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We consider quadrature formulae (q.f.) for the numerical evaluation of the Fourier, cosine, sine, and Laplace transformations. Three different approaches are used to construct our q.f.: we either integrate an appropriate spline interpolant to $f(x)$, require our q.f. to be exact for a particular sequence of so-called B-splines, or utilize a particular monospline. In any case, the generality and utility we achieve is due to the form of the splines we use, in particular to the components of these splines, the so-called B-splines.

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THE NUMERICAL EVALUATION BY SPLINES OF THE FOURIER TRANSFORM AND THE LAPLACE TRANSFORM

Sherwood D. Silliman

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ABSTRACT

We consider quadrature formulae (q. f.) for the numerical evaluation of the Fourier, cosine, sine, and Laplace transformations. Let S_n denote the class of spline functions of degree $n-1$ defined on the real line and having simple knots at the points $\nu + \frac{n}{2}$ for all integers ν. This means that $S(x) \in S_n$ provided that $S(x) \in C^{n-2}$ and that the restriction of $S(x)$ to any interval between consecutive knots is a polynomial of degree not exceeding $n-1$.

In Part I, we consider, for n a positive integer, a q. f. of the form

$$\int_{-\infty}^{\infty} f(x) e^{ixt} dx = \sum_{\nu=-\infty}^{\infty} H_{\nu,t}^{(n)} f(\nu) + Rf$$

where, for fixed t, the coefficients $H_{\nu,t}^{(n)}$ are bounded. We show that among all such q. f., there is a unique formula with the property of being exact, i. e. the remainder $Rf = 0$, whenever $f(x) \in S_n \cap L_1(R)$. We exhibit the explicit formula for arbitrary step length h and give a useful bound on the remainder Rf when n is even.

In Part II, we discuss the cosine and sine transforms, using derivative data at the origin. For the cosine case, we consider q. f. of the form

$$\int_0^\infty f(x)\cos xt\,dx = \sum_{\nu=0}^{\infty} H_{\nu,t}^{(n)} f(\nu) + \sum_{j=1}^{[\frac{n-1}{2}]} B_{2j-1,t}^{(n)} f^{(2j-1)}(0) + Rf$$

where, for fixed t, again the coefficients $H_{\nu,t}^{(n)}$ are bounded, or the similar case when $f^{(j)}(0)$ $(j = 0, 1, \ldots, [\frac{n-1}{2}])$ is known. We find that among all such q. f. there is a unique one that is exact when $f(x) \in S_n \cap L_1(R^+)$. We exhibit the explicit q. f. for arbitrary n, but have a proof only for specific n.

In Part III, the Laplace transform case, we use the weight function $e^{-x\rho}$ instead of $\cos xt$ or $\sin xt$ as in Part II, but otherwise proceed in much the same spirit as Part II. Part IV contains expressions for the remainder or error for the q. f. in the first three parts and explicit error bounds for the approximations of the first two parts. Two computational examples are also included.

We actually use three different approaches to construct our q. f.: we either integrate an appropriate spline interpolant to $f(x)$, require our q. f. to be exact for a particular sequence of so-called B-splines, or utilize a particular monospline. In any case, the generality and utility we achieve is due to the form of the splines we use, in particular to the components of these splines, the so-called B-splines.

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INTRODUCTION

In [10] I. J. Schoenberg generalized the construction of best quadrature formulae in two ways. He discusses integrals with an arbitrary pre-assigned weight function opening up the possibility of constructing quadrature formula (q. f.) for the numerical evaluation of Laplace transforms, Fourier integrals, and other special integral transforms. We pursue this possibility here; in particular we wish to discuss approximations to the integrals

(1) $\int_{-\infty}^{\infty} f(x)e^{ixt}dx,$ (2) $\int_{0}^{\infty} f(x)\cos xt\, dx,$

(3) $\int_{0}^{\infty} f(x)\sin xt\, dx,$ (4) $\int_{0}^{\infty} f(x)e^{-xp}dx$

which are the Fourier, cosine, sine, and Laplace transformations, respectively.

In 1949, A. Sard generalized the Newton-Cotes q. f. as follows: let $1 \leqq m \leqq n+1$ and let

(5) $$\int_0^n f(x)dx = \sum_{\nu=0}^{n} H_{\nu,n}^{(m)} f(\nu) + Rf$$

be the formula exact, i.e., $Rf = 0$, if $f(x) \in \Pi_{m-1}$, the class of polynomials of degree not exceeding $m-1$, and such that the functional Rf when written in Peano-fashion as an integral of the form $\int_0^n k(x)f^{(m)}(x)dx$ has the kernel $k(x)$ with least L_2-norm. It was

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shown by Schoenberg [8] that we can say the following: Sard's q. f. (5) is uniquely characterized by the requirement of being exact, hence $Rf = 0$, for the elements of the class $S_{2m}(0, 1, \ldots, n)$ of natural splines of degree $2m - 1$ having the knots $0, 1, \ldots, n$. The term "natural" indicates that the degree of the polynomial components of the spline function should drop from $2m - 1$ to $m - 1$ in each of the two intervals $(-\infty, 0)$ and (n, ∞).

In [10] Schoenberg discusses q. f. of the form

$$(6)\qquad \int_0^n w(x)f(x)dx = \sum_{\nu=0}^{n} H_{\nu,n}^{(m)} f(\nu) + \sum_{j=1}^{m-1} B_{j,n}^{(m)} f^{(j)}(0) + \sum_{j=1}^{m-1} C_{j,n}^{(m)} f^{(j)}(n) + Rf$$

where $w(x)$ is an arbitrary preassigned weight function and such that the q. f. (6) is exact for Π_{m-1} and has the property that the associated kernel $k(x)$ of the functional Rf has least L_2-norm. This q. f. he shows is uniquely characterized by requiring $Rf = 0$ if f is a spline function (not natural) of degree $2m - 1$ having the knots $1, \ldots, n-1$.

In the paper [12], Schoenberg discusses infinite analogues of Sard's q. f. (5) for the real line $\mathbb{R}$ and the half-line $\mathbb{R}^+$ or $(0, \infty)$. We first consider the entire line, the so-called cardinal case when all integers ν are nodes of the formula. Let n be a natural number and let S_n denote the class of spline functions of degree $n - 1$, or order n, defined on the real line and having simple knots at the integers ν if n is even, or at the halfway points $\nu + \frac{1}{2}$ if n is odd. This means that $S(x) \in S_n$ provided that $S(x) \in C^{n-2}$ (for $n = 1$ this condition is vacuous), and that the restriction of $S(x)$ to any interval between consecutive knots is identical with a polynomial of degree not exceeding $n-1$. Such functions are called <u>cardinal spline functions</u>.

Lemma 1 below (§1) shows that

$$S(x) \in S_n \cap L_1(\mathbb{R}) \text{ implies that } \sum_{\nu=-\infty}^{\infty} |S(\nu)| < \infty . \tag{7}$$

Let n be even, say $n = 2m$, and consider a q.f. of the form

$$\int_{-\infty}^{\infty} f(x)dx = \sum_{-\infty}^{\infty} H_\nu^{(2m)} f(\nu) + Rf \tag{8}$$

where the numerical coefficients $H_\nu^{(2m)}$ satisfy the condition that

$$|H_\nu^{(2m)}| < K \text{ for all } \nu \text{ and some appropriate } K . \tag{9}$$

The implication (7) shows that the functional Rf is well-defined by (8) if $f(x) \in S_{2m} \cap L_1(\mathbb{R})$. One of the results of [12] is as follows:

<u>Among all quadrature formulae (8), (9) the q.f.</u>

$$\int_{-\infty}^{\infty} f(x)dx = \sum_{-\infty}^{\infty} f(\nu) + Rf \tag{10}$$

<u>is characterized by the requirement that</u> $Rf = 0$ <u>if</u> $f \in S_{2m} \cap L_1(\mathbb{R})$.

Observe that (10) is none other than the Euler-Maclaurin q.f.

$$\int_{-\infty}^{\infty} f(x)dx = \sum_{-\infty}^{\infty} f(\nu) + \frac{(-1)^m}{m!} \int_{-\infty}^{\infty} \bar{B}_m(x) f^{(m)}(x)dx,$$

where, if $B_m(x)$ denotes the m^{th} Bernoulli polynomial, we have defined $\bar{B}_m(x)$ to be its periodic extension of period 1 from the interval $[0, 1]$.

In Part I, we consider the analog of q. f. (6) for the entire line $\mathbb{R}$ and we take $w(x) = e^{ixt}$, that is, we discuss approximations to the general Fourier transform (1). Let n be any positive integer and consider a q. f. of the form

$$(11) \qquad \int_{-\infty}^{\infty} f(x)e^{ixt}dx = \sum_{-\infty}^{\infty} H_{\nu,t}^{(n)} f(\nu) + Rf$$

where the coefficients $H_{\nu,t}^{(n)}$ satisfy the condition that

$$(12) \qquad |H_{\nu,t}^{(n)}| < K \text{ for fixed } t, \text{ for all } \nu \text{ and some } K.$$

Note that the coefficients $H_{\nu,t}^{(n)}$ are now functions of t. Again, the result (7) shows that the functional Rf, now given by (11), is well-defined if $f(x) \in S_n \cap L_1(\mathbb{R})$. We have the following

Theorem 1. Among all quadrature formulae of the form (11), (12), there is a unique formula with the property of being exact, i.e., $Rf = 0$, whenever $f(x) \in S_n \cap L_1(\mathbb{R})$.

This q. f. (11) could also be obtained by using Newton's fundamental idea: assuming the function f(x) to be given numerically at equidistant points of step 1, including the origin 0, we interpolate $f(x)$ by a function $S(x)$ at these points, and then construct the Fourier transform of $S(x)$. This idea has been used before, and often, for the integrals (1) - (4) [5].

In fact, for $n = 2$, the case of linear spline interpolation, the q. f. (11) can be found in [5, pp. 22, 23]. But for $n > 2$, similar

q. f. (11), (12) have not previously been developed. The generality and utility we achieve is due to the form of the interpolating functions we use, i.e., the splines, and, in particular, to the components of these splines, the so-called B-splines.

In the paper [12], Schoenberg also considers the analog of Sard's q. f. (5) for the half-line $\mathbb{R}^+$. Let S^+_{2m} denote the class of functions $S(x)$ satisfying the following four conditions.

1° $S(x) \in C^{2m-2}(\mathbb{R})$

2° $S(x) \in \Pi_{2m-1}$ in each interval $(\nu, \nu+1)$ for $\nu = 0, 1, \ldots$

3° $S(x) \in \Pi_{m-1}$ in the interval $(-\infty, 0)$

4° $S(x) \in L_1(\mathbb{R}^+)$.

We now consider a q. f. of the form

$$\int_0^\infty f(x)dx = \sum_{\nu=0}^{\infty} H_\nu^{(2m)} f(\nu) + Rf \tag{13}$$

whose coefficients satisfy the condition that

$$|H_\nu^{(2m)}| < K \qquad \text{for } \nu \geqq 0 \text{ and some } K. \tag{14}$$

By Lemma 5 of [12], $S(x) \in S^+_{2m}$ implies that $\sum_{\nu=0}^{\infty} |S(\nu)| < \infty$ so that the q. f. (13) is applicable whenever $f(x) \in S^+_{2m}$.

In [12], Schoenberg proved the following

(15) Among all q. f. of the form (13), (14), there is a unique formula with the property of being exact, i. e., $Rf = 0$,

whenever $f(x) \in S^{+}_{2m}$.

In Part II, we consider the analog of q.f. (6) for the half-line R^{+} and we take

(16) $$w(x) = \cos xt \text{ or } w(x) = \sin xt .$$

We want a q.f. of the form

(17) $$\int_0^\infty w(x)f(x)dx = \sum_{\nu=0}^{\infty} H^{(2m)}_{\nu,t} f(\nu) + \sum_{j=1}^{m-1} B^{(2m)}_{j,t} f^{(j)}(0) + Rf$$

where $w(x)$ is given by (16) and the coefficients $H^{(2m)}_{\nu,t}$ satisfy the condition that

(18) $$|H^{(2m)}_{\nu,t}| < K \text{ for fixed } t, \text{ for all integers } \nu \geq 0 \text{ and some } K.$$

Note that for m fixed, $H^{(2m)}_{\nu,t}$ is a function of t. Lemma 1 below (§ 1) shows that

(19) $$S(x) \in S_{2m} \cap L_1(R^{+}) \text{ implies that } \sum_{\nu=0}^{\infty} |S(\nu)| < \infty$$

so that the functional Rf is well-defined if $f(x) \in S_{2m} \cap L_1(R^{+})$. Similar to (15), then, for our endpoint derivative case, where $w(x)$ is given by (16), we have

Theorem 2, Among all quadrature formulae of the form (17), (18), there is a unique formula with the property of being exact, i.e., $Rf = 0$, whenever $f(x) \in S_{2m} \cap L_1(R^{+})$.

We also consider q.f. of the form

$$(20)\quad \int_0^\infty f(x)\cos xt\,dx = \sum_{\nu=0}^{\infty} H_{\nu,t}^{(n)} f(\nu) + \sum_{j=1}^{[\frac{n-1}{2}]} B_{j,t}^{(n)} f^{(2j-1)}(0) + Rf,$$

$$(21)\quad \int_0^\infty f(x)\sin xt\,dx = \sum_{\nu=0}^{\infty} H_{\nu,t}^{(n)} f(\nu) + \sum_{j=1}^{[\frac{n-1}{2}]} B_{j,t}^{(n)} f^{(2j)}(0) + Rf$$

where the coefficients $H_{\nu,t}^{(n)}$ again satisfy the condition (18). Lemma 1 will again assure us that the functional Rf given by (20) or (21) is well-defined if $f(x) \in S_n \cap L_1(R^+)$. The form of the $H_{\nu,t}^{(n)}$ and the $B_{j,t}^{(n)}$ of (20) and (21) is particularly simpler than the corresponding form of the coefficients for the q. f. (17) and for this reason we shall consider the q. f. (20) and (21) first. Our approach is the following: once the existence and unicity of the q. f. have been established, we shall exhibit the q. f. (20) and (21) that satisfy the requirement (18) and show that they are exact whenever $f(x) \in S_n \cap L_1(R^+)$.

We could also obtain the q. f. (20) and (21) by constructing the cosine or sine transform of the appropriate spline interpolant. Closest to this point of view is the paper [1] in which Einarsson approximates integrals of the form

$$(22)\qquad \int_a^b f(x)\cos wx\,dx, \qquad \int_a^b f(x)\sin wx\,dx$$

as follows: $f(x)$ is interpolated by a cubic spline with equidistant knots, the interpolation being at the knots, while the values $f'(a)$ and $f'(b)$ are matched by the cubic spline. Then he takes the transform of the spline. We could adopt this method also, and through the use of B-splines, achieve greater generality than Einarsson. However, we do not follow this approach because, in the general case, this method does not lead to the coefficients $H^{(n)}_{\nu,t}$, $B^{(n)}_{j,t}$ of (20) and (21) in a very simple form.

In Part III, we consider the Laplace transform (4) and establish the following

Theorem 3. *Among all q. f. of the form*

$$\int_0^\infty f(x)e^{-x\rho}dx = \sum_{\nu=0}^{\infty} H^{(m)}_{\nu,\rho}\, f(\nu) + \sum_{j=1}^{m-1} B^{(m)}_{j,\rho}\, f^{(j)}(0) + Rf \tag{23}$$

whose coefficients satisfy the condition

$$|H^{(m)}_{\nu,\rho}| < K\mu^{-\nu} \quad \text{for } \rho \text{ fixed, for all } \nu \geq 0, \text{ and some } K, \text{ some } \mu > 1, \tag{24}$$

there is a unique formula with the property of being exact whenever $f(x)$ *is a cardinal spline function of degree* $2m-1$ *such that* $f(x) = O(x^s)$ *as* $x \to \infty$ *for some* $s \geq 0$.

We do this in the same way we prove Theorem 2, by using a generating function approach similar to that used in [13].

Part IV contains expressions for the error for the approximations we make in the first three parts and estimates of error bounds for the first two parts. We acquire these expressions by showing that we could have constructed our q. f. in still a third way. This third approach utilizes a particular monospline, related to the so-called Rodrigues function of [10].

I. THE FOURIER TRANSFORM

1. Preliminaries. We first recall some known definitions and results [7]. Let n be a natural number and

$$(1.1) \qquad M(x) = M_n(x) = \frac{1}{(n-1)!} \delta^n x_+^{n-1}$$

where

$$x_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and where δ^n stands for the usual symbol of the n^{th} order central difference of step equal to 1. $M_n(x)$ is a spline function of degree $n-1$ having as knots the points ν (ν integer), or $\nu + \frac{1}{2}$, depending on whether $n-1$ is odd or even. $M_n(x)$ is positive in the interval $(-\frac{1}{2}n, \frac{1}{2}n)$ and vanishes elsewhere, and evidently $M_n(x) \in S_n$. It has the following Fourier transform:

$$(1.2) \qquad \int_{-\infty}^{\infty} M_n(x) e^{ixt} dx = \psi_n(t)$$

where

$$(1.3) \qquad \psi_n(t) = \left(\frac{2 \sin \frac{t}{2}}{t}\right)^n .$$

$M_n(x)$ is called a central B-spline or basis spline because of the following property: If $S(x) \in S_n$, then $S(x)$ admits a unique representation of the form

$$(1.4) \qquad S(x) = \sum_{-\infty}^{\infty} C_\nu M_n(x-\nu)$$

and conversely, any such series with arbitrarily prescribed $\{C_\nu\}$ converges and defines a cardinal spline function of degree $n-1$.

We also define a <u>forward B-spline</u> by

$$(1.5) \qquad Q_n(x) = M_n(x - \frac{n}{2}) = \frac{1}{(n-1)!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)_+^{n-1} .$$

$Q_n(x)$ has integer knots, is positive in $(0, n)$ and zero elsewhere.

With $\psi_n(t)$ defined by (1.3) we define

$$(1.6) \qquad \phi_n(t) = \sum_{j=-\infty}^{\infty} \psi_n(t + 2\pi j) .$$

$\phi_n(t)$ is a positive cosine polynomial of period 2π and order $[\frac{n+1}{2}] - 1$ that can be explicitly computed from the expression

$$(1.7) \qquad \phi_n(t) = \sum_{\nu=-\infty}^{\infty} M_n(\nu) e^{i\nu t} = \sum_{|\nu| \leq \frac{n}{2}} M_n(\nu) e^{i\nu t} .$$

By Lemma 6 of [9, p. 180], we have

$$(1.8) \qquad \max_t \phi_n(t) = \phi_n(0) = 1, \quad \min_t \phi_n(t) = \phi_n(\pi) > 0 .$$

By (1.7), we find

$$\phi_2(t) = 1$$

$$\phi_3(t) = \frac{3 + \cos t}{4}$$

$$\phi_4(t) = \frac{2 + \cos t}{3}$$

$$\phi_5(t) = \frac{115 + 76 \cos t + \cos 2t}{192}$$

$$\phi_6(t) = \frac{33 + 26 \cos t + \cos 2t}{60} .$$

We shall also define $\phi_0(t) = 1$ to make the notation in what follows more convenient.

In the Introduction, we referred to the following

<u>Lemma 1.</u> <u>If</u>

$$S(x) \in S_n \cap L_1(R) \tag{1.9}$$

<u>then</u>

$$\sum_{-\infty}^{\infty} |S(\nu)| < \infty \tag{1.10}$$

<u>If</u>

$$S(x) \in S_n \cap L_1(R^+) \tag{1.11}$$

<u>then</u>

$$\sum_{0}^{\infty} |S(\nu)| < \infty . \tag{1.12}$$

<u>Proof.</u> Let $n = 2m$. We reproduce the following remark of Louboutin [6, p. 1]. If $R_k(x) \in \Pi_k$, then by Markov's inequal.../

$$\max_{[0,1]} |R'_k(x)| \leq 2k^2 \max_{[0,1]} |R_k(x)| . \tag{1.13}$$

Let now $P(x) \in \Pi_{2m-1}$ and let

$$R_{2m}(x) = \int_0^x P(t)dt .$$

Applying (1.13) to this polynomial of degree $2m$ we find that

$$\max_{[0,1]} |P(x)| \leq 2(2m)^2 \max_{[0,1]} \left|\int_0^x P(t)dt\right| \leq 8m^2 \int_0^1 |P(x)|dx .$$

For a spline function $S(x)$ of degree $2m - 1$ with integer knots, we therefore have

$$\max_{[\nu,\nu+1]} |S(x)| \leq 8m^2 \int_{\nu}^{\nu+1} |S(x)|dx .$$

Assuming (1.9), we have

$$\sum_{-\infty}^{\infty} |S(\nu)| \leq \sum_{-\infty}^{\infty} \max_{[\nu,\nu+1]} |S(x)| \leq 8m^2 \|S(x)\|_{L_1(R)} < \infty$$

while (1.11) implies

$$\sum_{0}^{\infty} |S(\nu)| \leq \sum_{0}^{\infty} \max_{[\nu,\nu+1]} |S(x)| \leq 8m^2 \|S(x)\|_{L_1(R^+)} < \infty .$$

We have carried through the proof for $n = 2m$; however, we obtain the same result for $n = 2m - 1$ if we replace (1.13) by

$$\max_{[-1/2,1/2]} |R_k'(x)| \leq 2k^2 \max_{[-1/2,1/2]} |R_k(x)|$$

and for $P(x) \in \Pi_{2m-2}$ define

$$R_{2m-1}(x) = \int_{-1/2}^{x} P(t)dt .$$

This completes a proof of Lemma 1.

We also need to know just when a cardinal spline function of degree $n - 1$ is in $L_1(R)$ or in $L_1(R^+)$. The answers are given by

<u>Lemma 2. Suppose</u> $S(x) \in S_n$ <u>and</u> (1.4) <u>holds. Then</u>

(1.14) $\qquad S(x) \in L_1(R)$

<u>if and only if</u>

$$(1.15) \qquad \sum_{\nu=-\infty}^{\infty} |C_\nu| < \infty .$$

The inclusion

$$(1.16) \qquad S(x) \in L_1(R^+)$$

holds if and only if

$$(1.17) \qquad \sum_{\nu=0}^{\infty} |C_\nu| < \infty .$$

Proof. That (1.14) is equivalent to (1.15) is a special case of Theorem 12 of [9, p. 199], and is hereby established. (1.16) follows from (1.17) in precisely the same manner as (1.14) follows from (1.15) in the proof of Theorem 12 of [9, p. 199]. We now start with the assumption (1.16) and wish to show that (1.17) holds. This is derived from the previous case that has just been settled. Assuming (1.4), we consider the spline function,

$$(1.18) \qquad \bar{S}(x) = \sum_{\nu=-[\frac{n+1}{2}]+1}^{\infty} C_\nu M_n(x-\nu) .$$

Evidently

$$(1.19) \qquad \bar{S}(x) = \begin{cases} S(x) & \text{if } x \geqq 0 \\ 0 & \text{if } x \leqq -n+\frac{1}{2} . \end{cases}$$

From (1.16) we conclude that $\bar{S}(x) \in L_1(R)$ and the first part of Lemma 2 shows that (1.17) holds.

2. Proof of Theorem 1 of the Introduction. We adapt our proof from the proofs of Theorem 2 of [9] and Theorem 1 of [13]. For sim-

plicity we write $M(x) = M_n(x)$ and $H_\nu = H^{(n)}_{\nu,t}$. We require the q. f. (11) to be exact for $f(x) = M(x-j)$ for all integers j. This stipulation gives the following relations:

$$(2.1)\qquad \int_{-\infty}^{\infty} M(x-j)\, e^{ixt} dx = \psi_n(t)\, e^{ijt} = \sum_{\nu=-\infty}^{\infty} H_\nu M(\nu-j) \quad \text{for all } j$$

or, since $M(\nu-j) = M(j-\nu)$ as can be seen from (1.1), we have

$$(2.2)\qquad \sum_{\nu=-\infty}^{\infty} M(j-\nu) H_\nu = \psi_n(t) e^{ijt} \qquad \text{for all } j\,.$$

To invert this convolution transformation, we consider the positive cosine polynomial $\phi_n(t)$ as given by (1.6) and the expansion of its reciprocal in a Fourier series:

$$(2.3)\qquad \frac{1}{\phi_n(t)} = \sum_{k=-\infty}^{\infty} w_k^{(n)}\, e^{ikt}\,.$$

Lemma 11 of [9, p. 187], for $p = \infty$, implies that

$$(2.4)\qquad H_j = \sum_\nu w^{(n)}_{j-\nu}\, \psi_n(t)\, e^{i\nu t}$$

is a bounded linear transformation of ℓ_∞ into itself, whose inverse is given by (2.2). Since the sequence $\{\psi_n(t)\, e^{ijt}\}$ is in ℓ_∞, we conclude that the sequence $\{H_j\}$ defined by (2.4) also belongs to ℓ_∞. Since $w^{(n)}_{j-\nu} = w^{(n)}_{\nu-j}$ [9, p. 182], we know from (2.4) that

$$(2.5)\qquad H_j = \sum_{\nu=-\infty}^{\infty} w^{(n)}_{\nu-j}\, \psi_n(t)\, e^{i\nu t} = \psi_n(t)\, e^{ijt} \sum_{\nu=-\infty}^{\infty} w^{(n)}_{\nu-j}\, e^{i(\nu-j)t}$$

so that by (2.3) we get

$$(2.6) \qquad H_j = \frac{\psi_n(t)}{\phi_n(t)} e^{ijt} .$$

The sequence $\{H_j\}$ in ℓ_∞ is uniquely defined, by (2.4).

A proof will be complete as soon as we show that the functional

$$(2.7) \qquad Rf = \int_{-\infty}^{\infty} f(x)e^{ixt}dx - \sum_{\nu=-\infty}^{\infty} H_\nu f(\nu)$$

with the coefficients H_ν given by (2.6) has the property that

$$(2.8) \qquad Rf = 0 \text{ if } f \in S_n \cap L_1(R) .$$

Suppose $f(x)$ is such a function and let

$$(2.9) \qquad f(x) = \sum_{-\infty}^{\infty} C_\nu M(x-\nu)$$

be its expansion in terms of the central B-splines of degree $n-1$. By Lemma 2, we know that $f(x) \in L_1(R)$ implies that

$$(2.10) \qquad \sum_{-\infty}^{\infty} |C_\nu| < \infty .$$

The partial sums

$$(2.11) \qquad f_r(x) = \sum_{\nu=-r}^{r} C_\nu M(x-\nu) \qquad (r = 0, 1, 2, \ldots)$$

have the additional property that

$$(2.12) \qquad f_r(x) = 0 \qquad \text{if } |x| \geq \frac{n}{2} + r .$$

Since $f(x) \in S_n \cap L_1(R)$ and (2.12) holds, we conclude that $f_r(x) \in S_n \cap L_1(R)$. Using properties of the functional (2.7), we obtain

$$(2.13) \qquad \int_{-\infty}^{\infty} f_r(x)e^{ixt}dx = \sum_{\nu=-\infty}^{\infty} H_\nu f_r(\nu) .$$

Observing that each $f_r(x)$ is dominated by the function

$$\sum_{-\infty}^{\infty} |C_\nu| M(x-\nu)$$

which is summable on $\mathbb{R}$ by Lemma 2 and (2.10), we see that on letting $r \to \infty$, the relation (2.13) becomes the desired relation

$$\int_{-\infty}^{\infty} f(x)e^{ixt}dx = \sum_{\nu=-\infty}^{\infty} H_\nu f(\nu) .$$

This completes a proof of Theorem 1.

Substituting the coefficients H_ν as given by (2.6) gives us the unique q.f. of Theorem 1 in the following form:

$$(2.14) \qquad \int_{-\infty}^{\infty} f(x)e^{ixt}dx = \frac{\psi_n(t)}{\phi_n(t)} \sum_{\nu=-\infty}^{\infty} f(\nu)e^{i\nu t} + Rf .$$

Suppose now that $f(x)$ is a spline function of degree $n-1$ with knots at $(\nu + \frac{n}{2})h$, for all integers ν, that is also in $L_1(\mathbb{R})$. But then

$$S(x) = f(xh)$$

is a cardinal spline function of degree $n-1$ for the step 1. From Theorem 1 and (2.14) we have

$$\int_{-\infty}^{\infty} f(xh)e^{ixt}dx = \frac{\psi_n(t)}{\phi_n(t)} \sum_{\nu=-\infty}^{\infty} f(\nu h)e^{i\nu t} .$$

If we replace in the integral the variable x by x/h and then replace in the identity t by th, we obtain

$$(2.15) \qquad \int_{-\infty}^{\infty} f(x)e^{ixt}dx = \frac{\psi_n(th)}{\phi_n(th)} h \sum_{\nu=-\infty}^{\infty} f(\nu h)e^{i\nu th} .$$

If $f(x)$ is an arbitrary function, then this is no longer an identity. However, we can obtain information on the error made in using the approximation (2.15) if we consider n even, say $n = 2m$. In fact, in §14 below we shall prove the following

Theorem 4. *Suppose* $f(x) \in C^{2m} \cap L_1(R) \cap L_1^{2m}(R)$ *and* $\frac{2\pi}{h}$ *is a natural number. Then we can bound* $|Rf|$ *as given in the q.f.*

$$\int_{-\infty}^{\infty} f(x)e^{ixt}dx = \frac{\psi_{2m}(th)}{\phi_{2m}(th)} h \sum_{\nu=-\infty}^{\infty} f(\nu h)e^{i\nu th} + Rf \tag{2.16}$$

by

$$|Rf| \leqq 4\left(\frac{h}{\pi}\right)^{2m} \|f^{(2m)}\|_{L_1(R)} \quad \text{for all rational } t \neq 0 \text{ in } \left(-\frac{2\pi}{h}, \frac{2\pi}{h}\right). \tag{2.17}$$

In the theorem, $L_1^{2m}(R)$ denotes a particular choice of n and p for the set

$$L_p^n(R) = \{F(x) : F^{(n-1)} \text{ absolutely continuous, } F^{(n)} \in L_p(R)\}$$

where n is a natural number and $1 \leqq p \leqq \infty$. The set $L_p^n(R^+)$ is defined similarly.

II. THE COSINE AND SINE TRANSFORMS

For concreteness, we now take $w(x) = \cos xt$ and consider q. f. of the form (20). We shall also take $n = 2m$, and later will indicate the modifications necessary for different derivative data, for even degree splines $(n = 2m - 1)$, and for the weight function $\sin xt$.

3. A recurrence relation. For simplicity, we write $H_\nu = H^{(2m)}_{\nu,t}$, $B_j = B^{(2m)}_{j,t}$. We want to construct a q. f. of the form

$$(3.1) \qquad \int_0^\infty f(x)\cos xt\, dx = \sum_{\nu=0}^{\infty} H_\nu f(\nu) + \sum_{j=1}^{m-1} B_{2j-1} f^{(2j-1)}(0) + Rf$$

such that

$$(3.2) \qquad |H_\nu| < K \text{ for all integers } \nu \geqq 0 \text{ and some } K$$

and with the property that

$$(3.3) \qquad Rf = 0 \quad \text{if } f \in S_{2m} \cap L_1(R^+).$$

We do this by enforcing the requirement (3.3) for an appropriate sequence of elements of $S_{2m} \cap L_1(R^+)$. The sequence we require is the sequence of forward B-splines of degree $2m - 1$

$$(3.4) \qquad \{Q(x - r)\} \qquad (r = -2m+1, -2m+2, \ldots)$$

where, by substituting $n = 2m$ in (1.5) we have

$$(3.5) \qquad Q(x) = Q_{2m}(x) = \frac{1}{(2m-1)!} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} (x - i)_+^{2m-1}.$$

Since $Q(x)$ has support in $(0, 2m)$,

$$(3.6)\qquad Q(x-r)=0 \text{ outside the interval } (r,\ r+2m),$$

$$(r=-2m+1,\ -2m+2,\ \ldots)$$

$Q(x) \in \Pi_{2m-1}$ on any interval between consecutive knots so that $Q(x) \in S_{2m} \cap L_1(\mathbb{R}^+)$, and evidently we also have

$$(3.7)\qquad Q(x-r) \in S_{2m} \cap L_1(\mathbb{R}^+) \qquad (r=-2m+1,\ -2m+2,\ldots).$$

Substituting $f(x) = Q(x-r)$ in (3.1) and recalling (3.3) and (3.6), we have the sequence of relations

$$(3.8)\quad \int_0^{r+2m} Q(x-r)\cos xt\,dx = H_0Q(-r)+H_1Q(1-r)+\ldots+H_{r+2m-1}Q(2m+1)$$

$$+\sum_{j=1}^{m-1} B_{2j-1}Q^{(2j-1)}(-r), \quad (r=-2m+1,-2m+2,\ldots,-2,-1)$$

and

$$(3.9)\quad \int_0^{r+2m} Q(x-r)\cos xt\,dx = H_{r+1}Q(1)+H_{r+2}Q(2)+\ldots+H_{r+2m-1}Q(2m-1)$$

$$(r=0,1,2,\ldots).$$

4. <u>The summation of certain series.</u> We shall need the following lemma which deals with well-known power series.

<u>Lemma 3.</u> 1°. <u>The power series</u>

$$(4.1)\qquad \Phi_k(x) = \sum_{\nu=0}^{\infty} (\nu+1)^{k+1} x^{\nu} \qquad (k=0,1,2,\ldots)$$

<u>has the sum</u>

$$(4.2)\qquad \Phi_k(x) = \frac{P_k(x)}{(1-x)^{k+2}}$$

where $P_k(x)$ is a monic polynomial of degree k, with integer coefficients, that may be derived from the recurrence relation

(4.3) $P_k(x) = (1+kx)P_{k-1}(x) + x(1-x)P'_{k-1}(x)$, with $P_0(x) = 1$.

2°. The power series

(4.4) $$\bar{\Phi}_k(x) = \sum_{\nu=0}^{\infty} (2\nu+1)^k x^\nu$$

has the sum

(4.5) $$\bar{\Phi}_k(x) = \frac{T_k(x)}{(1-x)^{k+1}}$$

where $T_k(x)$ is a monic polynomial of degree k, with integer coefficients, that may be derived from the recurrence relation

(4.6) $T_k(x) = [1-(2k-1)x]T_{k-1}(x) + 2x(1-x)T'_{k-1}(x)$ with $T_0(x) = 1$.

The polynomials $P_k(x)$ are called Euler-Frobenius polynomials of degree k, while the $T_k(x)$ are called midpoint Euler-Frobenius polynomials of degree k. We omit the easy proof by induction which also furnishes the relations (4.3) and (4.6). We find readily that

(4.7)
$$\begin{aligned}
P_1(x) &= 1 + x \\
P_2(x) &= 1 + 4x + x^2 \\
P_3(x) &= 1 + 11x + 11x^2 + x^3 \\
P_4(x) &= 1 + 26x + 66x^2 + 26x^3 + x^4
\end{aligned}$$

and

$$
(4.8)\qquad
\begin{aligned}
T_1(x) &= 1 + x \\
T_2(x) &= 1 + 6x + x^2 \\
T_3(x) &= 1 + 23x + 23x^2 + x^3 \\
T_4(x) &= 1 + 76x + 230x^2 + 76x^3 + x^4
\end{aligned}
$$

and so on.

The form of the relations (3.8) and (3.9) suggest the use of generating functions for the determination of the H_ν and the B_j. The righthand side of (3.8) and (3.9) is equal to the coefficient of x^{r+2m-1} in

$$
(4.9)\quad \left(\sum_{\nu=0}^{\infty} H_\nu x^\nu\right)\left(\sum_{\nu=0}^{2m-2} Q(2m-1-\nu)x^\nu\right) + \sum_{\nu=0}^{2m-2}\left[\sum_{j=1}^{m-1} B_{2j-1}Q^{(2j-1)}(2m-1-\nu)\right]x^\nu .
$$

In order to simplify the two polynomials in (4.9), we note that

$$
(4.10)\qquad Q^{(k)}(x) = (-1)^k Q^{(k)}(2m-x) \qquad (k = 0, 1, \ldots, 2m-2)
$$

as can be verified from (3.5). With this substitution and the interchange of the order of summation in the second polynomial, (4.9) becomes

$$
(4.11)\quad \left(\sum_{\nu=0}^{\infty} H_\nu x^\nu\right)\left(\sum_{\nu=0}^{2m-2} Q(\nu+1)x^\nu\right) - \sum_{j=1}^{m-1} B_{2j-1}\left(\sum_{\nu=0}^{2m-2} Q^{(2j-1)}(\nu+1)x^\nu\right).
$$

We need the following result that is perhaps of independent interest:

<u>Theorem 5.</u> <u>The following identities hold</u>:

$$(4.12)\ 1^\circ.\quad \sum_{\nu=0}^{2m-2} Q^{(j)}(\nu+1)x^\nu = \frac{(1-x)^j P_{2m-2-j}(x)}{(2m-1-j)!} \qquad (j=0,1,\ldots,2m-2;\ m=1,2,\ldots)$$

$$(4.13)\ 2^\circ.\quad \sum_{\nu=0}^{m-1} Q^{(2m-1)}(\nu+1-0)x^\nu = (1-x)^{2m-1} \qquad (m=1,2,\ldots)$$

$$(4.14)\ 3^\circ.\quad \sum_{\nu=0}^{2m-2} M_{2m-1}^{(j)}(\nu+1-m)x^\nu = (\tfrac{1}{2})^{2m-2-j}\ \frac{(1-x)^j T_{2m-2-j}(x)}{(2m-2-j)!}$$

$$(j=0,1,\ldots,2m-3;\ m=1,2,\ldots)$$

$$4^\circ.\quad \sum_{\nu=0}^{2m-2} M_{2m-1}^{(2m-2)}(\nu+1-m-0)x^\nu = (1-x)^{2m-2} \qquad (m=1,2,\ldots).$$

We note that because of (1.5) we could also write (4.12) in the form

$$(4.15)\quad \sum_{\nu=0}^{2m-2} M_{2m}^{(j)}(\nu+1-m)x^\nu = \frac{(1-x)^j P_{2m-2-j}(x)}{(2m-1-j)!}$$

$$(j=0,1,\ldots,2m-2;\ m=1,2,\ldots).$$

Proof. First we show (4.12). Let

$$(4.16)\quad (1-x)^j P_{2m-2-j}(x) = \sum_{\nu=0}^{2m-2} A_\nu x^\nu .$$

From (4.1) and (4.2), for $k = 2m-2-j$, we find that

$$(1-x)^j P_{2m-2-j}(x) = (1-x)^{2m} \sum_{k=0}^{\infty} (k+1)^{2m-1-j} x^k$$

$$= \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} x^i \cdot \sum_{k=-\infty}^{\infty} (k+1)_+^{2m-1-j} x^k$$

so that

$$(4.17)\qquad A_\nu = \sum_{i+k=\nu} (-1)^i \binom{2m}{i} (k+1)_+^{2m-1-j} = \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} (\nu-i+1)_+^{2m-1-j} .$$

On the other hand, by differentiating (3.5) j times, we obtain

$$(4.18)\qquad Q^{(j)}(x) = \frac{1}{(2m-1-j)!} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} (x-i)_+^{2m-1-j} ,$$

and then

$$(4.19)\qquad (2m-1-j)!\, Q^{(j)}(\nu+1) = \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} (\nu-i+1)_+^{2m-1-j}$$

which is identical to (4.17) so that (4.12) follows.

To prove (4.13), we substitute $j = 2m-1$ in (4.18) and get

$$(4.20)\qquad Q^{(2m-1)}(x) = \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} (x-i)_+^0 .$$

$Q^{(2m-1)}(x)$ is a step function so that upon substitution of $x = (\nu+1-0)$ for $\nu = 0, 1, \ldots, 2m-1$, (4.20) becomes

$$(4.21)\qquad Q^{(2m-1)}(\nu+1-0) = \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} (\nu+1-i-0)_+^0 = \sum_{i=0}^{\nu} (-1)^i \binom{2m}{i}$$

$$(\nu = 0, 1, \ldots, 2m-1) .$$

But an easy induction shows

$$(4.22)\qquad \sum_{i=0}^{\nu} (-1)^i \binom{2m}{i} = (-1)^\nu \binom{2m-1}{\nu} \qquad (\nu = 0, 1, \ldots, 2m-1)$$

so that by (4.21) and (4.22), we have

$$\sum_{\nu=0}^{2m-1} Q^{(2m-1)}(\nu+1-0) x^\nu = \sum_{\nu=0}^{2m-1} (-1)^\nu \binom{2m-1}{\nu} x^\nu = (1-x)^{2m-1} .$$

The even degree spline case 3°, 4° is proved in the same manner as (4.12), (4.13) so we can omit this proof and the theorem follows.

With the substitution of (4.12), (4.11) becomes

$$(4.23)\qquad \Big(\sum_{\nu=0}^{\infty} H_\nu x^\nu\Big)\frac{P_{2m-2}(x)}{(2m-1)!} + \sum_{j=1}^{m-1} B_{2j-1}\frac{(x-1)^{2j-1}P_{2m-2j-1}(x)}{(2m-2j)!}\ .$$

We now turn our attention to the left side of relations (3.8) and (3.9) and define

$$(4.24)\qquad \begin{aligned} F_{r+2m-1} &= \int_0^{r+2m} Q(x-r)\cos xt\,dx \qquad (r=-2m+1,-2m+2,\ldots,-1,0) \\ F_{r+2m-1} &= \int_r^{r+2m} Q(x-r)\cos xt\,dx \qquad (r=1,2,\ldots)\ . \end{aligned}$$

If we integrate the righthand side of (4.24) for $(r=-2m+1,\ldots,-1,0)$ by parts $2m-1$ times, we obtain

$$(4.25)\qquad \begin{aligned} F_{r+2m-1} = {} & \Big[\frac{1}{t}Q(x-r)\sin xt + \frac{1}{t^2}Q'(x-r)\cos xt - \frac{1}{t^3}Q''(x-r)\sin xt \\ & - \frac{1}{t^4}Q'''(x-r)\cos xt + \ldots + \frac{(-1)^{m-1}}{t^{2m-1}}Q^{(2m-2)}(x-r)\sin xt\Big]\Big|_0^{r+2m} \\ & - \frac{(-1)^{m-1}}{t^{2m-1}}\int_0^{r+2m} Q^{(2m-1)}(x-r)\sin xt\,dx\ . \end{aligned}$$

Since $Q^{(2m-1)}(x)$ is a step function, we break up the interval of integration and writ^ the integral in (4.25) as

$$(4.26)\qquad \begin{aligned} \int_0^{r+2m} Q^{(2m-1)}(x-r)\sin xt\,dx &= \sum_{\ell=0}^{r+2m-1}\int_\ell^{\ell+1} Q^{(2m-1)}(x-r)\sin xt\,dx \\ &= \sum_{\ell=0}^{r+2m-1} Q^{(2m-1)}(\ell+1-r-0)\int_\ell^{\ell+1}\sin xt\,dx\ . \end{aligned}$$

After we integrate and collect terms, we obtain (4.26) in the form

$$(4.27) \quad -\frac{1}{t}\sum_{\ell=0}^{r+2m-1} Q^{(2m-1)}(\ell+1-r-0)\,[\cos(\ell+1)t - \cos \ell t]$$

$$= -\frac{1}{t}\{-Q^{(2m-1)}(1-r-0) + \sum_{\ell=1}^{r+2m-1}[Q^{(2m-1)}(\ell-r-0)-Q^{(2m-1)}(\ell+1-r-0)]$$

$$\cdot \cos \ell t + Q^{(2m-1)}(2m-0)\cos(r+2m)t\}.$$

By (4.21) we find

$$(4.28) \qquad Q^{(2m-1)}(\ell-r-0) - Q^{(2m-1)}(\ell+1-r-0) = -(-1)^{\ell-r}\binom{2m}{\ell-r}$$

and

$$(4.29) \qquad Q^{(2m-1)}(2m-0) = \sum_{i=0}^{2m-1}(-1)^i\binom{2m}{i} = -1$$

so that upon substitution in (4.27), we have

$$(4.30) \qquad \int_0^{r+2m} Q^{(2m-1)}(x-r)\sin xt\,dx$$

$$= \frac{1}{t}Q^{(2m-1)}(1-r-0) + \frac{1}{t}\sum_{\ell=1}^{r+2m}(-1)^{\ell-r}\binom{2m}{\ell-r}\cos \ell t\,.$$

If we let $i = 2m + r - \ell$ in the sum in (4.30) and substitute the result in (4.25), (4.25) becomes

$$(4.31) \qquad F_{r+2m-1} = -\left[\frac{1}{t^2}Q'(-r) - \frac{1}{t^4}Q'''(-r)+\ldots+\frac{(-1)^{m-2}}{t^{2m-2}}Q^{(2m-3)}(-r)\right]$$

$$+\frac{(-1)^m}{t^{2m}}Q^{(2m-1)}(1-r-0) + \frac{(-1)^m}{t^{2m}}\sum_{i=0}^{r+2m-1}(-1)^i\binom{2m}{i}\cos(r+2m-i)t$$

$$(r = -2m+1,\ldots,-1,0).$$

Let $j = r+2m-1$ and multiply each side of (4.31) by $(-1)^m t^{2m}$ to get

$$(4.32)\quad (-1)^m t^{2m} F_j = -[(-1)^m t^{2m-2} Q'(2m-1-j) - (-1)^m t^{2m-4} Q'''(2m-1-j) + \ldots + t^2 Q^{(2m-3)}(2m-1-j)] + Q^{(2m-1)}(2m-j-0) + \sum_{i=0}^{j} (-1)^i \binom{2m}{i} \cos(j+1-i)t \qquad (j = 0, 1, \ldots, 2m-1).$$

In view of the relations (4.10), we may write

$$(4.33)\quad (-1)^m t^{2m} F_j = \sum_{i=0}^{j} (-1)^i \binom{2m}{i} \cos(j+1-i)t - Q^{(2m-1)}(j+1-0) + t^2 Q^{(2m-3)}(j+1) - \ldots + (-1)^{m-1} t^{2m-4} Q'''(j+1) + (-1)^m t^{2m-2} Q'(j+1) \qquad (j = 0, 1, \ldots, 2m-1).$$

If we consider (4.24) for $r = 1, 2, \ldots$ and again integrate by parts $2m-1$ times, we obtain (4.25) with the lower limit of integration r instead of 0. Upon evaluation, the square bracket in (4.25) is 0 and we would get, by using (4.25), (4.26) and (4.27) that

$$(4.34)\quad F_{r+2m-1} = \frac{(-1)^m}{t^{2m-1}} \left\{ -\frac{1}{t} \sum_{\ell=r}^{r+2m-1} Q^{(2m-1)}(\ell+1-r-0)\,[\cos(\ell+1)t - \cos \ell t] \right\} \qquad (r = 1, 2, \ldots).$$

Following the same steps we did before, and noting by (4.21) that

$$Q^{(2m-1)}(1-0) = 1$$

allows us to use (4.30) to write

(4.35) $$F_{r+2m-1} = \frac{(-1)^m}{t^{2m}} \sum_{\ell=r}^{r+2m} (-1)^{\ell-r} \binom{2m}{\ell-r} \cos \ell t \ .$$

Letting $i = 2m + r - \ell$ in the sum in (4.35) and then $j = r + 2m - 1$, we obtain upon multiplication by $(-1)^m t^{2m}$

(4.36) $$(-1)^m t^{2m} F_j = \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \cos(j+1-i)t \qquad (j=2m, 2m+1, \ldots) \ .$$

Let us sum the series

(4.37) $$(-1)^m t^{2m} \sum_{j=0} F_j x^j$$

From (4.33) and (4.36) we find that

(4.38) $$(-1)^m t^{2m} \sum_{j=0}^{\infty} F_j x^j = \left\{ \sum_{j=0}^{2m-1} [\sum_{i=0}^{j} (-1)^i \binom{2m}{i} \cos(j+1-i)t] x^j \right.$$

$$\left. + \sum_{j=2m}^{\infty} [\sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \cos(j+1-i)t] x^j \right\} - \sum_{j=0}^{2m-1} Q^{(2m-1)}(j+1-0)x^j$$

$$+ t^2 \sum_{j=0}^{2m-2} Q^{(2m-3)}(j+1)x^j - \ldots + (-1)^m t^{2m-2} \sum_{j=0}^{2m-2} Q'(j+1)x^j \ .$$

To simplify the term in curly brackets on the right side of (4.38), we define

(4.39) $$\tau(x) = \sum_{\nu=0}^{\infty} [\cos(\nu+1)t] x^\nu$$

and note that

(4.40) $$(1-x)^{2m} \tau(x) = \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} x^i \cdot \sum_{\nu=0}^{\infty} [\cos(\nu+1)t] x^\nu$$

$$= \sum_{j=0}^{\infty} [\sum_{\substack{i+\nu=j \\ i=0,\ \nu \geq 0}}^{2m} (-1)^i \binom{2m}{i} \cos(\nu+1)t] x^j$$

$$= \sum_{j=0}^{2m-1} \left[\sum_{i=0}^{j} (-1)^i \binom{2m}{i} \cos(j+1-i)t \right] x^j + \sum_{j=2m}^{\infty} \left[\sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \cos(j+1-i)t \right] x^j .$$

Using (4.40) and (4.12) and (4.13) of Theorem 5, we may therefore write (4.38) as

$$(4.41) \quad (-1)^m t^{2m} \sum_{j=0}^{\infty} F_j x^j = (1-x)^{2m} \tau(x) - [(1-x)^{2m-1}$$

$$- \frac{t^2}{2!} P_1(x)(1-x)^{2m-3} + \frac{t^4}{4!} P_3(x)(1-x)^{2m-5} - \dots$$

$$+ (-1)^{m-1} \frac{t^{2m-2}}{(2m-2)!} P_{2m-3}(x)(1-x)] .$$

Equating the relation (4.23) and $\sum_{j=0}^{\infty} F_j x^j$ as determined from (4.41), we see that we require

$$(4.42) \quad \frac{(-1)^m}{t^{2m}} \{(x-1)^{2m} \tau(x) + (x-1)^{2m-1} - \frac{t^2}{2!} P_1(x)(x-1)^{2m-3} + \dots$$

$$+ (-1)^{m-1} \frac{t^{2m-2}}{(2m-2)!} P_{2m-3}(x)(x-1)\}$$

$$= \left(\sum_{\nu=0}^{\infty} H_\nu x^\nu\right) \frac{P_{2m-2}(x)}{(2m-1)!} + \sum_{j=1}^{m-1} B_{2j-1} \frac{(x-1)^{2j-1} P_{2m-2j-1}(x)}{(2m-2j)!} .$$

5. Determination of the coefficients $H_\nu = H^{(2m)}_{\nu,t}$, $B_j = B^{(2m)}_{j,t}$.

Solving (4.42) for $\sum_{\nu=0}^{\infty} H_\nu x^\nu$ gives the final relation

$$(5.1) \quad \sum_{\nu=0}^{\infty} H_\nu x^\nu = \frac{(2m-1)!}{P_{2m-2}(x)} \left\{ \frac{(-1)^m}{t^{2m}} [(x-1)^{2m} \tau(x) + (x-1)^{2m-1} + \right.$$

$$+\sum_{j=1}^{m-1}(-1)^j \frac{t^{2j}}{(2j)!}P_{2j-1}(x)(x-1)^{2m-2j-1}] - \sum_{j=1}^{m-1} B_{2j-1} \frac{(x-1)^{2j-1}P_{2m-2j-1}(x)}{(2m-2j)!}\Bigg\}.$$

Our derivation of (5.1) establishes the following

<u>Proposition 1.</u> <u>The coefficients</u> $H_\nu = H^{(2m)}_{\nu,t}$, $B_{2j-1} = B^{(2m)}_{2j-1,t}$ <u>of the most general functional</u>

$$(5.2) \qquad Rf = \int_0^\infty f(x)\cos xt\, dx - \sum_{\nu=0}^{\infty} H_\nu f(\nu) - \sum_{j=1}^{m-1} B_{2j-1} f^{(2j-1)}(0)$$

<u>vanishing for the functions</u>

$$(5.3) \qquad Q_{2m}(x-r) \qquad (r = -2m+1, -2m+2, \ldots)$$

<u>are the expansion coefficients of the rational function</u> (5.1) <u>where the</u> B_{2j-1} $(j = 1, \ldots, m-1)$ <u>are chosen arbitrarily.</u>

We want to investigate the functionals (5.2) further, and, in particular, determine the unique functional having bounded coefficients. Let $R_{2m}(x)$ denote the right side of (5.1), where the B_{2j-1} $(j=1,\ldots,m-1)$ are as yet undetermined. To use $R_{2m}(x)$ effectively, we need information on its poles. To this end, in view of (4.12) of Theorem 5 for $j = 0$, we may write $P_{2m-2}(x)$ in terms of the central B-spline (1.5) as

$$(5.4) \qquad P_{2m-2}(x) = (2m-1)! \sum_{\nu=0}^{2m-2} M_{2m}(\nu-m+1)x^\nu.$$

By Lemma 8 of [9, p. 182] we know that this reciprocal polynomial has only simple and negative zeros so that we may label them to satisfy the conditions

(5.5) $$\lambda_{2m-2} < \dots < \lambda_m < -1 < \lambda_{m-1} < \dots < \lambda_1 < 0$$

and

(5.6) $$\lambda_1\lambda_{2m-2} = \lambda_2\lambda_{2m-3} = \dots = \lambda_{m-1}\lambda_m = 1 .$$

From the form of $\tau(x)$ as given by (4.39), we note that $\tau(x)$ converges for $|x| < 1$. Observing that for $R_{2m}(x)$ the poles $\lambda_1, \dots, \lambda_{m-1}$ are inside the unit circle while $\lambda_m, \dots, \lambda_{m-2}$ are outside, in view of (5.5), we shall have the coefficients H_ν bounded, if and only if the coefficients B_{2j-1} $(j = 1, \dots, m-1)$ can be chosen so that the $m-1$ poles $\lambda_1, \dots, \lambda_{m-1}$ of $R_{2m}(x)$ have vanishing residues. By (5.1) this will occur if and only if the B_j satisfy the equations

(5.7) $$\sum_{j=0}^{m-1} B_{2j-1} \frac{(\lambda_\nu-1)^{2j-1} P_{2m-2j-1}(\lambda_\nu)}{(2m-2j)!}$$

$$= \frac{(-1)^m}{t^{2m}} [(\lambda_\nu-1)^{2m}\tau(\lambda_\nu)+(\lambda_\nu-1)^{2m-1}+\sum_{j=1}^{m-1}(-1)^j \frac{t^{2j}}{(2j)!} \cdot$$

$$\cdot P_{2j-1}(\lambda_\nu)(\lambda_\nu-1)^{2m-2j-1}] \qquad (\nu = 1, \dots, m-1) .$$

In the system (5.7), we have $m-1$ equations in the $m-1$ unknowns B_{2j-1} $(j=1, \dots, m-1)$. To show that the system is nonsingular, it is sufficient to show that the determinant

(5.8) $$|A_{\nu j}| = \left|\frac{(\lambda_\nu-1)^{2j-1} P_{2m-2j-1}(\lambda_\nu)}{(2m-2j)!}\right| \neq 0$$

$$(\nu = 1, 2, \dots, m-1;\ j=1, \dots, m-1) .$$

In order to accomplish this, we shall consider a related problem, a special case of which will solve our problem. Let

$$(5.9) \qquad S_{2m}^{0} = \{S(x) : S(x) \in S_{2m},\ S(\nu) = 0 \text{ for all integers } \nu\}.$$

In [9, p. 194] Schoenberg shows that every element of S_{2m}^{0} admits a unique representation

$$(5.10) \qquad S(x) = \sum_{k=1}^{2m-2} a_k S_k(x)$$

for appropriate values of the coefficients a_k, where the $S_k(x)$ are the so-called <u>eigensplines</u> of the class S_{2m}^{0} and are defined in terms of the zeros (5.5) by

$$(5.11) \qquad S_k(x) = S_{2m}(x; \lambda_k) = \sum_{j=-\infty}^{\infty} \lambda_k^j M_{2m}(x-j) \qquad (k=1,2,\ldots,2m-2).$$

In [9, §9] Schoenberg proved a Theorem 11, a special case of which asserts the following:

$$(5.12) \qquad S(x) \in S_{2m}^{0} \cap L_1^{s}(\mathbb{R}) \qquad \text{for some } s = 0, 1, \ldots, 2m-1$$

<u>implies that</u>

$$(5.13) \qquad S(x) = 0 \qquad \underline{\text{for all}}\ x.$$

The first half of the proof actually establishes the following:

<u>Every</u> $S(x) \in S_{2m}^{0} \cap L_1^{s}(\mathbb{R}^{+})$ <u>for some</u> $s = 0, 1, \ldots, 2m-1$ <u>may be uniquely represented in the form</u>

$$(5.14) \qquad S(x) = \sum_{k=1}^{m-1} a_k S_k(x)$$

<u>for appropriate values of the coefficients</u> a_k.

In particular, the $S_k(x)$ for $k = 1, 2, \ldots, m-1$ are linearly independent.

We determine the set

(5.15) $$I \subset \{1, 2, \ldots, m-1\}$$

the null set also being allowed, its complement

$$I^c = \{1, 2, \ldots, m-1\} - I$$

and in terms of I^c, the set

(5.16) $$I' = \{2m-2-i : i \in I^c\} .$$

Notice that while I is a subset of $\{1, 2, \ldots, m-1\}$, the new set I' is a subset of $\{m, m+1, \ldots, 2m+2\}$ so that the intersection $I \cap I'$ is empty. The particular set we will be concerned with in connection with the system (5.7) is $I \cup I' = \{1, 3, 5, \ldots, 2m-3\}$.

Suppose that $S(x)$ is of the form

(5.17) $$S(x) = \sum_{k=1}^{m-1} a_k S_k(x) .$$

We want to be able to choose the a_k in such a way that

(5.18) $$S^{(i)}(0) = y_0^{(i)} \qquad i \in I \cup I'$$

where the righthand side has arbitrarily prescribed values. In other words, we want the a_k to satisfy

(5.19) $$\sum_{k=1}^{m-1} a_k S_k^{(i)}(0) = y_0^{(i)} \qquad i \in I \cup I' .$$

This then is the related problem, a solution to which will enable us to show (5.8).

In order to show the existence and uniqueness of the a_k, we consider the corresponding homogeneous system and prove the following

Lemma 4. *If* $S(x) \in S^0_{2m} \cap L^s_1(R^+)$ *for some* $s = 0, 1, \ldots, 2m-1$ *and*

(5.20) $$S^{(i)}(0) = 0 \qquad \text{if } i \in I \cup I'$$

then

(5.21) $$S(x) = 0 \quad \text{for all } x.$$

Remark. This is a special case of a result of Schoenberg [10, Lemma 2] concerning a finite interval that we have extended to the infinite interval $(0, \infty)$. We follow his proof which was in turn based originally on a proof of Greville [4, p. 4].

Proof of Lemma 4. Suppose (5.14) is the canonical representation of $S(x) \in S^0_{2m} \cap L^s_1(R^+)$ for some $s = 0, 1, \ldots, m-1$. Note that, since

(5.22) $$S_k^{(s)} \in L_1(R^+) \text{ for } (s = 0, 1, \ldots, 2m-1) \text{ and } k = 1, 2, \ldots, m-1$$

by the nature of the representation (5.14), we also have

(5.23) $$S^{(s)}(x) \in L_1(R^+) \qquad \text{for } s = 0, 1, \ldots, 2m-1.$$

We let

(5.24) $$\Omega = \int_0^\infty [S^{(m)}(x)]^2 dx = \lim_{b\to\infty} \int_0^b [S^{(m)}(x)]^2 dx$$

and wish to show that

(5.25) $$\Omega = 0.$$

We write $\Omega = \int S^{(m)}S^{(m)}dx$ and integrate by parts successively to obtain

$$(5.26) \qquad \Omega = \lim \int_0^b S^{(m)}dS^{(m-1)} = -\lim \int_0^b S^{(m+1)}dS^{(m-2)} = \dots$$

$$= \pm \lim \int_0^b S^{(\beta)}\,dS^{(\alpha)}$$

where the integers α and β satisfy the conditions

$$(5.27) \qquad 0 \leqq \alpha \leqq m-1, \quad m \leqq \beta \leqq 2m-1, \quad \alpha + \beta = 2m-1.$$

Notice that we have written (5.26) as if all the "finite parts" drop out at each end of the interval of integration and at each step of the successive integrations. That this is indeed the case follows thus: for each pair of numbers (α, β) satisfying (5.27) we have

$$(5.28) \qquad S^{(\alpha)}(0)\,S^{(\beta)}(0) = 0$$

because either $\alpha \in I$ or $\beta \in I'$ so that (5.20) implies that (5.28) holds. For each (α, β) we also have

$$\lim_{b \to \infty} S^{(\alpha)}(b)\,S^{(\beta)}(b) = 0$$

in view of (5.23).

The integrations can be continued all the way down until we reach

$$(5.29) \qquad \Omega = \lim \int_0^b S^{(2m-1)}dS = \lim_{b \to \infty} \pm \sum_{\nu=0}^{[b]} \int_{x_\nu}^{x_{\nu+1}} S^{(2m-1)}(x)S'(x)dx$$

where we have written $x_\nu = \nu$ for $\nu = 0, 1, \dots, [b]$ and $x_{[b]+1} = b$. Since $S^{(2m-1)}(x)$ is a step function, the integrals in the sum (5.29) vanish if $0 \leqq \nu < [b]$, since $S(x) \in S^0_{2m}$. There remains to show that also

$$(5.30) \qquad \lim_{b \to \infty} \int_{[b]}^{b} S^{(2m-1)}(x) S'(x)dx = 0 .$$

If b is an integer, (5.30) is true. If b is not an integer, since $S^{(2m-1)}(x)$ is a step function,

$$(5.31) \qquad \int_{[b]}^{b} S^{(2m-1)}(x)S'(x)dx = S^{(2m-1)}(b)\int_{[b]}^{b} S'(x)dx$$

$$= S^{(2m-1)}(b)[S(b) - S([b])] .$$

$S(x) \in S_{2m}^{0}$ implies $S([b]) = 0$ and, upon letting $b \to \infty$ on the right side of (5.31), we obtain (5.30) by virtue of (5.23).

We have just established (5.25), and therefore that

$$(5.32) \qquad S(x) \in \Pi_{m-1} .$$

But $S(x) \in S_{2m}^{0}$ implies that $S(x) \equiv 0$. This completes a proof of Lemma 4.

In view of Lemma 4, $S(x)$ as given by (5.17) for the homogeneous system

$$(5.33) \qquad \sum_{k=1}^{m-1} a_k S_k^{(i)}(0) = 0 \qquad i \in I \cup I'$$

must vanish for all x. Then, since the $S_k(x)$ for $k = 1, \ldots, m-1$ are linearly independent, we must have $a_k = 0$, $k = 1, \ldots, m-1$. This shows the existence and uniqueness of the a_k for the system (5.19), so that we must have that the determinant

$$(5.34) \qquad |\bar{A}_{ik}| = |S_k^{(i)}(0)| \neq 0 \qquad (i \in I \cup I', \ k = 1, 2, \ldots, m-1) .$$

We claim that

(5.35) $$S_k^{(i)}(0) = \frac{(\lambda_k-1)^i P_{2m-2-i}(\lambda_k)}{(2m-1-i)!} \frac{1}{\lambda_k^{m-1}}$$

$(i=0, 1, \ldots, 2m-2;\ k=1, \ldots, m-1).$

Indeed, if we differentiate (5.11) i times and substitute $x = 0$, we obtain

(5.36) $$S_k^{(i)}(0) = \sum_{j=-\infty}^{\infty} \lambda_k^j M_{2m}^{(i)}(-j).$$

$M_{2m}(x)$ has support in $(-m, m)$ and

(5.37) $$M_{2m}^{(i)}(-x) = (-1)^i M_{2m}^{(i)}(x)$$

as can be seen from (4.10), so that we can write (5.36) in the form

(5.38) $$S_k^{(i)}(0) = (-1)^i \sum_{j=-(m-1)}^{m-1} \lambda_k^j M_{2m}^{(i)}(j).$$

If we let $\nu = j + m - 1$ in the sum in (5.38), we obtain

(5.39) $$S_k^{(i)}(0) = \frac{(-1)^i}{\lambda_k^{m-1}} \sum_{\nu=0}^{2m-2} \lambda_k^\nu M_{2m}^{(i)}(\nu+1-m).$$

Then (4.15) of Theorem 5 establishes our claim (5.35).

If we multiply the k^{th} column of the determinant in (5.34) by λ_k^{m-1} $k = 1, \ldots, m-1$ and take the transpose of this resulting determinant, we have by (5.34) that the determinant

(5.40) $$\left| \frac{(\lambda_k-1)^i P_{2m-2-i}(\lambda_k)}{(2m-1-i)!} \right|_{(k,i)} \neq 0 \qquad (k=1, 2, \ldots, m-1;\ i \in I \cup I').$$

If we consider the special case $I \cup I' = \{1, 3, 5, \ldots, 2m-3\}$, then the determinant

(5.41) $$\left|\frac{(\lambda_k-1)^i P_{2m-2-i}(\lambda_k)}{(2m-1-i)!}\right|_{(k,i)} \neq 0$$

$(k=1,2,\ldots,m-1;\ i=1,3,5,\ldots,2m-3).$

This is precisely the relation (5.8), so that we have established the first part of the following

Theorem 6. 1°. Among all functionals

(5.2) $$Rf = \int_0^\infty f(x)\cos xt\,dx - \sum_{\nu=0}^{\infty} H_{\nu,t}^{(2m)} f(\nu) - \sum_{j=1}^{m-1} B_{2j-1,t}^{(2m)} f^{(2j-1)}(0)$$

vanishing for the sequence of spline functions

(5.3) $$Q_{2m}(x-r) \qquad (r=-2m+1,\ -2m+2,\ \ldots)$$

there is a unique one such that the sequence $\{H_\nu\}$ is bounded.

2°. In fact, this unique functional Rf can be given explicitly by

(5.42) $$Rf = \int_0^\infty f(x)\cos xt\,dx - \frac{\psi_{2m}(t)}{\phi_{2m}(t)}\left\{\frac{1}{2}f(0) + \sum_{\nu=1}^{\infty} f(\nu)\cos\nu t\right\}$$

$$+ \sum_{j=1}^{m-1} \frac{(-1)^j}{t^{2j}}\left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j}(t)}{\phi_{2m}(t)}\right] f^{(2j-1)}(0)$$

(where $\psi_n(t)$ and $\phi_n(t)$ are defined in (1.3) and (1.6), respectively) for $m=1,2,3,4$.

Proof of 2° of Theorem 6. We observe that the functional Rf of (5.42) is of the proper form (5.2) where

(5.43) $$H_0 = \frac{1}{2}\frac{\psi_{2m}(t)}{\phi_{2m}(t)}, \quad H_\nu = \cos \nu t \frac{\psi_{2m}(t)}{\phi_{2m}(t)} \quad (\nu = 1, 2, \ldots).$$

By (1.3) and (1.8),

(5.44) $$|H_\nu| \leq \left|\frac{\sin t/2}{t/2}\right|^{2m} \frac{1}{\phi_{2m}(\pi)} \leq \frac{1}{\phi_{2m}(\pi)} < \infty$$

$$(\nu = 0, 1, 2, \ldots)$$

so the sequence $\{H_\nu\}$ as given by (5.43) is bounded. Once we show that Rf as given by (5.42) vanishes for the sequence of splines (5.3), the unicity established in part 1° will establish part 2°.

We accomplish this in two steps, one for $(r = 0, 1, 2, \ldots)$ and the other for $(r = -2m+1, -2m+2, \ldots, -1)$. We remark that we prove the first case for general m but the latter only for the special cases of $m = 1, 2, 3, 4$. So far, the latter general case still eludes us; it is a matter of showing the validity of one necessary identity. This same dilemma prevents us from claiming explicit versions of other q.f. to follow as well as (5.42) for general m.

We first show that $Rf = 0$ for $f(x) = Q_{2m}(x-r)$ for $r = 0, 1, 2, \ldots$. Since (1.2) holds, we get

$$\int_{-\infty}^{\infty} M_n(x-j) e^{ixt} dx = \psi_n(t) e^{ijt} .$$

Taking real and imaginary parts, we find

(5.45) $$\int_{-\infty}^{\infty} M_n(x-j) \cos xt = \psi_n(t) \cos jt$$

and

$$(5.46) \qquad \int_{-\infty}^{\infty} M_n(x-j) \sin jt = \psi_n(t) \sin jt .$$

By (1.5) and (5.45) for $n = 2m$ and since $M_{2m}(x) = 0$ if $|x| > m$, we obtain

$$(5.47) \qquad \int_0^{\infty} Q_{2m}(x-r)\cos xt\, dx = \int_0^{\infty} M_{2m}(x-m-r)\cos xt\, dx$$

$$= \psi_{2m}(t)\cos(m+r)t \qquad (r = 0, 1, 2, \ldots).$$

Since $f^{(2j-1)}(0) = 0$ for $f(x) = Q_{2m}(x-r)$ and $r = 0, 1, 2, \ldots$ we need only show that

$$(5.48) \qquad \phi_{2m}(t)\cos(m+r)t = \sum_{\nu=r+1}^{r+2m-1} M_{2m}(\nu-m-r)\cos \nu t$$

$$(r = 0, 1, 2, \ldots).$$

By (1.7) we have

$$\phi_{2m}(t) = \sum_{k=-(m-1)}^{m-1} M_{2m}(k)\cos kt$$

which upon multiplying by $\cos(m+r)t$ and letting $\nu = m+r+k$ becomes

$$(5.49) \qquad \phi_{2m}(t)\cos(m+r)t = \sum_{\nu=r+1}^{r+2m-1} M_{2m}(\nu-m-r)\cos(\nu-m-r)t \cdot \cos(m+r)t .$$

By using the identity

$$\cos at \cos bt = \cos(a+b)t + \sin at \sin bt$$

we get

$$(5.50) \qquad \cos(\nu-m-r)t \cos(m+r)t = \cos \nu t + \sin(\nu-m-r)t \sin(m+r)t .$$

But

$$\sum_{\nu=r+1}^{r+2m-1} M_{2m}(\nu-m-r)\sin(\nu-m-r)t - \sin(m+r)t =$$

$$= \sin(m+r)t \sum_{k=-(m-1)}^{m-1} M_{2m}(k)\sin kt = 0$$

since $M_{2m}(-k) = M_{2m}(k)$, so that upon substituting (5.50) into (5.49) we obtain (5.48) as we wished.

What remains then is the case $r = -2m+1, -2m+2, \ldots, -1$. By (1.5) we can write (5.3) as $M_{2m}(x-m-r)$ and need to show that $Rf = 0$ for

$$M_{2m}(x-m-r) \qquad (r = -2m+1, -2m+2, \ldots, -1)$$

or for

$$(5.51) \qquad M_{2m}(x-j) \qquad (j = -m+1, -m+2, \ldots, m-1).$$

By the symmetry of $M_{2m}(x)$ and (5.45) for $n = 2m$, we find

$$(5.52) \qquad \int_0^\infty M_{2m}(x-j)\cos xt\, dx = \psi_{2m}(t)\cos jt - \int_0^\infty M_{2m}(x+j)\cos xt\, dx$$

and

$$(5.53) \qquad \int_0^\infty M_{2m}(x)\cos xt\, dx = \frac{1}{2}\psi_{2m}(t).$$

Since $M^{(2k-1)}(-j) = -M^{(2k-1)}(j)$ and $M^{(2k-1)}(0) = 0$ for $(k=1, 2, \ldots, m-1)$, and by the previous case for $(r = 0, 1, 2, \ldots)$, we need only show that $Rf = 0$ for (5.51) for $(j = -m+1, -m+2, \ldots, -1)$. For $m = 1, 2, 3, 4$ this is just a matter of computation. For instance, for $m = 2$, the cubic case, we need show only that

$$(5.54) \qquad \int_0^\infty M_4(x+1)\cos xt\, dx = \frac{\psi_4(t)}{\phi_4(t)}\left\{\frac{1}{2}M_4(1)\right\} - \frac{1}{t^2}\left[1 - \frac{\phi_2(t)\psi_2(t)}{\phi_4(t)}\right]M'(1).$$

By integrations by parts, the left side becomes

$$\frac{1}{t^4}[\cos t - 1 + \frac{t^2}{2!}]$$

and the right side, upon substituting $M_4(1) = 1/6$, $M_4'(+1) = -1/2$, $\phi_2(t) = 1$, $\phi_4(t) = \frac{2+\cos t}{3}$ and using (1.3) and trigonometric manipulations, agrees. This establishes Theorem 6.

We can now prove the following

Theorem 7. Among all q. f. of the form

$$\int_0^\infty f(x)\cos xt\,dx = \sum_{\nu=0}^{\infty} H_{\nu,t}^{(2m)} f(\nu) + \sum_{j=0}^{m-1} B_{2j-1,t}^{(2m)} f^{(2j-1)}(0) + Rf$$

where the $H_{\nu,t}^{(2m)}$ satisfy the condition

$$|H_{\nu,t}^{(2m)}| < K \quad \text{for fixed } t, \text{ for all integer } \nu \geqq 0 \text{ and some } K,$$

there is a unique q. f. with the property of being exact, i. e., $Rf = 0$, whenever $f(x) \in S_{2m} \cap L_1(R^+)$. This unique q. f. is given by (5.42) for $m = 1, 2, 3, 4$.

Proof. The proof is modeled after the proof of Theorem 1. We want to show that the functional

$$Rf = \int_0^\infty f(x)\cos xt\,dx - \sum_{\nu=0}^{\infty} H_\nu f(\nu) - \sum_{j=1}^{m-1} B_{2j-1} f^{(2j-1)}(0)$$

with the coefficients $H_\nu = H_{\nu,t}^{(2m)}$, $B_{2j-1} = B_{2j-1,t}^{(2m)}$ as given in (5.42) [or as given as the expansion coefficients of the rational function (5. 1) where the B_{2j-1} are defined by (5. 7)} has the property

$$Rf = 0 \qquad \text{if } f \in S_{2m} \cap L_1(R^+)$$

Suppose $f \in S_m \cap L_1(R^+)$ and let

$$f(x) = \sum_{r=-\infty}^{\infty} c_r Q_{2m}(x-r)$$

be the expansion in terms of the forward B-splines of degree $2m - 1$. By Lemma 2, we know that $f(x) \in L_1(R^+)$ implies that

$$\sum_{r=0}^{\infty} |c_r| < \infty. \tag{5. 55}$$

The partial sums

$$f_k(x) = \sum_{r=-\infty}^{k} c_r Q_{2m}(x-r) \qquad (k = 0, 1, 2, \ldots)$$

have the additional property that

$$f_k(x) = 0 \qquad \text{if } x \geqq 2m + k. \tag{5. 56}$$

Moreover, $f_k(x) = f(x)$ if $x \leqq 0$ so that since $f(x) \in S_{2m} \cap L_1(R^+)$ and (5. 56) holds, we conclude that $f_k(x) \in S_{2m} \cap L_1(R^+)$ for integer $k \geqq 0$. Using the properties of the functional (5. 2) we obtain

$$\int_0^{\infty} f_k(x) \cos xt\, dx = \sum_{\nu=0}^{\infty} H_\nu f_k(\nu) + \sum_{j=1}^{m-1} B_{2j-1} f_k^{(2j-1)}(0). \tag{5. 57}$$

Observing that each $f_k(x)$ is dominated by the function

$$\sum_{r=-\infty}^{\infty} |c_r| Q_{2m}(x-r)$$

which is summable on R^+ by Lemma 2 and (5.55) we see that on letting $k \to \infty$, the relation (5.57) goes over into the desired relation

$$\int_0^\infty f(x)\cos xt\,dx = \sum_{\nu=0}^{\infty} H_\nu f(\nu) + \sum_{j=1}^{m-1} B_{2j-1}\, f^{(2j-1)}(0).$$

This completes a proof of Theorem 7.

6. The sine transform (21) for $n = 2m$. In §3-5, we've considered $w(x) = \cos xt$ and $n = 2m$. We now want to consider the weight function $w(x) = \sin xt$ and indicate the modifications in these previous sections that allow us to prove the following

Theorem 8. Among all q. f. of the form

$$(6.1) \qquad \int_0^\infty f(x)\sin xt\,dx = \sum_{\nu=0}^{\infty} H_{\nu,t}^{(2m)} f(\nu) + \sum_{j=1}^{m-1} B_{2j,t}^{(2m)} f^{(2j)}(0) + Rf$$

where the coefficients satisfy

$$|H_{\nu,t}^{(2m)}| < K \text{ for fixed } t, \text{ for all } \nu \geq 0 \text{ and some } K,$$

there is a unique q. f. with the property of being exact, $Rf = 0$, whenever $f(x) \in S_{2m} \cap L_1(R^+)$. This unique q. f. is given by

$$(6.2) \qquad \int_0^\infty f(x)\sin xt\,dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)} \left\{ \sum_{\nu=1}^{\infty} f(\nu)\sin \nu t \right\}$$

$$+ \sum_{j=0}^{m-1} \frac{(-1)^j}{t^{2j+1}} \left[1 - \frac{\phi_{2j}(t)\,\psi_{2m-2j-1}(t)\cos\frac{t}{2}}{\phi_{2m}(t)} \right] f^{(2j)}(0) + Rf,$$

for $m = 1, 2.$

For simplicity, we write $H_\nu = H^{(2m)}_{\nu,t}$, $B_{2j} = B^{(2m)}_{2j,t}$. We again shall attempt to show that

$$(6.3) \qquad Rf = 0 \qquad \text{if } f \in S_{2m} \cap L_1(R^+)$$

by enforcing this requirement for the sequence of forward B-splines of degree $2m-1$ given by (3.4). Upon substituting $f(x) = Q(x-r)$ in (6.1) we have the sequence of relations

$$(6.4) \qquad \int_0^{r+2m} Q(x-r)\sin xt\,dx = H_0Q(-r) + H_1Q(1-r) + \ldots$$
$$+ H_{r+2m-1}Q(2m-1) + \sum_{j=1}^{m-1} B_{2j}Q^{(2j)}(0),$$
$$(r = -2m+1, -2m+2, \ldots, -2, -1)$$

and

$$(6.5) \qquad \int_r^{r+2m} Q(x-r)\sin xt\,dx = H_{r+1}Q(1) + H_{r+2}Q(2) + \ldots + H_{r+2m-1}Q(2m-1)$$
$$(r = 0, 1, 2, \ldots)$$

which are the analogues of (3.8) and (3.9), respectively. Again, we use a generating function approach and observe that the righthand side of (6.4) and (6.5) is equal to the coefficient of x^{r+2m-1} in

$$(6.6) \qquad \left(\sum_{\nu=0}^{\infty} H_\nu x^\nu\right)\left(\sum_{\nu=0}^{2m-2} Q(2m-1-\nu)x^\nu\right)$$
$$+ \sum_{\nu=0}^{2m-2}\left[\sum_{j=1}^{m-1} B_{2j}Q^{(2j)}(2m-1-\nu)\right]x^\nu .$$

Similar to our approach in §4 then, we use (4.10) and (4.12) of Theorem 5 to obtain (6.6) in the form

$$(6.7)\qquad \left(\sum_{\nu=0}^{\infty} H_\nu x^\nu\right)\frac{P_{2m-2}(x)}{(2m-1)!} + \sum_{j=1}^{m-1} \frac{(x-1)^{2j}P_{2m-2j-2}(x)}{(2m-2j-1)!}\ .$$

This is the analog of (4.23).

We consider the left side of relations (6.4) and (6.5), and define

$$(6.8)\qquad \begin{aligned} F_{r+2m-1} &= \int_0^{r+2m} Q(x-r)\sin xt\,dx \qquad (r=-2m+1,-2m+2,\ldots,-1,0)\\ F_{r+2m-1} &= \int_r^{r+2m} Q(x-r)\sin xt\,dx \qquad (r=1,2,\ldots). \end{aligned}$$

If we integrate the right side of (6.8) for $(r=-2m+1,\ldots,-1,0)$ by parts $2m-1$ times and follow the same procedure we used in §4, we get the following analog of (4.31)

$$(6.9)\qquad F_{r+2m-1} = -\left[-\frac{1}{t}Q(-r)+\frac{1}{t^3}Q''(-r)-\ldots+\frac{(-1)^m}{t^{2m-1}}Q^{(2m-2)}(-r)\right]$$

$$-\frac{(-1)^m}{t^{2m}}\sum_{i=0}^{r+2m-1}(-1)^i\binom{2m}{i}\sin(r+2m-i)t \qquad (r=-2m+1,\ldots,-1,0)$$

or by letting $j=r+2m-1$ and using (4.10), we obtain

$$(6.10)\qquad (-1)^m t^{2m}F_j = -\sum_{i=0}^{j}(-1)^i\binom{2m}{i}\sin(j+1-i)t - tQ^{(2m-2)}(j+1)$$

$$+t^3Q^{(2m-4)}(j+1)-\ldots-(-1)^m t^{2m-3}Q''(j+1)+(-1)^m t^{2m-1}Q(j+1)$$

$$(j=0,1,\ldots,2m-1).$$

If we consider (6.8) for $r=1,2,\ldots$ and again integrate by parts $2m-1$ times, we get analogous to (4.36) the relation

$$(6.11)\qquad (-1)^m t^{2m} F_j = -\sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \sin(j+1-i)t$$

$$(j = 2m,\ 2m+1, \ldots).$$

From (6.10) and (6.11) we find that

$$(6.12)\qquad -(-1)^m t^m \sum_{j=0}^{\infty} F_j x^j = \left\{ \sum_{j=0}^{2m-1} \left[\sum_{i=0}^{j} (-1)^i \binom{2m}{i} \sin(j+1-i)t \right] x^j \right.$$

$$\left. + \sum_{j=2m}^{\infty} \left[\sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \sin(j+1-i)t \right] x^j \right\}$$

$$+ t \sum_{j=0}^{2m-2} Q^{(2m-2)}(j+1)x^j - t^3 \sum_{j=0}^{2m-2} Q^{(2m-4)}(j+1)x^j + \ldots$$

$$- (-1)^m t^{2m-1} \sum_{j=0}^{2m-2} Q(j+1)x^j .$$

We define

$$(6.13)\qquad \bar{\tau}(x) = \sum_{\nu=0}^{\infty} [\sin(\nu+1)t] x^\nu$$

and note that $(1-x)^{2m}\bar{\tau}(x)$ is precisely the term in curly brackets on the right side of (6.12), so that by using (4.12) of Theorem 5, we can write (6.12) as

$$(6.14)\qquad -(-1)^m t^{2m} \sum_{j=0}^{\infty} F_j x^j = (1-x)^{2m}\bar{\tau}(x) + t(1-x)^{2m-2} P_0(x)$$

$$- \frac{t^3}{3!}(1-x)^{2m-4} P_2(x) + \ldots - (-1)^m \frac{t^{2m-1}}{(2m-1)!} P_{2m-2}(x) .$$

Equating (6.7) and $\sum_{j=0}^{\infty} F_j x^j$ as determined from (6.14), we see that we require

$$(6.15)\quad \frac{(-1)^{m-1}}{t^{2m}}\{(x-1)^{2m}\bar{\tau}(x) + t(x-1)^{2m-2} - \frac{t^3}{3!}P_2(x)(x-1)^{2m-4} + \dots$$

$$+ (-1)^{m-1}\frac{t^{2m-1}}{(2m-1)!}P_{2m-2}(x)\}$$

$$= (\sum_{\nu=0}^{\infty} H_\nu x^\nu)\frac{P_{2m-2}(x)}{(2m-1)!} + \sum_{j=1}^{m-1} B_{2j}\frac{(x-1)^{2j}P_{2m-2j-2}(x)}{(2m-2j-1)!}\ .$$

Solving (6.15) for $(\sum_{\nu=0}^{\infty} H_\nu x^\nu)$ gives the final relation

$$(6.16)\quad \sum_{\nu=0}^{\infty} H_\nu x^\nu = \frac{(2m-1)!}{P_{2m-2}(x)}\Bigg\{\frac{(-1)^{m-1}}{t^{2m}}[(x-1)^{2m}\bar{\tau}(x)$$

$$+\sum_{j=0}^{m-1}(-1)^j\frac{t^{2j+1}}{(2j+1)!}P_{2j}(x)(x-1)^{2m-2j-2}]$$

$$-\sum_{j=1}^{m-1} B_{2j}\frac{(x-1)^{2j}P_{2m-2j-2}(x)}{(2m-2j-1)!}\Bigg\}.$$

This is the analog of (5.1). Our derivation of (6.16) evidently establishes the following

Proposition 2. The coefficients $H_\nu = H^{(2m)}_{\nu,t}$, $B_j = B^{(2m)}_{j,t}$ of the most general functional

$$(6.17)\qquad Rf = \int_0^\infty f(x)\sin xt\,dx - \sum_{\nu=0}^{\infty} H_\nu f(\nu) - \sum_{j=1}^{m-1} B_{2j}f^{(2j)}(0)$$

vanishing for the function (5.3) are the expansion coefficients of the rational function (6.16) where the B_{2j} $(j=1,\dots,m-1)$ are chosen arbitrarily.

We again wish to determine the unique functional (6.17) having bounded coefficients H_ν. From the form of $\bar{\tau}(x)$ given by (6.13), we note that $\bar{\tau}(x)$ converges for $|x| < 1$. Then in a similar manner to the discussion in §5 we observe that the coefficients H_ν will be bounded, if and only if the coefficients B_{2j} $(j = 1, \ldots, m-1)$ can be chosen to satisfy the equations

$$(6.18)\quad \sum_{j=1}^{m-1} B_{2j} \frac{(\lambda_\nu - 1)^{2j} P_{2m-2j-2}(\lambda_\nu)}{(2m-2j-1)!} = \frac{(-1)^{m-1}}{t^{2m}} [(\lambda_\nu - 1)^{2m} \bar{\tau}(x)$$

$$+ \sum_{j=0}^{m-1} \frac{(-1)^j t^{2j+1}}{(2j+1)!} P_{2j}(\lambda_\nu)(\lambda_\nu - 1)^{2m-2j-2}] \qquad (\nu = 1, \ldots, m-1)$$

where the λ_ν $(\nu = 1, \ldots, m-1)$ are the zeros of $P_{2m-2}(x)$ less than one in absolute value. So we need only show that the determinant

$$(6.19)\quad |A_{\nu j}| = \left| \frac{(\lambda_\nu - 1)^{2j} P_{2m-2j-2}(\lambda_\nu)}{(2m-2j-1)!} \right| \neq 0$$

$$(\nu = 1, \ldots, m-1;\ j = 1, \ldots, m-1).$$

That this is the case is evident from the expression (5.40) if we choose the special case $I \cup I' = \{2, 4, 6, \ldots, 2m-2\}$. This establishes the existence of a unique functional $F\cdot$ of the form (6.17) of Proposition 2 such that the sequence $\{H_\nu\}$ is bounded. The remainder of a proof of the first part of Theorem 8 is essentially the same as the proof of Theorem 7 so we may omit it.

We observe that the functional Rf determined from (6.2) is of the proper form (6.17) where

$$(6.20)\qquad H_0 = \frac{1}{t}\Big[1 - \frac{\psi_{2m-1}(t)\cos\frac{t}{2}}{\phi_{2m}(t)}\Big],\quad H_\nu = \sin \nu t\,\frac{\psi_{2m}(t)}{\phi_{2m}(t)}$$

$$(\nu = 1, 2, \ldots).$$

By (1.3) and (1.8) we find

$$(6.21)\qquad |H_\nu| \leq \left|\frac{\sin t/2}{t/2}\right|^{2m} \frac{1}{\phi_{2m}(\pi)} \leq \frac{1}{\phi_{2m}(\pi)} < \infty \qquad (\nu = 1, 2, \ldots)$$

so, for fixed t, the sequence $\{H_\nu\}$ as given by (6.20) is bounded. Once we show that Rf as given by (6.2) vanishes for the sequence of splines (5.3), the unicity established in the first part of the theorem will complete the proof of Theorem 8. This latter task is accomplished similar to the $\cos xt$ case of §5, showing that $Rf = 0$ for any $m = 1, 2, \ldots$ for $(r = 0, 1, 2, \ldots)$ and looking at the particular cases of $m = 1, 2$ to show that $Rf = 0$ for $(r = -2m+1, -2m+2, \ldots, -1)$. We omit the details.

7. The even degree spline case, $n = 2m-1$. In §3-5 and §6 we considered $w(x) = \cos xt$ and $w(x) = \sin xt$, respectively, for the odd degree spline case, $n = 2m$. Here we shall consider the same weight functions but take $n = 2m-1$ and prove the following

Theorem 9. Let $m = 2$ or 3. Among all q.f. of the form

$$(7.1)\qquad \int_0^\infty f(x)\cos xt\,dx = \sum_{\nu=0}^{\infty} H_{\nu,t}^{(2m-1)} f(\nu) + \sum_{j=1}^{m-1} B_{2j-1,t}^{(2m-1)} f^{(2j-1)}(0) + Rf$$

$$(7.2)\qquad \int_0^\infty f(x)\sin xt\,dx = \sum_{\nu=0}^{\infty} H_{\nu,t}^{(2m-1)} f(\nu) + \sum_{j=1}^{m-1} B_{2j,t}^{(2m-1)} f^{(2j)}(0) + Rf$$

where the coefficients satisfy

$$|H_{\nu,t}^{(2m-1)}| < K \quad \text{for fixed } t, \text{ for all } \nu \geqq 0 \text{ and some } K$$

there is a unique q. f. with the property of being exact, $Rf = 0$, whenever $f(x) \in S_{2m-1} \cap L_1(R^+)$. This unique q. f. is given by

$$(7.3) \quad \int_0^\infty f(x)\cos xt\,dx = \frac{\psi_{2m-1}(t)}{\phi_{2m-1}(t)}\left\{\frac{1}{2}f(0) + \sum_{\nu=1}^{\infty} f(\nu)\cos \nu t\right\}$$

$$\sum_{j=1}^{m-1} \frac{(-1)^j}{t^{2j}}\left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j-1}(t)}{\phi_{2m-1}(t)}\right]f^{(2j-1)}(0) + Rf$$

$$(7.4) \quad \int_0^\infty f(x)\sin xt\,dx = \frac{\psi_{2m-1}(t)}{\phi_{2m-1}(t)}\left\{\sum_{\nu=1}^{\infty} f(\nu)\sin \nu t\right\}$$

$$+ \sum_{j=0}^{m-1} \frac{(-1)^j}{t^{2j+1}}\left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j-2}(t)\cos\frac{t}{2}}{\phi_{2m-1}(t)}\right]f^{(2j)}(0) + Rf.$$

We shall indicate the modifications in §3-5 necessary to prove this theorem for the weight function $\cos xt$. The case of $\sin xt$ then follows as §6 did for the case $n = 2m$. For simplicity, we write $H_\nu = H_{\nu,t}^{(2m-1)}$, $B_{2j-1} = B_{2j-1,t}^{(2m-1)}$. We again attempt to show for (7.1) that

$$(7.5) \qquad Rf = 0 \quad \text{if } f \in S_{2m-1} \cap L_1(R^+)$$

by enforcing this requirement for a sequence of B-splines. This time we choose to use the sequence of central B-splines of degree $2m - 2$

(7.6) $$\{M_{2m-1}(x-m-r)\} = \{M(x-m-r)\} \qquad (r=-2m+1,-2m+2,\ldots)$$

whose knots are at the points $(\nu+\frac{1}{2})$, ν an integer. We write these B-splines in the form (7.6) to make the analogy with §3-5 clearer. By (1.5) we have the explicit expression

(7.7) $$M_{2m-1}(x) = \frac{1}{(2m-2)!}\sum_{i=0}^{2m-1}(-1)^i\binom{2m-1}{i}\left(x+\frac{2m-1}{2}-i\right)_+^{2m-2}.$$

Upon substituting $f(x) = M(x-m-r)$ in (7.1) and noting the requirement (7.5), we obtain the sequence of relations

(7.8) $$\int_0^{r+2m-\frac{1}{2}} M(x-m-r)\cos xt\,dx = H_0 M(-m-r) + H_1 M(1-m-r) + \ldots$$

$$+ H_{r+2m-1} M(m-1) + \sum_{j=1}^{m-1} B_{2j-1} M^{(2j-1)}(-m-r)$$

$$(r=-2m+1,\ldots,-1)$$

and

(7.9) $$\int_{r+\frac{1}{2}}^{r+2m-\frac{1}{2}} M(x-m-r)\cos xt\,dx = H_{r+1} M(1-m) + H_{r+2} M(2-m) + \ldots$$

$$+ H_{r+2m-1} M(m-1) \qquad (r=0,1,2,\ldots)$$

which are the analogs of (3.8) and (3.9), respectively. We again employ a generating function approach and note that the right side of (7.8) and (7.9) is equal to the coefficient of x^{r+2m-1} in

(7.10) $$\left(\sum_{\nu=0}^{\infty} H_\nu x^\nu\right)\left(\sum_{\nu=0}^{2m-2} M(m-1-\nu)x^\nu\right) + \sum_{\nu=0}^{2m-2}\left[\sum_{j=0}^{m-1} B_{2j-1} M^{(2j-1)}(m-1-\nu)\right]x^\nu.$$

In order to simplify the two polynomials in (7.10) we note that

$$(7.11)\qquad M^{(k)}(x) = (-1)^k M^{(k)}(-x)$$

as can be verified from (7.7). With this substitution and (4.14) of Theorem 5, we obtain (7.10) in the form

$$(7.12)\qquad \left(\sum_{\nu=0}^{\infty} H_\nu x^\nu\right)\left(\frac{1}{2}\right)^{2m-2} \frac{T_{2m-2}(x)}{(2m-2)!} - \sum_{j=1}^{m-1} B_{2j-1}\left(\frac{1}{2}\right)^{2m-2j-1} \cdot \frac{(1-x)^{2j-1}\, T_{2m-2j-1}(x)}{(2m-2j-1)!} .$$

This is the analog of (4.23).

We consider the left side of relations (7.8) and (7.9), and define

$$(7.13)\qquad \begin{aligned} F_{r+2m-1} &= \int_0^{r+2m-\frac{1}{2}} M(x-m-r)\cos xt\, dx \qquad (r=-2m+1, -2m+2, \ldots, -2, -1) \\ F_{r+2m-1} &= \int_{r+\frac{1}{2}}^{r+2m-\frac{1}{2}} M(x-m-r)\cos xt\, dx \qquad (r = 0, 1, 2, \ldots). \end{aligned}$$

If we integrate the right side of (7.13) for $(r = -2m+1, -2m+2, \ldots, -1)$ by parts $2m-2$ times and follow the same procedure we used in §4, we get the following analog of (4.31)

$$(7.14)\qquad \begin{aligned} F_{r+2m-1} = &- \left[\frac{1}{t^2} M'(-m-r) - \frac{1}{t^4} M'''(-m-r) + \ldots \right. \\ &\left. + \frac{(-1)^{m-2}}{t^{2m-2}} M^{(2m-3)}(-m-r)\right] \\ &- \frac{(-1)^{m-2}}{t^{2m-1}} \sum_{i=0}^{r+2m-1} (-1)^i \binom{2m-1}{i} \sin\left(2m+r-1+\frac{1}{2}-i\right)t \\ &\qquad\qquad (r = -2m+1, \ldots, -1) \end{aligned}$$

or by letting $j = r + 2m - 1$ and using (7.11) we obtain

$$(7.15)\qquad (-1)^{m-1} t^{2m-1} F_j = \sum_{i=0}^{j} (-1)^i \binom{2m-1}{i} \sin(j + \tfrac{1}{2} - i)t - tM^{(2m-3)}(j+1-m)$$

$$+ t^3 M^{(2m-5)}(j+1-m) - \ldots + (-1)^{m-1} t^{2m-3} M'(j+1-m)$$

$$(j = 0, 1, \ldots, 2m-2).$$

If we consider (7.13) for $(r = 0, 1, 2, \ldots)$ and again integrate by parts $2m-2$ times, we get analogous to (4.36) the relation

$$(7.16)\qquad (-1)^{m-1} t^{2m-1} F_j = \sum_{i=0}^{2m-1} (-1)^i \binom{2m-1}{i} \sin(j + \tfrac{1}{2} - i)t$$

$$(j = 2m-1, 2m, \ldots).$$

From (7.15) and (7.16), we find that

$$(7.17)\qquad (-1)^{m-1} t^{2m-1} \sum_{j=0}^{\infty} F_j x^j =$$

$$= \left\{ \sum_{j=0}^{2m-2} \left[\sum_{i=0}^{j} (-1)^i \binom{2m-1}{i} \sin(j + \tfrac{1}{2} - i)t \right] x^j \right.$$

$$\left. + \sum_{j=2m-1}^{\infty} \left[\sum_{i=0}^{2m-1} (-1)^i \binom{2m-1}{i} \sin(j + \tfrac{1}{2} - i)t \right] x^j \right\}$$

$$- t \sum_{j=0}^{2m-2} M^{(2m-3)}(j+1-m)x^j + t^3 \sum_{j=0}^{2m-3} M^{(2m-5)}(j+1-m)x^j - \ldots$$

$$+ (-1)^{m-1} t^{2m-3} \sum_{j=0}^{2m-2} M'(j+1-m)x^j.$$

We define

$$U(x) = \sum_{\nu=0}^{\infty} [\sin(\nu+\tfrac{1}{2})t]x^{\nu} \tag{7.18}$$

and note that $(1-x)^{2m-1}U(x)$ is precisely the term in curly brackets on the right side of (7.17), so that by using (4.14) of Theorem 5, we can write (7.17) as

$$(-1)^{m-1}t^{2m-1}\sum_{j=0}^{\infty} F_j x^j = (1-x)^{2m-1}U(x) - (\tfrac{t}{2})\frac{(1-x)^{2m-3}T_1(x)}{1!} \tag{7.19}$$

$$+ (\tfrac{t}{2})^3 \frac{(1-x)^{2m-5}T_3(x)}{3!} - \cdots$$

$$+ (-1)^{m-1}(\tfrac{t}{2})^{2m-3}\frac{(1-x)T_{2m-3}(x)}{(2m-3)!}.$$

Equating (7.10) and $\sum_{j=0}^{\infty} F_j x^j$ as determined from (7.19) and then solving for $\sum_{\nu=0}^{\infty} H_{\nu}x^{\nu}$ gives the final relation

$$\sum_{\nu=0}^{\infty} H_{\nu}x^{\nu} = \frac{(2m-2)!\,2^{2m-2}}{T_{2m-2}(x)}\left\{\frac{(-1)^{m-1}}{t^{2m-1}}[(x-1)^{2m-1}U(x) \right. \tag{7.20}$$

$$+ \sum_{j=1}^{m-1}(-1)^j(\tfrac{t}{2})^{2j-1}\frac{1}{(2j-1)!}T_{2j-1}(x)(x-1)^{2m-2j-1}]$$

$$\left. - \sum_{j=1}^{m-1} B_{2j-1}(\tfrac{1}{2})^{2m-2j-1}\frac{(x-1)^{2j-1}T_{2m-2j-1}(x)}{(2m-2j-1)!}\right\}.$$

This is the analog of (5.1). Our derivation of (7.20) evidently establishes

Proposition 3. The coefficients $H_{\nu} = H_{\nu,t}^{(2m-1)}$, $B_{2j-1} = B_{2j-1,t}^{(2m-1)}$

of the most general functional

$$\text{(7.21)} \qquad Rf = \int_0^\infty f(x)\cos xt\,dx - \sum_{\nu=0}^{\infty} H_\nu f(\nu) - \sum_{j=1}^{m-1} B_{2j-1}\, f^{(2j-1)}(0)$$

vanishing for the functions (7.6) are the expansion coefficients of the rational function (7.20) where the B_{2j-1} $(j = 1, \ldots, m-1)$ are chosen arbitrarily.

We again wish to determine the unique functional (7.21) having bounded coefficients H_ν. From the form of $U(x)$ given by (7.18), we see that $U(x)$ converges for $|x| < 1$. So that just as in §5 we note that the coefficients H_ν will be bounded if and only if the coefficients B_{2j-1} $(j = 1, \ldots, m-1)$ can be chosen to satisfy

$$\text{(7.22)} \quad \sum_{j=1}^{m-1} B_{2j-1}\left(\frac{1}{2}\right)^{2m-2j-1} \frac{(\lambda_\nu - 1)^{2j-1} T_{2m-2j-1}(\lambda_\nu)}{(2m-2j-1)!}$$

$$= \frac{(-1)^{m-1}}{t^{2m-1}}\left[(\lambda_\nu - 1)^{2m-1} U(\lambda_\nu) + \sum_{j=1}^{m-1} (-1)^j \left(\frac{t}{2}\right)^{2j-1} \frac{T_{2j-1}(\lambda_\nu)(\lambda_\nu - 1)^{2m-2j-1}}{(2j-1)!}\right.$$

$$(\nu = 1, 2, \ldots, m-1)$$

where the λ_ν $(\nu = 1, \ldots, m-1)$ are the zeros of $T_{2m-2}(x)$ less than one in absolute value. Lemma 8 of [9, p. 182] had guaranteed that $T_{2m-2}(x)$ was a reciprocal polynomial which had only simple and negative zeros $\lambda_1, \lambda_2, \ldots, \lambda_{2m-2}$ that we may label to satisfy the conditions (5.5) and (5.6). So we need only show that the determinant

$$(7.23) \qquad |A_{\nu j}| = \left|\left(\frac{1}{2}\right)^{2m-2j-1} \frac{(\lambda_\nu-1)^{2j-1} T_{2m-2j-1}(\lambda_\nu)}{(2m-2j-1)!}\right| \neq 0 .$$

We haven't proved a lemma similar to Lemma 4 of §5, but note that the determinant in (7.23) for the cases $m = 2$ and $m = 3$ takes the forms

$$(7.24) \qquad \frac{1}{2}(\lambda-1)(\lambda+1)$$

and

$$(7.25) \qquad \frac{1}{4}(\lambda_1-1)(\lambda_2-1)(\lambda_1+1)(\lambda_2+1)(\lambda_1-\lambda_2)(1-\lambda_1\lambda_2)$$

respectively, and so by (5.5) the condition (7.23) is satisfied. This establishes the existence of a unique functional Rf of the form (7.21) of Proposition 3 such that the sequence $\{H_\nu\}$ is bounded. The remainder of the proof of the first part of Theorem 9 is essentially the same as the proof of Theorem 7 and we omit it.

We observe that the functional Rf determined from (7.3) is of the appropriate form (7.21) where

$$(7.26) \qquad H_0 = \frac{1}{2} \frac{\psi_{2m-1}(t)}{\phi_{2m-1}(t)} , \quad H_\nu = \frac{\psi_{2m-1}(t)}{\phi_{2m-1}(t)} \cos \nu t \qquad (\nu = 1, 2, \ldots)$$

and by (1.3) and (1.8) we again have this sequence $\{H_\nu\}$ bounded. It is a straightforward procedure to show for $m = 2$, $m = 3$ that Rf as given by (7.3) vanishes for the sequence of splines (7.6), so that the unicity established in the first part of Theorem 9 completes the proof of the theorem.

8. <u>Explicit forms for the q. f. of Theorem 2.</u> Theorem 2 is established similar to the way the first parts of Theorem 7 and 8 were proved. The only change needed is to take $I \cup I' = \{1, 2, \ldots, m-1\}$ instead of what we used before. For instance, for the $\cos xt$ case, we use the same left sides of (3.8) and (3.9), but we have to modify the right sides. Theorem 5 readily allows us to do this and we get a similar expression to (4.42) but now in the coefficients $B_1, B_2, \ldots, B_{m-1}$. Lemma 4 for the choice $I \cup I' = \{1, 2, \ldots, m-1\}$ enables us to establish Theorem 2 just as the first part of Theorem 7 was proved.

For the weight function $\cos xt$, the q. f. (5.42) gives explicit expressions for our present cases $m = 1$ and $m = 2$. We want to get a q. f. similar to (5.42) for $m = 3$, the quintic spline case, in a form particularly amenable to computation. We shall find that the form we do obtain is precisely (5.42) with the exception that $f'''(0)$ is replaced by $S'''(0)$, the third derivative of a particular interpolating spline to $f(x)$, evaluated at 0. The expression for $S'''(0)$ involves the values $f''(0)$, $f'(0)$, $f(0)$, $f(1)$, $\ldots$, but <u>not</u> $f'''(0)$.

For the $\sin xt$ case, the q. f. (6.2) gives our desired q. f. when $m = 1$. For $m = 2$, the only change we make in (6.2) is to replace $f''(0)$ by $S''(0)$, the second derivative of a particular spline interpolant to $f(x)$, evaluated at 0. Here $S''(0)$ is expressed in terms of the values $f'(0)$, $f(0)$, $f(1)$, $\ldots$ but not $f''(0)$. There is a similar q. f.

when $m = 3$ that we get from (6.2) by replacing $f^{(iv)}(0)$ by $S^{(iv)}(0)$.

We summarize the foregoing in the following

Theorem 2'. *Suppose* $f \in L_1(R^+)$. *Let* $S(x) \in L_1(R^+)$ *be the unique spline of degree* $2m-1$ *for* $x \geq 0$ *with knots at* $x = 1, 2, 3, \ldots$ *satisfying the conditions*

$$S(\nu) = f(\nu) \qquad (\nu = 0, 1, 2, \ldots) \tag{8.1}$$

$$S^{(j)}(0) = f^{(j)}(0) \qquad (j = 1, 2, \ldots, m-1). \tag{8.2}$$

For $m = 1, 2, 3,$ *the unique q.f. of Theorem* 2 *are given explicitly by*

$$\int_0^\infty f(x)\cos xt\,dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)}\left\{\frac{1}{2}f(0) + \sum_{\nu=1}^{\infty} f(\nu)\cos\nu t\right\} \tag{8.3}$$

$$+ \sum_{j=1}^{[m/2]} \frac{(-1)^j}{t^{2j}}\left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j}(t)}{\phi_{2m}(t)}\right] f^{(2j-1)}(0)$$

$$+ \sum_{j=[m/2]+1}^{m-1} \frac{(-1)^j}{t^{2j}}\left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j}(t)}{\phi_{2m}(t)}\right] S^{(2j-1)}(0) + Rf$$

and

$$\int_0^\infty f(x)\sin xt\,dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)}\left\{\sum_{\nu=1}^{\infty} f(\nu)\sin\nu t\right\} \tag{8.4}$$

$$+ \sum_{j=0}^{[\frac{m-1}{2}]} \frac{(-1)^j}{t^{2j+1}}\left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j-1}(t)\cos\frac{t}{2}}{\phi_{2m}(t)}\right] f^{(2j)}(0)$$

$$+ \sum_{j=[\frac{m-1}{2}]+1}^{m-1} \frac{(-1)^j}{t^{2j+1}}\left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j-1}(t)\cos\frac{t}{2}}{\phi_{2m}(t)}\right] S^{(2j)}(0) + Rf.$$

We also state here for reference in §10 the following

<u>Corollary 1.</u> <u>Let</u> $f(x)$ <u>and</u> $S(x)$ <u>be as given in Theorem 2'. Then the q. f.</u>

$$\int_0^\infty f(x)e^{itx}dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)}\left\{\frac{1}{2}f(0) + \sum_{\nu=1}^{\infty} f(\nu)e^{it\nu}\right\}$$

$$+ \sum_{j=1}^{[m/2]} \frac{(-1)^j}{t^{2j}}\left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j}(t)}{\phi_{2m}(t)}\right]f^{(2j-1)}(0)$$

$$+ \sum_{j=[m/2]+1}^{m-1} \frac{(-1)^j}{t^{2j}}\left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j}(t)}{\phi_{2m}(t)}\right]S^{(2j-1)}(0)$$

$$+ i\left\{\sum_{j=0}^{[(m-1)/2]} \frac{(-1)^j}{t^{2j+1}}\left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j-1}(t)\cos\frac{t}{2}}{\phi_{2m}(t)}\right]f^{(2j)}(0)\right.$$

$$\left. + \sum_{j=[(m-1)/2]+1}^{m-1} \frac{(-1)^j}{t^{2j+1}}\left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j-1}(t)\cos\frac{t}{2}}{\phi_{2m}(t)}\right]S^{(2j)}(0)\right\} + Rf$$

<u>is the unique q. f. exact whenever</u> $f(x) \in S_{2m} \cap L_1(\mathbb{R}^+)$.

We discuss in detail the cubic case, $m = 2$, of the q. f. (8. 4), that is, the q. f.

$$(8.5) \quad \int_0^\infty f(x)\sin xt\, dx = \frac{\psi_4(t)}{\phi_4(t)} \sum_{\nu=1}^{\infty} f(\nu)\sin \nu t + \frac{1}{t}\left[1 - \frac{\psi_3(t)\cos t/2}{\phi_4(t)}\right]f(0)$$

$$- \frac{1}{t^3}\left[1 - \frac{\psi_1(t)\cos t/2}{\phi_4(t)}\right]S''(0) + Rf .$$

How to extend the method to $m = 3$, and in fact larger m, and (8.3) will be clear. We observe that by the second part of Theorem 8, the exactness of (8.4) for $f(x) \in S_{2m} \cap L_1(\mathbb{R}^+)$ is already established since for such f we have $S''(0) = f''(0)$. What remains to show is that the coefficients $H_{\nu,t}^{(2m)}$ of the q.f. (17) of the Introduction are bounded.

An important point is that we do not want to use the "natural" semi-cardinal cubic spline interpolant. This is of course best in the sense that it minimizes

$$\int_0^{\infty} [F''(x)]^2 dx$$

among all functions $F(x)$ that interpolate $f(x)$ at $x = 0, 1, 2, \ldots$ but it is not a good approximation. Rather we use the "complete" semi-cardinal spline approximation where also $f'(0)$ is assumed known and is matched by the cubic spline, that is, (8.2) holds.

We note that Lemma 4 guarantees the unicity of the interpolating spline. The interpolating spline $S(x)$ is given by the spline interpolation formula

$$S(x) = \sum_{\nu=0}^{\infty} f(\nu)L_\nu(x) + f'(0)\Lambda(x) \tag{8.6}$$

where the fundamental functions $L_\nu(x)$ and $\Lambda(x)$ satisfy

$$\begin{aligned} &L_\nu(\nu) = 1 \\ &L_\nu(\mu) = 0 \qquad \text{if } \nu \neq \mu \qquad (\nu = 0, 1, 2, \ldots) \\ &L_\nu'(0) = 0 \end{aligned} \tag{8.7}$$

and

$$\text{(8.8)} \qquad \begin{aligned} \Lambda(\nu) &= 0 \\ \Lambda'(0) &= 1 \end{aligned} \qquad (\nu = 0, 1, 2, \ldots)$$

respectively. In order to construct $L_\nu(x)$ and $\Lambda(x)$ we shall use two other important cubic splines.

One is the fundamental function $L(x)$ of cardinal cubic spline interpolation, i.e., $L(x)$ is a cardinal cubic spline satisfying

$$\text{(8.9)} \qquad L(\nu) = \begin{cases} 1 & \text{if } \nu = 0 \\ 0 & \text{if } \nu \neq 0. \end{cases}$$

In terms of the cubic B-spline $M(x) = M_4(x)$, we have explicitly

$$\text{(8.10)} \qquad L(x) = \sqrt{3} \sum_{j=-\infty}^{\infty} \lambda^{|j|} M(x-j)$$

where

$$\text{(8.11)} \qquad \lambda = -2 + \sqrt{3} = -.268\ldots$$

is the root of least absolute value of

$$P_2(\lambda) = \lambda^2 + 4\lambda + 1 = 0.$$

To prove (8.10) we note that, for $\nu \geqq 1$

$$L(\nu) = \sqrt{3} \sum_{j=\nu-1}^{\nu+1} \lambda^{|j|} M(\nu-j) = \sqrt{3}\,\frac{1}{6}(\lambda^{\nu+1} + 4\lambda^{\nu} + \lambda^{\nu-1})$$

$$= \sqrt{3} \cdot \frac{1}{6} \cdot \lambda^{\nu-1}(\lambda^2 + 4\lambda + 1) =)$$

and

$$L(0) = \sqrt{3} \cdot \frac{1}{6} \cdot (\lambda+4+\lambda) = \frac{\sqrt{3}}{3}(2+\lambda) = 1$$

so that the unicity of a bounded $L(x)$ satisfying (8.9) implies that (8.10) is correct.

The other cubic spline we use is the decreasing cubic eigenspline

$$(8.12) \qquad S_1(x) = \sum_{j=-\infty}^{\infty} \lambda^j M(x-j)$$

which, from §5, is a cardinal spline satisfying

$$(8.13) \qquad S_1(x) = O(|\lambda|^x) \qquad \text{as } x \to \infty$$

and

$$(8.14) \qquad S_1(\nu) = 0 \qquad \text{for all } \nu .$$

Now if we write

$$(8.15) \qquad \Lambda(x) = \frac{1}{S_1'(0)} S_1(x)$$

and use (8.14), we see that (8.8) is satisfied. From (8.12),

$$(8.16) \qquad S_1'(0) = \Sigma \lambda^j M'(-j) = \frac{1}{2}(\lambda - \lambda^{-1}) = \sqrt{3}$$

because $M'(-1) = \frac{1}{2}$, $M'(0) = 0$, $M'(1) = -\frac{1}{2}$ and $M'(j) = 0$ for all other j. From (8.15) and (8.12)

$$(8.17) \qquad \Lambda(x) = \frac{1}{\sqrt{3}} \sum_{j=-1}^{\infty} \lambda^j M(x-j) \qquad \text{for } x \geqq 0 .$$

Writing

$$(8.18) \qquad \Lambda(x) = \sum_{j=-1}^{\infty} \gamma_j M(x-j)$$

we see that

$$(8.19) \qquad \gamma_j = \frac{1}{\sqrt{3}} \lambda^j \qquad (j = -1, 0, 1, \ldots).$$

We shall use the notation

$$(8.20) \qquad L_\nu(x) = \sum_{j=-1}^{\infty} c_{j,\nu} M(x-j) \qquad \text{for } x \geqq 0 .$$

We have

$$(8.21) \qquad L_0(x) = L(x) \qquad \text{for } x \geqq 0$$

since by (8.8) and $L'(0) = 0$, (8.7) is satisfied. Using (8.10) and (8.21), we find

(8.22) $$c_{j,0} = \sqrt{3}\,\lambda^{|j|} \qquad (j = -1, 0, 1, \ldots).$$

It is easily verified that

(8.23) $$L_\nu(x) = L(x-\nu) - \frac{L'(-\nu)}{S_1'(0)} S_1(x) \qquad (\nu = 1, 2, \ldots)$$

satisfies the conditions (8.7). From (8.10) and (8.16)

$$-\frac{L'(\nu)}{S_1'(0)} = \frac{\sqrt{3}(\lambda^{\nu+1} - \lambda^{\nu-1})}{\lambda - \lambda^{-1}} = \sqrt{3}\,\lambda^\nu \qquad (\nu = 1, 2, \ldots)$$

and therefore by (8.23), (8.10) and (8.12)

$$L_\nu(x) = \sqrt{3}\sum_j \lambda^{|j|} M(x-\nu-j) + \sqrt{3}\,\lambda^\nu \sum_j \lambda^j M(x-j)$$

$$= \sqrt{3}\sum_j \lambda^{|j-\nu|} M(x-j) + \sqrt{3}\,\lambda^\nu \sum_j \lambda^j M(x-j) = \sum_{j=-1}^{\infty} c_{j,\nu} M(x-j)$$

for $x \geqq 0$

where

(8.24) $$c_{j,\nu} = \sqrt{3}(\lambda^{|j-\nu|} + \lambda^{j+\nu}) \qquad (\nu \geqq 1,\ j = -1, 0, 1, \ldots).$$

We can now use the interpolating spline $S(x)$ given in (8.6) to determine from the q. f. (8.5) what the form of the coefficients $H^{(4)}_{\nu,t} = H_\nu$ of q. f. (17) is. Differentiating each side of (8.6) twice and substituting 0 for x gives

(8.25) $$S''(0) = \sum_{\nu=0}^{\infty} f(\nu) L_\nu''(0) + f'(0)\Lambda''(0)$$

so that

$$(8.26)\qquad H_\nu = \frac{\psi_4(t)}{\phi_4(t)}\sin \nu t - \frac{1}{t^3}\left[1 - \frac{\psi_1(t)\cos\frac{t}{2}}{\phi_4(t)}\right] L_\nu''(0) \qquad (\nu = 1, 2, \dots).$$

From (8.20) we have

$$(8.27)\qquad L_\nu''(0) = \sum_{j=-1}^{\infty} c_{j,\nu} M''(-j) = c_{-1,\nu} - 2c_{0,\nu} + c_{1,\nu}$$

since $M''(-1) = M''(1) = +1$, $M''(0) = -2$ and $M''(j) = 0$ for all other j. From (8.24) and (8.27), then, after simplification we get

$$(8.28)\qquad L_\nu''(0) = -12\sqrt{3}\,\lambda^\nu \qquad (\nu = 1, 2, \dots).$$

Therefore, for fixed t, by (6.21), (8.11), (8.26) and (8.28) there exists a constant K such that

$$|H_\nu| < K \qquad \text{for all } \nu \geqq 0.$$

By the unicity established in Theorem 2, Theorem 2' is established for $m = 2$.

We also note that in terms of the functions $L_\nu(x)$ and $\Lambda(x)$ just defined, we have the following

Corollary 2. The following identities hold

$$\int_0^\infty L_\nu(x)\cos xt\,dx = \frac{\psi_4(t)}{\phi_4(t)}\cos \nu t \qquad (\nu = 1, 2, \dots)$$

$$\int_0^\infty L_\nu(x)\sin xt\,dx = \frac{\psi_4(t)}{\phi_4(t)}\sin \nu t \qquad (\nu = 1, 2, \dots)$$

$$\int_0^\infty \Lambda(x)\cos xt\,dx = -\frac{1}{t^2}\left[1 - \frac{\psi_2(t)}{\phi_4(t)}\right]$$

$$\int_0^\infty \Lambda(x) \sin xt \, dx = -\frac{1}{t^3}\left[1 - \frac{\psi_1(t)\cos\frac{t}{2}}{\phi_4(t)}\right]$$

$$\int_0^\infty L_0(x)\cos xt \, dx = \frac{1}{2}\frac{\psi_4(t)}{\phi_4(t)}, \quad \int_0^\infty L_0(x)\sin xt \, dx = \frac{1}{t}\left[1 - \frac{\psi_3(t)\cos\frac{t}{2}}{\phi_4(t)}\right].$$

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III. THE LAPLACE TRANSFORM

We follow the same kind of generating function approach we used earlier, the only modifications coming from the enlarged class of functions for which we can find transforms now that the weight function is $e^{-x\rho}$, $\rho > 0$.

9. Proof of Theorem 3 of the Introduction. We first define, for each $\gamma \geqq 0$, the class of functions

$$F_\gamma = \{F(x) : F(x) \in C(\mathbb{R}^+) \text{ and } F(x) = O(x^\gamma) \text{ as } x \to +\infty\}$$

and the class of sequences

$$Y_\gamma = \{y = \{y_\nu\}_{\nu=1}^{\infty} : y_\nu = O(\nu^\gamma) \text{ as } \nu \to \infty\}.$$

We note that

(9.1) $\quad S(x) \in S_{2m} \cap F_\gamma$ for some $\gamma \geqq 0$ implies $\{S(\nu)\} \in Y_\gamma$

so that by (24) the functional Rf given by (23) is well-defined if $f(x) \in S_{2m} \cap F_\gamma$. We also need to know just when a cardinal spline function of degree $2m-1$ is in F_γ. The answer is given by

Lemma 5. If $S(x) \in S_{2m}$ and

(9.2) $$S(x) = \sum_\nu c_\nu M_{2m}(x-\nu)$$

then

(9.3) $$S(x) \in F_\gamma$$

if and only if

(9.4) $$\{c_\nu\} \in Y_\gamma .$$

Proof. First, we assume (9.4), so there exist constants A, N such that

$$|c_\nu| < A\nu^\gamma \qquad \text{for } \nu > N. \tag{9.5}$$

Let $n > m + N$. From

$$|S(x)| \leqq \sum_\nu |c_\nu| M_{2m}(x-\nu) \leqq M_{2m}(0) \sum_{\nu=n-m}^{n+m} |c_\nu| \quad \text{if } n \leqq x \leqq n+1$$

and (9.4), we find

$$|S(x)| \leqq M_{2m}(0) \cdot A \sum_{\nu=n-m}^{n+m} \nu^\gamma \leqq [M_{2m}(0) \cdot A \cdot (2m+1)\bar{K}] n^\gamma \leqq K|x|^\gamma \tag{9.6}$$

$$\text{if } n \leqq x \leqq n+1$$

where K represents the quantity in square brackets in (9.6) and does not depend on n. So for any $x > n > m+N$

$$|S(x)| \leqq K|x|^\gamma$$

and (9.3) holds.

Now, we assume (9.3) and are to derive (9.4). We adapt the proof of Theorem 4 of [11, p. 18-19] to our particular situation. We observe first that in Theorem 5 of [11, p. 7] Schoenberg explicitly expresses the c_ν of (9.2) in the form

$$c_\nu = \sum_{r=0}^{m-1} (-1)^r \gamma_{2r}^{(m)} S^{(2r)}(\nu) \tag{9.7}$$

where the $\{\gamma_{2r}^{(m)}\}$ is a sequence of rational numbers generated by the expression

$$\left[\frac{u}{2\sin u/2}\right]^{2m} = \sum_{r=0}^{\infty} \gamma_{2r}^{(m)} u^{2r}$$

By (9.7) then in order to prove (9.4) it is sufficient to show that

$$(9.8) \qquad \{S^{(2r)}(\nu)\} \in Y_\gamma \qquad (r = 0, 1, \ldots, m-1).$$

By (9.3) there exist constants C, D such that

$$(9.9) \qquad |S(x)| \leqq Cx^\gamma \qquad \text{for } x > D.$$

Let $R(x)$ be a polynomial of degree k in the interval $[0, 1]$. By Markov's theorem we obtain the string of inequalities

$$\max|R'| \leqq 2k^2 \max|R|$$

$$\max|R''| \leqq 2(k-1)^2 \max|R'|$$

$$\vdots$$

$$\max|R^{(t)}| \leqq 2(k-t+1)^2 \max|R^{(t-1)}| \qquad (t \leqq k,$$

and putting them together we obtain

$$(9.10) \qquad \max|R^{(t)}| \leqq A(k,t) \max|R| \qquad (t = 1, \ldots, k)$$

where $A(k,t) = 2^t[k(k-1) \ldots (k-t+1)]^2$. Applying (9.9) with $k = 2m-1$, $t = 2r$, to each of the polynomial components of $S(x)$ in each of the successive intervals $[\nu, \nu+1]$ for $\nu > B+1$, we conclude that

$$|S^{(2r)}(\nu)| \leqq \max_{[\nu-1,\nu]} |S^{(2r)}(x)| \leqq A(2m-1, 2r) \max_{[\nu-1,\nu]} |S(x)|$$

$$\leqq A(2m-1, 2r) \cdot C \max_{[\nu-1,\nu]} x^\gamma = A(2m-1, 2r)C \cdot \nu^\gamma$$

so that (9.8) and the lemma are proved.

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We now turn to the proof of Theorem 3 and again shall indicate the modifications in a previous proof, this time Theorem 2, that leads us to our desired conclusion. We again require exactness for the B-splines (3.4) and are led to the sequence of relations

$$(9.11)\quad \int_0^{r+2m} Q(x-r)e^{-x\rho}\,dx = H_0 Q(-r) + H_1 Q(1-r) + \dots + H_{r+2m-1} Q(2m-1) + \sum_{j=1}^{m-1} B_j Q^{(j)}(-r) \qquad (r = -2m+1, -2m+2, \dots, 1)$$

and

$$(9.12)\quad \int_0^{r+2m} Q(x-r)e^{-x\rho}\,dx = H_{r+1} Q(1) + H_{r+2} Q(2) + \dots + H_{r+2m-1} Q(2m-1) \qquad (r = 0, 1, 2, \dots).$$

The righthand sides of (9.11) and (9.12) are the same as in the proof of Theorem 2, but the left sides now have the weight function $e^{-x\rho}$. Following the same procedure as before, we arrive at the following analog of (5.1)

$$(9.13)\quad \sum_{\nu=0}^{\infty} H_\nu x^\nu = \frac{(2m-1)!}{P_{2m-2}(x)} \left\{ \frac{1}{\rho^{2m}} \left[(x-1)^{2m} \frac{e^{-\rho}}{1-e^{-\rho}x} + \sum_{j=0}^{2m-1} \frac{\rho^j}{j!} \cdot P_{2j-1}(x)(x-1)^{2m-1-j} \right] - \sum_{j=1}^{m-1} B_j \frac{(x-1)^j P_{2m-2-j}(x)}{(2m-1-j)!} \right\}.$$

Let $R_{2m}(x)$ denote the right side of (9.13), where the B_j $(j = 1, 2, \dots, m-1)$ are as yet undetermined. We recall that $P_{2m-2}(x)$ has the simple zeros $\lambda_1, \lambda_2, \dots, \lambda_{2m-2}$ satisfying (5.5) and (5.6).

Observing that for $R_{2m}(x)$ the poles $\lambda_1, \ldots, \lambda_{m-1}$ are inside the unit circle while e^ρ and $\lambda_m, \ldots, \lambda_{2m-2}$ are outside, in view of (5.5) and (9.13), we note that the coefficients H_ν will satisfy the condition (24) if and only if the coefficients B_j $(j = 1, 2, \ldots, m-1)$ can be chosen so that the $m-1$ poles $\lambda_1, \ldots, \lambda_{m-1}$ have vanishing residues. By (9.13) this will occur if and only if the B_j satisfy the equations

$$(9.14)\quad \sum_{j=1}^{m-1} B_j \frac{(\lambda_\nu - 1)^j P_{2m-2-j}(\lambda_\nu)}{(2m-1-j)!} = \frac{1}{\rho^{2m}}\Big[(\lambda_\nu - 1)^{2m} \frac{e^{-\rho}}{1-e^{-\rho}\lambda_\nu} + \sum_{j=0}^{2m-1} \frac{\rho^j}{j!} P_{2j-1}(\lambda_\nu)(\lambda_\nu - 1)^{2m-1-j}\Big] \qquad (\nu = 1, 2, \ldots, m-1).$$

The determinant of the system (9.14) though is not zero as is evident from the expression (5.40) if we choose the special case $I \cup I' = \{1, 2, \ldots, m-1\}$. This establishes the existence of a unique functional Rf defined by (23) whose coefficients satisfy (24) and which vanishes for the functions (5.3).

The remainder of a proof of Theorem 3 follows the same procedure as the proof of Theorem 7, where now we use F_γ and $e^{-x\rho}$ instead of $L_1(R^+)$ and $\cos xt$ respectively, and so we omit it.

10. An explicit version of the g. f. of Theorem 3. In terms of the central B-spline $M_n(x)$ of degree $n-1$, we define

(10.1) $$\int_{-\infty}^{\infty} M_n(x)e^{-x\rho}\,dx = \bar{\psi}_n(\rho)$$

which upon evaluation gives the relation

(10.2) $$\bar{\psi}_n(\rho) = \left[\frac{2\sinh\rho/2}{\rho}\right]^n .$$

We also define

(10.3) $$\bar{\phi}_n(\rho) = \sum_{\nu=-\infty}^{\infty} M_n(\nu)e^{-\nu\rho} = \sum_{|\nu|\leq\frac{n}{2}} M_n(\nu)e^{-\nu\rho}$$

and note that $\bar{\phi}_n(\rho)$ is positive. From (10.3) we find

$$\bar{\phi}_2(\rho) = 1$$

$$\bar{\phi}_3(\rho) = \frac{3+\cosh\rho}{4}$$

$$\bar{\phi}_4(\rho) = \frac{2+\cosh\rho}{3}$$

$$\bar{\phi}_5(\rho) = \frac{115+76\cosh\rho+\cosh 2\rho}{192}$$

$$\bar{\phi}_6(\rho) = \frac{33+26\cosh\rho+\cosh 2\rho}{60}$$

and observe that $\bar{\phi}_n(\rho)$ has the form of $\phi_n(\rho)$ given by (1.7) if we replace the cosine function by the cosh function. Similarly, if we replace the sine function in the expression of $\psi_n(\rho)$ in (1.3) by the sinh function, we get precisely $\bar{\psi}_n(\rho)$ as given by (10.2).

Now we can state the following

<u>Theorem 3'.</u> <u>Suppose</u> $f(x) \in F_\gamma$ <u>for some</u> $\gamma \geqq 0$. <u>Let</u> $S(x) \in F_\gamma$ <u>be the unique spline of degree</u> $2m-1$ <u>for</u> $x \geqq 0$ <u>with knots at</u>

$x = 1, 2, 3, \ldots$ satisfying the conditions

$$(10.4) \qquad S(\nu) = f(\nu) \qquad (\nu = 0, 1, 2, \ldots)$$

$$(10.5) \qquad S^{(j)}(0) = f^{(j)}(0) \qquad (j = 1, 2, \ldots, m-1).$$

For $m = 1, 2, 3$ the unique q. f. of Theorem 3 is given explicitly by

$$(10.6) \qquad \int_0^\infty f(x)e^{-x\rho}\,dx = \frac{\bar{\psi}_{2m}(\rho)}{\bar{\phi}_{2m}(\rho)}\left[\frac{1}{2}f(0) + \sum_{\nu=1}^{\infty} e^{-\nu\rho} f(\nu)\right]$$

$$+ \sum_{j=1}^{[m/2]} \frac{1}{\rho^{2j}}\left[1 - \frac{\bar{\phi}_{2j}(\rho)\bar{\psi}_{2m-2j}(\nu)}{\bar{\phi}_{2m}(\rho)}\right] f^{(2j-1)}(0)$$

$$+ \sum_{j=[\frac{m}{2}]+1}^{m-1} \frac{1}{\rho^{2j}}\left[1 - \frac{\bar{\phi}_{2j}(\rho)\bar{\psi}_{2m-2j}(\rho)}{\bar{\phi}_{2m}(\rho)}\right] S^{(2j-1)}(0)$$

$$+ \sum_{j=0}^{[\frac{m}{2}]} \frac{1}{\rho^{2j+1}}\left[1 - \frac{\bar{\phi}_{2j}(\rho)\bar{\psi}_{2m-2j-1}(\rho)\cosh\frac{\rho}{2}}{\bar{\phi}_{2m}(\rho)}\right] f^{(2j)}(0)$$

$$+ \sum_{j=[\frac{m-1}{2}]+1}^{m-1} \frac{1}{\rho^{2j+1}}\left[1 - \frac{\bar{\phi}_{2j}(\rho)\bar{\psi}_{2m-2j-1}(\rho)\cosh\frac{\rho}{2}}{\bar{\phi}_{2m}(\rho)}\right] S^{(2j)}(0) + Rf$$

where we've written $\bar{\phi}_0(\rho) = 1$ for notational convenience.

We first remark that as a result of Lemma 2 of [11, p. 12] we have the following:

Every $S(x) \in S^0_{2m} \cap F_\gamma$ may be uniquely represented in the form

$$(10.7) \qquad S(x) = \sum_{k=1}^{m-1} a_k S_k(x)$$

for appropriate values of the coefficients a_k.

Then, by (5.22) each such $S(x)$ also satisfies

$$S(x) \in S^0_{2m} \cap L^r_1(R^+) \qquad \text{for some } r = 0, 1, \ldots, 2m-1$$

and Lemma 4 applies. This guarantees the unicity of the interpolating spline of Theorem 3'.

We could follow the same type of procedure we used for Theorems 8 and 10 to show that (10.6) actually is the unique q. f. of Theorem 3, but we shall not. Instead, we note that (10.6) follows formally from Corollary 1 to Theorem 8 by the substitution of $i\rho$ for t where $i^2 = -1$. In particular, since

$$\cosh x = \cos ix, \qquad \sinh x = -i \sin ix$$

we have

$$\psi_n(i\rho) = \bar{\psi}_n(\rho), \qquad \phi_n(i\rho) = \bar{\phi}_n(\rho)$$

formally. Precisely the same type of proofs used for the $\sin xt$ and $\cos xt$ cases establishes the exactness of (10.6) for the functions (5.3), where here we need the sinh and cosh functions instead of the sine and cosine functions, respectively. Where before (see (8.26) and (8.28)) we had

$$|H_\nu| \leq K_1 |e^{it\nu}| + K_2 \mu_1^{-\nu} \qquad \mu_1 > 1,\ K_1, K_2 \text{ constants}$$

now we obtain

$$|H_\nu| \leq K_3 e^{-\nu\rho} + K_4 \mu_1^{-\nu} \qquad \mu_1 > 1,\ K_3, K_4 \text{ constants}$$

so that (24) is satisfied.

IV. EXPRESSIONS FOR THE ERROR

11. <u>An explicit expression for the remainder Rf.</u> In the introduction we mentioned that there was still a third approach to our particular q. f. for the case of odd-degree splines. This way lay through the use of the so-called Rodrigues function $H(x)$ of the Peano kernel of the q. f. [10]. We consider an interval of integration $[a, b]$ and assume n interior nodes x_ν such that

$$a < x_1 < \ldots < x_n < b.$$

Let $w(x) \in L(a, b)$ be a given weight function and let $w^{(-2m)}(x)$, m a fixed integer $\geqq 1$, denote any $2m$-fold integral of $w(x)$. Suppose $I \cup I'$ is defined, <u>not</u> as in (5.15) and (5.16), but as follows: Let I be a subset of $\{0, 1, \ldots, m-1\}$, $I^c = \{0, 1, \ldots, m-1\} - I$ and $I' = \{2m-1-i : i \in I^c\}$. Define $J \cup J'$ similarly. Schoenberg in [10, §7] discusses so-called <u>complete</u> quadrature formulae of the form

$$\int_a^b w(x)f(x)dx = \sum_{\nu=1}^{n} C_\nu f(x_\nu) + \sum_{i \in I \cup I'} A_i f^{(i)}(a) + \sum_{j \in J \cup J'} B_j f^{(j)}(b) + Rf \tag{11.1}$$

where

$$Rf = \int_a^b H(x) f^{(2m)}(x)dx, \tag{11.2}$$

Under suitable conditions on the sets I and J, q. f. of the form (11.1) and (11.2) exist for any choice of weight function $w(x)$. In this event the $H(x)$ of (11.2) is a unique monospline of the form

(11.3) $$H(x) = w^{(-2m)}(x) - S_{2m,n}(x)$$

where $S_{2m,n}(x)$ is a spline function of degree $2m-1$ with the simple knots $x_1, \ldots, x_n$, satisfying

$$H^{(i)}(a) = 0 \qquad \text{if } i \in I \cup I'$$

(11.4) $$H(x_\nu) = 0 \qquad \nu = 1, 2, \ldots, n$$

$$H^{(j)}(b) = 0 \qquad \text{if } j \in J \cup J'.$$

The coefficients C_ν, A_i and B_j are given by

$$C_\nu = H^{(2m-1)}(x_\nu - 0) - H^{(2m-1)}(x_\nu + 0) \quad \nu = 1, 2, \ldots, n$$

(11.5) $$A_i = -(-1)^i H^{(2m-1-i)}(a) \qquad \text{if } i \in I \cup I'$$

$$B_j = (-1)^j H^{(2m-1-j)}(b) \qquad \text{if } j \in J \cup J'.$$

We want a related expression for the interval $[0, \infty)$ and the following choices of weight functions $w_t(x)$ and set $I \cup I'$.

(11.6) $$w_t(x) = \cos xt, \qquad I \cup I' = \{0, 1, 3, 5, \ldots, 2m-3\}$$

(11.7) $$w_t(x) = \sin xt, \qquad I \cup I' = \{0, 2, 4, \ldots, 2m-4, 2m-2\}.$$

To obtain an approximation like (11.3) for $H(x)$, we first consider spline interpolants to $w_t(x)$ on the whole line $\mathbb{R}$. Because

$$\sup_\nu |\Delta^{2m-1} w_t(\nu)| \leq 2^{2m-1} < \infty,$$

by Theorems 1 and 2 of [9, p. 169] we know that there exists a unique cardinal spline function $S_t(x)$ satisfying

(11.8) $$S_t(\nu) = w_t(\nu) \quad \text{for all integers } \nu$$

and

$$(11.9) \qquad S_t(x) \in S_{2m} \cap L_\infty^{2m-1}(\mathbb{R}).$$

On the other hand, since the sequence $\{w_t(\nu)\}$ is bounded, by Theorem 1 of [11] there also exists a unique cardinal spline function $\hat{S}_t(x)$ satisfying

$$(11.10) \qquad \hat{S}_t(\nu) = w_t(\nu) \qquad \text{for all } \nu$$

and

$$(11.11) \qquad \hat{S}_t(x) \in S_{2m} \cap L_\infty(\mathbb{R}).$$

We want to show that $\hat{S}_t(x)$ is the same as $S_t(x)$. From the nature of the data (11.6), (11.7), we know that

(11.12) $S_t(x)$ and $\hat{S}_t(x)$ are even or odd as the sequence $\{w_t(\nu)\}$ is

Let $S(x) = S_t(x) - \hat{S}_t(x)$, so that (11.8) and (11.10) imply that

$$(11.13) \qquad S(\nu) = 0 \qquad \text{for all integers } \nu.$$

Evidently (11.9) and (11.11) require $S(x) \in S_{2m}$ so that we also have $S(x) \in S^0_{2m}$. We wish to show now that $S(x)$ can grow by at most some power of x, so by (11.12) it is sufficient to consider $S(x)$ for $x \geqq 0$. In particular, we can write $S_t(x)$ in the form

$$(11.14) \qquad S_t(x) = a_0 + a_1 x + \ldots + \frac{a_{2m-1}}{(2m-1)!} x^{2m-1} + \frac{1}{(2m-1)!} \sum_{\nu=1}^{\infty} C_\nu (x-\nu)_+^{2m-1}$$

where the coefficients are to be determined. Taking $2m-1$ derivatives in (9.14) gives the relations

$$S_t^{(2m-1)}(x) = a_{2m-1} + \sum_{\nu=1}^{\infty} C_\nu (x-\nu)_+^0$$

or

$$S_t^{(2m-1)}(x) = \begin{cases} a_{2m+1} + \sum_{\nu=1}^{x-1} C_\nu & \text{if } x \text{ is an integer} \\ a_{2m+1} + \sum_{\nu=1}^{[x]} C_\nu & \text{if not.} \end{cases}$$

If the sequence $\{C_\nu\}$ is not bounded, then $S_t^{(2m-1)}(x) \notin L_\infty(\mathbb{R})$ which contradicts (11.9). Hence we have the following necessary condition

(11.15) $$S_t^{(2m-1)}(x) \in L_\infty(\mathbb{R}) \ \underline{\text{implies}} \ \{C_\nu\} \ \underline{\text{is bounded.}}$$

Suppose $|C_\nu| < K$ for $\nu \geqq 1$; then by (11.14) we have

(11.16) $$|S_t(x)| \leqq |a_0 + a_1 x + \ldots + \frac{a_{2m-1}}{(2m-1)!} x^{2m-1}| + \frac{K}{(2m-1)!} \sum_{\nu=1}^{\infty} (x-\nu)_+^{2m-1} .$$

But the sum in (11.16) is bounded by x^{2m-1} so that we get

$$S_t(x) = O(x^{2m-1}) \qquad \text{as } x \to \infty .$$

This, with (11.11), (11.12) and the definition of $S(x)$, implies that

(11.17) $$S(x) = O(|x|^{2m-1}) \qquad \text{as } x \to \pm\infty .$$

A special case of Lemma 2 of Schoenberg's [11, p. 12] states that if $S(x) \in S_{2m}^0$ and satisfies (11.17), then $S(x)$ is identically zero. Thus $\hat{S}_t(x) = S_t(x)$ and we have

(11.18) $$S_t(x) \in \mathcal{S}_{2m} \cap L_\infty(\mathbb{R})$$

In terms of $S_t(x)$, we define

(11.19) $$H_t(x) = \frac{(-1)^m}{t^{2m}} (w_t(x) - S_t(x)).$$

From the form of $w_t(x)$ in (11.6) and (11.7) and by (11.9) and (11.18), we obtain

(11.20) $$H_t(x) \in L_\infty(\mathbb{R}) \cap L_\infty^{2m-1}(\mathbb{R}).$$

We also seek the analog of (11.4) for our half-line case. Indeed, since $S_t(x) \in C^{2m-2}$ and (11.12) holds, we have

(11.21) $$S_t^{(i)}(0) = 0 \qquad \text{if } i \in I \cup I'$$

which implies, by (9.19), that

(11.22) $$H_t^{(i)}(0) = 0 \qquad \text{if } i \in I \cup I'.$$

Evidently, (11.19) also enables us to write

(11.23) $$H_t(\nu) = 0 \qquad \text{for all } \nu.$$

We can state the following

<u>Theorem 10. Suppose</u> $H_t(x)$ <u>is given by</u> (11.19), <u>and</u> $w_t(x)$ <u>and</u> $I \cup I'$ <u>are as in</u> (11.6) <u>and</u> (11.7). <u>If</u>

(11.24) $$f(x) \in C^{2m}(\mathbb{R}^+) \cap L_1^{2m}(\mathbb{R}^+) \cap L_1(\mathbb{R}^+)$$

<u>then</u>

(11.25) $$\int_0^\infty f(x)w_t(x)dx = \sum_{\nu=0}^{\infty} H_{\nu,t}^{(2m)} f(\nu) + \sum_{i \in I \cup I'} B_i f^{(i)}(0) + Rf$$

<u>where</u>

$$(11.26)\qquad Rf = \int_0^\infty H_t(x)\, f^{(2m)}(x)dx\,.$$

<u>Proof.</u> Since (11.20) and (11.24) hold, we write, letting $H(x) = H_t(x)$,

$$(11.27)\qquad \int_0^\infty H_t(x)f^{(2m)}(x)dx = \lim_{b\to\infty} \int_0^{[b]} H_t(x)f^{(2m)}(x)dx.$$

By successive integrations by parts, we find

$$(11.28)\qquad \int_0^{[b]} H(x)f^{(2m)}(x)dx = [Hf^{(2m-1)} - H'f^{(2m-2)} + \dots$$

$$+ (-1)^{2m-2} H^{(2m-2)}f']\Big|_0^{[b]} + (-1)^{2m-1} \int_0^{[b]} H^{(2m-1)}(x)f'(x)dx\,.$$

But $H^{(2m-1)}(x)$ is a step function, so we split up the interval of integration, and find

$$(11.29)\qquad -\int_0^{[b]} H^{(2m-1)}(x)f'(x)dx = -\sum_{\nu=0}^{[b]-1} \int_\nu^{\nu+1} H^{(2m-1)}(x)f'(x)dx$$

$$= -\sum_{\nu=0}^{[b]-1} [H^{(2m-1)}(x)f(x)\Big|_{\nu+0}^{\nu+1-0} - \int_\nu^{\nu+1} f(x)H^{(2m)}(x)dx].$$

We note from (11.19) that we can substitute $w_t(x)$ for $H^{(2m)}(x)$ in (11.29), so that after summing and rearranging, (11.29) becomes

$$(11.30)\qquad -\{-H^{(2m-1)}(0+0)f(0) + \sum_{\nu=1}^{[b]-1} [H^{(2m-1)}(\nu-0) - H^{(2m-1)}(\nu+0)]f(\nu)$$

$$+ H^{(2m-1)}([b]-0)\, f([b])\} + \int_0^{[b]} f(x)w_t(x)dx\,.$$

If we substitute (11.30) for (11.29) in (11.28) we find that (11.28) upon evaluation of the term in brackets, becomes

$$(11.31)\quad \int_0^{[b]} H(x)f^{(2m)}(x)dx = \sum_{\nu=0}^{2m-2} (-1)^\nu H^{(\nu)}([b])f^{(2m-1-\nu)}([b])$$

$$- \sum_{\nu=0}^{2m-2} (-1)^\nu H^{(\nu)}(0)f^{(2m-1-\nu)}(0)$$

$$+ (-1)\{-H^{(2m-1)}(0+0)f(0) + \sum_{\nu=1}^{[b]-1} [H^{(2m-1)}(\nu-0)-H^{(2m-1)}(\nu+0)]f(\nu)$$

$$+ H^{(2m-1)}([b]-0)f([b])\} + \int_0^{[b]} f(x)w_t(x)dx .$$

By (11.31), then, (11.20) and (11.24) give us, on letting $b \to \infty$, that

$$(11.32)\quad \int_0^{\infty} H(x)f^{(2m)}(x)dx = - \sum_{\nu=0}^{2m-2} (-1)^\nu H^{(\nu)}(0)f^{(2m-1-\nu)}(0)$$

$$+ H^{(2m-1)}(0+0)f(0) - \sum_{\nu=1}^{\infty} [H^{(2m-1)}(\nu-0)-H^{(2m-1)}(\nu+0)]f(\nu)$$

$$+ \int_0^{\infty} f(x)w_t(x)dx .$$

If we use (11.22) in (11.32) and solve for $\int f(x)w_t(x)dx$, we obtain

$$(11.33)\quad \int_0^{\infty} f(x)w_t(x)dx = \sum_{\nu \notin I \cup I'} (-1)^\nu H^{(\nu)}(0)f^{(2m-1-\nu)}(0)$$

$$- H^{(2m-1)}(0+0)f(0) + \sum_{\nu=1}^{\infty} [H^{(2m-1)}(\nu-0)-H^{(2m-1)}(\nu+0)]f(\nu)$$

$$+ \int_0^{\infty} H(x)f^{(2m)}(x)dx .$$

Upon interchanging the order of summation in the first sum on the right of (11.33), we get

$$(11.34)\qquad \int_0^\infty f(x)w_t(x)dx = -H^{(2m-1)}(0+0)f(0) - \sum_{i\epsilon I \cup I'} (-1)^i H^{(2m-1-i)}(0)f^{(i)}(0)$$

$$+ \sum_{\nu=1}^{\infty} [H^{(2m-1)}(\nu-0)-H^{(2m-1)}(\nu+0)]f(\nu) + \int_0^\infty H(x)f^{(2m)}(x)dx.$$

If we define

$$H_{0,t}^{(2m)} = -H^{(2m-1)}(0+0)$$

$$(11.35)\qquad H_{\nu,t}^{(2m)} = H^{(2m-1)}(\nu-0) - H^{(2m-1)}(\nu+0) \qquad \nu = 1,2,\ldots$$

$$B_{i,t}^{(2m)} = -(-1)^i H^{(2m-1-i)}(0) \qquad i \epsilon I \cup I'$$

we obtain the desired form for our q. f. (11.25) and (11.26) and so establish the theorem.

12. <u>The remainder Rf for the cosine transform (5.51).</u> We specialize to the case (11.6) and now want to establish that the q. f. in Theorem 10 is the same as the q. f. in Theor⁻ ɹ 7. We do this by examining the form that the function $H_t(x)$ as defined by (11.19) must take. Enforcing the requirements (11.22) and (11.23) will lead us by a generating function approach to tne same coefficients $H_{\nu,t}^{(2m)}$, $B_{2j-1,t}^{(2m)}$ of Theorem 7.

We attempt to find an expression for the cardinal spline function

$S_t(x)$ satisfying (11.8) and (11.9). Since we need only consider the half-line R^+, $S_t(x)$ can be written in the form (11.14) where the coefficients a_i, C_ν are to be determined. We want to express the a_i, C_ν in terms of $H^{(2m)}_{\nu,t} = H_\nu$ and $B^{(2m)}_{i,t} = B_i$. From the definition of $H_t(x)$ (11.19) or from

$$(12.1) \qquad (-1)^m t^{2m} H_t(x) = \cos xt - S_t(x)$$

we find by differentiating $2m-1$ times and employing (11.29) that

$$(12.2) \qquad C_\nu = (-1)^m t^{2m} H_\nu, \quad a_{2m-1} = (-1)^m t^{2m} H_0 .$$

If we differentiate (12.1) $(2m-1-i)$ times and use (11.29), we get

$$(-1)^m t^{2m} B_i = (-1)^{\frac{2m-1-i}{2}} t^{2m-1-i} - a_{2m-1-i} \qquad i \in I \cup I'$$

or solving for a_{2m-1-i} that

$$(12.3) \qquad a_{2m-1-i} = (-1)^{\frac{2m-1-i}{2}} t^{2m-1-i} - (-1)^m t^{2m} B_i \qquad i \in I \cup I'.$$

Enforcing (11.22) in (12.1) gives us

$$(12.4) \qquad a_i = 0 \quad \text{if } i \in I \cup I'$$

so that by employing (12.2), (12.3) and (12.4) we may rewrite (11.14) as

$$(12.5) \qquad S_t(x) = 1 + \frac{[-t^2-(-1)^m t^{2m} B_{2m-3}]}{2!} x^2 + \frac{[t^4-(-1)^m t^{2m} B_{2m-5}]}{4!} x^4 +$$

$$\ldots + [(-1)^{m-1} t^{2m-2} - (-1)^m t^{2m} B_1] \frac{x^{2m-2}}{(2m-2)!}$$

$$+\frac{[(-1)^m t^{2m} H_0]}{(2m-1)!} x^{2m-1} + \frac{(-1)^m t^{2m}}{(2m-1)!} \sum_{\nu=1}^{\infty} H_\nu (x-\nu)_+^{2m-1} .$$

By virtue of (12.2) and (11.15) we have the following necessary condition:

(12.6) $\quad S_t^{(2m-1)}(x) \in L_\infty(R)$ <u>implies</u> $\{H_\nu\}$ <u>is bounded.</u>

If we solve (12.1) for $\cos xt$ and then require (11.23) for k a positive integer, we obtain the sequence of relations

$$\text{(12.7)} \quad \cos kt = 1 + \sum_{j=1}^{m-1} [(-1)^j t^{2j} - (-1)^m t^{2m} B_{2m-1-2j}] \frac{k^{2j}}{(2j)!}$$

$$+ \frac{(-1)^m t^{2m}}{(2m-1)!} \sum_{\nu=0}^{\infty} H_\nu (k-\nu)_+^{2m-1} \qquad (k = 1, 2, \ldots).$$

The form of these relations suggests the use of generating functions for the determination of the coefficients. The righthand side of (12.7) in view of Lemma 3 is equal to the coefficient of x^{k-1} in

$$\text{(12.8)} \quad \frac{1}{1-x} + \sum_{j=1}^{m-1} \frac{[(-1)^j t^{2j} - (-1)^m t^{2m} B_{2m-1-2j}]}{(2j)!} \frac{P_{2j-1}(x)}{(1-x)^{2j+1}}$$

$$+ \frac{(-1)^m t^{2m}}{(2m-1)!} \left(\sum_{\nu=0}^{\infty} H_\nu x^\nu \right) \frac{P_{2m-2}(x)}{(1-x)^{2m}} .$$

The lefthand side of (12.7) is the coefficient of x^{k-1} in what we called $\tau(x)$, defined in (4.39). Equating $\tau(x)$ and the expression in (12.8) and then solving for $\Sigma H_\nu x^\nu$ gives

$$(12.9)\quad \sum_{\nu=0}^{\infty} H_\nu x^\nu = \frac{(2m-1)!}{P_{2m-2}(x)} \Bigg\{ \frac{(-1)^m}{t^{2m}} \Big[(1-x)^{2m} \tau(x) - (1-x)^{2m-1}$$

$$- \sum_{j=1}^{m-1} \frac{(-1)^j t^{2j}}{(2j)!} P_{2j-1}(x)(1-x)^{2m-1-2j}\Big]$$

$$+ \sum_{j=1}^{m-1} \frac{B_{2m-1-2j}}{(2j)!} P_{2j-1}(x)(1-x)^{2m-1-2j} \Bigg\}.$$

A change in the order of summation in the last sum on the right side of (12.9) leads to the final relation

$$(12.10)\quad \sum_{\nu=0}^{\infty} H_\nu x^\nu = \frac{(2m-1)!}{P_{2m-2}(x)} \Bigg\{ \frac{(-1)^m}{t^{2m}} \Big[(x-1)^{2m} \tau(x) + (x-1)^{2m-1}$$

$$+ \sum_{j=1}^{m-1} (-1)^j \frac{t^{2j}}{(2j)!} P_{2j-1}(x)(x-1)^{2m-1-2j}\Big]$$

$$- \sum_{j=1}^{m-1} B_{2j-1} \frac{(x-1)^{2j-1} P_{2m-1-2j}(x)}{(2m-2j)!} \Bigg\}.$$

This is precisely the same relation as (5.1)! The analysis in §5 led to a unique choice of the sequence $\{H_\nu\}$ under the stipulation that this sequence be bounded. By (12.6) and the existence and unicity of an $S_t(x)$ satisfying (11.8) and (11.9), we conclude that the H_ν and B_j as determined in §5 are the required coefficients for $S_t(x)$ as given in (12.5). So this approach through the use of the function $H_t(x)$ leads to precisely the same q. f. as that of Theorem 7, and in particular leads to an expression for the remainder Rf in Theorem 7. We have thereby established

Theorem 11. The remainder Rf in the q. f. (5. 51) of Theorem 7 under the stipulation (5. 52) for

$$f(x) \in C^{2m}(R^+) \cap L_1^{2m}(R^+) \cap L_1(R^+) \tag{12.11}$$

is given by

$$Rf = \int_0^\infty H_t(x) f^{(2m)}(x)dx = \frac{(-1)^m}{t^{2m}} \int_0^\infty [\cos xt - S_t(x)] f^{(2m)}(x)dx \tag{12.12}$$

where $S_t(x)$ is the unique, bounded $(2m-1)$st degree cardinal spline interpolating $\cos xt$ at the integers.

13. A bound on the remainder Rf of (5. 51). We now examine the problem of expressing the above cosine transform (5. 51) in steps of length h. Let $f(x) \in S_{2m} \cap L_1(R^+)$ so that by Theorem 7 and (5. 42) we can write

$$\int_0^\infty f(x)\cos xt\, dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)} \left\{ \frac{1}{2} f(0) + \sum_{\nu=0}^\infty f(\nu)\cos \nu t \right\} + \sum_{j=1}^{m-1} \frac{(-1)^j}{t^{2j}} \left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j}(t)}{\phi_{2m}(t)}\right] f^{(2j-1)}(0). \tag{13.1}$$

Let now $F(x)$ be a $(2m-1)$st degree spline function in $(0, \infty)$ having its knots in $x = h, 2h, 3h, \ldots$ where $h > 0$. We want to express the cosine transform in terms of the values

$$F'(0), F'''(0), \ldots, F^{(2m-3)}(0), F(0), F(h), F(2h), \ldots \; . \tag{13.2}$$

If we let

$$f(x) = F(xh) \qquad (0 \leqq x < \infty) \tag{13.3}$$

we see that $f(x)$ is a semi-cardinal $(2m-1)$st degree spline function for which the data are

$$(13.4) \qquad f'(0) = hF'(0), \ldots, f^{(2m-3)}(0) = h^{2m-3}F^{(2m-3)}(0),$$
$$f(0) = F(0),\ f(1) = F(h),\ \ldots .$$

From (13.1) we therefore obtain the relation

$$(13.5) \qquad \int_0^\infty F(xh)\cos xt\, dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)} \cdot \left\{ \frac{1}{2}F(0) + \sum_{\nu=1}^{\infty} F(\nu h)\cos \nu t \right\}$$
$$+ \sum_{j=1}^{m-1} \frac{(-1)^j}{t^{2j}} \left[1 - \frac{\phi_{2j}(t)\psi_{2m-2j}(t)}{\phi_{2m}(t)}\right] h^{2j-1} F^{(2j-1)}(0).$$

Replacing in the integral x by x/h and replacing afterwards in this identity t by th, we

$$(13.6) \qquad \int_0^\infty F(x)\cos xt\, dx = \frac{\psi_{2m}(th)}{\phi_{2m}(th)}\, h \left\{ \frac{1}{2}F(0) + \sum_{\nu=0}^{\infty} F(\nu h)\cos \nu th \right\}$$
$$+ \sum_{j=1}^{m-1} \frac{(-1)^j}{t^{2j}} \left[1 - \frac{\phi_{2j}(th)\psi_{2m-2j}(th)}{\phi_{2m}(th)}\right] F^{(2j-1)}(0).$$

Suppose

$$(13.7) \qquad F(x) \in C^{2m} \cap L_1^{2m}(R^+) \cap L_1(R^+)$$

then (13.6) is no longer an identity. However, the righthand side will give us the desired approximation to the cosine transform of $F(x)$ for reasonably small h. We now want to see how good an approximation this is.

(13.5) is still valid if we use (12.15) and (13.3) and add the term

$$(13.8) \qquad \frac{(-1)^m}{t^{2m}} h^{2m} \int_0^\infty [\cos xt - S_t(x)] F^{(2m)}(xh)\,dx$$

to the right side of (13.5). If we denote (13.8) by R and again first replace x by x/h and then t by th, we obtain the expression (13.6) with the added term

$$(13.9) \qquad R = \frac{(-1)^m}{t^{2m}} \int_0^\infty [\cos xt - S_{th}(x/h)] F^{(2m)}(x)\,dx$$

on the right side. Here $S_{th}(x/h)$ is the unique, bounded $(2m-1)$st degree spline interpolating $\cos xt$ at νh for all integers ν. We get a bound on R by bounding what we shall call

$$(13.10) \qquad M(t,h) = \max_{x \geqq 0} \left| \frac{1}{t^{2m}} [\cos xt - S_{th}(x/h)] \right|.$$

Let $z = \frac{x}{2\pi}$ so that $S_{th}(\frac{2\pi z}{h})$ is the unique, bounded $(2m-1)$st degree spline interpolating $\cos 2\pi zt$ for $z = 0, \pm\frac{h}{2\pi}, \pm\frac{2h}{2\pi}, \ldots$

We also let h be of the form

$$(13.11) \qquad h = \frac{2\pi}{n} \quad \text{for } n \text{ a natural number}$$

so that the spline agrees with $\cos 2\pi zt$ for $z = 0, \pm\frac{1}{n}, \pm\frac{2}{n}, \ldots$

If we consider $t = 1, 2, \ldots, n-1$, then $\cos 2\pi zt$ is periodic on the interval $[0,1]$ and so is $S_{th}(2\pi z/h)$, by Theorem 6 of [11]. We require a special case of Lemma 6.3 of Golomb's [3] which we state as

Lemma 6 (Golomb). *Let* $b_s(u)$ *denote the unique bounded* $(2m-1)$st *degree spline that interpolates the function* $e^{2\pi isu}$ *at* k/n $(k = 0, \pm 1, \pm 2, \ldots)$. *Then*

$$(13.12) \qquad |e^{2\pi isu} - b_s(u)| \leqq 4 \cdot 2^{2m} s^{2m} n^{-2m} \qquad s = 0, \pm 1, \ldots, \pm(n-1).$$

In [3, p. 13] Golomb remarks that $\operatorname{Re} b_s(u)$ and $\operatorname{Im} b_s(u)$ are the corresponding spline interpolants to $\cos 2\pi su$ and $\sin 2\pi su$, respectively, so that we also have

$$(13.13) \qquad |\cos 2\pi su - \operatorname{Re} b_s(u)| \leqq 4 \cdot 2^{2m} s^{2m} n^{-2m} \qquad s = 0, \pm 1, \ldots, \pm(n-1)$$

where $\operatorname{Re} b_s(u)$ is the unique, bounded $(2m-1)$st degree spline that interpolates $\cos 2\pi su$ at the points k/n $(k = 0, \pm 1, \pm 2, \ldots)$.

We can therefore employ (13.13) to get the bound

$$(13.14) \qquad \max_{z \geqq 0} |\cos 2\pi tz - S_{th}(2\pi z/h)| \leqq 4 \cdot 2^{2m} t^{2m} n^{-2m}$$

$$t = 1, 2, \ldots, n-1.$$

Combining (13.14) and (13.11) with (13.10) and the definition that $z = x/2\pi$, we obtain

$$(13.15) \qquad M(t,h) = \max_{z \geqq 0} \left| \frac{1}{t^{2m}} [\cos 2\pi tz - S_{th}(2\pi z/h)] \right| \leqq 4 \frac{h^{2m}}{\pi^{2m}}$$

$$t = 1, 2, \ldots, n-1.$$

We now consider t in the form

$$(13.16) \qquad t = p/q \qquad \text{where } p, q \in Z \text{ and } p = 1, 2, \ldots, n-1.$$

Now let $w = z/q$ and we get from the equality in (13.15) that

$$(13.17)\qquad M(t,h) = \max_{w \geqq 0} \left|\frac{1}{t^{2m}}\left[\cos 2\pi pw - S_{th}\left(\frac{2\pi qw}{h}\right)\right]\right|$$

where $S_{th}(\frac{2\pi qw}{h})$ interpolates $\cos 2\pi pw$ for $w = 0, \pm h/2\pi q, \pm 2h/2\pi q, \ldots$ If we let h be of the form

$$(13.18)\qquad h = 2\pi q/h \qquad \text{for } n \text{ a natural number}$$

and consider $p = 1, 2, \ldots, n-1$, then $S_{th}(\frac{2\pi qw}{h})$ is periodic for $0 \leqq w \leqq 1$. By (13.13) then we have

$$(13.19)\qquad \max_{w \geqq 0} \left|\cos 2\pi pw - S_{th}\left(\frac{2\pi qw}{h}\right)\right| \leqq 4 \cdot 2^{2m} p^{2m} n^{-2m}$$

$$(p = 1, \ldots, n-1).$$

Substituting (13.19) in (13.17) in view of (13.16) and (13.18) we find

$$(13.20)\qquad M(t,h) \leqq 4\frac{h^{2m}}{\pi^{2m}} \qquad \left(t = \frac{1}{q}, \frac{2}{q}, \ldots, \frac{n-1}{q}\right).$$

Suppose now we fix h of the form

$$(13.21)\qquad h = \frac{2\pi}{N} \qquad \text{for } N \text{ a natural number.}$$

We let q be any positive integer and choose n such that

$$(13.22)\qquad \frac{q}{n} = \frac{1}{N}.$$

Then (13.22), (13.18) and (13.20) enable us to show that

$$(13.23)\qquad M(t,h) \leqq 4\frac{h^{2m}}{\pi^{2m}} \qquad \left(t = \frac{1}{q}, \frac{2}{q}, \ldots, N - \frac{1}{q}\right).$$

By (13.19), (13.10), (13.21) and (13.23), then, we have the bound

$$(13.24)\qquad |R| \leqq M(t,h)\|F^{(2m)}\|_{L_1(\mathbb{R}^+)} \leqq 4\frac{h^{2m}}{\pi^{2m}}\|F^{(2m)}\|_{L_1(\mathbb{R}^+)}$$

$$\left(t = \frac{1}{q}, \frac{2}{q}, \ldots, \frac{2\pi}{h} - \frac{1}{q}\right)$$

and the following

<u>Theorem 12. Suppose</u> $f \in C^{2m}(\mathbb{R}^+) \cap L_1(\mathbb{R}^+) \cap L_1^{2m}(\mathbb{R}^+)$ <u>and</u> $\frac{2\pi}{h}$ <u>is a natural number. Then we can bound</u> $|Rf|$ <u>as given in the q. f.</u>

$$\int_0^\infty f(x)\cos xt\,dt = \frac{\psi_{2m}(th)}{\phi_{2m}(th)}\, h\left\{\frac{1}{2}f(0) + \sum_{\nu=1}^{\infty} f(\nu h)\cos \nu th\right\} \tag{13.25}$$

$$+ \sum_{j=1}^{m-1} \frac{(-1)^j}{t^{2j}}\left[1 - \frac{\phi_{2j}(th)\,\psi_{2m-2j}(th)}{\phi_{2m}(th)}\right] f^{(2j-1)}(0) + Rf$$

by

$$|Rf| \leq 4\left(\frac{h}{\pi}\right)^{2m} \|f^{(2m)}\|_{L_1(\mathbb{R}^+)} \quad \text{for all rational } t \text{ in } \left(0, \frac{2\pi}{h}\right). \tag{13.26}$$

14. <u>Proof of Theorem 4 of §2</u>. We also have an analogous theorem to Theorem 11 for $\sin xt$, which we shall only state as

<u>Theorem 13. The remainder</u> Rf <u>in the q. f.</u>

$$\int_0^\infty f(x)\sin xt\,dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)}\left\{\sum_{\nu=1}^{\infty} f(\nu)\sin \nu t\right\} \tag{14.1}$$

$$+ \sum_{j=0}^{m-1} \frac{(-1)^j}{t^{2j+1}}\left[1 - \frac{\phi_{2j}(t)\,\psi_{2m-2j-2}(t)}{\phi_{2m}(t)}\,\frac{\sin t}{t}\right] f^{(2j)}(0) + Rf$$

<u>for</u> $f \in C^{2m}(\mathbb{R}^+) \cap L_1^{(2m)}(\mathbb{R}^+) \cap L_1(\mathbb{R}^+)$ <u>is given by</u>

$$Rf = \frac{(-1)^m}{t^{2m}} \int_0^\infty [\sin xt - S_{t,s}(x)]\, f^{(2m)}(x)\,dx \tag{14.2}$$

<u>where</u> $S_{t,s}(x)$ <u>is the unique, bounded</u> $(2m-1)$st <u>degree cardinal spline interpolating</u> $\sin xt$ <u>at the integers.</u>

Let $S_{t,c}(x)$ denote the spline $S_t(x)$ of Theorem 11 interpolating $\cos xt$.

Toward a proof of Theorem 4, let us assume that

$$f \in C^{2m} \cap L_1^{2m}(\mathbb{R}) \cap L_1(\mathbb{R}). \tag{14.3}$$

Then we can write

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)e^{ixt}dx &= \int_{-\infty}^{\infty} f(x)\cos xt\,dx + i\int_{-\infty}^{\infty} f(x)\sin xt\,dx \\ &= \int_{-\infty}^{\infty} [f(x) + f(-x)]\cos xt\,dx + i\int_{0}^{\infty} [f(x) - f(-x)]\sin xt\,dx. \end{aligned} \tag{14.4}$$

Applying (13.25) for $h = 1$ and (14.1) to $f(x) + f(-x)$ and $f(x) - f(-x)$, respectively, we find that (14.4) becomes

$$\begin{aligned} &\frac{\psi_{2m}(t)}{\phi_{2m}(t)}\Big[\sum_{\nu=-\infty}^{\infty} f(\nu)\cos \nu t\Big] + \sum_{j=1}^{m-1} \frac{(-1)^j}{(t^{2j})}\Big[1 - \frac{\phi_{2j}(t)\,\psi_{2m-2j}(t)}{\phi_{2m}(t)}\Big]\cdot \\ &\quad\cdot [f^{(2j-1)}(0^+) - f^{(2j-1)}(0^-)] + R_c f \\ &+ i\left\{\frac{\psi_{2m}(t)}{\phi_{2m}(t)}\Big[\sum_{\nu=1}^{\infty} f(\nu)\sin \nu t + \sum_{\nu=-1}^{-\infty} f(\nu)\sin \nu t\Big]\right. \\ &\left.+\sum_{j=0}^{m-1} \frac{(-1)^j}{t^{2j+1}}\Big[1 - \frac{\phi_{2j}(t)\,\psi_{2m-2j-2}(t)}{\phi_{2m}(t)}\,\frac{\sin t}{t}\Big][f^{(2j)}(0^+) - f^{(2j)}(0^-)] \quad R_s f\right\} \end{aligned} \tag{14.5}$$

where

$$R_c f = \int_0^{\infty} \frac{(-1)^m}{t^{2m}}[\cos xt - S_{t,c}(x)][f^{(2m)}(x) + f^{(2m)}(-x)]dx \tag{14.6}$$

$$R_s f = \int_0^{\infty} \frac{(-1)^m}{t^{2m}}[\sin xt - S_{t,s}(x)][f^{(2m)}(x) - f^{(2m)}(-x)]dx.$$

We've taken $\phi_0(t) = 1$ for convenience of notation. By (14.3) the derivative terms in (14.5) are zero so that substituting (14.5) in (14.4) gives

$$(14.7) \qquad \int_{-\infty}^{\infty} f(x)e^{ixt}dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)} \sum_{\nu=-\infty}^{\infty} f(\nu)e^{i\nu t} + R_c f + iR_s f.$$

From (14.6)

$$R_c f = \int_0^{\infty} \frac{(-1)^m}{t^{2m}} [\cos xt - S_{t,c}(x)]f^{(2m)}(x)dx$$

$$+ \int_0^{-\infty} \frac{(-1)^m}{t^{2m}} [\cos(-ty) - S_{t,c}(-y)]f^{(2m)}(y)(-dy)$$

where we've used $y = -x$ to get the second integral. But $S_{t,c}(-y) = S_{t,c}(y)$ so that

$$(14.8) \qquad R_c f = \int_{-\infty}^{\infty} \frac{(-1)^m}{t^{2m}} [\cos xt - S_{t,c}(x)]f^{(2m)}(x)dx.$$

We can argue the same way, using the fact that $S_{t,s}(-y) = -S_{t,s}(y)$ and obtain

$$(14.9) \qquad R_s f = \int_{-\infty}^{\infty} \frac{(-1)^m}{t^{2m}} [\sin xt - S_{t,s}(x)] f^{(2m)}(x)dx.$$

By (14.7), (14.8) and (14.9), the remainder Rf of Theorem 4 therefore has the form

$$(14.10) \quad Rf = R_c f + iR_s f = \int_{-\infty}^{\infty} \frac{(-1)^m}{t^{2m}} [e^{ixt} - (S_{t,c}(x) + iS_{t,s}(x))]f^{(2m)}(x)dx.$$

We note that $S_{t,c}(x) + iS_{t,s}(x)$ is the unique, bounded (2m-1)st degree cardinal spline that interpolates e^{ixt} at the integers. By precisely the same argument that led to the bound (13.26) of Theorem 10 except that now we use (13.12) of Lemma 6 where before we used (13.13), we reach the desired bound (2.17) of Theorem 4.

In the foregoing proof in (14.10) we determined the explicit form of Rf, a result we state as a

Corollary. *For* $f(x) \in C^{2m} \cap L_1^{2m}(R) \cap L_1(R)$, *the remainder in the q.f.*

$$\int_{-\infty}^{\infty} f(x)e^{ixt}dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)} \sum_{\nu=-\infty}^{\infty} f(\nu)e^{i\nu t} + Rf \tag{14.11}$$

is given by

$$Rf = \frac{(-1)^m}{t^{2m}} \int_{-\infty}^{\infty} [e^{ixt} - b_t(x)]\, f^{(2m)}(x)dx \tag{14.12}$$

where $b_t(x)$ *is the unique, bounded cardinal spline interpolant of the function* e^{ixt} *at the integers.*

15. An explicit expression for the remainder Rf in Theorem 3. The same type of approach we took in §11, 12 will lead us to an explicit form for the remainder Rf in Theorem 3. For our case of the weight function $e^{-x\rho}$ and $I \cup I' = \{0, 1, \ldots, m-1\}$ we require the unique semi-cardinal spline $S_\rho(x) \in S_{2m} \cap L_1(R^+)$ satisfying the conditions

$$S_\rho(\nu) = e^{-\nu\rho} \qquad (\nu = 1, 2, \ldots) \tag{15.1}$$

and

$$S_\rho^{(j)}(0) = (-\rho)^j \qquad j \in I \cup I'. \tag{15.2}$$

Here (15.2) reflects the fact that the first $m-1$ derivatives of $e^{-x\rho}$ and $S_\rho(x)$ agree. In terms of $S_\rho(x)$ we can state the following

<u>Theorem 14.</u> <u>Let</u> $f(x) \in C^{2m}$ <u>and</u> $f^{(j)}(x) \in F_{\gamma_j}$ <u>for some</u> γ_j, $j = 0, 1, \ldots, 2m-1$. <u>Then the remainder</u> Rf <u>of Theorem 3 may be expressed as</u>

$$Rf = \int_0^\infty H_\rho(x) f^{(2m)}(x)dx \tag{15.3}$$

where

$$H_\rho(x) = \frac{1}{\rho^{2m}} [e^{-x\rho} - S_\rho(x)]. \tag{15.4}$$

In order to obtain the appropriate form (23) of Theorem 3 we want an analog of Theorem 10 to hold. By (15.1) and (15.2), we find

$$H_\rho^{(j)}(0) = 0 \qquad \text{if } j \in I \cup I' \tag{15.5}$$

and

$$H_\rho(\nu) = 0 \qquad (\nu = 1, 2, \ldots) \tag{15.6}$$

so that an appropriate analog can be proved in the same way that Theorem 10 was proved if we can show

$$\lim_{b \to \infty} H_\rho^{(\alpha)}(b) f^{(\beta)}(b) = 0 \qquad \text{if } \alpha + \beta = 2m-1. \tag{15.7}$$

By the assumptions of this theorem and (15.4), this amounts to showing

$$S_\rho^{(k)}(x) \quad \text{for } k=0, 1, \ldots, 2m-1 \text{ is of exponential decay as } x \to \infty. \tag{15.8}$$

Just as in §8, we shall discuss only the cubic case $m = 2$; the extension to higher m is very similar. By using (8.6) we can write $S_\rho(x)$ as

$$(15.9) \qquad S_\rho(x) = \sum_{\nu=0}^{\infty} e^{-\nu\rho} L_\nu(x) - \rho\Lambda(x)$$

where $L_\nu(x)$ and $\Lambda(x)$ are the fundamental functions discussed in §8. By using (8.15), (8.21) and (8.23) in (15.9), we obtain

$$(15.10) \qquad S_\rho(x) = L(x) + \sum_{\nu=1}^{\infty} e^{-\nu\rho} L(x-\nu) + \sum_{\nu=1}^{\infty} e^{-\nu\rho}(\sqrt{3}\,\lambda^\nu) S_1(x) - \frac{\rho}{\sqrt{3}} S_1(x).$$

But for the special case $m = 2$ of Theorem 2 of [11, p. 5], we have

$$(15.11) \qquad |L(x)| < Ce^{-\gamma_*|x|} \qquad \text{for all real } x,$$

where C and γ_* are positive constants. This, in conjunction with (8.13), and the expression (15.10) guarantees that $S_\rho(x)$ decays exponentially as $x \to \infty$.

Similarly, by differentiating each side of (8.10) and (8.12) three times, we obtain

$$(15.12) \qquad L'''(x) = -6\sqrt{3}\,(\lambda-1)\lambda^k \qquad \text{if } k < x < k+1, \quad k \geqq 1$$

and

$$(15.13) \qquad S_1'''(x) = -6(\lambda-1)\lambda^k \qquad \text{if } k < x < k+1$$

so that $S_\rho'''(x)$ as determined from (15.10) also decays exponentially as $x \to \infty$. Therefore (15.8) holds and we obtain the desired analog to Theorem 10.

Now we employ the same procedure used in §12 to show that the q.f. we've obtained is indeed the same q.f. as given by Theorem 3. The approach in §12 leads us to the relation (9.13) so we can continue as in the analysis of §9 to finally obtain our desired q.f. and in this way a proof of Theorem 14. The actual procedure is too similar to repeat.

16. Computational examples. In [2], Einarsson compares several methods for computing cosine transforms for the special case of $f(x) = e^{-x}$. One method he uses and the reason for the paper is based on the approximation of $f(x)$ by its cubic spline approximation. This q. f., precisely the same one as (13.25) for $m = 2$, is

$$(16.1)\quad \int_0^\infty f(x)\cos xt\,dx = \frac{\psi_4(th)}{\phi_4(th)}\,h\left\{\frac{1}{2}f(0) + \sum_{\nu=1}^{\infty} f(\nu h)\cos \nu th\right\}$$

$$-\frac{1}{t^2}\left[1 - \frac{\psi_2(th)}{\phi_4(th)}\right]f'(0) + Rf$$

where we've used $\phi_2(t) = 1$. Einarsson's main conclusion is that this spline q. f. is superior to Filon's formula, a q. f. based on approximation of the function by a quadratic in each double interval and one of the most used formulae for the calculation of Fourier integrals. One of the other methods Einarsson uses for comparison is the so-called Filon-Trapezoidal rule found in Tuck [14], which for the interval $(0, \infty)$ is merely (13.25) for $m=1$, that is, the linear spline case.

Einarsson's calculations indicate that for small values of t, the q. f. (16.1) gives a relative error that is four times less than the Filon formula. For large values of t, the relative error of the Filon formula increases rapidly, while the spline method (16.1) gives a surprisingly small error growth. This same phenomenon we found to be the case for the following q. f., obtained from (13.25) and (8.3), respectively, for m=3

$$(16.2)\qquad \int_0^\infty f(x)\cos xt\,dx = \frac{\psi_6(th)}{\phi_6(th)}\,h\left\{\frac{1}{2}f(0) + \sum_{\nu=1}^{\infty} f(\nu h)\cos \nu th\right\}$$

$$+\frac{1}{t^2}\left[1-\frac{\psi_4(th)}{\phi_6(th)}\right]f'(0) + \frac{1}{t^4}\left[1-\frac{\phi_4(th)\,\psi_2(th)}{\phi_6(th)}\right]f'''(0) + Rf,$$

$$(16.3)\qquad \int_0^\infty f(x)\cos xt\,dx = \frac{\psi_6(th)}{\phi_6(th)}\,h\left\{\frac{1}{2}f(0) + \sum_{\nu=1}^{\infty} f(\nu h)\cos \nu th\right\}$$

$$-\frac{1}{t^2}\left[1-\frac{\psi_4(th)}{\phi_6(th)}\right]f'(0) + \frac{1}{t^4}\left[1-\frac{\phi_4(th)\,\psi_2(th)}{\phi_6(th)}\right]S'''(0) + Rf$$

two q. f. corresponding to spline approximation, the first using $I \cup I' = \{0, 1, 3\}$ and the second using $I \cup I' = \{0, 1, 2\}$.

We now consider the absolute error and concern ourselves with the q. f. (16.1), (16.2) and (16.3) and the bounds we obtained for the error in (16.1) and (16.2) for two examples. We first remark that as the step h gets small, it appears that $S'''(0)$ of (16.3) approaches $f'''(0)$ of (16.2) so that the difference in these approximations becomes very small. An instance of this we indicate below.

In Figures 1-6 the absolute values of the absolute error for the calculation of the cosine transform with (16.1) and one of (16.2) or (16.3) is given as a function of t for the stepsize h at 80 different places from $t = .25$ to $t = 50$. The cubic curve is represented by x's and the quintic curve for (16.2) or (16.3) is solid. The dips are at points where the absolute error changes sign. We also point out that each axis is scaled logarithmically and a lower bound for the error in each graph is 2×10^{-9}. Along the vertical axis we indicate by a 3 or a 5 where the computed error bounds of (13.26) fall.

In Figure 1, we consider

$$\int_0^\infty e^{-x} \cos xt \, dx = \frac{1}{1+t^2}$$

and the stepsize $h = \frac{2\pi}{32} \approx .2$. Here $S'''(0) = -.99994$ versus $f'''(0) = -1$ and since the corresponding graphs arising from (16.2) and (16.3) were virtually indistinguishable we only need consider one, (16.2). We also compute by (13.26) the bounds on $|Rf|$ and find that

$$(16.4) \qquad |Rf|_3 \le 6.1 \times 10^{-5}, \quad |Rf|_5 \le 2.4 \times 10^{-7} \text{ for all rational } t \text{ in } (0, 32)$$

where the subscript indicates the case (16.1) or (16.2), respectively. We note that the bound for $|Rf|_5$ is actually less than the computed transform corresponding to the cubic case.

In Figures 2 and 3 we consider

$$(16.5) \qquad \int_0^\infty \frac{1}{1+x^2} \cos xt \, dx = \frac{\pi}{2} e^{-t}$$

for $h = \frac{2\pi}{16} \approx .4$ to examine the difference between the q.f. (16.2) and (16.3). We've plotted the cubic case (16.1) also to serve as a reference.

ABSOLUTE VALUE OF ABSOLUTE ERROR

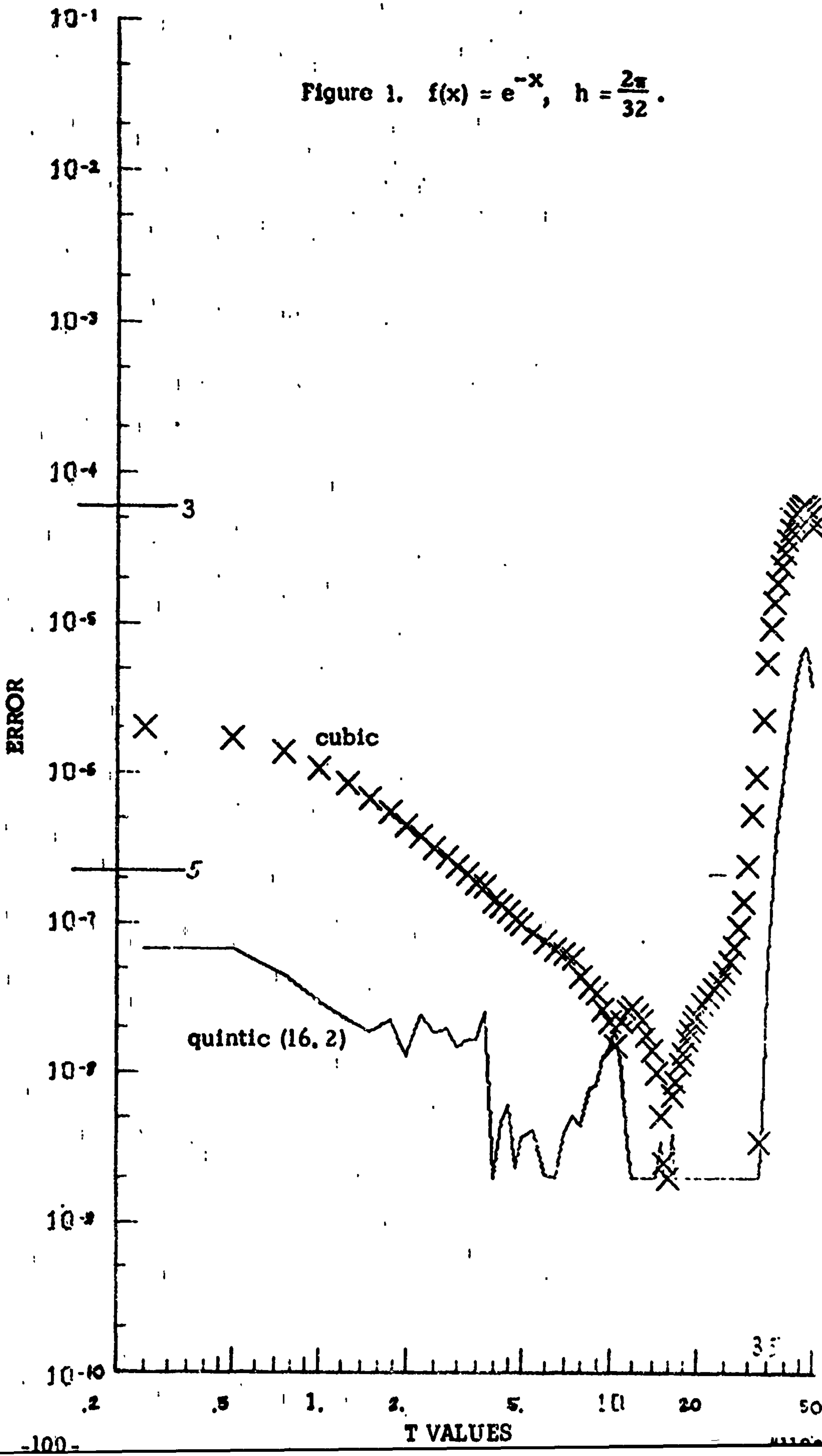
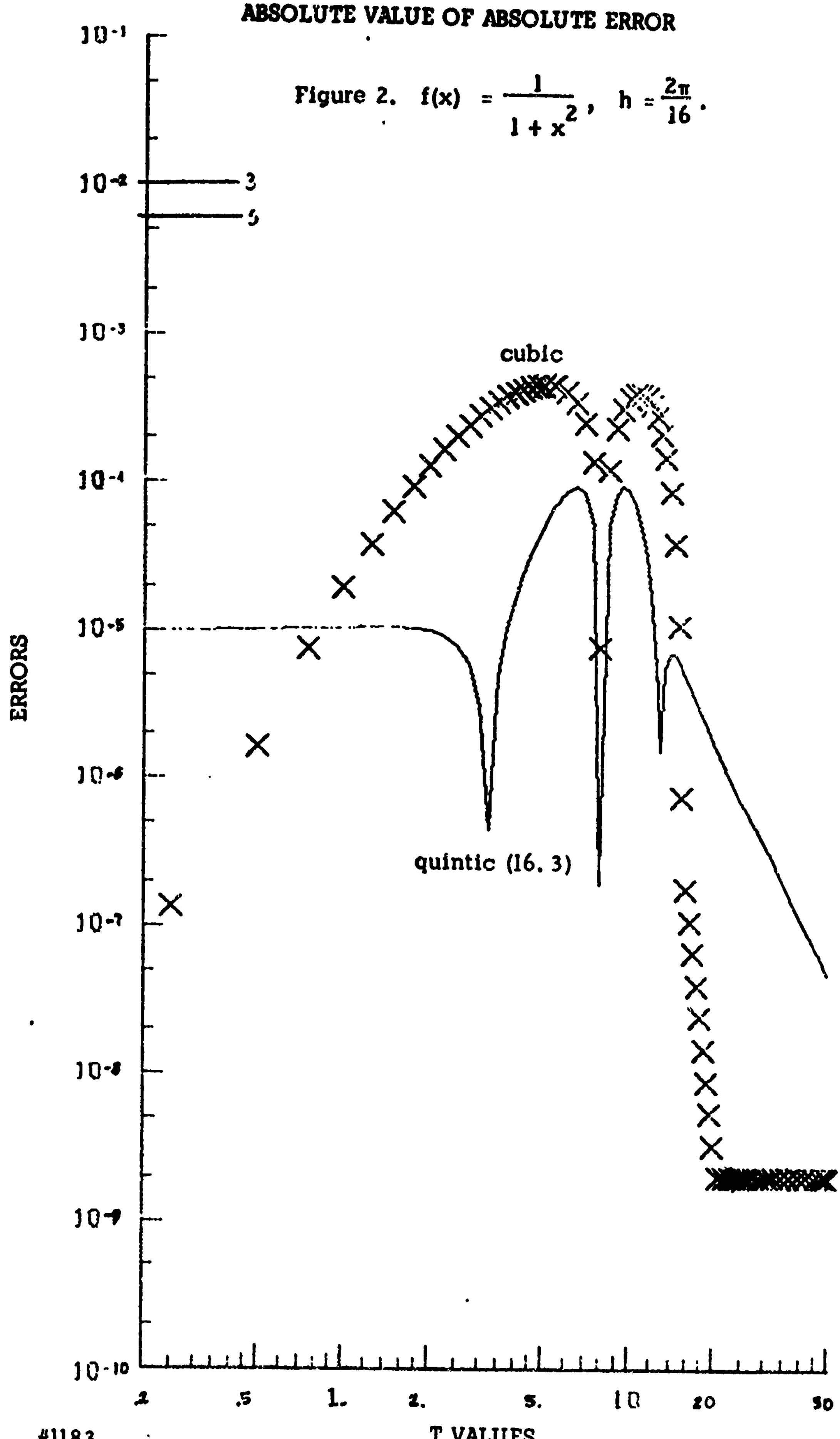


Figure 1. $f(x) = e^{-x}$, $h = \frac{2\pi}{32}$.


ABSOLUTE VALUE OF ABSOLUTE ERROR
Figure 2. $f(x) = \frac{1}{1+x^2}$, $h = \frac{2\pi}{16}$.
10^{-1}
10^{-2}
3
5
10^{-3}
cubic
10^{-4}
10^{-5}
ERRORS
10^{-6}
quintic (16, 3)
10^{-7}
10^{-8}
10^{-9}
10^{-10}
.2
.5
1.
2.
5.
10
20
50
T VALUES

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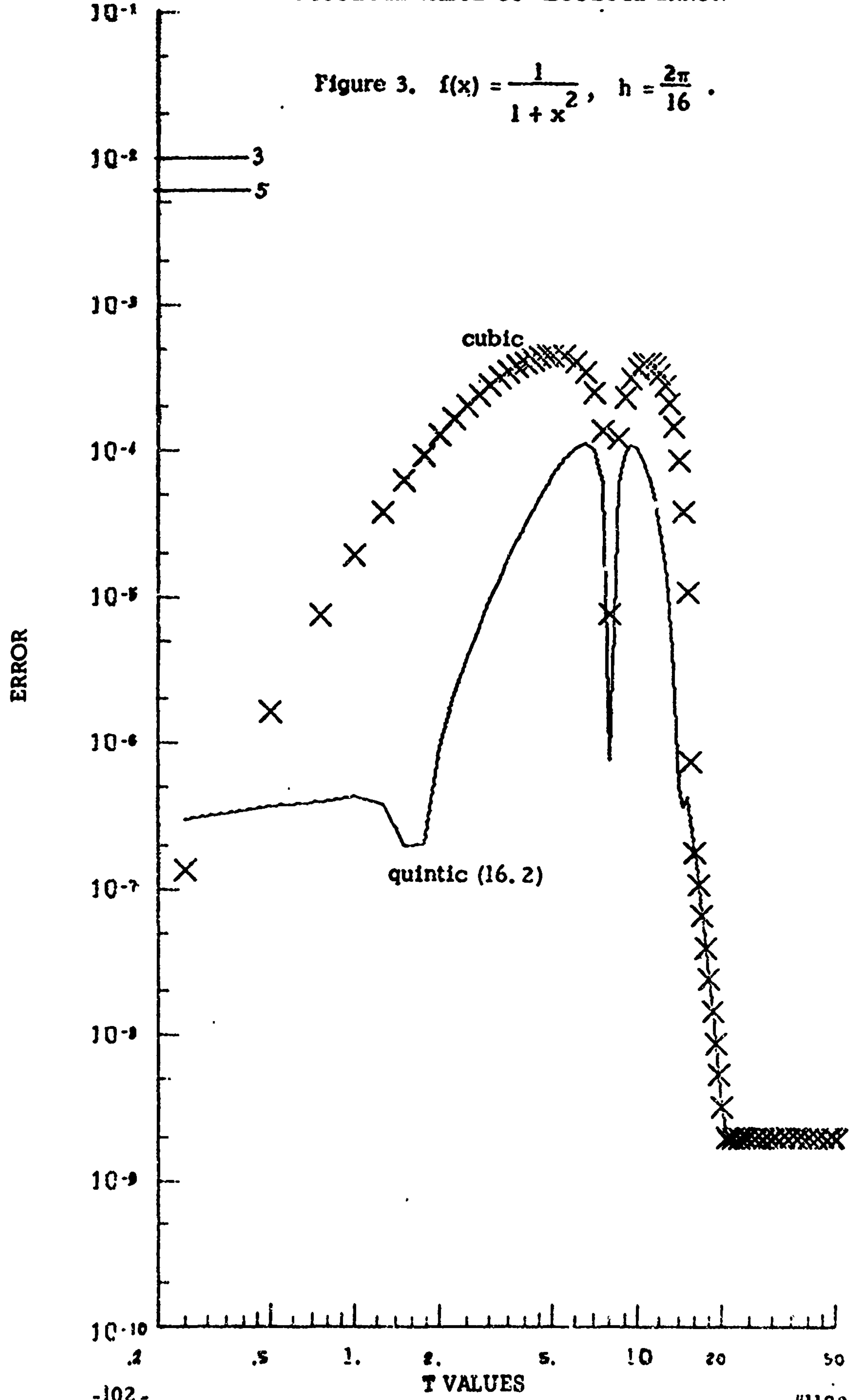
ABSOLUTE VALUE OF ABSOLUTE ERROR

Figure 3. $f(x) = \frac{1}{1+x^2}$, $h = \frac{2\pi}{16}$.



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We see that for small values of t the q.f. (16.3) (Figure 2) gives a considerably worse approximation than (16.2). Here $S'''(0) \approx -.295$ compared with $f'''(0) = 0$. From (13.26) we find

$$(16.6) \quad |Rf|_3 \leq 1.0 \times 10^{-2}, \quad |Rf|_5 \leq 5.86 \times 10^{-3} \text{ for all rational } t \text{ in } (0,16), \quad h = \frac{2\pi}{16}.$$

In Figures 3, 4, and 5 we again consider (16.5), but now for $h = \frac{2\pi}{16}, \frac{2\pi}{32}$ and $\frac{2\pi}{64}$ respectively to consider the q.f. (16.1) and (16.2) as h decreases. From (13.26), we obtain

$$(16.7) \quad |Rf|_3 \leq 6.214 \times 10^{-4}, \quad |Rf|_5 \leq 9.16 \times 10^{-5}$$

$$\text{for all rational } t \text{ in } (0,32), \quad h = \frac{2\pi}{32}$$

$$(16.8) \quad |Rf|_3 \leq 3.884 \times 10^{-5}, \quad |Rf|_5 \leq 1.43 \times 10^{-6}$$

$$\text{for all rational } t \text{ in } (0,64), \quad h = \frac{2\pi}{64}.$$

The figures indicate that we do seem to have errors of order h^4 for (16.1) and h^6 for (16.2).

If we consider Figures 5 and 6 where we calculate (16.5) for $h = \frac{2\pi}{64}$ from q.f. (16.2) and (16.3), respectively, we can see how much closer these quintic curves are than they were in Figures 2 and 3. From (16.3) for $h = \frac{2\pi}{64}$, we find that $S'''(0) = -.00674$ compared to $f'''(0) = 0$ and the $S'''(0) = -.295$ we computed above.

ABSOLUTE VALUE OF ABSOLUTE ERROR

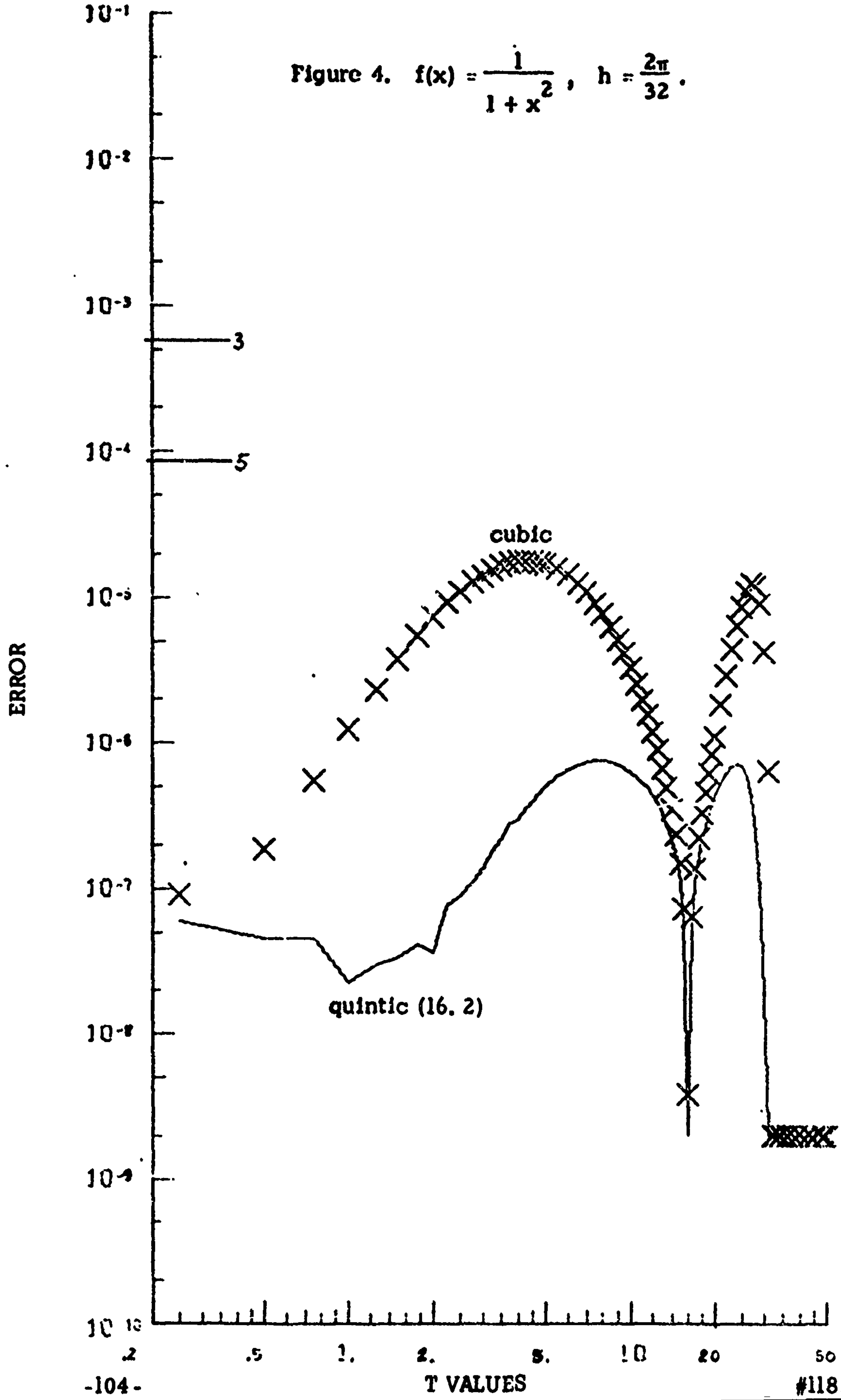

Figure 4. $f(x) = \frac{1}{1+x^2}$, $h = \frac{2\pi}{32}$.

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ABSOLUTE VALUE OF ABSOLUTE ERROR

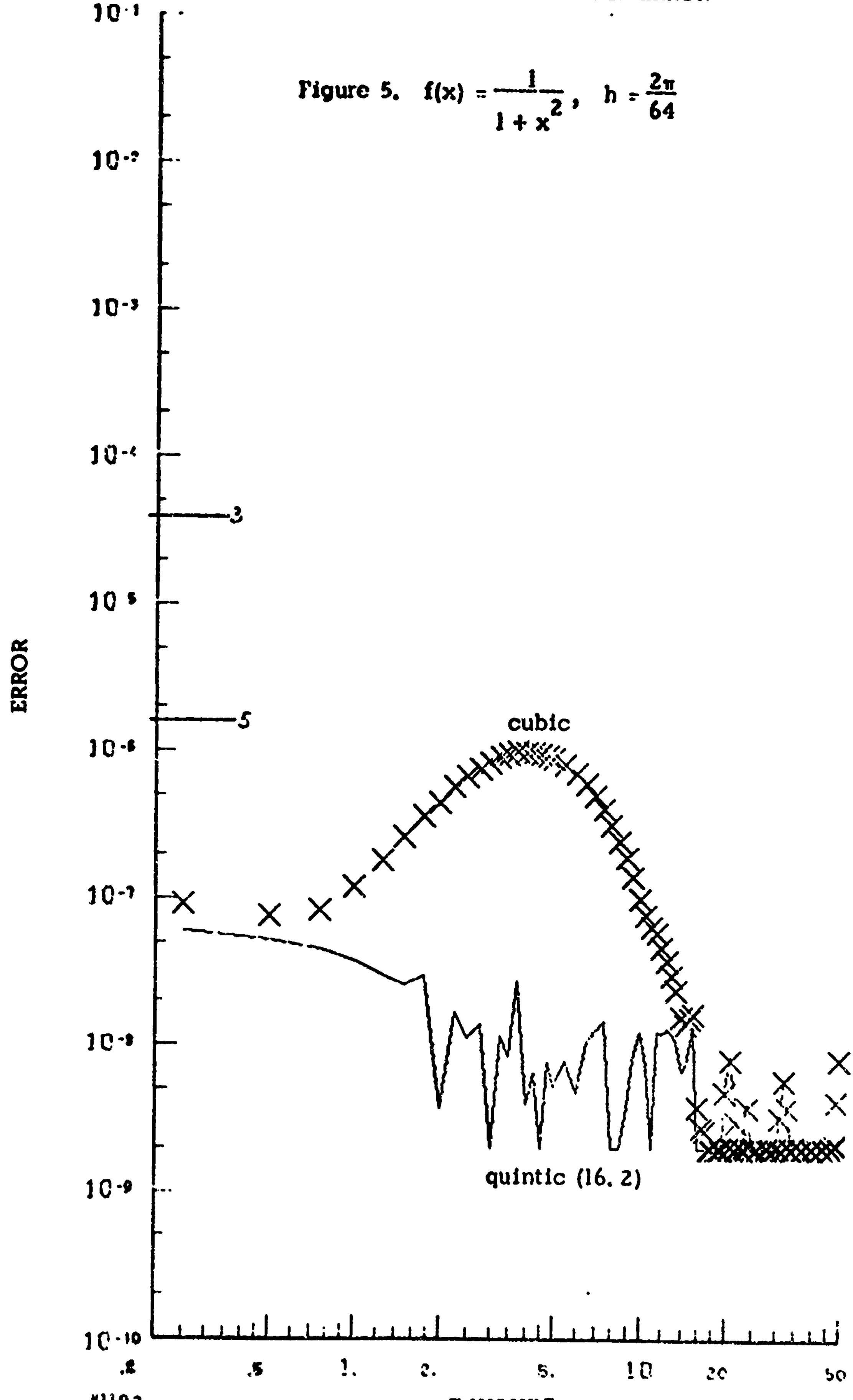

Figure 5. $f(x) = \frac{1}{1 + x^2}$, $h = \frac{2\pi}{64}$

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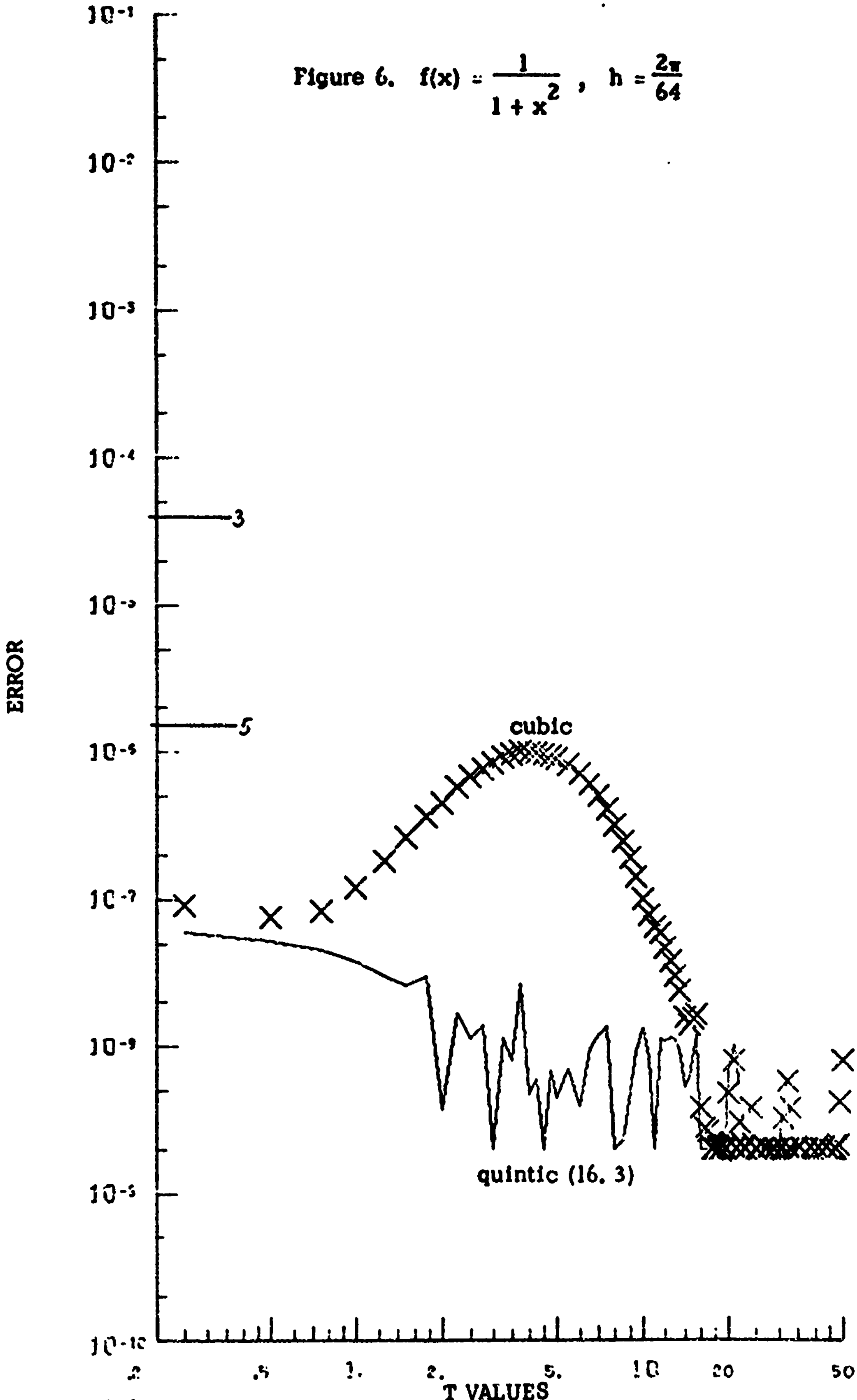
ABSOLUTE VALUE OF ABSOLUTE ERROR

Figure 6. $f(x) = \frac{1}{1+x^2}$, $h = \frac{2\pi}{64}$



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