, m_i x_j x_k^2 - 3m_k x_i x_j x_k\Big] \tag{4.7}$$

Next, the strains are found to be

$$E_{ii}^1 = \frac{1}{8\pi\mu(1-\nu)r^5}\Big[4(\nu-1)\, m_i x_i^3 + m_j x_j^3 + m_k x_k^3 - 2m_j x_i^2 x_j - 2m_k x_i^2 x_k + (4\nu-1)\, m_i\left(x_i x_j^2 + x_i x_k^2\right) + m_j x_j x_k^2 + m_k x_j^2 x_k\Big] \quad . \tag{4.8}$$

For the off-diagonal terms,

$$E^1_{ij} = \frac{1}{8\pi\mu(1-\nu)r^5}\Big[(2\nu-1)\,m_j x_i^3 + (2\nu-1)\,m_i x_j^3 + (2\nu-4)\,m_i x_i^2 x_j$$

$$+ (2\nu-4)\,m_j x_i x_j^2 + (2\nu-1)\,m_j x_i x_k^2 + (2\nu-1)\,m_i x_j x_k^2 - 3m_k x_i x_j x_k\Big] = E^1_{ji} \; . \tag{4.9}$$

Finally, the rotations are

$$\Omega^1_{ij} = \frac{1}{8\pi\mu(1-\nu)r^5}\Big[4(1-\nu)\,m_j x_i^3 + 4(\nu-1)\,m_i x_j^3 + 4(\nu-1)\,m_i x_i^2 x_j$$

$$+ 4(1-\nu)\,m_j x_i x_j^2 + 4(1-\nu)\,m_j x_i x_k^2 + 4(\nu-1)\,m_i x_j x_k^2\Big] = -\Omega^1_{ji} \; . \tag{4.10}$$

So, the displacements, displacement gradients, strains, and rotations for a state of stress with a first order isolated singularity at the origin have been defined in terms of m_i, μ, ν, and the coordinates of the point.

It is now possible to obtain higher order isolated singularities by a [illegible] superposition technique. Flexibility influence coefficients $A_{ij}(\underline{x}, \underline{\xi})$ can be used to describe the effects of a force Q_j/ϵ acting at a point $\underline{\xi}$ on the displacement u_j at some other point $\underline{x}$ (Figure 1). In fact, the displacement is simply

$$U^1_i(x)\Big|_{\underline{\xi}} = A_{ij}(\underline{x}, \underline{\xi})\,\frac{Q_j}{\epsilon}\,(\underline{\xi}) \; . \tag{4.11}$$

Under conditions shown in Figure 1, Eq. (4.11) will be identical to Eq. (4.5). This represents the fundamental or first order singular solution. Similarly, the displacement at $\underline{x}$ because of a concentrated force $-\,Q_j/\epsilon$ acting at $\underline{\xi} + \underline{\epsilon}$ is

$$U^1_i(\underline{x})\Big|_{\underline{\xi}+\underline{\epsilon}} = -\,A_{ij}(\underline{x}, \underline{\xi} + \underline{\epsilon})\,\frac{Q_j}{\epsilon} \; . \tag{4.12}$$

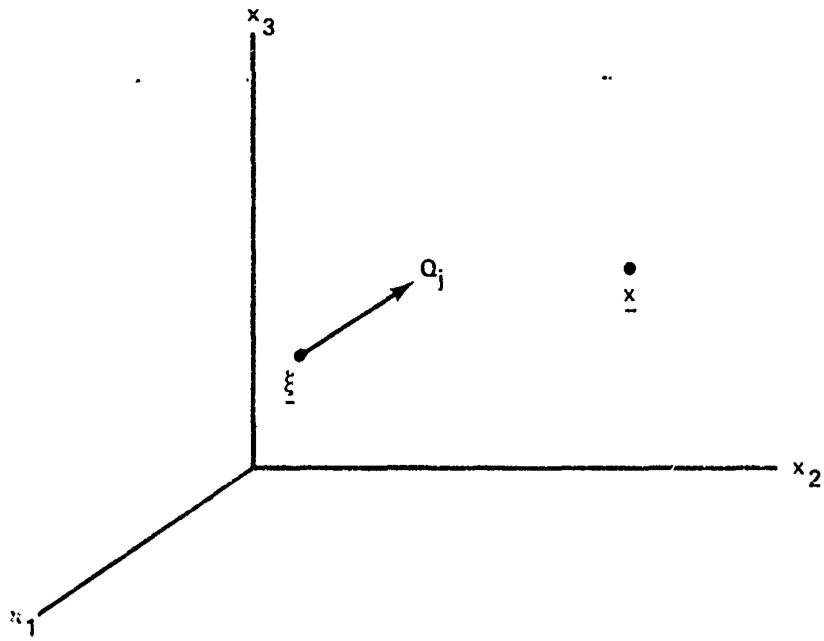

Figure 1. Concentrated Load at $\underline{\xi}$, First Order Singularity

Because the force-displacement relationship is considered linear, the displacements created by Q_j/ϵ at $\underline{\xi}$ and $-Q_i/\epsilon$ at $\underline{\xi} + \underline{\epsilon}$ are simply the superposition of Eqs. (4.11) and (4.12) (Figure 2). Taking the limit of the resulting expression yields a second order singular solution

$$U_i^2(\underline{x}) = \lim_{\epsilon \to 0} \left[U_i^1(\underline{x})\Big|_{\underline{\xi}} + U_i^1(\underline{x})\Big|_{\underline{\xi}+\underline{\epsilon}} \right] = \lim_{\epsilon \to 0} \frac{A_{ij}(\underline{x}, \underline{\xi}) - A_{ij}(\underline{x}, \underline{\xi} + \underline{\epsilon})}{\epsilon} Q_j \tag{4.13}$$

which is recognized as the first derivative of $U_i^1(\underline{x})$.

Specifically, in Figure 3, Q is parallel to the x_1 axis and $\underline{\epsilon}$ is measured along the x_1 axis. Therefore, let $\bar{U}_i^1(\underline{x})\Big|_{\underline{\xi}}$ and $\bar{U}_i^1(x)\Big|_{\underline{\xi}+\underline{\epsilon}}$ be represented by Eq. (4.5) where only m_1 is nonzero. For convenience, the point ξ is designated at the origin. Because the influence coefficients are dependent upon the vector $R(\underline{x}, \underline{\xi})$, the displacements at $\underline{x}$ caused by the forces shown are

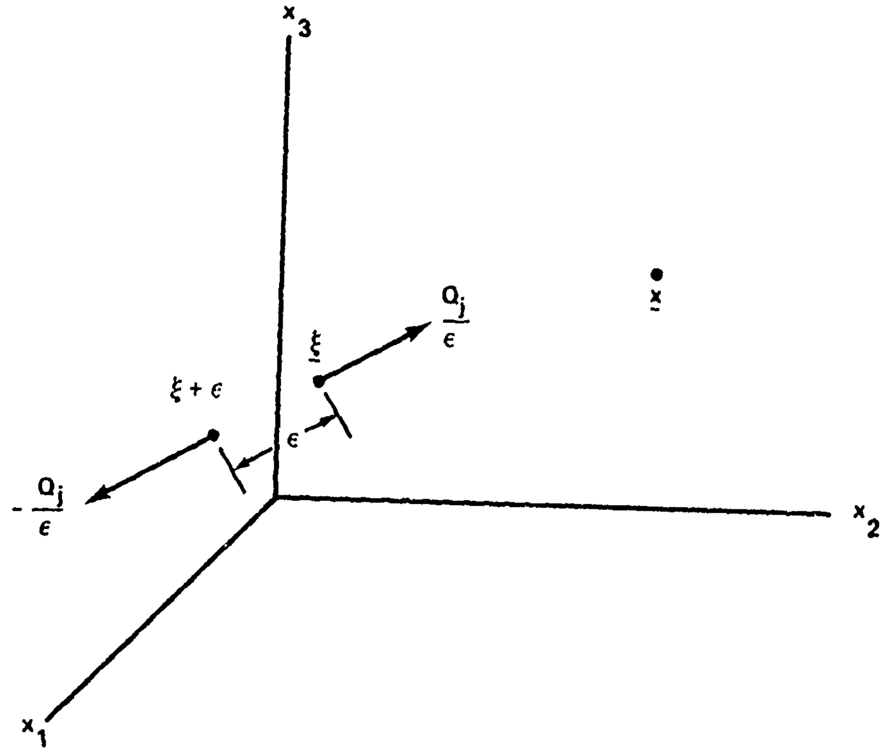

Figure 2. Second Order Singularity

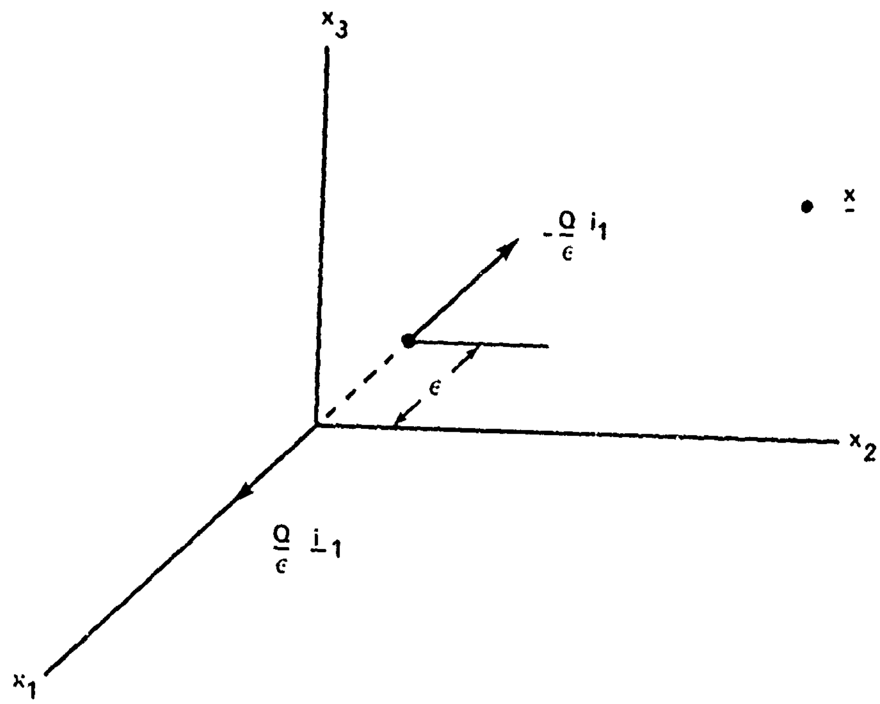

Figure 3. Force Doublet

$$U_i^2(\underline{x}) = \lim_{\epsilon \to 0} \left[\bar{U}_i^1(\underline{x}) \Big|_{\xi} + \bar{U}_i^1(\underline{x}) \Big|_{\xi+\epsilon} \right]$$

$$= \lim_{\epsilon \to 0} \frac{A_i\left(x_1, x_2, x_3\right) - A_i\left[\left(x_1 - \epsilon\right), x_2, x_3\right]}{\epsilon} Q = U^1_{i,1}(\underline{x}) \quad . \qquad (4.14)$$

This is referred to as a force doublet.* If the forces are arranged as shown in Figure 4 then the displacement field is given by

$$U_i^2(x) = \lim_{\epsilon \to 0} \frac{A_i\left(x_1, x_2, x_3\right) - A_i\left[\left(x_1, x_2 - \epsilon\right), x_3\right]}{\epsilon} Q = \hat{U}^1_{i,2}(\underline{x}) \qquad (4.15)$$

where $\hat{U}_i^1(\underline{x})$ is represented by Eq. (4.5) if only m_1 is nonzero and is called a force couplet.

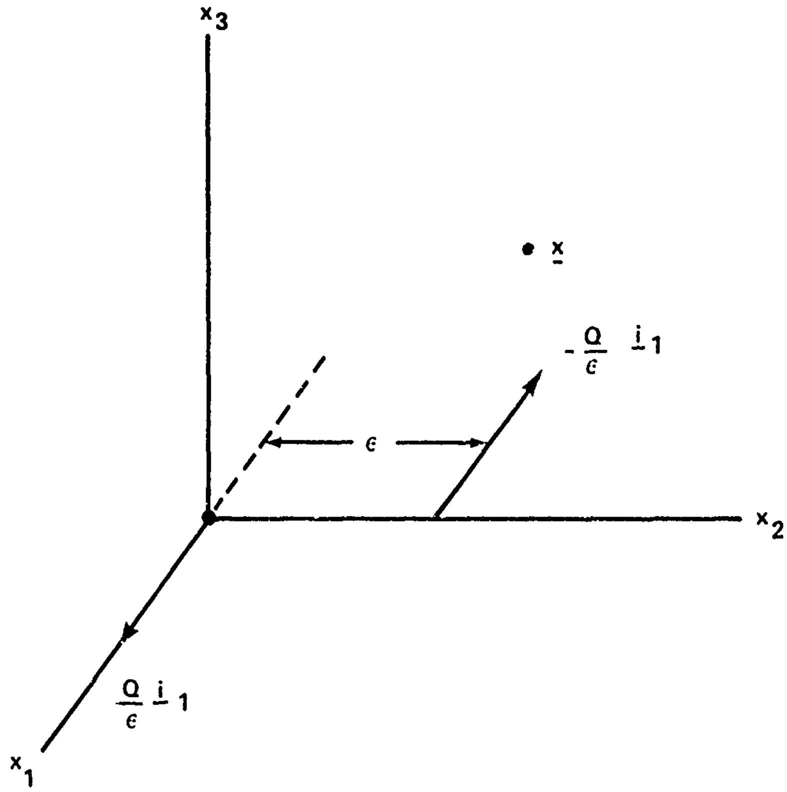

Figure 4. Force Couplet

*Three force doublets of equal magnitude acting along the x_1, x_2, and x_3 axes comprise a center of dilatation.

The generalization of this process is intuitively simple. Doublets and couplets of any magnitude may be generated in various directions or may themselves be superposed to form third or higher order singularities. This procedure is, however, artificial and tedious. In Paragraphs 5 and 6, the limit solution technique will be employed to form expressions for all possible second and third order isolated singular solutions.

5. Second Order Singularities

It is now observed that the terms $\overset{100}{\Phi}(r)$, $\overset{010}{\Phi}(r)$, $\overset{001}{\Phi}(r)$ and $\underline{M}^{100}$, $\underline{M}^{010}$, $\underline{M}^{001}$ can be treated as components of first and second order tensors. Making the following definitions

$$\phi_j(r) = -\frac{\alpha x_j}{r} , \quad m_{ij} = \lim_{n \to \infty} \int \rho F_i^{(n)}(\underline{\xi})\, \xi_j \, dV^{(n)}(\underline{\xi}) \quad . \tag{5.1}$$

The Galerkin vector corresponding to second order singularities may be written as

$$G_i^2(\underline{x}) = \phi_j(r)\, m_{ij} \tag{5.2}$$

which is identical to the second term of Eq. (3.10).

To examine the physical characteristics of various second order singularities, one may first allow all m_{ij} except m_{11} to be zero. In this case, Eq. (5.2) becomes

$$\underline{G}^2(\underline{x}) = \left[\phi_1(r)\, m_{11},\ 0,\ 0\right] \quad . \tag{5.3}$$

One may say that the resulting displacement pertains to a force doublet along the x_1 axis.* Similarly if m_{22} or m_{33} are the only nonzero components of m_{ij}, then the doublets are along the x_2 or x_3 axis, respectively. The significance of the off-diagonal terms is shown by setting all m_{ij} equal to zero except m_{23}. Now the Galerkin vector is

---

*If only m_{11}, m_{22}, m_{33} are nonzero and $m_{11} = m_{22} = m_{33}$, a center of dilatation is produced.

$$\underline{G}^2(\underline{x}) = \left[0, \phi_3(r)\, m_{23}, 0\right] \quad . \tag{5.4}$$

This corresponds to a force couplet about the x_1 axis with the forces in the x_2 direction.

If $\rho F_i^{(n)}$ are symmetric about all axes then all m_{ij} are zero and, hence, there are no second order singular terms. If $\rho F_i^{(n)}$ are symmetric about one or two axes then certain components of the tensor m_{ij} will be zero. For example, if $\rho F_i^{(n)}$ is symmetric about the plane perpendicular to the x_3 axis then $m_{13} = m_{23} = m_{33} = 0$ and the resulting Galerkin vector is

$$\underline{G}^2(\underline{x}) = \left\{\left[\phi_1(r)m_{11} + \phi_2(r)m_{12}\right], \left[\phi_1(r)m_{21} + \phi_2(r)m_{22}\right], \left[\phi_1(r)m_{31} + \phi_2(r)m_{32}\right]\right\} . \tag{5.5}$$

This indicates doublets on the x_1 and x_2 axes as well as couplets about the x_1, x_2, and x_3.

If $\rho F_i^{(n)}$ are symmetric relative to planes along two axes (for example, the x_2 and x_3 axes) then only the m_{11}, m_{21}, and m_{31} components are nonzero. Hence, the Galerkin vector becomes

$$\underline{G}^2(\underline{x}) = \left[\phi_1(r)m_{11}, \phi_1(r)m_{21}, \phi_1(r)m_{31}\right] \tag{5.6}$$

which corresponds to a doublet on the x_1 axis and couplets about the x_2 and x_3 axes.

Every second order tensor may be expressed as the sum of a symmetric and antisymmetric tensor. Therefore, m_{ij} may be written as

$$m_{ij} = m_{ij}^S + m_{ij}^A \tag{5.7}$$

where

$$m_{ij}^S = \frac{1}{2}\left(m_{ij} + m_{ji}\right), \quad m_{ij}^S = m_{ji}^S \tag{5.8}$$

$$m_{ij}^A = \frac{1}{2}\left(m_{ij} - m_{ji}\right), \quad m_{ij}^A = - m_{ji}^A \quad . \tag{5.9}$$

Now, the products of $\phi_j(r)$ and m_{ij}^S are given below:

$$G_i^{2S}(\underline{x}) = \phi_j(r)\, m_{ij}^S = -\frac{\alpha}{r}\, x_j m_{ij}^S \quad . \tag{5.10}$$

Similarly, the products of $\phi_j(r)$ and m_{ij}^A are

$$G_i^{2A}\ (\underline{x}) = \phi_j(r)\, m_{ij}^A = -\frac{\alpha}{r}\, x_j m_{ij}^A \quad . \tag{5.11}$$

The displacement fields corresponding to Eqs. (5.10) and (5.11) are

$$U_i^{2S} = \frac{1}{16\pi\mu(1 - \nu)r^5} \Bigg\{ \left[4(\nu - 1)\, m_{ii}^S + m_{jj}^S + m_{kk}^S\right] x_i^3 - (4\nu - 2)\, m_{ij}^S x_j^3$$

$$+ (4\nu - 2)\, m_{ik}^S x_k^3 + 4(\nu - 1)\, m_{ij}^S x_i^2 x_j + 4(\nu - 1)\, m_{ik}^S x_i^2 x_k$$

$$+ \left[(4\nu - 1)\, m_{ii}^S - 2m_{jj}^S + m_{kk}^S\right] x_i x_j^2 + \left[(4\nu - 1)\, m_{ii}^S + m_{jj}^S - 2m_{kk}^S\right] x_i x_k^2$$

$$+ (4\nu - 2)\, m_{ik}^S x_j^2 x_k + (4\nu - 2)\, m_{ij}^S x_j x_k^2 - 6m_{jk}^S x_i x_j x_k \Bigg\}$$

$$\text{--- no sum, } i \neq j \neq k \tag{5.12}$$

$$U_i^A = \frac{1}{16\pi\mu(1 - \nu)r^5} \Bigg\{ 4(1 - \nu)\, m_{ij}^A x_j^3 + 4(1 - \nu)\, m_{ik}^A x_k^3 + 4(1 - \nu)\, m_{ij}^A x_i^2 x_j$$

$$+ 4(1 - \nu)\, m_{ik}^A x_i^2 x_k + 4(1 - \nu)\, m_{ik}^A x_j^2 x_k + 4(1 - \nu)\, m_{ij} x_j x_k^2 \Bigg\},$$

$$\text{--- no sum, } i \neq j \neq k \quad . \tag{5.13}$$

It is easily shown from Eqs. (4.8) and (5.12) that if m_1 is the only nonzero component in Eq. (4.8) and if $m_{11}^S = m_1$ and is the only nonzero element in the m_{ij}^S tensor, then

$$U_1^{2S} \quad E_{11}^1 \quad . \tag{5.14}$$

This can, of course, be generalized for the second and third components of $\underline{U}^{2S}$ as well. Thus, under these conditions U_i^{2S} and E_{ii}^1 correspond to a force doublet in the x_i direction.

Additionally, if $m_1 = m_2 = m_3$ in Eq. (4.8) and $m_{11}^S = m_{22}^S = m_{33}^S$ and are the only nonzero terms in m_{ij}^S, then

$$U_1^{2S} + U_2^{2S} + U_3^{2S} = E_{11}^1 + E_{22}^1 + E_{33}^1 \qquad (5.15)$$

and represent a center of dilatation. Further, from Eqs. (5.10) and (5.13) it can be shown that each component of the antisymmetric part of the second order singularity is identical to the rotation vector about its axis. That is (if the signs of m_{ij}^A are dictated by the right hand rule),

$$U_i^{2A} = \Omega_{jk}^1 \; , \quad i \neq j \neq k \qquad (5.16)$$

and corresponds to a force couplet about the x_i axis.

6. Third Order Singularities

The third term in Eq. (3.10) can also be expanded and specialized for particular cases. The expansion is shown as

$$G_i^3(\underline{x}) = \Phi_{kj}(r)\; m_{ijk} \qquad (6.1)$$

where

$$\Phi_{ij}(r) = \alpha\left[\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}\right] , \quad m_{ijk} = \lim_{n \to \infty} \int \rho F_i^{(n)}(\underline{\xi})\, \xi_j \xi_k \, dV^{(n)}(\underline{\xi}) . \qquad (6.2)$$

Only one particular form of third order singularity will be presented; however, it is obvious that one can easily express any specific type. For example, the following Galerkin vector corresponds to a singularity at the origin which can be called a double center of dilatation on the x_3 axis if

$$\underline{G}^3(\underline{x}) = \left[\phi_{13}(r)m_{131},\ \phi_{23}(r)m_{232},\ \phi_{33}(r)m_{333}\right] \tag{6.3}$$

where $2m_{131} = 2m_{232} = m_{333}$.

7. Spherical Biharmonics

The series represented by Eq. (3.10) provides the foundation for a development analogous to that of the class of Newtonian potentials called spherical harmonics. It may be recalled that a Newtonian potential is a solution to Laplace's equation at points external to the body. For a body of density $\sigma(\xi_\alpha)$ occupying the region $V + \delta V$ in space, the Newtonian potential at $\underline{x}$ is

$$N(\underline{x}) = \int \frac{\sigma(\underline{\xi})}{R(\underline{x}, \underline{\xi})}\, dV(\underline{\xi}) \quad . \tag{7.1}$$

By expanding $1/R(\underline{x}, \underline{\xi})$ about $\underline{x}$ in a Taylor Series, N may be written as

$$N = \sum_{i,j,k=0}^{\infty} B_{ijk}\left(\frac{1}{r}\right) I_{ijk} \tag{7.2}$$

where

$$B_{ijk}\left(\frac{1}{r}\right) = \frac{(-1)^{i+j+k}}{i!j!k!}\, D_1^i D_2^j D_3^k \left(\frac{1}{r}\right) \tag{7.3}$$

$$I_{ijk} = \int \sigma(\underline{\xi})\, \xi_1^i \xi_2^j \xi_3^k \, dV(\underline{\xi}) \quad . \tag{7.4}$$

It is easily seen that $B_{ijk}(1/r)$ is homogeneous of degree $-(i+j+k+1)$. Further, $1/r$ is harmonic. Therefore, each $B_{ijk}(1/r)$ must be harmonic because ∇^2 operates only on x_i. Thus, by definition, $B_{ijk}(1/r)$ are spherical harmonics of degree $-(i+j+k+1)$.

A comparison of Eqs. (7.2) and (3.9) enables one to make several observations. First, Eq. (7.2) represents the potential function of an attractive force on $\underline{x}$ by the mass occupying region V, while Eq. (3.9) is a potential function of the displacement at a point $\underline{x}$ resulting from a force or moment acting at the origin. It was shown that the coefficients $B_{ijk}(1/r)$ are spherical harmonics and by similar reasoning it is noted that $\Phi_{ijk}(r)$ are biharmonic and homogeneous of degree (1-i-j-k), a function which will be called a spherical biharmonic of degree (1-i-j-k).

Thus, with the analogy clearly established, an interesting point may be investigated by first noting that the expansion of Eq. (7.1) yields

$$N = \frac{1}{r}\int \sigma \, dV(\underline{\xi}) + \frac{x_1}{r}\int \sigma\xi_1 \, dV(\underline{\xi}) + \frac{x_2}{r}\int \sigma\xi_2 \, dV(\underline{\xi})$$

$$+ \frac{x_3}{r}\int \sigma\xi_3 \, dV(\underline{\xi}) + 0\left(\xi_i^2\right) + 0\left(\xi_i^3\right)$$

$$+ 0\left(\xi_i^4\right) + \text{---} \quad . \qquad (7.5)$$

All terms except the fundamental singularity may be eliminated by prescribing that

$$\int \sigma\xi_1^i \xi_2^j \xi_3^k \, dV(\underline{\xi}) = 0 \quad \text{for} \quad i \neq 0,\ j \neq 0,\ k \neq 0 \quad . \qquad (7.6)$$

This step is sufficient to insure that N is the fundamental singularity, but is it necessary? One suspects that it is not when considering the following: A complete, homogeneous, polynomial of degree n contains $(n + 1)(n + 2)/2$ coefficients, only $2n + 1$ of which are arbitrary [9] (assuming that all coefficients are harmonic). For example, the $0(\xi_i^2)$ terms are

$$\frac{1}{2r^5}\left[\left(2x_1^2 - x_2^2 - x_3^2\right)\int \sigma\xi_1^2 \, dV(\underline{\xi}) + \left(-x_1^2 + 2x_2^2 - x_3^2\right)\int \sigma\xi_2^2 \, dV(\underline{\xi})\right.$$

$$+ \left(-x_1^2 - x_2^2 + 2x_3^2\right)\int \sigma\xi_3^2 \, dV(\underline{\xi}) + 6x_1x_2 \int \sigma\xi_1\xi_2 \, dV(\underline{\xi})$$

$$\left. + 6x_1x_3 \int \sigma\xi_1\xi_3 \, dV(\underline{\xi}) + 6x_2x_3 \int \sigma\xi_2\xi_3 \, dV(\underline{\xi})\right] \quad . \tag{7.7}$$

The second degree complete, homogeneous, harmonic polynomial inside the brackets has six terms.* The fact that there are dependent coefficients in terms of order greater than 1 suggests that conditions other than Eq. (7.6) will result in the elimination of higher order terms. This lack of uniqueness can be illustrated by allowing

$$\int \sigma\xi_1^2 \, dV(\underline{\xi}) = \int \sigma\xi_2^2 \, dV(\underline{\xi}) = \int \sigma\xi_3^2 \, dV(\underline{\xi}) \tag{7.8}$$

in Eq. (7.7). In this case the first three terms in Eq. (7.5) are zero. The $O(\xi_i^2)$ terms may be written in powers of x_i; that is,

$$\frac{1}{2r^5}\left[\left(2\bar{M}_1 - \bar{M}_2 - \bar{M}_3\right) x_1^2 + \left(-\bar{M}_1 + 2\bar{M}_2 - \bar{M}_3\right) x_2^2 + \left(-\bar{M}_1 - \bar{M}_2 + 2\bar{M}_3\right) x_3^2\right.$$

$$\left. + \bar{M}_{12}x_1x_2 + \bar{M}_{13}x_1x_3 + \bar{M}_{23}x_2x_3\right] \tag{7.9}$$

where

$$\bar{M}_{ij} = \int \sigma\xi_i\xi_j \, dV(\underline{\xi}), \quad i \neq j \tag{7.10}$$

---

*The bracketted expression may be discussed with generality because if B_n is a spherical harmonic, then B_n/r^{2n+1} is also a spherical harmonic [9].

and

$$\overline{M}_i \equiv \int \sigma \xi_i^2 \, dV(\underline{\xi}) \quad . \tag{7.11}$$

Now the first three terms of Eq. (7.9) are zero if

$$\{\overline{M}\}[K] \equiv \begin{Bmatrix} \overline{M}_1 \\ \overline{M}_2 \\ \overline{M}_3 \end{Bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} = 0 \quad . \tag{7.12}$$

Thus, a condition other than Eq. (7.6) exists which eliminates the $O(\xi_i^2)$ terms.

The $O(\xi_i^3)$ term has 10 coefficients. One can write

$$\frac{1}{2r^7} \Bigg\{ \int \sigma \left(2\xi_1^3 - 3\xi_1\xi_2^2 - 3\xi_1\xi_3^2\right) dV(\underline{\xi})\, x_1^3 + \int \sigma \left(-3\xi_1^3 + 12\xi_1\xi_2^2 - 3\xi_1\xi_3^2\right) dV(\underline{\xi})\, x_1 x_2^2$$

$$+ \int \sigma \left(-3\xi_1^3 - 3\xi_1\xi_2^2 + 12\xi_1\xi_3^2\right) dV(\underline{\xi})\, x_1 x_3^2 + \int \sigma \left(2\xi_2^3 - 3\xi_2\xi_1^2 - 3\xi_2\xi_3^2\right) dV(\underline{\xi})\, x_2^3$$

$$+ \int \sigma \left(-3\xi_2^3 + 12\xi_1^2\xi_2 - 3\xi_2\xi_3^2\right) dV(\underline{\xi})\, x_1^2 x_2 + \int \sigma \left(-3\xi_2^3 - 3\xi_1\xi_2^2 + 12\xi_2\xi_3^2\right) dV(\underline{\xi})\, x_2 x_3^2$$

$$+ \int \sigma \left(2\xi_3^2 - 3\xi_3\xi_1^2 - 3\xi_3\xi_2^2\right) dV(\underline{\xi})\, x_3^3 + \int \sigma \left(-3\xi_3^3 + \xi_3\xi_1^2 - 3\xi_3\xi_2^2\right) dV(\underline{\xi})\, x_1^2 x_3$$

$$+ \int \sigma \left(-3\xi_3^3 - 3\xi_3\xi_1^2 + 12\xi_3\xi_2^2\right) dV(\underline{\xi})\, x_2^2 x_3 + \int 30\sigma \xi_1\xi_2\xi_3 \, dV(\underline{\xi})\, x_1 x_2 x_3 \Bigg\} \quad . \tag{7.13}$$

It may be seen that the entire term vanishes if all integrals are zero, or if

$$\int \frac{\sigma}{3} \xi_i^3 \, dV(\underline{\xi}) = \int \sigma \xi_i \xi_j^2 \, dV(\underline{\xi}) = \int \sigma \xi_i \xi_k^2 \, dV(\underline{\xi}) = \alpha_i , \quad i \neq j \neq k$$

$$\int \sigma \xi_1 \xi_2 \xi_3 \, dV(\underline{\xi}) = 0 \tag{7.14}$$

and so on for all orders.

To visualize the theoretical possibility of such an occurrence, imagine a sphere of radius c centered at 0 and a spherical shell with an outer radius a and an inner radius b also centered at 0. If $b > c$ the sphere is encapsulated inside the shell. The potential of attraction of the sphere relative to a point $\underline{x}$ lying outside the sphere is

$$N_0 = \frac{4\pi\sigma}{r(\underline{x})} \int_0^c [r(\underline{\xi})]^2 \, dr(\underline{\xi}) = \frac{4\pi\sigma}{3r(\underline{x})} c^3 \quad . \tag{7.15}$$

Similarly, the potential of the shell relative to a point $\underline{x}$ lying inside of the inner wall is

$$N_I = 4\pi\sigma' \int_b^a r(\underline{\xi}) \, dr(\underline{\xi}) = 2\pi\sigma' \left(a^2 - b^2\right) \quad . \tag{7.16}$$

Of course, the potential of a point lying between the sphere and shell is $N = N_0 + N_I$.

If one prescribes

$$\sigma' = - \frac{2c^3 \sigma}{3\left(a^2 - b^2\right)} \tag{7.17}$$

then N is zero without requiring that all integrals vanish.

The preceding investigation was conducted with a harmonic function f. Its results, however, can be applied to a biharmonic function g by noting that an arbitrary harmonic function β can be defined so that

$$\nabla^2 g = \beta \quad . \tag{7.18}$$

Thus, in the expansion of the biharmonic vector $\underline{g}$ there will be dependencies among coefficients beginning with the $O(r_i^4)$ term, because this corresponds to the $O(r_i^2)$ term in the expansion of β.

8. Betti's Reciprocal Work Theorem

In Paragraph 2, Galerkin's vector $\underline{g}(x_i)$ was used in discussing the development of a particular solution of the Cauchy's equations. In general, these functions are analogous to potential functions and, as stated previously, several techniques and theorems of potential theory are applicable in the treatment of the equations of elasticity. A lucid explanation of the application of potential theory to the solution of elliptical equations is found in Courant and Hilbert [3]. One of the most important aspects of potential theory is the theorem and identities of Green. Betti [1], in 1872, adapted Green's identities to elasticity and thus formulated the reciprocal work theorm. In this section the reciprocal work theorem and a particular application will be discussed.

Consider two stress states $S(\underline{x})$ and $S'(\underline{x})$ which are, for the moment, assumed to be regular throughout an elastic body V. If the unprimed state has a zero body force, the equations of equilibrium for $S(\underline{x})$ and $S'(\underline{x})$ are

$$H_{ijkl}\, u_{k,lj} = 0 \tag{8.1}$$

$$H_{ijkl}\, u'_{k,lj} + \rho F'_i = 0 \quad . \tag{8.2}$$

Multiplying Eq. (8.1) by u'_i and Eq. (8.2) by u_i, respectively, and subtracting the resulting expression yields a relationship between the primed and unprimed states. Integrating this relationship over V results in the following:

$$\int H_{ijkl} \left[u_{k,lj}\, u'_i - u'_{k,lj}\, u_i \right] dV + \int \rho F'_i u_i \, dV = 0 \quad . \tag{8.3}$$

Finally, employing Green's theorem and noting that

$$\int H_{ijkl}\, u_{k,l}\, n_j\, u_i'\, d\partial V = \int T_i u_i'\, d\partial V \quad , \tag{8.4}$$

one may write the reciprocal work theorem as

$$\int \left[T_i u_i' - T_i' u_i \right] d\partial V + \int \rho F_i' u_i \, dV = 0 \quad . \tag{8.5}$$

Imagine now that lying within the region V is the sequence of regions $\{V^{(n)}\}$ having the properties described in Paragraph 3. Suppose further that the primed stress state corresponds to $S^{(n)}(\underline{x})$, where $\rho F_i^{(n)}(\underline{x})$ is zero for $\underline{x}$ not in $V^{(n)}$. For the region $V^{(n)}$, Eq. (8.5) becomes

$$\int \left[T_i u_i^{(n)} - T_i^{(n)} u_i \right] d\partial V^{(n)} - \int \rho F_i^{(n)} u_i \, dV^{(n)} = 0 \quad . \tag{8.6}$$

Similarly, applying Eq. (8.5) over the region $V - V^{(n)}$ yields

$$- \int \left[T_i u_i^{(n)} - T_i^{(n)} u_i \right] d\partial V + \int \left[T_i u_i^{(n)} - T_i^{(n)} u_i \right] d\, V^{(n)}$$

$$+ \int \rho F_i^{(n)} u_i d \left(V - V^{(n)} \right) = 0 \quad . \tag{8.7}$$

Note that the tractions over $V^{(n)}$ as well as V must be considered. The terms $T_i^{(n)}$ and T_i involve an outer normal in Eq. (8.6) and an inner normal in the second integral of Eq. (8.7); therefore, the signs are opposite. The addition of these two equations results in

$$\int \left[T_i u_i^{(n)} - T_i^{(n)} u_i \right] d\, V - \int F_i^{(n)} u_i dV^{(n)} = 0 \quad . \tag{8.8}$$

Expanding u_i about the origin in a Taylor's series and taking the limit as n approaches infinity yields

$$0 = \lim_{n\to\infty} \int \left[T_i u_i^{(n)} - T_i^{(n)} u_i\right] d\partial V + u_i(0) \lim_{n\to\infty} \int \rho F_i^{(n)} dV^{(n)}$$

$$+ u_{i,j}(0) \lim_{n\to\infty} \int \rho F_i^{(n)} \xi_j dV^{(n)} + u_{i,jk}(0) \lim_{n\to\infty} \int \rho F_i^{(n)} \xi_j \xi_k dV^{(n)}$$

$$+ \text{---} \tag{8.9}$$

or

$$0 = \int \left[T_i U_i - \tau_i u_i\right] d\partial V + u_i(0) m_i + u_{i,j}(0) m_{ij} + u_{i,jk}(0) m_{ijk} + \text{---} \tag{8.10}$$

where

$$\tau_i = \lim_{n\to\infty} T_i^{(n)} \quad .$$

And so, by choosing the appropriate order singularity (by choosing values of m_i, m_{ij}, etc.), the displacements or any order derivatives of the displacement field of a regular stress state acting throughout can be reproduced if the displacements and tractions are known on ∂V. Note that by choosing a particular isolated singular solution U_i, one correspondingly chooses the form of m_i, m_{ij}, m_{ijk}, etc.

9. Discussion

As stated previously, the object of this report is to offer a physical interpretation of various first and higher order isolated singular solutions of Cauchy's equations using Galerkin's vector. These singularities are constructed by means of a limit solution proposed by Sternberg and Eubanks [8]. Some forms of first, second, and third order isolated singularities are investigated and will be followed by a discussion of uniqueness and an application of Betti's reciprocal theorem. In the following paragraphs, various salient points will be reviewed.

In Paragraph 2 Cauchy's equations are rewritten in a form more easily solved. This form is a nonhomogeneous biharmonic equation in terms of an auxiliary vector called Galerkin's vector. The solution is derived by employing a technique analogous to one used in potential theory.

Next, the limit solution is discussed and a series is developed which contains all first order isolated singular solutions to Cauchy's equations in the first set of terms, all second order singularities in the second set of terms, etc. An investigation of the first order singularity leads to its employment in a superposition method to construct various second order singularities.

Paragraph 5 contains an examination of the second term of Eq. (4.10) and hence, the development of the second order singularities.

It is shown in Paragraph 4 that the first term of Eq. (4.10) may be thought of as corresponding to a concentrated load at the origin. An examination of the second term in Eq. (4.10) revealed the following facts:

a) Each of the diagonal components m_{ii} (no sum) produces a force doublet along axis x_i

b) Each of the off-diagonal components $m_{ij}(i \neq j \neq k)$ produces a force couplet about the x_k axis

c) Symmetry of $\rho F_i^{(n)}$ about one or two axes insures that certain components of m_{ij} vanish.

By dividing m_{ij} into its symmetric and antisymmetric parts m^S_{ij} and m^A_{ij}, it is found that the solution U^{2S}_i derived from m^S_{ij} is identical to the strain E^1_{ii} (no sum) corresponding to the first order singularity and represents a force doublet in the x_i direction. Further, it is noted that $U^{2S}_1 + U^{2S}_2 + U^{2S}_3 = E^1_{11} + E^1_{22} + E^1_{33}$ and is a center of dilatation. Next, it is shown that the antisymmetrical part of the second order singularity produces U^A_i which is identical to the rotation about the x_i axis ω^1_{jk} corresponding to the fundamental singularity and represents a force couplet about the x_i axis. Finally, the Galerkin vector producing a third order singular solution is expressed as $\phi_{kj}(r)m_{ijk}$ and $G^3_i(\underline{x})$ corresponding to a double center of dilatation is presented.

Next, an analogy is made between the series representing the Galerkin vector pertaining to all isolated singularities and the spherical harmonics series of potential theory. It is found that the $G_i(\underline{x})$ represented by $!_{ijk}(r)\underline{M}_{ijk}$ can be defined as a spherical biharmonic. Further, it is demonstrated that, because there are dependencies among coefficients of $G_i(\underline{x})$ beginning with the $O(\cdot_i^4)$ term, there are an infinite number of combinations which will yield any given singular solution.

Finally, furthering the analogy with potential theory, Betti's reciprocal theorem is presented. Two stress states are chosen, one with a zero body force field and one with an isolated singularity at the origin. Applying the reciprocal work theorem, an expression is presented which enables one to reproduce any order derivatives of the displacement field of a regular stress state acting throughout V if the displacements and tractions are known on ·V by simply choosing the appropriate isolated singular solution.

A physical interpretation of the most important isolated singularities (i.e., concentrated force, force doublet, force couplet, center of dilatation, and double center of dilatation) has been given for isotropy. The same approach could undoubtedly be used in the case of anisotropy. As stated previously, proof that the derivatives of the limit value of the auxiliary function must be identical to the limit of the derivatives of the auxiliary function must be given.

REFERENCES

1. Betti, E., Il Nuovo Cimento, Ser. 2, Vol. 6-10 (1872ff).

2. Simmons, J. A., de Wit, Bollough, and Harwell, R.; Fundamental Aspects of Dislocation Theory; NBS Special Bulletin Publication; Vol. I; 1969.

3. Courant, R. and Hilbert D., "Methods of Mathematical Physics," Vol. II, Interscience Publishers, 1966.

4. Somigliana, C., Il Nuovo Cimento, Ser. 3, Vol. 36, 1894, pp. 28-39.

5. Chen, Y., Vibrations: Theoretical Methods, Addison-Wesley Publishing Company, 1966.

6. John, Fritz, "Plane Waves and Spherical Means," Vol. 2, Interscience Tracts in Applied Mathematics, Interscience Press, 1955.

7. Richardson, J., "Point Singularities in Linear Elasticity," PhD Thesis, University of Illinois, 1972.

8. Sternberg, E. and Eubanks, R. A., "On the Concept of Concentrated Loads and an Extension of the Uniqueness Theorem in the Linear Theory of Elasticity," J. Rat. Math. and Mech., Vol. IV, p. 135-168, 1955.

9. MacMillian, W. D., The Theory of the Potential, Dover Publications, Inc., 1958.

BIBLIOGRAPHY

Boresi, A. P., Elasticity in Engineering Mechanics, Prentice-Hall, 1965.

Bross, V. H., Zeit. fur Angewandte Math. and Phys., Vol. 19, 1968, p. 434.

Buck, R. C., Advanced Calculus, Second Edition, McGraw-Hill Book Company, 1965.

Burgers, J. M., Proc. Kon. Nederland Acadamie Wetensh., Vol. 42, 1939, p. 378.

Dedericks, P. H., and Leibfried, G., Phys. Review, Vol. 188, 1969, p. 1175.

Dougall, J., Edinbourgh Math. Soc. Proc., Vol. 16, 1898.

Duffin, R. J., and Noll, W., "On Exterior Boundary Value Problems in Linear Elasticity," Archive for Rational Mechanics and Analysis, Vol. 2, 1958, pp. 191-196.

Duffin, R. J., "Continuation of Biharmonic Functions by Reflection," AMA, 1954.

Flugge, W., Handbuch der Physik, Vol. 6, Berlin, Springer, 1958.

Fredholm, I., "Sur Les Equations De L'Equilibre D'Un Corps Solide Elastique," Acta Mathematica, Vol. 23, 1898, pp. 1-42.

Galerkin, B. G., Comptes Rendus Hebdomadaires des Seances de L'Academie des Sciences, Paris, Vol. 190, 1930, p. 1047.

Gebbia, M., Ann. di Math. Pura ed Applicata, Vol. 10, 1904, p. 157.

Gibbs, J. W., Vector Analysis, Dover Publications Inc., 1960.

Green, A. E., and Zerna, W., Theoretical Elasticity, Second Edition, Clarendon Press, Oxford, 1968.

Gurtin, M. E., and Sternberg, E., "Theorems in Linear Elastostatics for Exterior Domains," Arkiv Rat. Mech. and Anal., Vol. 8, 1969, pp. 99-119.

John, Fritz, Communications on Pure and Applied Math., Vol. 2, 1949, p. 209.

Kroner, E., Z. Phys., Vol. 151, 1958, p. 504.

Lifschitz, I. M., and Rosenzweig, L. N., J. Exp. and Theor. Phys., Vol. 17, 1947, p. 783.

Love, A. E. H., A Treatise on the Mathematical Theory of Elasticity, Dover Publications, 1944.

MacRobert, T. M., Spherical Harmonics, Third Edition, Pergamon Press, 1967.

Mann, E., Jan, R. V., and Seeger, A., Phys. Stat. Sold, Vol. 1, 1961, p. 17.

Maxwell, C., Electricity and Magnetism, Oxford, 1904.

Mindlin, R. D., Bull. Amer. Math. Soc., Vol. 42, 1936, p. 373.

Naghi, P. and Hsu, C. S., J. Math. and Mech., Vol. 10, 1961, p. 233.

Papkovich, P. F., Comptes Rendus, Paris, Vol. 195, 1932, p. 513.

Pearson, C. E., Theoretical Elasticity, Harvard University Press, 1959.

Sneddon, I. N., Fourier Transforms, McGraw-Hill Book Company, 1951.

Sokolnikoff, I. S., Mathematical Theory of Elasticity, McGraw-Hill Book Company, 1956.

Somigliana, C., Il Nuovo Cimento, Ser. 3, Vol. 36, 1894, pp. 28-36.

Stippes, M., "A Unified Derivation of Displacement Potentials in Classical Elasticity," Unpublished Note.

Thomson, W. (Lord Kelvin), Cambridge and Dublin Math. Jour., Vol. 6, 1847, p. 61.

Thomson, W. (Lord Kelvin), Cambridge and Dublin Math. Jour., Vol. 7, 1848.

Thomson, W. (Lord Kelvin), Math. and Phys. Papers, Vol. 1, p. 97.

Westergaard, H. M., Bull. Amer. Math. Soc., Vol. 41, 1935, p. 695.

Zeilon, N., Ark. fur Math. Astr. Och Fysik, Vol. 6, 1911, p. 23.