APPROXIMATIONS FOR CONVERTING GEODETIC TO CARTESIAN COORDINATES

Herbert R. Lotze

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AIR FORCE WEAPONS LABORATORY
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Detachment 1
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FOREWORD

This document has been prepared to describe work done under the Short Range Attack Missile (SRAM) project of AFSC, Weapon System 140A to support the flight test evaluation of B-52 missions and in particular to evaluate onboard targeting computations.

This technical report has been reviewed and is approved.

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SECTION I

INTRODUCTION

Approximations for Converting Geodetic to Cartesian Coordinates

The purpose of a study related to the SRAM project was to simplify calculations for the post-flight evaluation of trajectory data and to reduce the number of numerical operations so that a programmable desk calculator could be used for the computations. Of primary interest was the process used to convert geodetic data of latitude, longitude and height to cartesian coordinates in a plane tangent to the surface of the earth. The Marchant Calculator, Model PR 1016, was available for these calculations and it was required that position fixes and target range computations performed during the flight of a B-52 aircraft be examined on the ground to obtain "quick-look" information revealing the accuracy of the onboard computations. This calculator had to be utilized to save the "turn-around" time spent in waiting for the results of a large electronic computer (CDC-3600).

To find an adequate procedure, conversion methods had to be compared with respect to their accuracy and a compromise made between the complexity of the calculations and the accuracy of the final results. Known approximations had to be examined and their limitations with respect to the maximum range between the fixed point or target and the aircraft had to be determined. The goal was to implement the calculator with a program capable of converting geodetic data to tangent plane data with an accuracy equivalent to a few feet compared to the solution of using a spheroidal earth model which was assumed to yield the exact solution. A specific question was then, up to what maximum range the spherical earth model would be adequate in this respect.

Target ranges in the White Sands test area barely exceed 100 nautical miles; however, because of the significance of this subject, errors caused by some approximations were calculated for ranges up to 700 nautical miles. These ranges or distances between the aircraft and targets are calculated to determine the "range-to-go", which is derived from the output of the inertial platform sensors and the known target coordinates.

Since information about approximations for the required conversion process are scattered in the literature, the more important expressions related to this subject are reviewed and, in some cases, derived.

SECTION II

METHODS FOR THE CONVERSION PROCESS

It is standard practice to calculate position differences of an aircraft with respect to a ground target in cartesian coordinates. To relate the two positions to each other, it is necessary to convert the geodetic survey data of the target into cartesian coordinates of the platform system or vice versa. Normally a north/east orientation is used as a reference, and a deviation of the platform axes from this orientation is taken into account; results are presented with components in north and east direction counted positive which are called R_N or X and R_E or Y hereafter. Of less interest here is the Z component which may be computed but can also be found from the altitude measurement of the onboard radar altimeter combined with information of terrain altitude.

In the following, it is assumed, unless otherwise stated, that the point on the earth's surface is at a mean sea level. The models investigated with respect to their accuracy are:

Circular arcs in north/south and east/west direction and

Spherical earth with geocentric radius derived from the mean latitude of the two points.

The spheroidal earth model with major and minor axes specified for the basic ellipse serves as a reference to determine the error of the results derived both from circular arcs and spherical earth. It is noted that the spheroidal earth model is also an approximation, but it best approximates the actual earth shape.

a. Circular-arc Approximations

Circular arcs can be approximated using either geodetic or geocentric quantities for the earth radius and for latitude. Based on geodetic quantities the equations for the north range R_N and for the east range R_E in the local tangent plane are

$$R_N = X = R_g \Delta\Phi_g \qquad (1a)$$

$$R_E = Y = R_g \cos\Phi_g \Delta\lambda \qquad (1b)$$

where

R_g = geodetic earth radius

$\Delta\Phi_g$ = $(\Phi_{g2} - \Phi_{g1})$ radians

$\Delta\lambda$ = $(\lambda_2 - \lambda_1)$ radians

Φ_g = $(\Phi_{g1} + \Phi_{g2})/2$

Φ_{g1}, Φ_{g2} are the geodetic latitudes and λ_1, λ_2 the longitudes of point 1 and 2 respectively.

From standard text books (for instance, reference 7):

$$R_g = R_{eq}(1 - \epsilon^2 \sin^2 \Phi_g)^{-\frac{1}{2}}$$

where

R_{eq} = equatorial radius of the earth

ϵ = eccentricity of the earth ellipse

Equation (1b) is a valid approximation and can be used for quick estimates; (1a) should not be used because it is not a valid expression and leads to significant errors for values above 200 feet. In this context a few feet are considered as a tolerable error. Equation (1a) is mentioned here because it is occasionally used for quick-look estimates, with $\Delta\Phi_g$ and R_g being readily available. The error is shown for some typical values of R_N in the section on "Numerical Examples"; values of $X(=R_N)$ computed from (1a) are designated as "approximation 1".

Selecting a certain earth model, the equatorial radius R_{eq} and the polar radius R_{pol} of the earth are given and can be used to compute the eccentricity ϵ which is defined by

$$\epsilon^2 = 1 - (R_{pol}/R_{eq})^2 \qquad (2)$$

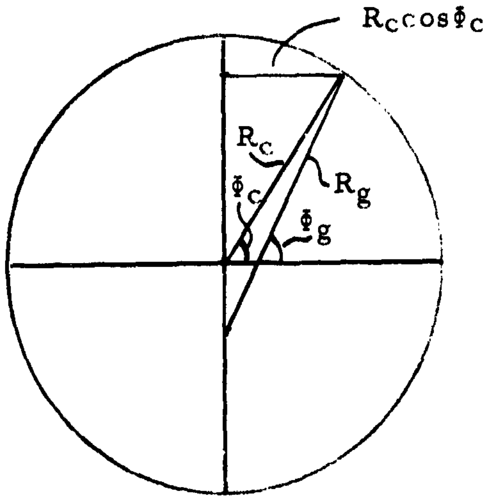
A second approximation for X uses the mean geocentric earth radius R_c and geocentric latitudes Φ_c:

$$X = R_c \Delta\Phi_c \qquad (3a)$$

where

$\Delta\Phi_c$ = $\Phi_{c2} - \Phi_{c1}$

(3a) is a valid approximation of X and is adequate for a wider range compared to X computed from (1a), provided the two points are on the same meridian.



Geocentric and Geodetic Earth Radius

Figure 1

The corresponding expression for the east/west component R_E

$$R_E = Y = R_{c2} \cos \Phi_{c2} \Delta\lambda \qquad (3b)$$

is equivalent to equation (1b). This can be seen from figure 1, since

$$R_c \cos \Phi_c = R_g \cos \Phi_g$$

Both (1b) and (3b) represent valid approximations of the east range R_E for two points on the same parallel. The accuracy of the approximations is limited by:

$$\Delta Y = R_{c2} \cos \Phi_{c2} (\Delta\lambda - \sin\Delta\lambda)$$

The equation

$$Y = R_{c2} \cos \Phi_{c2} \sin\Delta\lambda \qquad (3c)$$

is exact for all earth models described here.

The computation of X from (3a) and Y from (3b) or (3c) requires the values of R_c and Φ_c: they can be obtained from equations in standard textbooks. For instance, from reference 4:

$$R_c = R_{eq}[\cos^2 \Phi_c + (R_{eq}/R_{pol})^2 \sin^2 \Phi_c]^{-\frac{1}{2}}$$

$$= R_{eq}\{1 + \sin^2 \Phi_c [(R_{eq}/R_{pol})^2 - 1]\}^{-\frac{1}{2}} \qquad (4a)$$

and using (2) we obtain

$$R_c = R_{eq}(1 + \frac{\epsilon^2}{1-\epsilon^2} \sin^2 \Phi_c)^{-\frac{1}{2}} \qquad (4b)$$

The geocentric latitude Φ_c can be computed from

$$\Phi_c = \arctan[(R_{pol}/R_{eq})^2 \tan\Phi_g] \tag{5a}$$

Equation (5a) is exact and is found in reference 4. Φ_c can also be obtained by applying the small angle approximation to $\tan(\Phi_c - \Phi_g)$ and then expanding in a power series. This is done in appendix A and the first two terms of the expansion are

$$\Phi_c - \Phi_g = -0.19390737\sin 2\Phi_g - 0.00131249\sin 2\Phi_g \sin^2\Phi_g \tag{5b}$$

where Φ_c and Φ_g are in degrees.

Two other forms of this series are given in references 2 and 3. One can derive the series of reference 3 by manipulating equation (5b) so that the second term in (5b) disappears for one particular value of Φ_g. This is described in appendix A. Choosing 33° for this particular value of Φ_g yields:

$$\Phi_c - \Phi_g = -0.19429670\sin 2\Phi_g + 0.00038933(1-3.371184\sin^2\Phi_g)\sin 2\Phi_g \tag{5c}$$

(Corresponding to equation (A7) in appendix A).

If one neglects the second term of (5c) one finds an improved approximation of $(\Phi_c - \Phi_g)$ compared to a one-term-only solution of (5b). This improvement is obtained of course for the specific value of Φ_g and to some degree also for values of Φ_g in the neighborhood of the specific value. Since the mean geodetic latitude at WSMR was 33° or close to 33° for those target ranges, which were evaluated, this number was considered adequate as basis for the one-term approximation.

$$\Phi_c - \Phi_g = -0.19429670\sin 2\Phi_g \tag{5d}$$

which follows from (5c).

The values of X and Y from equations (3a) and (3b) can now be computed provided the geodetic latitudes Φ_g are given. Examples are discussed in the section "Numerical Examples". Equation (5d) was used to compute Φ_c in equation (3a) and in (3b) and the approximations for X and Y found in this way are referred to as "approximation 2". The errors caused by the one-term approximation are plotted in figure 2 for 30° and 33°.

In practice, another procedure is more frequently used to estimate the length

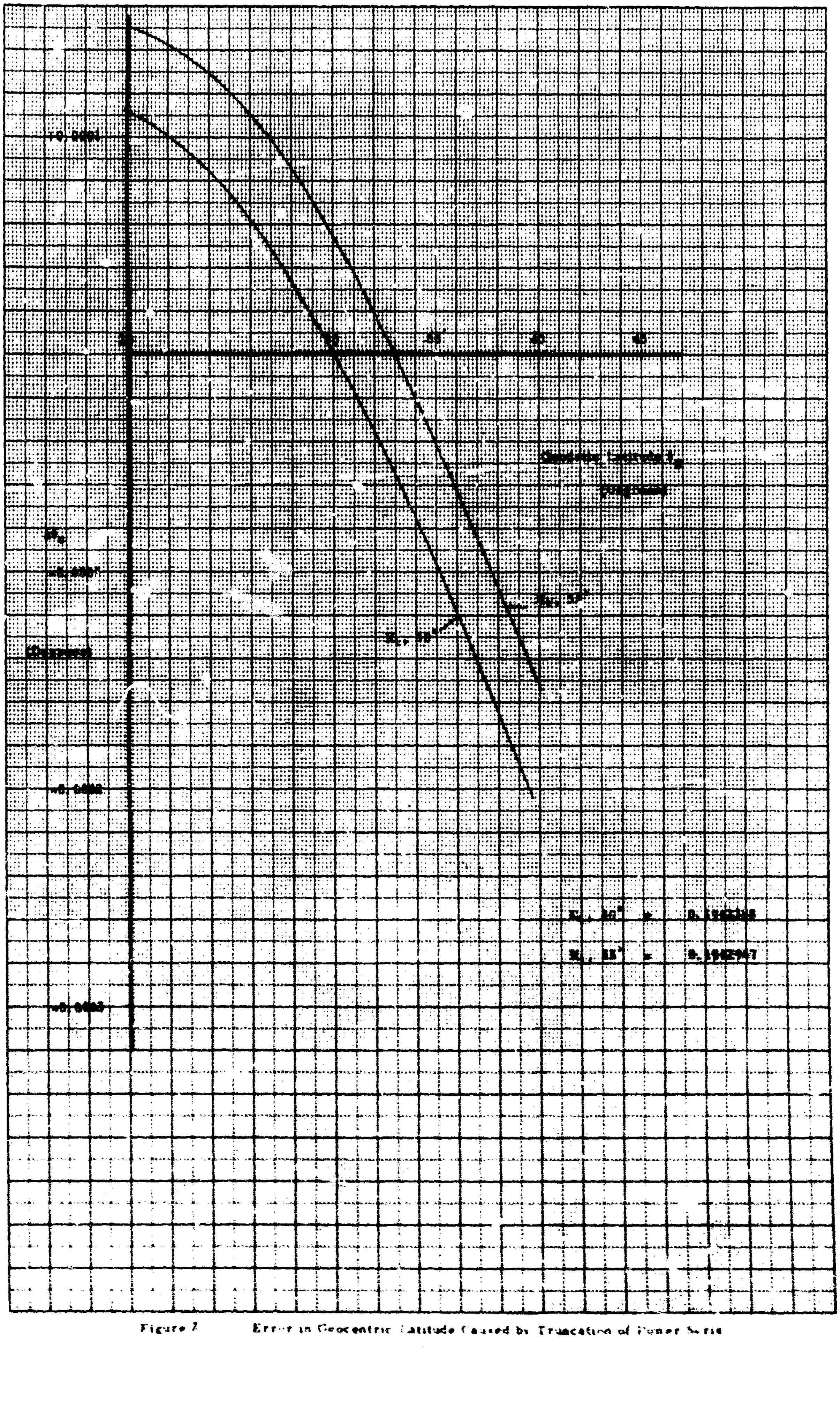


Figure 2 Error in Geocentric Latitude Caused by Truncation of Power Series

of an arc of a meridian or a parallel, and this estimate in turn is used for approximating north or east components in the local tangent plane. This is the method of using precalculated scaled factors of distance in feet per angular degrees or per arc minute. These scale factors can be derived from a spheroidal model of the earth and computed as functions of geodetic latitude. In figure 3 and tables 1 and 2, these factors are presented both for arcs in latitude and longitude based on Clarke's spheroid of 1866. They were found by converting a one-arc-minute difference in latitude or longitude respectively to tangent plane coordinates. As a result, the following expressions can be used for a geodetic latitude of 33° to estimate the north/south (X) and east/west (Y) coordinates in a tangent plane system.

$$X = 6064.1\Delta\Phi_g \tag{6a}$$

$$Y = 6093.2\Delta\lambda\cos 33^\circ \tag{6b}$$

where $\Delta\Phi_g$ and $\Delta\lambda$ are in minutes of arc, or

$$X = 363848\cdot\Delta\Phi_g \tag{6c}$$

$$Y = 365592\Delta\lambda\cos 33^\circ \tag{6d}$$

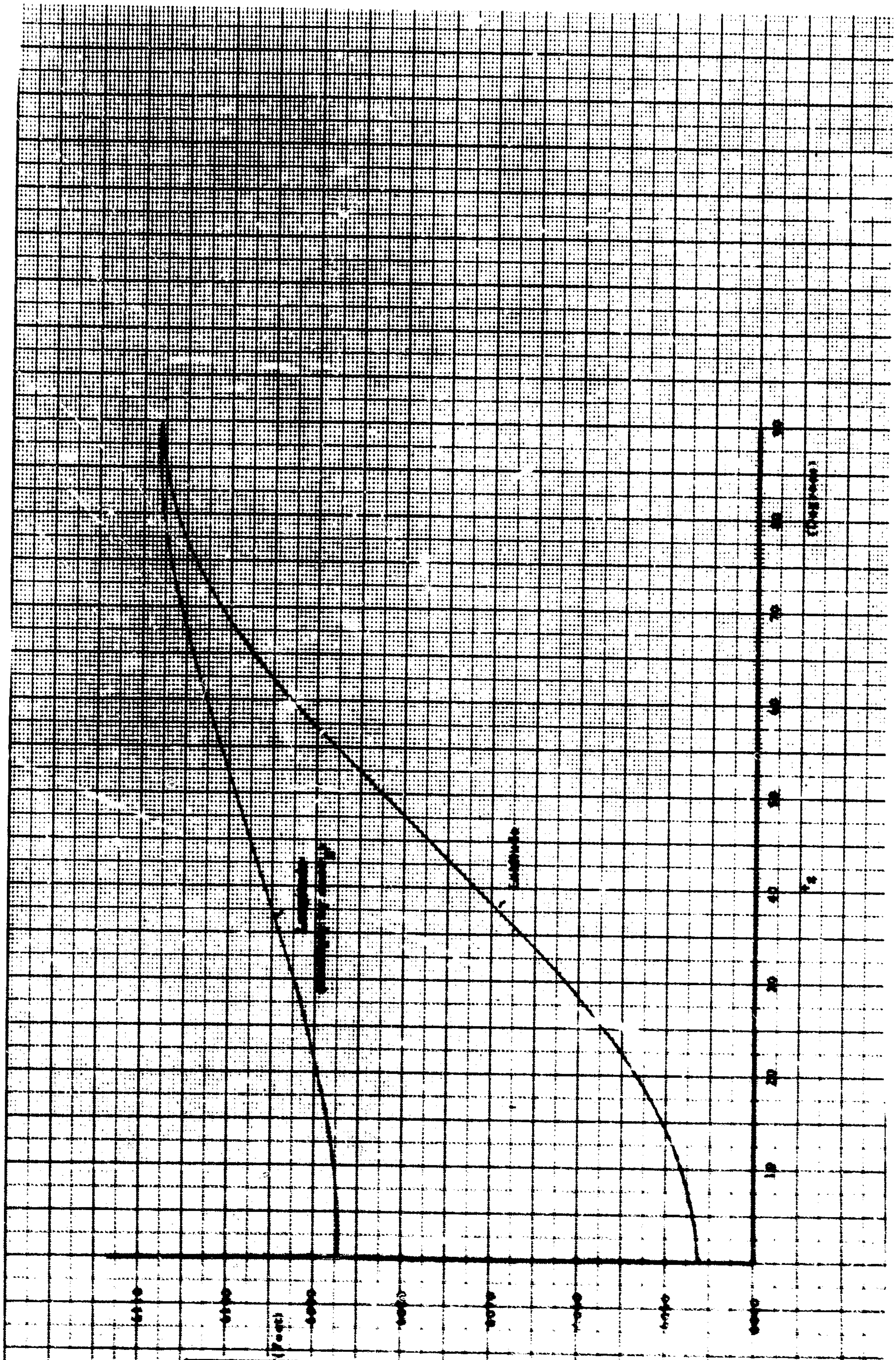
$$= 306611\Delta\lambda$$

where $\Delta\Phi_g$ and $\Delta\lambda$ are in degrees.

It is emphasized here that equations (3a), (3b), (6a), (6b), (6c) and (6d) are valid approximations only if the two points of interest are located on the same meridian, or on the same parallel, respectively. The error caused by not being on the same meridian or parallel depends on the value of the angular difference. This limitation is related to the subject of the next section.

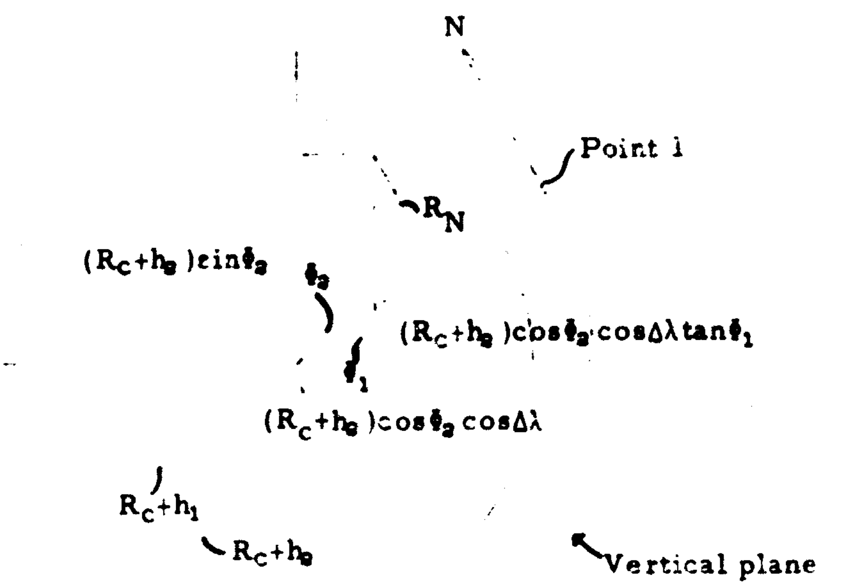
b. Spherical Earth Approximation

A better approximation is obtained by using a spherical earth model with a radius R_C equal to the geocentric radius which is computed as the mean of the geocentric radii of the two particular points of the surface of the earth. The expressions for the north and east components of the distance between two points in the local tangent plane are then derived from figure 4 as follows:



Feet per Arc Minute of Latitude and Longitude versus Latitude

Figure [illegible]



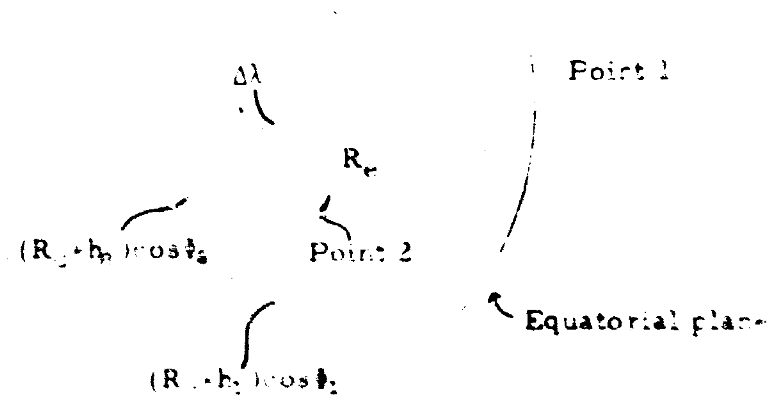


Figure 4. Spherical Earth Model

TABLE I

FEET PER DEGREE AND PER ARC MINUTE AS FUNCTION OF LATITUDE FOR CLARKE'S 1866 SPHEROID

Degree	Feet per Degree of Geodetic Latitude	Feet per Arc Minute of Geodetic Latitude
0	362,760	6046.0
10	362,874	6047.9
20	363,192	6053.2
30	363,684	6061.4
33	363,848	6064.1
40	364,290	6071.5
50	364,932	6082.2
60	365,538	6092.3
70	366,036	6100.6
80	366,360	6106.0
90	366,474	6107.9

Table II

FEET PER DEGREE AND PER ARC MINUTE OF LONGITUDE
FOR CLARKE'S 1866 SPHEROID

Φ Degree	Feet per Degree of Longitude	Feet per Arc Minute of Longitude
0	365,225	6087.1
10	359,713	5995.2
20	343,334	5722.3
30	316,562	5276.0
33	306,611	5110.2
40	280,170	4669.5
50	235,230	3920.5
60	183,078	3051.3
70	125,289	2088.1
80	63,650	1060.6
90	0	0

It can be seen in this figure that R_N is the base and

$$(R_C + h_2)\sin\Phi_2 - (R_C + h_2)\cos\Phi_2\cos\Delta\lambda\tan\Phi_1$$

is the hypotenuse of a right triangle with the angle Φ_1 between base and hypotenuse. Note that the angles Φ_1 and Φ_2 represent geocentric latitudes and R_C is the mean geocentric radius:

Therefore

$$X = R_N = \cos\Phi_1[(R_C + h_2)\sin\Phi_2 - (R_C + h_2)\tan\Phi_1\cos\Phi_2\cos\Delta\lambda]$$

where

$$\Delta\lambda = \lambda_2 - \lambda_1$$

or

$$R_N = (R_C + h_2)(\cos\Phi_1\sin\Phi_2 - \sin\Phi_1\cos\Phi_2\cos\Delta\lambda) \quad (7a)$$

Setting $\Delta\lambda = 0$, under the assumption that the two points are on the same meridian, one obtains as can be expected

$$R_N = (R_C + h_2)\sin(\Phi_2 - \Phi_1) \quad (7b)$$

Only for small angular differences $(\Phi_2 - \Phi_1)$ and $h_2 = 0$ does this equation yield approximately the same results as equation (3a) which indicates the limitation of the circular-arc approximation. The east component R_E is

$$Y = R_E = (R_{C2} + h_2)\cos\Phi_2\sin\Delta\lambda \quad (7c)$$

Where R_{C2} is the geocentric radius of point 2.

Equation (7c) yields an approximation for Y with errors depending on the value of h_2. The exact value of Y is found from

$$Y = R_E = (R_{C2}\cos\Phi_2 + h_2\cos\Phi_{g,2})\sin\Delta\lambda \quad (7d)$$

It is noted here that the expression for the north component (7a) is an approximation because of the use of a spherical earth model to represent an elliptical earth; but (7d) is an exact expression for the calculation of the east component of the range. It is noteworthy also that (7d) uses the geocentric radius at point 2, R_{C2}, instead of the mean geocentric radius R_C which is used in (7a).

In many practical cases, equation (7a) is accurate enough and its simplicity makes computation possible on a small programmable electronic desk calculator. An example of the program which was prepared for the Marchant 1016 PR is listed in appendix B. The program consists of four parts each of which is recorded on a separate magnetic tape*. These tapes are read into the core memory sequentially and require four quantities as manual input:

Φ_{g1} = Geodetic latitude (in degrees) of point 1

Φ_{g2} = Geodetic latitude (in degrees) of point 2

$\Delta\lambda$ = $\lambda_2 - \lambda_1$ (in degrees)

h_2 = Height of point 2 in feet (multiplied by a scale factor of 10^{-8}).

Typical values of X and Y were calculated from equations (7a) and (7c) or (7d) respectively and the results are discussed in the section "Numerical Examples". When the approximated geocentric latitudes Φ_C from equation (5d) are used to compute X and Y the approximations are referred to as "approximation 3"; when the exact value of Φ_C from equation (5a) is inserted into equations (7a), (7c) or (7d) the values of X and Y are referred to as "approximation 4". With this arrangement the approximations are easy to distinguish. Those designated with higher numbers should produce more accurate results with respect to the earth spheroid.

A simple relationship (7e) can be used to approximate the Z component of the vector connecting point 1 and 2:

$$Z \simeq [(R_{C2} + h_2)^2 - X^2 - Y^2]^{\frac{1}{2}} - (R_{C1} + h_1) \qquad (7e)$$

$$X = R_N$$

$$Y = R_E$$

Note that (7e) is not an exact equation since the heights h_1 and h_2 do not form a straight line with R_{C1} and R_{C2} respectively; however, this approximation is accurate enough for the applications considered here. For higher accuracy requirements the Z component also referred to as R_Z, should be derived from the spheroidal earth model.

c. Spheroidal Earth Model

The most accurate estimate of position differences is obtained by applying a spheroidal or ellipsoidal model of the earth in the coordinate transformation process. Frequently used models are Clarke's spheroid of 1866 and Hayford's

* With minor manipulations explained in appendix B the program can be recorded on 3 tapes.

spheroid of 1910, also called the international spheroid. Improved earth model parameters were recently (1966) derived by the Smithsonian Astrophysical Observatory (SAO) from earth satellite data. The model is referred to later in the text as SAO spheroid. The equatorial and polar radius for these three models are as follows:

	R_{eq}(ft)	R_{pol}(ft)
Clarke's spheroid	20,925,832	20,854,892
Hayford's spheroid	20,926,470	20,856,010
SAO spheroid	20,925,738	20,855,576

Derived from R_{pol} and R_{eq} are by definition the ellipticity $E = 1 - R_{pol}/R_{eq}$ and the eccentricity $\epsilon = [1 - (R_{pol}/R_{eq})^2]^{\frac{1}{2}}$, which are listed below together with ϵ^2 and the frequently used term $\frac{\epsilon^2}{1-\epsilon^2}$

	E	ϵ	ϵ^2	$\epsilon^2/(1-\epsilon^2)$
Clarke's spheroid	1/295	.082271770	.0067686441	.0068147708
Hayford's spheroid	1/297	.08199218	.0067227183	.0067681701
SAO spheroid	1/298.25	.08182018	.0066945419	.0067396608

Geodetic survey data of targets, impact locations and radar sites, etc., at WSMR are in general based on Clarke's spheroid of 1866 and the latter is used in connection with numerical examples discussed later. There are two ways to compute the north and east components of the range between two points from given geodetic data. First, the "classical" or conventional procedure consisting of the conversion from geodetic to earth-centered coordinates, followed by a translation and rotation of the coordinates. Second, relatively simple explicit formulae can be derived for X, Y and Z, also called R_N, R_E and R_Z in the text. The classical method is described as follows:

The given geodetic coordinates of point one and two are converted to earth-centered cartesian system (ecs) which is a left-handed system. The coordinates of point two are translated to point one as new origin which yields:

$$X_D = X_{ecs2} - X_{ecs1}$$

$$Y_D = Y_{ecs2} - Y_{ecs1}$$

$$Z_D = Z_{ecs2} - Z_{ecs1}$$

These coordinates are rotated three times to obtain the final set of cartesian coordinates in the local tangent plane. The conversion process is described in more detail in appendix C. Statements of a FORTRAN program based on this procedure are included in appendix D.

The other procedure, mentioned above, for converting geodetic data to local tangent plane coordinates is described next. It utilizes in a straightforward manner the geometry of the ellipse representing the earth and of the cartesian coordinates connecting the two points of interest. Similarly as for the spherical earth model in figure 4, point 2 and straight line connections through point 2 are shown in figure 5 projected into the vertical plane through the meridian on which point 1 is located. In the derivation of the expression for R_N, the two equations of the vertical and horizontal cartesian coordinates R_V and R_H are used as follows:

$$R_V = R_g(1 - \epsilon^2)\sin\Phi_g \qquad (9a)$$

$$R_H = R_g\cos\Phi_g \qquad (9b)$$

Equation (9b) can be derived from figure 5 readily and equation (9a) is found by inserting (9b) into the expression for $\tan\Phi_c$

$$\tan\Phi_c = \frac{R_V}{R_H} = \frac{b^2}{a^2}\tan\Phi_g \qquad (9c)$$

which yields

$$R_V = \frac{b^2}{a^2} R_g \sin\Phi_g$$

The expressions for R_N and R_Z can be found from figure 5 which shows that R_N is the base and

$$H = (R_2 + h_2)\sin\Phi_2 - \epsilon^2(R_2\sin\Phi_2 - R_1\sin\Phi_1) - (R_2 + h_2)\cos\Phi_2\cos\Delta\lambda\tan\Phi_1 \qquad (9d)$$

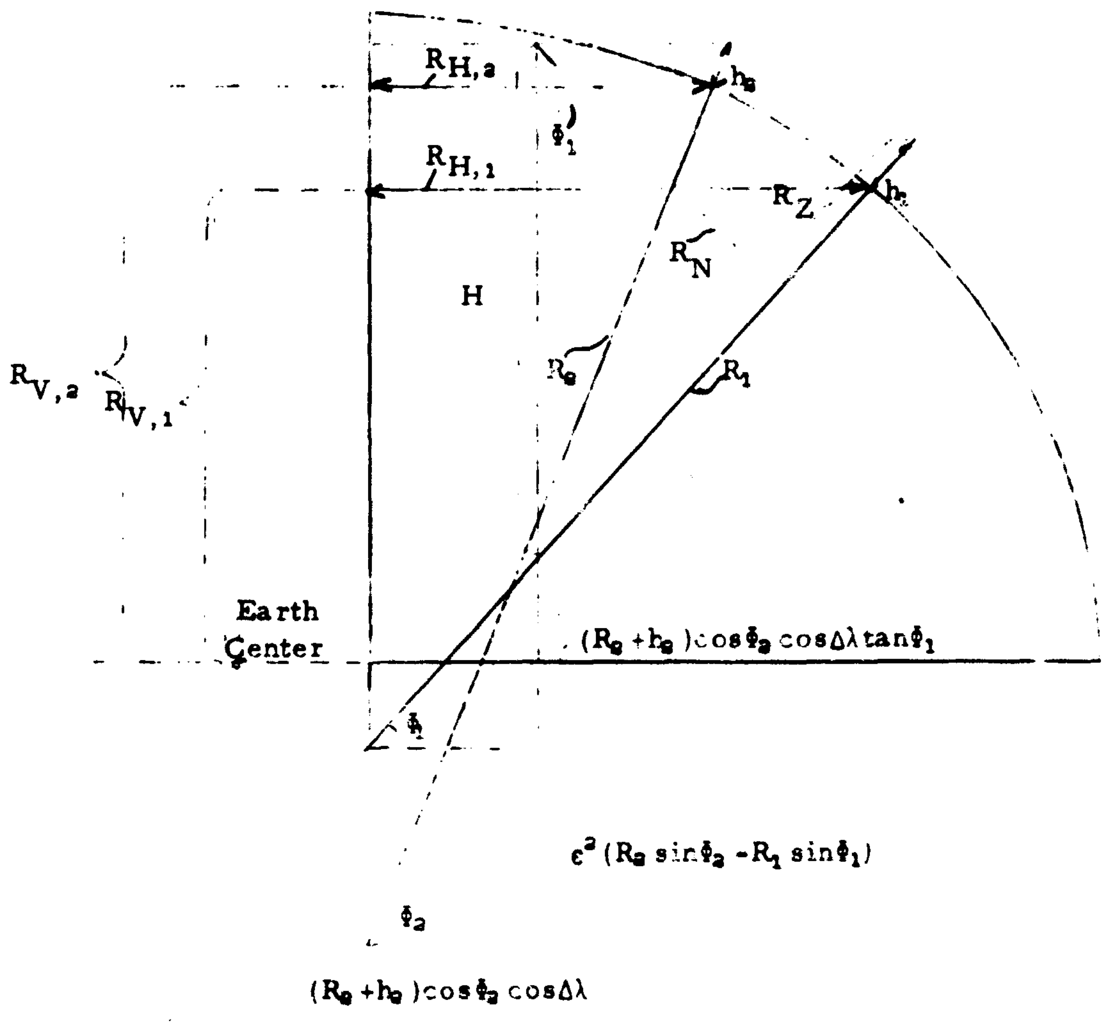


Figure 5. Spheroid Geometry

is the hypotenuse of a right triangle with Φ_1 being the angle between base and hypotenuse. Therefore

$$R_N = (R_2 + h_2)(\cos\Phi_1 \sin\Phi_2 - \sin\Phi_1 \cos\Phi_2 \cos\Delta\lambda)$$

$$- \epsilon^2 (R_2 \sin\Phi_2 - R_1 \sin\Phi_1) \cos\Phi_1 \quad (9e)$$

As can be expected, this formula is similar to the expression for R_N which was derived from spherical earth (7c) using the mean geocentric radius. Since (9e) is based on the elliptical earth model utilizing the geodetic radii R_1 and R_2 and the geodetic latitude Φ_1 and Φ_2, an additional term must be applied. This term, the second part of (9e), accounts for the distance between the centers of the radii $\epsilon^2 (R_2 \sin\Phi_2 - R_1 \sin\Phi_1)$. The expression for R_Z is found as

$$R_Z = R_1 + h_1 - [(R_2 + h_2)\cos\Phi_2 \cos\Delta\lambda / \cos\Phi_1 + R_N \tan\Phi_1] \quad (9f)$$

The equation for R_E (9g) is equivalent to equation (7c) with the geodetic radius R_2 and geodetic latitude Φ_2 used instead of the corresponding geocentric data:

$$R_E = (R_2 + h_2)\cos\Phi_2 \sin\Delta\lambda \quad (9g)$$

To compare the two procedures described above for the computation of the LTP* components, one may count the numerical operations (products, divisions, additions, subtractions and squaring) and the subroutine entries. Under the assumption that in either case, as is usual, the geodetic coordinates of the two points are given, it is found in this way that the second procedure requires

32 operations including 7 trigonometric functions
and 2 square roots to compute R_N and R_E.

and in addition

8 operations to compute R_Z.

The former procedure requires a minimum of

45 operations including 8 trigonometric functions
and 2 square roots to compute R_N and R_E

and in addition

5 operations to compute R_Z.

* Local Tangent Plane

A double precision version of the former program was used as reference for investigating the accuracy of the two programs. It was found that the second program, the new version, yields slightly more accurate results because fewer operations are involved. The difference was more pronounced for the Z component than for the X component, and more for the X component than for the Y component. Results for R_Z obtained from the new version agree to 9 decimal places with the double precision results, whereas the older procedure leads to an agreement of 8 places.

By manipulating the matrix product in equation (C4) of appendix C representing the classical procedure it can be shown that (C4) leads to equations identical with (9e), (9f) and (9g).

SECTION III

NUMERICAL EXAMPLES

Accuracy limitations of the circular arc and of the spherical earth approximations become evident from numerical examples discussed in this section. The errors ΔR_N in R_N are presented in figures 6 through 10 which are made by the 4 approximations defined as follows:

Approximation 1	circular arcs $R_g \Delta \Phi_g$
Approximation 2	circular arcs $R_c \Delta \Phi_c$
Approximation 3	spherical earth using (5d) for Φ_c
Approximation 4	spherical earth using (5a) for Φ_c

Those values of R_N served as a reference for all presented errors ΔR_N which were computed from equation (9e) based on Clarke's spheroid of 1866 for various ranges and azimuth angles. These reference values are listed in tables 3 and 4 as functions of the range R_N (or R_E) and of the azimuth.

The input for the computation is assumed to be given in geodetic coordinates, latitude Φ and longitude λ in degrees and height h in feet above mean sea level. For applications considered in this context, it is adequate to carry six decimal places for Φ and λ and to use a rounded integer number for h considering a one-foot accuracy as well within accuracy requirements.

Some details of the computation which apply to approximations 3 and 4 are clarified first. Neither version computes the mean geocentric radius R_c as the mean of the two geocentric radii R_{c_1} and R_{c_2} which would appear necessary from a theoretical standpoint; instead R_c is derived from the mean geocentric latitude $\bar{\Phi}_c$ as

$$R_c = R_{eq}\left(1 + \frac{\epsilon^2}{1-\epsilon^2}\sin^2\bar{\Phi}_c\right)^{-\frac{1}{2}} \tag{10a}$$

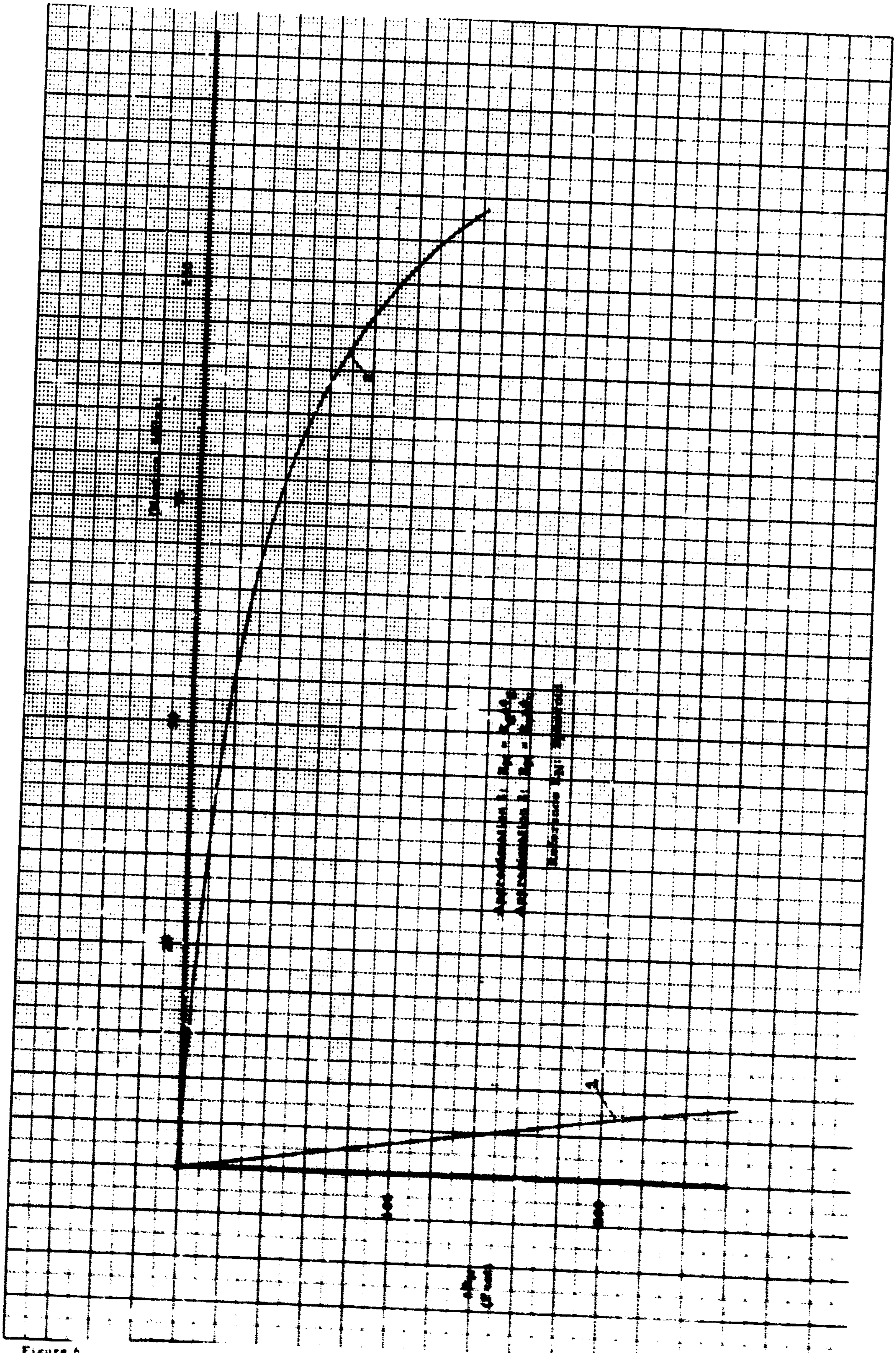


Figure 6 Approximations 1 and 2 Circular Arc

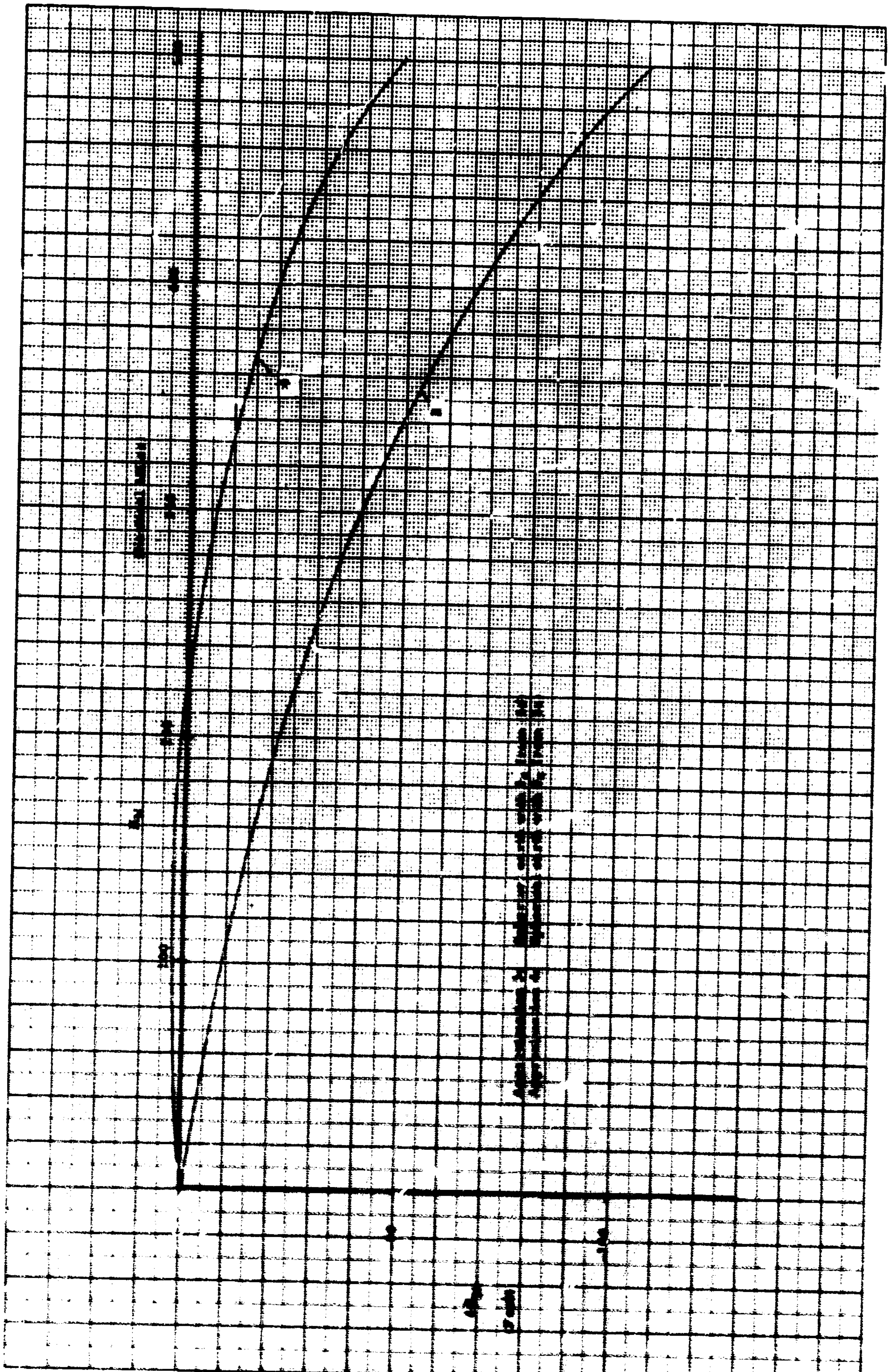


Figure [illegible] R Approximations 3 and 4 Azimuth = [illegible]

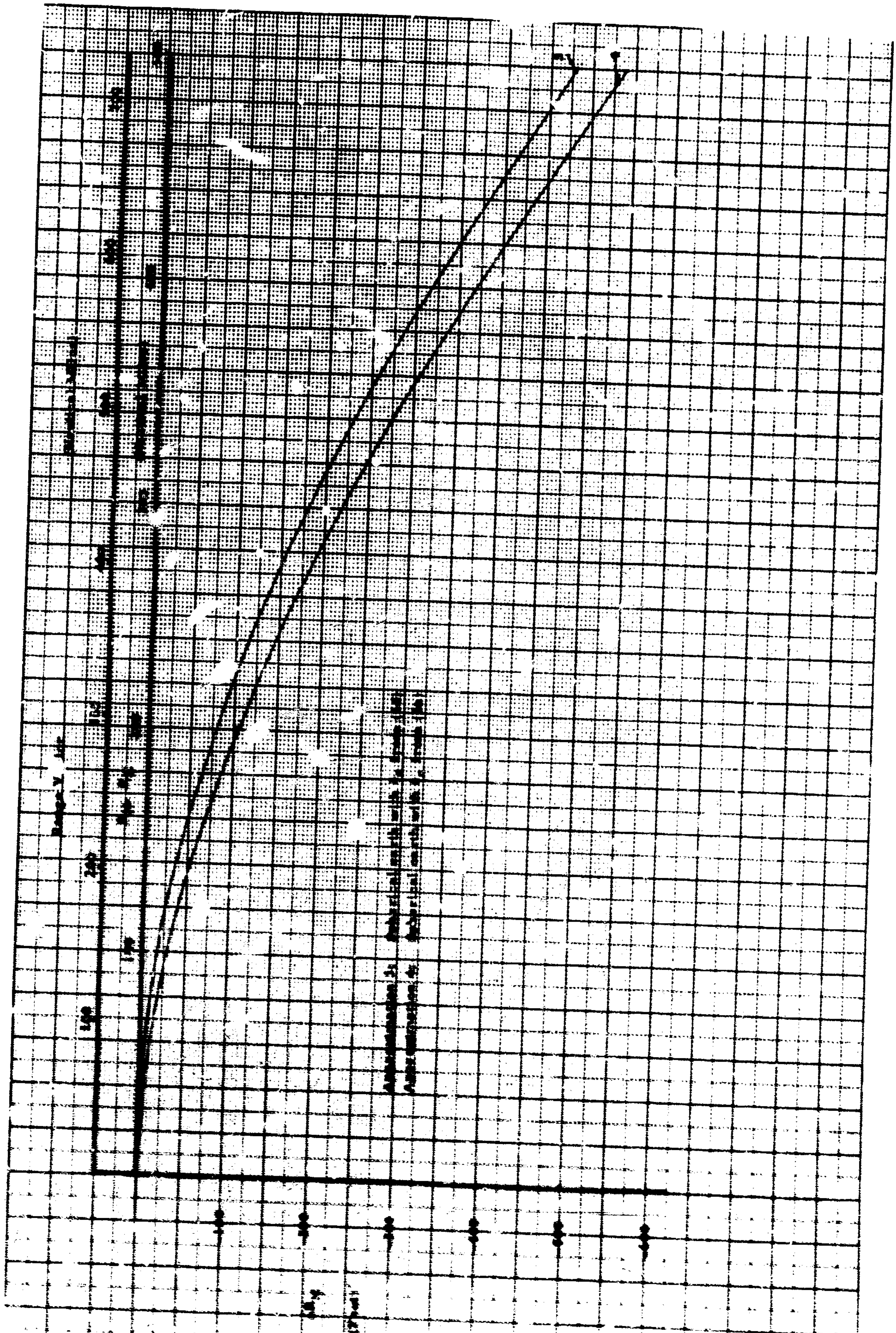


Figure 4a

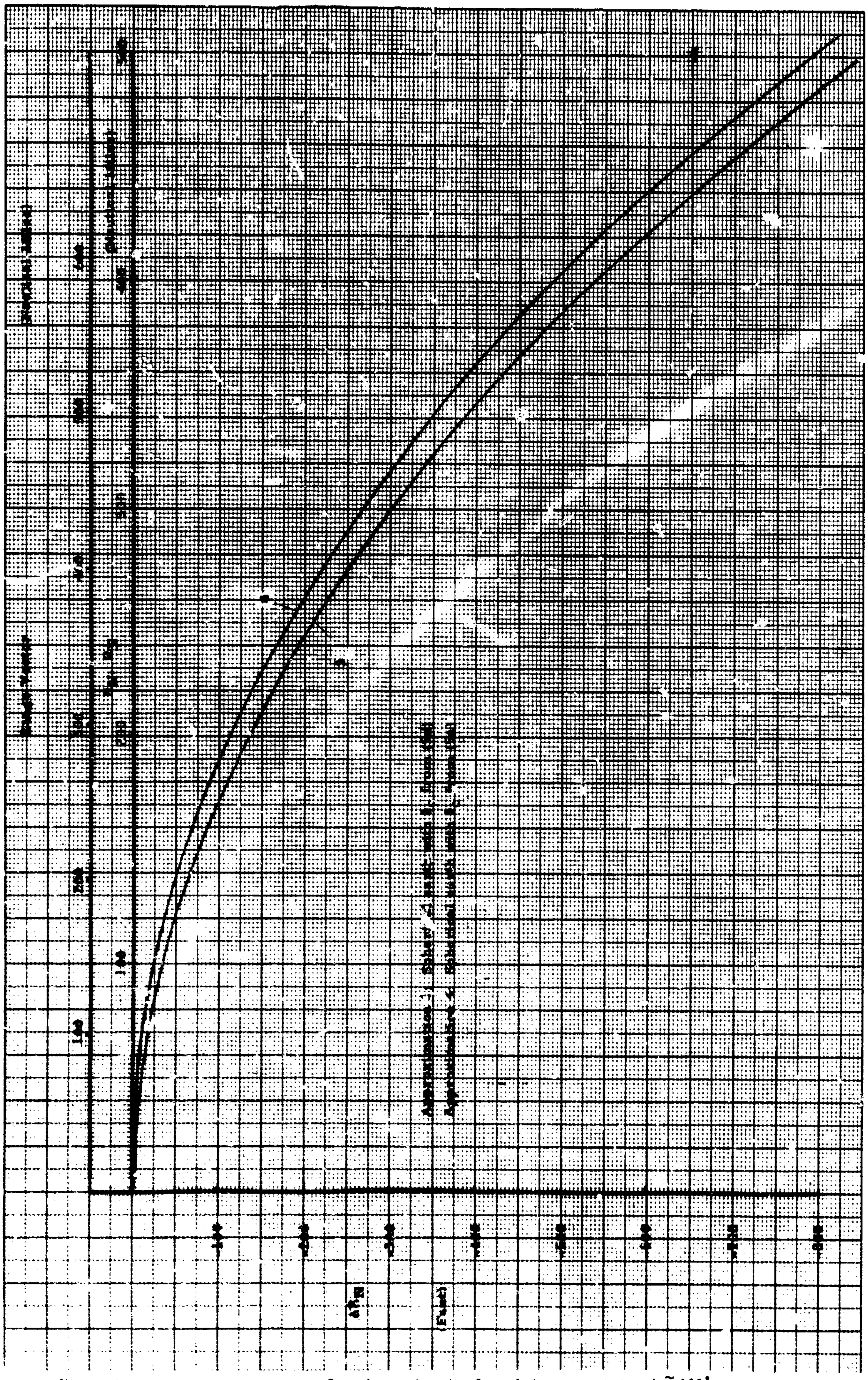


Figure 5[illegible] ΔR_N Approximation 3 and 4 Azimuth $\cong$ 135°

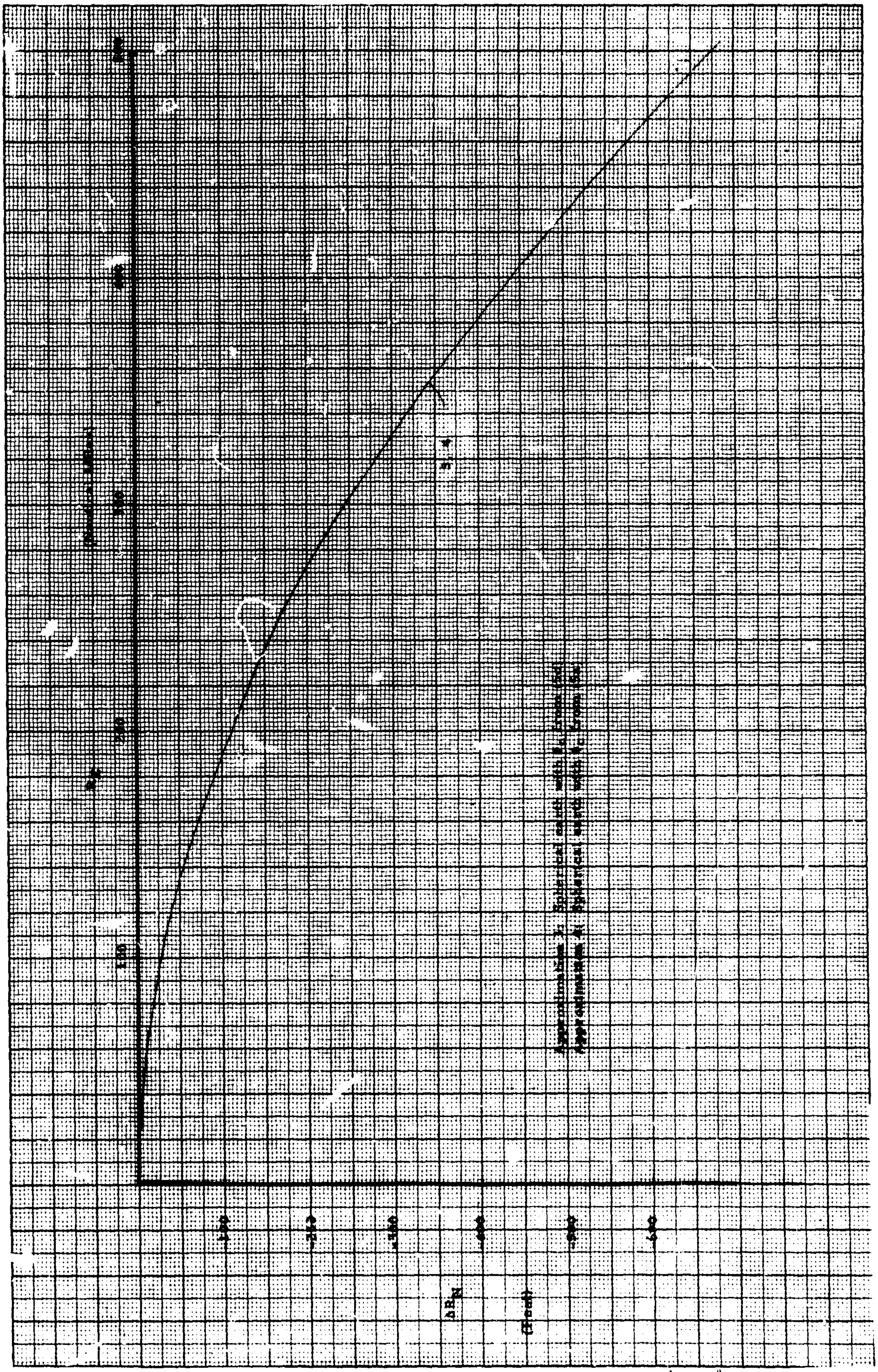


Figure 9   25   Approximation 3 and 4   Azimuth 89° to 91°

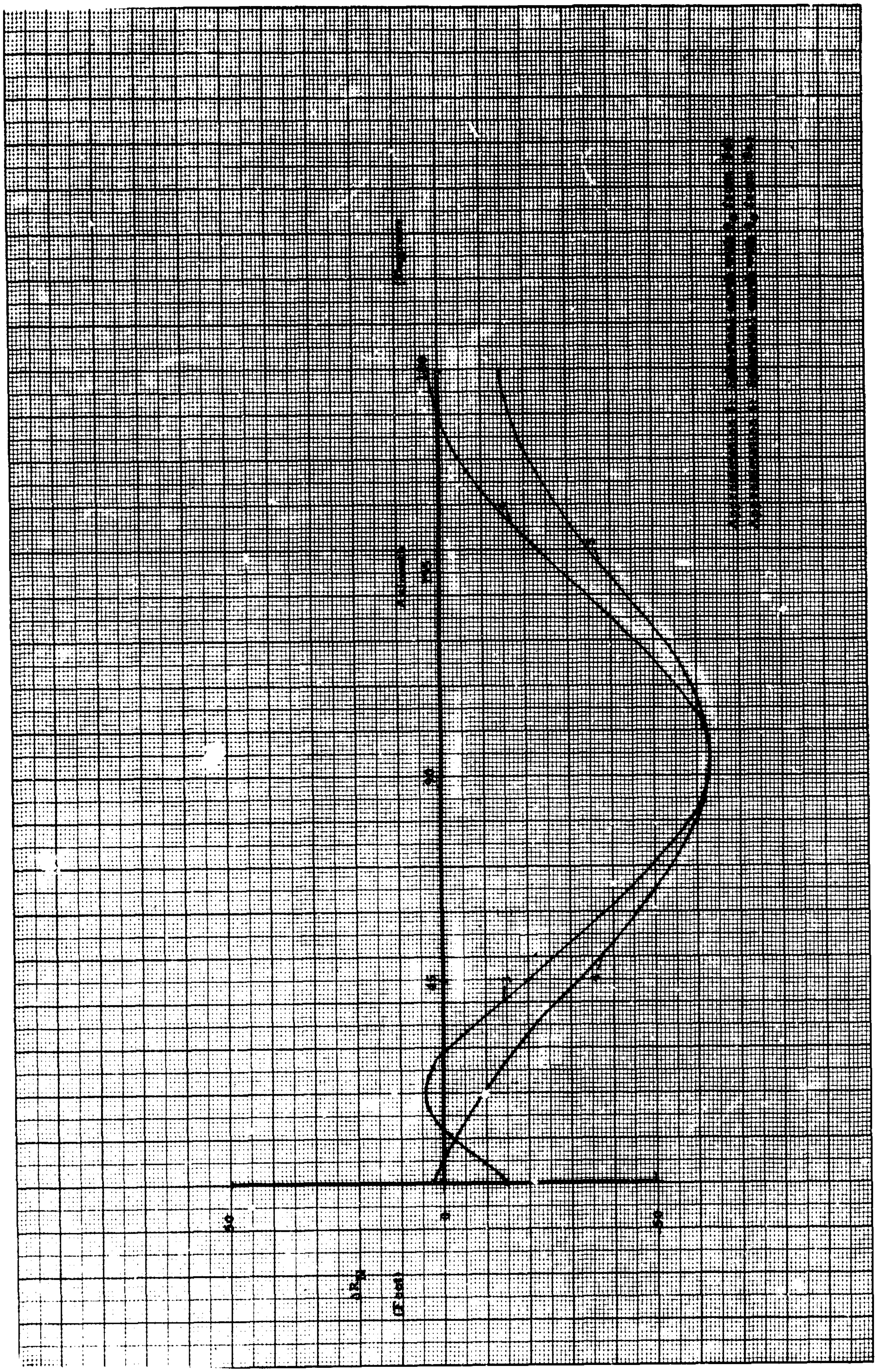


Figure 10 ΔR_N versus Azimuth Approximation 3 and 4 $R_{Vector} \cong$ 150 Nautical Miles

Table III

R_N AND R_E FOR CLARKE'S SPHEROID 1866 AS FUNCTION OF RANGE AND AZIMUTH

Azimuth: 0°, 45°, 135°

R_N* Nautical Miles	Azimuth: 0° $\Delta\Phi°$ / $\Delta\lambda°$	R_N (ft) / R_E (ft)	Azimuth: 45° $\Delta\Phi°$ / $\Delta\lambda°$	R_N (ft) / R_E (ft)	Azimuth: 135° $\Delta\Phi°$ / $\Delta\lambda°$	R_N (ft) / R_E (ft)
60	1.0	363829	1.0	363637	-1.0	-362744
	0	0	1.185	361245	1.185	365344
100	1.666	606085	1.666	612588	-1.666	-600221
	0	0	1.975	599684	1.975	611062
150	2.5	909332	2.5	922911	-2.5	-895867
	0	0	2.97	897177	2.97	922844
200	3.32	1207300	3.32	1228462	-3.32	-1183256
	0	0	3.94	1183989	3.94	1229189
400	6.64	2410358	6.64	2489249	-6.64	-2309093
	0	0	7.8684	2310904	7.8684	2490940
500	8.33	3020213	8.33	[illegible]	-8.33	-2956144
	0	0	9.871	2962274	9.871	[illegible]

* Approximate values for nautical miles.

Exact values for $\Delta\Phi$ and $\Delta\lambda$

$$\Delta\Phi = \Phi_{g2} - \Phi_{g1} \qquad \Delta\lambda = \lambda_2 - \lambda_1$$

Table IV

R_N AND R_E FOR CLARKE'S SPHEROID 1866 AS FUNCTION OF RANGE AND AZIMUTH

Azimuth: 89°, 90°, 91°

R_E * Nautical Miles	Azimuth 89° $\Delta\Phi^\circ$ / $\Delta\lambda^\circ$	R_N(ft) / R_E(ft)	Azimuth 90° $\Delta\Phi^\circ$ / $\Delta\lambda^\circ$	R_N(ft) / R_E(ft)	Azimuth 91° $\Delta\Phi^\circ$ / $\Delta\lambda^\circ$	R_N(ft) / R_E(ft)
60	0.0175	8430	0	2064	-0.0175	- 4304
	1.18982	364750	1.19	364841	1.18982	354822
100	0.02917	16340	0	5280	-0.02917	- 4880
	1.98303	607798	1.983	607889	1.98303	607998
150	0.04375	28799	0	12891	-0.04375	- 3017
	2.97455	911396	2.9745	911605	2.97455	911845
200	0.05834	44115	0	22913	-0.05834	1712
	3.96606	1214667	3.966	1215049	3.96606	1215465
400	0.11668	133794	0	91541	-0.11668	49294
	7.93212	2422718	7.932	2424279	7.93212	2425901
500	0.14585	195578	0	142904	-0.14585	90239
	9.91515	3022454	9.915	3024899	9.91515	3027419

* Approximate values for nautical miles

Exact values for $\Delta\Phi$ and $\Delta\lambda$

$\Delta\Phi = \Phi_{g2} - \Phi_{g1}$ $\qquad$ $\Delta\lambda = \lambda_2 - \lambda_1$

This simplification produces an error which is insignificant compared to other errors, but it substantially reduces the numerical effort. The geocentric earth radius at point 2, R_{c2}, is used to compute the east/west component R_E from

$$R_E = R_{c2} \cos\Phi_{c2} \sin\Delta\lambda \qquad (10b)$$

This expression is exact and values of R_E computed from (10b) are identical with those derived from (9g) if the exact value of Φ_{c2} is used. But even for approximated geocentric angles Φ_{c2} the value of R_E computed from (10b) agreed for all examples very closely (the maximum difference found was 3.4 feet) with the value of R_E resulting from approximation 4. R_E values are therefore omitted from further discussion.

To make various results comparable, the assumption was made in all cases that the center of the range between the aircraft and the ground target was at 33° geodetic latitude. Basically, two parameters have been varied in the numerical examples. These are, first, the length of the range vector for a given constant azimuth and, second, the azimuth for a given constant range vector.

The azimuth A_z is the angle between north and the range vector, counted positive in clockwise direction. It is computed as an approximation from angular data ($\Delta\lambda$, $\Delta\Phi_g$) instead of from the cartesian coordinates.

$$A_z = \arctan \frac{306611\Delta\lambda}{363848\Delta\Phi_g}$$

Six different azimuth angles are considered to demonstrate the general behavior of R_N as function of the range vector length. They are 0°, 45°, 135°, 89°, 90° and 91°. For these angles, R_N was computed from equation (9e) as function of the range; the results are listed in Tables 3 and 4.

As can be expected, R_N grows nonlinearly with the length of one of the vector coordinates which is listed in nautical miles. The sign of R_N is positive or negative (same as the sign of $\Delta\Phi$) and stays so, independent of the vector length. An exception to this rule is found for an azimuth slightly larger than 90° as can be seen at an angle of 91°. (Table 4). This unusual situation can be explained as follows:

The sign of R_N depends in this special case on where point 2 is located with respect to a borderline which is defined by

$$\cos\Phi_1 \sin\Phi_2 = \cos\Phi_2 \sin\Phi_1 \cos\Delta\lambda \tag{11}$$

This equation is found from (7a) by setting $R_N = 0$.

If this equation is satisfied, R_N vanishes; if it is not satisfied, the location of point 2 north or south determines the sign. For a 90° azimuth therefore, R_N is a relatively small positive or negative quantity, depending on whether point 2 is west or east of point 1, respectively. Increasing the vector length and, as a result, also increasing R_E for the case of a 91° azimuth, point 2 changes its position from one side to the other side of this border line and R_N changes its sign accordingly. This situation has some effect on an application discussed in the next section.

Considering the accuracy of the circular arc approximation first, it is found from figure 6 that approximation 1 is not adequate for ranges R_N exceeding 0.6 nautical miles and approximation 2 is restricted to ranges R_N up to 33 nautical miles provided, of course, that the two points are located on the same meridian; for larger values of R_N, an error of at least 10 feet is made. If the two points are not on the same meridian, approximations 3 and 4 can be used over a comparatively wide range to estimate R_N with tolerable errors. This is demonstrated in figures 7 through 10.

Inspecting these figures, we find that as a general trend for both approximations of R_N (approximations 3 and 4) the error ΔR_N is negative and its absolute value grows as a nonlinear function with the vector length. Its value varies between 20 feet and 135 feet at 200 nautical miles depending on the azimuth angle and reaches more than 600 feet at 500 nautical miles. In the neighborhood of 90°, the two approximations, 3 and 4, yield almost identical values; the differences are negligible for practical purposes between 89° and 91°. Of particular interest is the fact that for the azimuth of 45° the "poorer" approximation (3), representing the spherical earth model, leads to a value of R_N more closely resembling the value corresponding to the spheroidal earth. This, of course, is an effect of the bias error in the geocentric latitudes derived from the truncated equation mentioned earlier.

More information on the behavior of approximations 3 and 4 is found from figure 10; it shows for a constant vector length of about 150 nautical miles the errors R_N as functions of the azimuth angle. The maximum is found at 90°. From this diagram, it is also evident that within some interval between 10° and 90°, approximation 3 yields more accurate values of R_N than approximation 4. This includes the case of 45° mentioned earlier.

It is noted that all results of ΔR_N shown in figures 6 through 10 were obtained for an altitude of the target h_2 equal to zero. An experiment showed that the value of h_2 has no significant effect on ΔR_N. Two values of h_2 were considered: $h_2 = 0$ and $h_2 = 4000$ feet MSL, which is approximately the elevation of the basin of White Sands Missile Range. The effect was 2 feet or less when h_2 was changed from 0 to 4000 feet. As can be seen in equation (9e), h_2 does have an effect on R_N itself, whereas the height of the airplane h_1 does not.

SECTION IV

IMPROVEMENTS OF THE SPHERICAL EARTH APPROXIMATIONS

An improvement of the approximations based on a spherical earth model (approximations 3 and 4 in the previous sections) is desirable in the sense that R_N, derived from a slightly modified model, approaches more closely the value of R_N which is based on a spheroidal earth model. This is desired over a certain interval of R_N values. There may be various ways of achieving this improvement. As it has been mentioned earlier, approximation 3 may yield better results in this respect than approximation 4; the improvement is caused in this case by a bias error affecting geocentric latitudes. One way which appears practical and efficient on account of some experiments is described in this section.

Considering equation (7a), and rewriting it as

$$R_N = F_1 F_2 \tag{12a}$$

where

$$F_1 = R_C + h_s$$

$$F_2 = \cos\Phi_1 \sin\Phi_2 - \cos\Phi_2 \sin\Phi_1 \cos\Delta\lambda$$

it is evident that a systematic or bias type error either in F_1 or F_2 or in both may affect the value of R_N. Introducing a bias in a controlled way may lead to the improvement of the approximation. This has been done by adding a correction term ΔR_c to F_1 using the following equations:

$$F_1 = (R_c + h_s) + \Delta R_c$$

where

$$\Delta R_c = \Delta R_{N_0} \frac{R_c}{R_{N_0}} \frac{\Delta\Phi(R_N)}{\Delta\Phi_0} \tag{12b}$$

$\Delta\Phi(R_N)$	difference of geodetic latitudes between point 1 and 2.
$\Delta\Phi_0$	one selected $\Delta\Phi$, for which ΔR_N is to be reduced to zero by adjustment.
R_c	mean geocentric earth radius.
ΔR_{N_0}	the error to be adjusted or compensated.
R_{N_0}	the north component of the total range for which ΔR_{N_0} is to be adjusted.

The following numerical example explains the procedure for finding the value of ΔR_c. The intention is to make ΔR_N zero at a range R_N of 200 nautical miles for an azimuth angle of 135°. The corresponding, uncompensated error ΔR_N for approximation 3 is found from figure 8b as

$$\Delta R_N = -135 \text{ feet}$$

To compensate this error, R_c must be reduced by a proportionate value

$$\Delta R_{co} = -135 \frac{R_c}{R_{No}} \text{ feet}$$

With $R_c = 20,904,910$ feet from equation (4b) for $\Phi_g = 33°$ and with $R_{No} =$ 1,183,256 feet from table 3 for 200 nautical miles ($A_2 = 135°$), we obtain

$$\Delta R_{co} = -2,367 \text{ feet}$$

For values of R_N other than 200 nautical miles

$$\Delta R_c = \Delta R_{co} \frac{\Delta\Phi(R_N)}{\Delta\Phi_o}$$

For instance, for $R_N = 100$ nautical miles, using the value of $\Delta\Phi = 1.666°$ from table 3 and $\Delta\Phi_o = 3.32$ for 200 nautical miles, we find

$$\Delta R_c = -2367 \frac{1.666}{3.32}$$

$$= -1187 \text{ feet}$$

In the same way, additional correction factors can be determined between $R_N = 0$ and $R_N = 500$ nautical miles.

For later reference (12b) is rewritten as

$$\Delta R_c = C_1 \Delta\Phi(R_N) \qquad ([illegible])$$

where C_1 is a constant which depends on the azimuth and the range for which ΔR_N is compensated. For one particular vector length, the error ΔR_N is compensated and for values above and below it, ΔR_N is reduced to some extent. Numerical values of ΔR_c were used in connection with the Marchant Calculator Program described in appendix B to compute improved approximations of R_N and the results are referred to in this report as "approximation 5". Results obtained by this error compensation are shown in figures 11, 12a, 12b, 13 and 14*. For C

* These figures also contain the "unreduced error" ΔR_N as obtained from approximation 4.

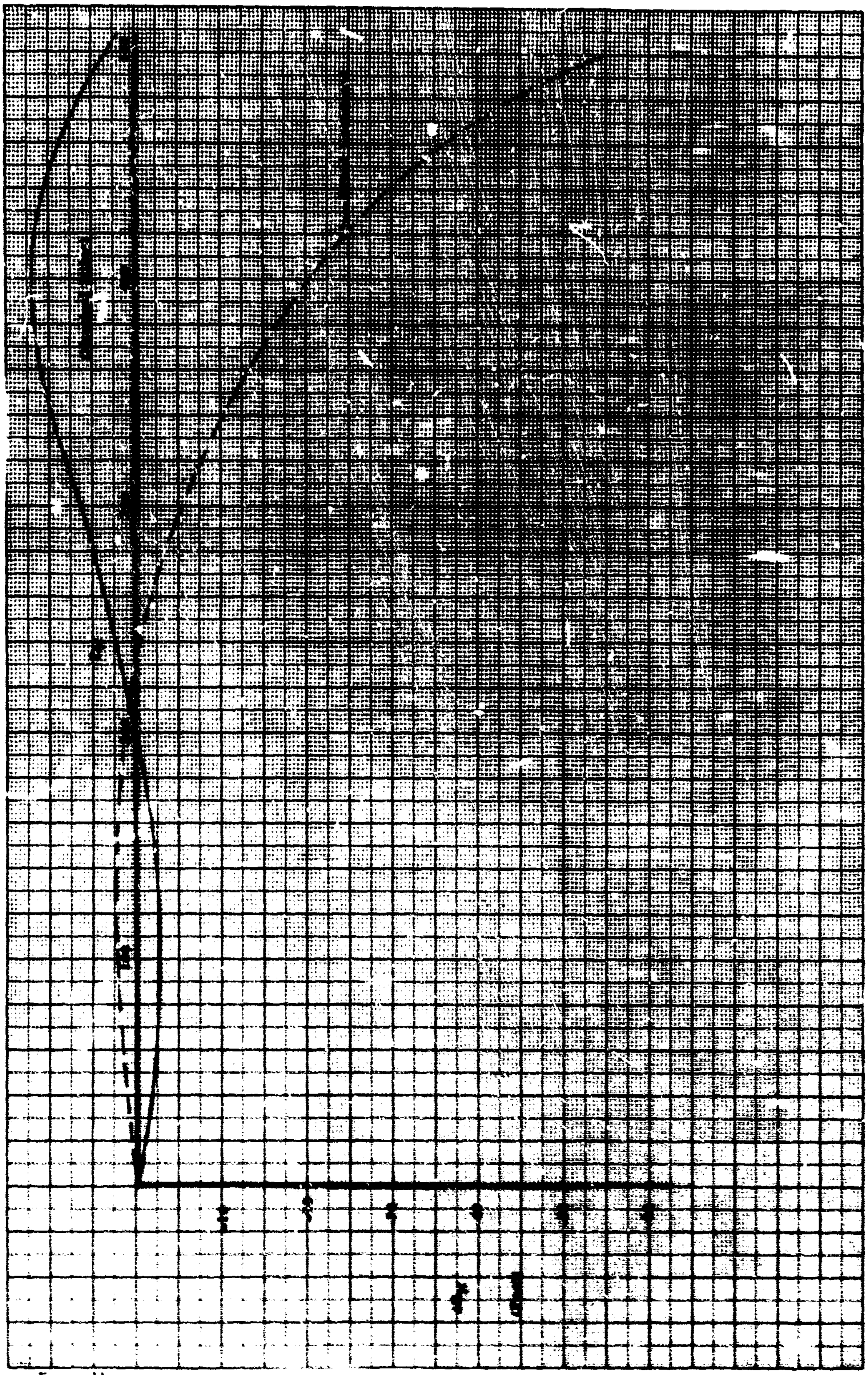


Figure 11   Reduced Error δR_N versus R_N Azimuth = 0°

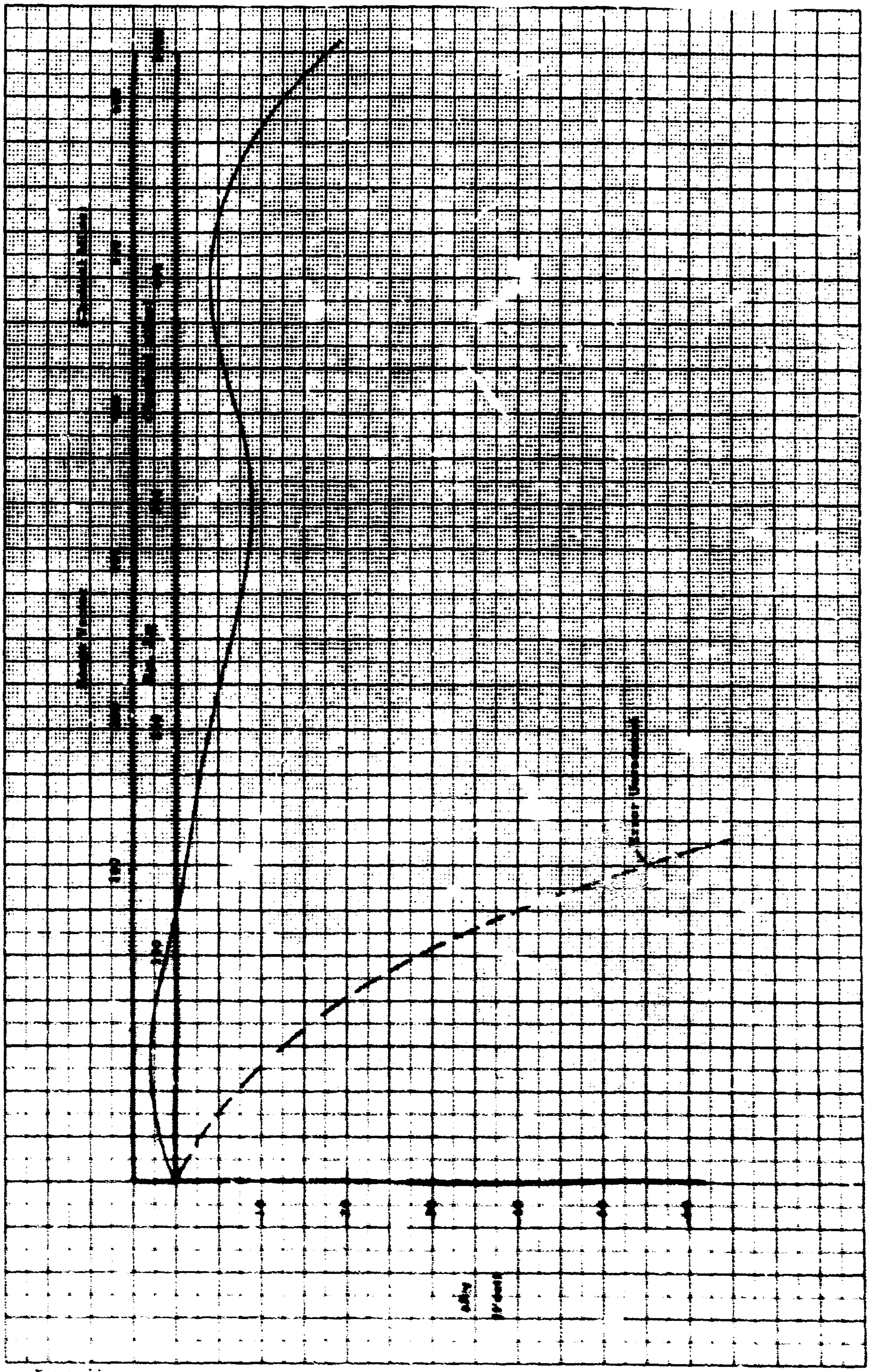


Figure 11a

Reduced Error ΔR_1 versus R_1, R_3 Azimuth = 45°

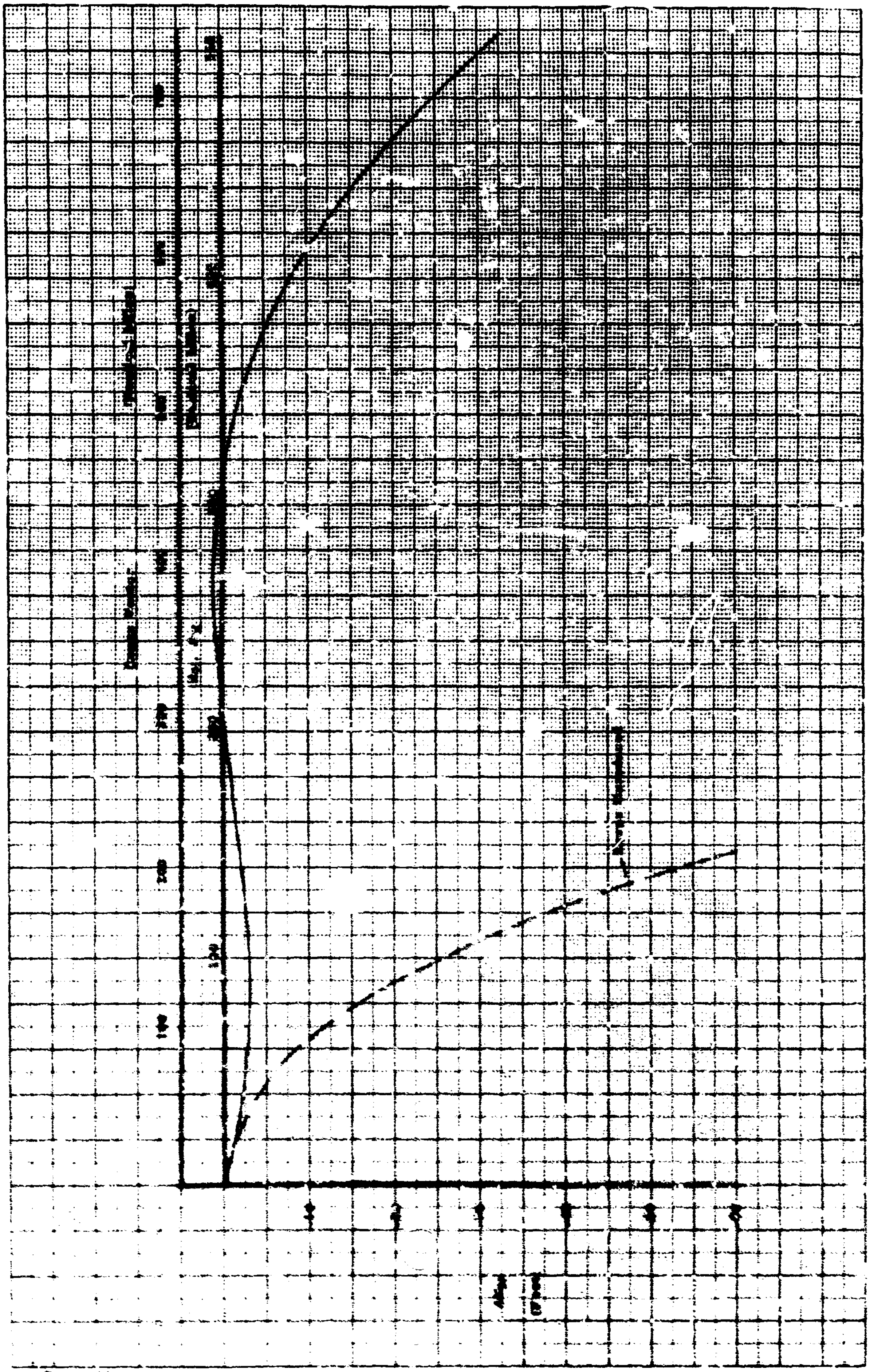


Figure …

Required … R_x versus R_y … Azimuth …

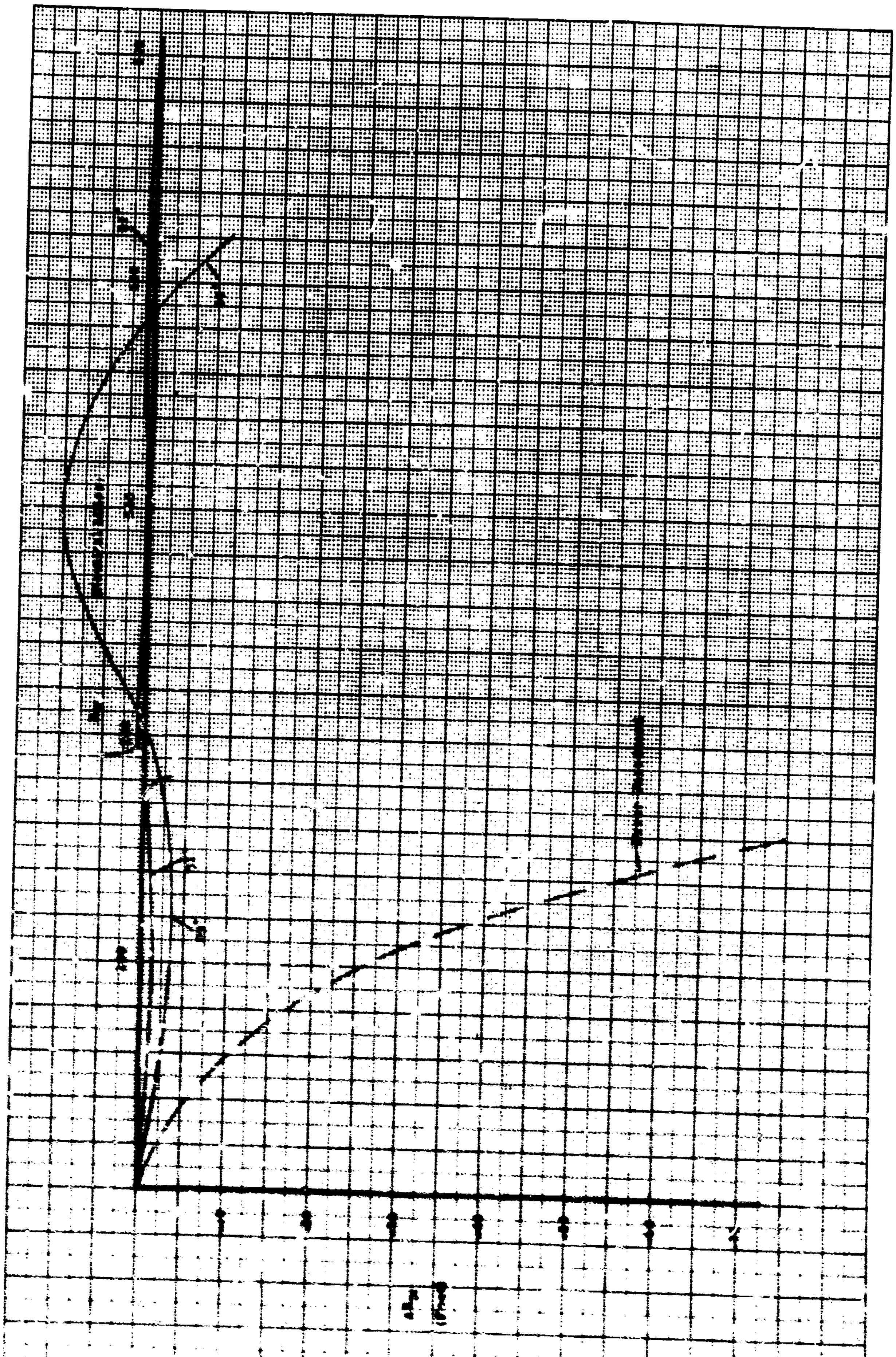


Figure 11

Reduced Error ... Azimuth ... and ...

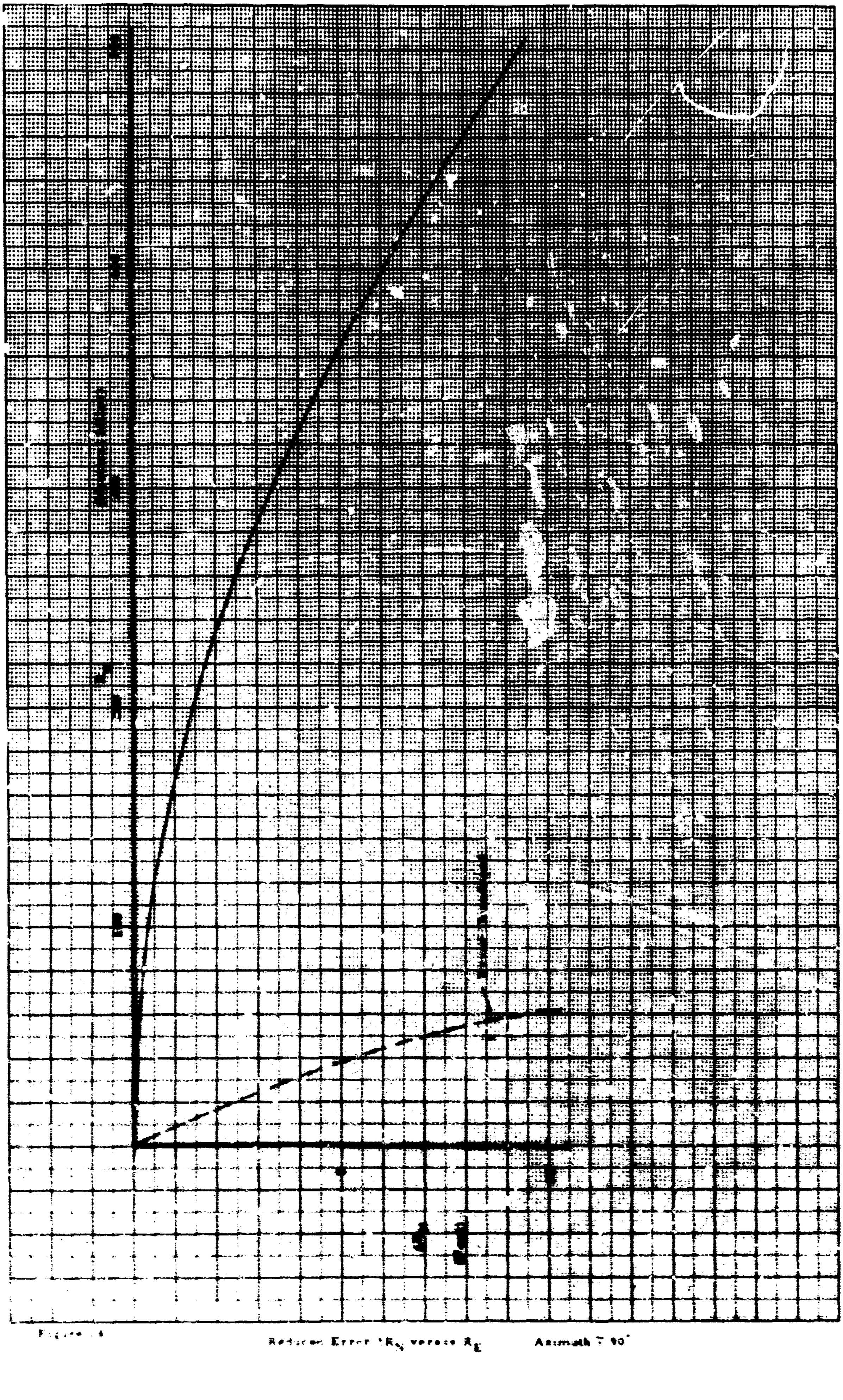


Figure 14

Relative Error ΔR_s versus R_E Azimuth = 90°

azimuth (figure 11), the error adjustment is of practical value only at large values of R_N(R_N>300 nautical miles), simply because approximation 4 leads to comparatively small errors ΔR_N at lower R_N values. For azimuth angles other than 0°, a significant improvement is found if R_N (or R_E) exceeds 100 nautical miles. A similar reduction of ΔR_N as for 45° and 135° (figures 12a and 12b) was obtained for 67.5° azimuth also by applying equation (13). This equation can be used in general for reducing the error except near 90° azimuth. For 90° itself, a very simple and effective error correction was achieved by adding a constant amount of

$$\Delta R_c = 98925 \text{ feet}$$

to the mean geocentric earth radius R_c. The result is shown in figure 14. It is noted here that the value of this constant and of those listed below depend on the mean geodetic latitude which in the case considered here was always 33°.

The special situation for azimuth angles slightly bigger or smaller than 90° requires a slightly different adjustment. Considering first the case of an azimuth slightly bigger than 90° (e.g. 91°), the value of ΔR_c was computed from

$$\Delta R_c = C_1 \frac{\Delta \Phi}{\Delta \Phi - C_0} \qquad (14)$$

and ΔR_c is used again to adjust the factor F_1 in (12a) as

$$F_1 = R_e + h_e + \Delta R_c$$

The resulting reduced error ΔR_N is shown in figure 13. In a small region which is close to 186 nautical miles, equation (14) cannot be used because the denominator becomes zero. To circumvent this difficulty, another set of coefficients C_0 and C_1 must be chosen *.

In the other case, if the azimuth is slightly below 90° (e.g. 89°), a different equation is used to compute ΔR_c. A second degree approximation was found adequate for the error adjustment:

$$\Delta R_c = C_1 \Delta \Phi + C_2 \Delta \Phi^2 \qquad (15)$$

* A better solution is probably to use two sets of coefficients for equation (13) corresponding to the positive and negative values of R_N.

For the examples shown in figures 11, 12a, 12b and 13, the following coefficients have been used in equations (13), (14) and (15):

Azimuth (degrees)			
0	C_1	=	90.030 feet/degree
45	C_1	=	399.24 feet/degree
135	C_1	=	713.08 feet/degree
89	C_1	=	1,219,609 feet/degree
	C_2	=	5,691,402 feet/degree^2
91	C_0	=	0.0543 degree
	C_1	=	100,305.5 feet

For an azimuth of 67.5° the coefficient C_1 was 713.08 feet/degree.

SECTION V

CONCLUSIONS

The study of methods and of numerical expressions for the conversion of geodetic data to local tangent plane coordinates leads to the following conclusions:

1. The equations for computing the north and east components R_N and R_E, based on a spherical earth model can be mechanized on the programmable Marchant desk calculator 1016 PR and stored on a minimum of three magnetic tapes with a storage capacity of 100 bits each. The program uses an approximation for computing the geocentric latitude.

2. Depending on the azimuth of the local tangent plane vector, errors of up to 37 feet in R_N are caused by this program for a 100 nautical mile distance in north (or east) direction using the spheroidal earth model as reference. This error grows rapidly with increasing length of the vector. The error in R_E is negligible.

3. The error in R_N as mentioned can be reduced by using precalculated correction factors which depend on the azimuth of the local tangent plane vector. Further investigation is needed to determine the optimum correction factors as function of azimuth.

4. The effect of the height of the target point (point 2) in the interval from 0 to 4,000 feet (MSL) is found insignificant with respect to the error ΔR_N. Discrepancies in the results did not exceed 2 feet for all ranges investigated.

5. A simplified equation is derived to determine R_N, R_E and R_Z based on the spheroidal earth model: the numerical effort for solving it is reduced compared to the effort associated with the standard procedure.

6. Because of the reduction of the numerical effort, the results and simplifications presented are applicable for quick look postflight evaluation and to some extent for real time data processing.

SECTION VI

ACKNOWLEDGEMENTS

Capt J. D. Hopkins, SRAM Project Officer at Detachment 1, Air Force Weapons Laboratory, during the time this work was done, recommended preparing this report based on a memorandum on the same subject submitted by the author; he also gave advice and guidance to the latter in discussions during the report writing period. The support given by Mr. D. H. Liston was valuable; his suggestions for modifying and rearranging the text and the tables and for omitting some details helped to enhance the clarity of the report.

APPENDIX A

COMPUTATION OF GEOCENTRIC LATITUDE

Based on equation (5a) in the basic report, a power series is derived to approximate Φ_c and a truncated version of this series can be used to compute Φ_c for angles in the neighborhood of a given value of Φ_g. To find the power series, equations (5a) and (2) are inserted into

$$\tan(\Phi_g - \Phi_c) = \frac{\tan\Phi_g - \tan\Phi_c}{1 + \tan\Phi_c \tan\Phi_g}$$

which yields, after manipulation of trigonometric relationships,

$$\tan(\Phi_g - \Phi_c) = \frac{\epsilon^2}{2} \frac{\sin 2\Phi_g}{1 - \epsilon^2 \sin^2\Phi_g} \qquad \text{(A1)}$$

Assuming that $\Phi_g - \Phi_c$ is a small quantity and applying the binominal expansion, one finds the difference $\Phi_g - \Phi_c$ in degrees as

$$\Phi_g - \Phi_c \cong \frac{180}{\pi} \frac{\epsilon^2}{2} \sin 2\Phi_g (1 + \epsilon^2 \sin^2\Phi_g + \epsilon^4 \sin^4\Phi_g + \ldots)$$

Solving for Φ_c and carrying only one term yields

$$\Phi_c \simeq \Phi_g - \frac{180}{\pi} \left(\frac{\epsilon^2}{2} + \frac{\epsilon^4}{2} \sin^2\Phi\right) \sin 2\Phi_g \qquad \text{(A2)}$$

One may rewrite (A2) so that the term containing $\sin^2\Phi_g$ disappears for one particular value of Φ_g, and that the equation for Φ_c is of the form

$$\Phi_c = \Phi_g - K_1 \sin 2\Phi_g + K_2 (1 - K_3 \sin^2\Phi_g) \sin 2\Phi_g \ldots \qquad \text{(A3)}$$

To determine the value of the coefficients K_1, K_2 and K_3, one may then proceed as follows: assuming a certain special value for Φ_g called $\Phi_{g,s}$ is chosen, one substitutes this value into equation (A2) and finds

$$K_1 = \frac{180}{\pi} \left(\frac{\epsilon^2}{2} + \frac{\epsilon^4}{2} \sin^2\Phi_{g,s}\right)$$

The two following equations, derived from (A2) and (A3), are then compared to determine the coefficients K_2 and K_3

$$\phi_c \simeq \phi_g - (K_1 - K_2)\sin 2\phi_g - K_2 K_3 \sin^2\phi_g \sin 2\phi_g$$

$$\phi_c \simeq \phi_g - \frac{180}{\pi}\left(\frac{\epsilon^2}{2} + \frac{\epsilon^4}{2}\sin^2\phi_g\right)\sin 2\phi_g$$

To satisfy both equations, it is found that

$$K_2 = \frac{180}{\pi}\frac{\epsilon^4}{2}\sin^2\phi_{g,s}$$

$$K_3 = 1/\sin^2\phi_{g,s}$$

Inserting the numerical value for $\sin^2\phi_g$, for $\phi_g = 30°$ equation (A2) is changed to be

$$\phi_c \simeq \phi_g - \frac{180}{\pi}\left(\frac{\epsilon^2}{2} + \frac{\epsilon^4}{8}\right)\sin 2\phi_g + \frac{180}{\pi}\frac{\epsilon^4}{8}(1 - 4\sin^2\phi_g)\sin 2\phi_g \tag{A4}$$

In a similar way for $\phi_g = 33°$

$$\phi_c \simeq \phi_g - \frac{180}{\pi}\left(\frac{\epsilon^2}{2} + \frac{\epsilon^4}{2}\,0.29663168\right)\sin 2\phi_g$$

$$+ \frac{180}{\pi}\frac{\epsilon^4}{2}\,0.29663168\,(1 - 3.371184\sin^2\phi_g) \tag{A5}$$

Applying a value for ϵ corresponding to Clarke's spheroid 1866, finally for $\phi_{g,s} = 30°$

$$\phi_c \simeq \phi_g - \{0.19423549 - 0.00032812\,(1 - 4\sin^2\phi_g)\}\sin 2\phi_g \tag{A6}$$

and for $\phi_{g,s} = 33°$

$$\phi_c \simeq \phi_g - \{0.19429670 - 0.00038933\,(1 - 3.371184\sin^2\phi_g)\}\sin 2\phi_g \tag{A7}$$

Using equations (A6) and (A7), the geocentric latitude can be calculated accurately enough for many purposes in the neighborhood of 30° or 33° geodetic latitude. Equation (A6) is found in reference 3 with rounded-off coefficients. If only one term is used, e.g., for $\phi_{g,s} = 30°$

$$\phi_c - \phi_g \simeq -0.19423549\sin 2\phi_g \tag{A8}$$

and for $\phi_{g,s} = 33°$

$$\phi_c - \phi_g \simeq -0.19429670\sin 2\phi_g \tag{A9}$$

a still relatively small error is made. It is depicted in figure 2. Equation (A9) is used in the Marchant 1016 PR computer program described in appendix B.

APPENDIX B

COMPUTER PROGRAM FOR THE MARCHANT 1016 PR CALCULATOR

Purpose of the program was to compute from given geodetic data (longitude, latitude and altitude) the tangent plane coordinates R_N and R_E of point B with respect to the origin at point A based on a spherical earth model. A minimum of memory space of the calculator had to be used without significant loss of accuracy. The program consists of 4 sections each recorded on a magnetic tape, read into the calculator from the IOTA tape unit.

Tape 1:

This part of the program computes the geocentric latitudes Φ_c for point 1 and point 2 using the approximation

$$\Phi_c = \Phi_g - 0.1942967 \sin 2\Phi_g \;^{*}$$

where

Φ_g = geodetic latitude in degrees,

and it computes the mean geocentric latitude $\bar{\Phi}_c$.

Input:

$\Phi_{g,1}$, $\Phi_{g,2}$, (in degrees) are loaded manually into the keyboard register

Output:

$\Phi_{c,1}$ in W1, $\Phi_{c,2}$ in W2, $\bar{\Phi}_c = (\Phi_{c,1} + \Phi_{c,2})/2$ in W3

All three angles in degrees.

Code**

```
♯ W2 M X .034906585 04 X   W5 M T W5

1 M N X W6 R ÷ 1 • S ÷ 1 • S W5 A BO [illegible] BX 27

T R X   -1 • T √ X .1942967   -W2 R • S M

W1 R BO 93 • T ÷ 2   W3 M K T M K BX 00
```

* This expression is derived in appendix A.

** Marchant symbols are used for the code except for the negative sign. In the code list above, N stands for negative sign. Marchant uses ♯ for plus and negative sign.

Controls set before input values are read into the keyboard:

10 digits [*], clear all registers, round-off switch on.

Explanations:

$\Phi_{g,1}$ is inserted in the keyboard before the "run" key is operated and the value of $\Phi_{g,1}$ is printed immediately before the actual computation starts. Insertions of $\Phi_{g,2}$ follows when the program comes to the first stop.

Tape 2:

This part of the program computes the cosines of $\Phi_{c,1}$, $\Phi_{c,2}$, $\bar{\Phi}$ and $\Delta\lambda$.

Input:

$\Phi_{c,1}$ in W1, $\Phi_{c,2}$ in W2, $\bar{\Phi}_c$ in W3 (all in degrees), $\Delta\lambda$ in degrees manually into the keyboard register at the first stop.

Output:

$\cos\Phi_{c,1}$ in W1, $\cos\Phi_{c,2}$ in W2, $\cos\bar{\Phi}_c$ in W3, $\cos\Delta\lambda$ in W4

Code:

```
÷ X .01745329252 X = W6 M W5 1 M N X
W6 R ÷ 1 + S : 1 + S W5 A BO 47 BX 25 T R +
W1 R BO 76 W2 R BO 84 W3 R BO 92 T W4 M
K T M W2 BX 00 T M W3 BX 00 T M K # BX 01
```

Controls set before running section 2 of the program:

10 digits, round-off switch on, select W'.

Tape 3:

This part computes the mean geocentric earth radius R_c and the north coordinate R_c of point 2 in a tangent plane with point 1 as the origin based on the equations (B1) and (B2).

$$R_c = A \frac{1}{(B - \cos^2 \Phi_c)^{\frac{1}{2}}} \qquad (B1)$$

[*] 12 digit precision avoids an error of 2 feet, which occurs if the function $\sin^2$ is computed with 10 digits only, for $\Delta\lambda \approx 0.01$ degree.

where

$$A = \frac{R_{eq}}{[\epsilon^2/(1-\epsilon^2)]^{\frac{1}{2}}}$$

$$B = \frac{1-\epsilon^2}{\epsilon^2} + 1$$

with

$$\epsilon^2 = 0.0067686441$$

and

$$R_{eq} = 20,925,832$$

$$A = 2.5348783$$

$$B = 147.7401$$

Note that R_c, A, R_{eq}, and R_N are all scaled down by a factor of 10^7. The formulation of (B1) reduces the memory space requirement to a minimum.

$$R_N = (R_c + h_e)(\sin\Phi_{c,2}\cos\Phi_{c,1} - \sin\Phi_{c,1}\cos\Phi_{c,2}\cos\Delta\lambda) \qquad (B2)$$

Input:

$\cos\Phi_{c,1}$ in W1, $\cos\Phi_{c,2}$ in W2, $\cos\bar{\Phi}$ in W3, $\cos\Delta\lambda$ in W4, h_e into keyboard at the first stop.

Printed Output:

R_N in feet.

Code:

```
W3 ÷ X -147.7401 + T√ ÷ 2.5348783
÷ T A K ♦ A W2 R X -1 + T√ X W1 R ÷ M T X
-1 + T√ X W2 R X W4 R ÷ -W1 R + T X W3 R ÷ ♦ K
```

Controls set before running the programs:

10 digits, round-off switch on.

<u>Tape 4</u>:

To compute the east component R_E, first the calculation of the geocentric angles is repeated using tape 1 and tape 2 programs. Then tape 4 is read into the calculator to compute

$$R_{c,2} = A\frac{1}{(B-\cos^2\Phi_{c,2})^{\frac{1}{2}}}$$

where as before:

$$A = \frac{R_{eq}}{[\epsilon^2/(1-\epsilon^2)]^{\frac{1}{2}}}$$

$$B = \frac{1-\epsilon^2}{\epsilon^2} + 1$$

and

$$R_E = (R_{c,2} + h_2)\cos\Phi_{c,2}\sin\Delta\lambda$$

Input:

$\cos\Phi_{c,2}$ in W2, $\cos\Delta\lambda$ in W4, h_2 into keyboard at the first stop.

Printed Output:

R_E in feet

Code:

W2 R X = -147.7401 + T√ + 2.5348783 ÷ T

W3 A K # A W4 R X = -1 + T√ X W3 X W2 R = # K

Controls for running the program: 10 digits, round-off switch on.

The codes listed above are not necessarily the shortest possible codes.

Remarks on Program Operation With Three Tapes Instead of Four.

With very little extra effort, tape 4 can be eliminated and both components, R_N and R_E, can be computed using only 3 tapes. Tape 3 has enough unused space to carry the program for computing R_E. After R_N has been computed tapes 1 and 2 have to be rerun to generate $\cos\Phi_{c,2}$ unless the value of this quantity was saved at the end of the program 2 run by printing the contents of W2. Before running tape 3 again $\cos\Phi_{c,2}$ must be transferred into W3 (and also kept in W2). The code of the additional, second part of tape 3 for computing R_E is listed below:

W4 R X = -1 + T√ X W3 R X W2 R = # K

The stop code K at the end of the first part of the tape 3 program (see page 47) must be replaced by the first letter of the second part, which is W. If this operation is performed with only 3 tapes, then the result of R_E printed at the end of the first path and the result of R_N printed at the end of the second path are disregarded.

APPENDIX C

TRANSLATING AND ROTATING GEOCENTRIC COORDINATES TO A LOCAL TANGENT PLANE

This appendix derives formulas which are used in a standard computer program for coordinate transformation under the option for converting geodetic to local tangent plane coordinates. The pertinent FORTRAN statements are listed in appendix D. First, the expressions for earth centered (ecs) coordinates (equation 8)*

$$X_{ecs} = R_C \cos\Phi_C \cos\lambda \quad \text{(C1a)}$$

$$Y_{ecs} = R_C \cos\Phi_C \sin\lambda \quad \text{(C1b)}$$

$$Z_{ecs} = R_C \sin\Phi_C \quad \text{(C1c)}$$

are manipulated by inserting the equation for the geocentric earth radius R_C (equation 4b)* and replacing in the expression for R_C the term $\epsilon^2/(1-\epsilon^2)$ by $(a^2-b^2)b^2$

where

a = equatorial radius

b = polar radius

We find

$$X_{ecs} = a\left(1 + \frac{a^2}{b^2}\tan^2\Phi_C\right)^{-\frac{1}{2}}\cos\lambda$$

and using (5a)*

$$X_{ecs} = a^2(a^2 + b^2\tan^2\Phi_g)^{-\frac{1}{2}}\cos\lambda \quad \text{(C2a)}$$

In a similar way we find

$$Y_{ecs} = a^2(a^2 + b^2\tan^2\Phi_g)^{-\frac{1}{2}}\sin\lambda \quad \text{(C2b)}$$

$$Z_{ecs} = b^2(a^2 + b^2\tan^2\Phi_g)^{-\frac{1}{2}}\tan\Phi_g \quad \text{(C2c)}$$

Note that in the equations (C2), the coordinates are functions of the geodetic latitude which is more readily available than the geocentric latitude Φ_C in equations (C1).

* equation in the basic report

If the altitude h, which was assumed as zero so far, has a finite value, we obtain

$$X_{ecs} = [a^2(a^2 + b^2\tan^2\Phi_g)^{-\frac{1}{2}} + h\cos\Phi_g]\cos\lambda$$

$$Y_{ecs} = [a^2(a^2 + b^2\tan^2\Phi_g)^{-\frac{1}{2}} + h\cos\Phi_g]\sin\lambda$$

$$Z_{ecs} = b^2(a^2 + b^2\tan^2\Phi_g)^{-\frac{1}{2}}\tan\Phi_g + h\sin\Phi_g \qquad (C3)$$

The following steps are then performed: First, the ecs coordinates are computed for point 1 using (C3) and the new coordinates serve as coordinates of the origin of the new system. The ecs coordinates of point 2 are computed and then translated to the new origin:

$$X_D = X_{ecs,2} - X_{ecs,1}$$

$$Y_D = Y_{ecs,2} - Y_{ecs,1}$$

$$Z_D = Z_{ecs,2} - Z_{ecs,1}$$

The coordinates X_D, Y_D, Z_D are rotated in three steps. First, X_D and Y_D are rotated by an amount of $(+\lambda_1)$ degrees around the vertical Z_D axis so that the X axis is in the vertical plane through the meridian of point 1. The new coordinates are called X, Y, Z. Second, Y and Z are rotated by 180° around X.

Finally X and Z are rotated in north direction by an amount of $(90 + \Phi_1)$ degrees around the Y axis which points into east direction. As a result, X points north and Z upwards along the local vertical at point 1. The 3 rotations are expressed in the following equation

$$\begin{bmatrix} R_N \\ R_E \\ R_Z \end{bmatrix} = \begin{bmatrix} -\sin\Phi & 0 & -\cos\Phi \\ 0 & 1 & 0 \\ \cos\Phi & 0 & -\sin\Phi \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_D \\ -Y_D \\ Z_D \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\Phi\cos\lambda & \sin\Phi\sin\lambda & \cos\Phi \\ -\sin\lambda & -\cos\lambda & 0 \\ \cos\Phi\cos\lambda & -\cos\Phi\sin\lambda & \sin\Phi \end{bmatrix} \begin{bmatrix} X_D \\ -Y_D \\ Z_D \end{bmatrix} \qquad (C4)$$

The final result represents a left-handed system with X replaced by R_N, Y by R_E and Z by R_Z.

Note: The component Y_D in west direction in (C4) has a negative sign to obtain positive values along the Y_{ecs} axis from negative longitudes. In the corresponding FORTRAN program in appendix D the second column of the 3 x 3 matrix is negative instead of Y_D:

$$\begin{bmatrix} R_N \\ R_E \\ R_Z \end{bmatrix} = \begin{bmatrix} -\sin\Phi\cos\lambda & -\sin\Phi\sin\lambda & \cos\Phi \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\Phi\cos\lambda & \cos\Phi\sin\lambda & \sin\Phi \end{bmatrix} \begin{bmatrix} X_D \\ Y_D \\ Z_D \end{bmatrix}$$

APPENDIX D

FORTRAN STATEMENTS OF CONVERSION PROGRAMS

1. Conventional Program

The FORTRAN statements listed here are an excerpt of a program for coordinate transformation which was coded for a variety of applications by D. Dickinson and D. Walter at AFMDC, HAFB in 1969. Symbols have been changed to aid the understanding.

Before execution of the program at the computer (CDC 3600/3800), one card is read to insert the origin of the LTP system with coordinates:

Geodetic Latitude	$\Phi_{g,1}$	=	PH1 in degrees
Longitude	λ_1	=	LAM1 in degrees
Altitude	h_1	=	H1 in feet

The next card contains the geodetic coordinates of the point 2:

Geodetic Latitude	$\Phi_{g,2}$	=	PH2 in degrees
Longitude	λ_2	=	LAM2 in degrees
Altitude	h_2	=	H2 in feet

Input data are based on Clarke's spheroid 1866.

Data (DTR = 0.017453292519), (REQ = 20925832),
(RPØL = 20854892)

Statement
Number

```
1    TYPE REAL LAM1, LAM2
2    PH1 = PH1*DTR $ LAM1 = LAM1*DTR
3    PH2 = PH2*DTR $ LAM2 = LAM2*DTR
4    SL2 = SINF(LAM2) $ CL2 = CØSF(PH2)
5    SP2 = SINF(PH2) $ CP2 = CØSF(PH2) $ TP2 = SP2/CP2
```

```
Statement
Number
    6       A1 = SQRTF((RPØL*TP2)**2 + REQ**2)
    7       R = REQ**2/A1 + H2*CP2
    8       XECS2 = R*CL2
    9       YECS2 = -R*SL2
   10       ZECS2 = RPØL**2*TP2/A1 + H2*SP2

   11       A21 = -SINF(LAM1) $ A22 = -CØSF(LAM1)
   12       A13 = CØSF(PH1) $ A33 = SINF(PH1)
   13       A11 = A22*A33 $ A12 = -A21*A33
   14       A31 = -A13*A22 $ A32 = A13*A21
   15       A01 = SQRTF((RPØL*A33/A13)**2 + REQ**2)

   16       R0 = REQ**2/A01 + H1*A13
   17       XECS1 = -R0*A22
   18       YECS1 = +(R0*A21)
   19       ZECS1 = RPØL**2*A33/(A13*A01) + H1*A33
   20       DX = XECS2 - XECS1

   21       DY = YECS2 - YECS1
   22       DZ = ZECS2 - ZECS1
   23       RN = A11*DX + A12*DY + A13*DZ
   24       RE = A21*DX + A22*DY
   25       RZ = A31*DX + A32*DY + A33*DZ
```

2. New Program:

```
DATA (DTR = 0.01745329251 9), (REQ = 20925832)
         (E2 = 0.006768644065)
```

Statement
Number

```
 1    TYPE REAL LAM1, LAM2
 2    PH1 = PH1*DTR $ PH2 = PH2*DTR
 3    DL = (LAM1 - LAM2)*DTR
 4    S1 = SINF(PH1) $ C1 = CØSF(PH1) $ T1 = S1/C1
 5    S2 = SINF(PH2) $ C2 = CØSF(PH2)

 6    SDL = SINF(DL) $ CDL = CØSF(CDL)
 7    R1 = REQ/SQRTF(1 - E2*S1**2)
 8    R2 = REQ/SQRTF(1 - E2*S2**2)
 9    RN = (R2 + H2) * (C1*S2 - S1*C2*CDL) - E2*(R2*S2 - R1*S1)*C1
10    RE = (R2 + H2)*C2*SDL

11    RZ = R1 + H1 - (R2 + H2)*C2*CDL/C1 - RN*T1
```

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