

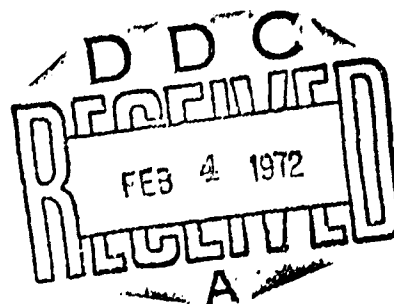
AD736166

DEPARTMENT
OF
MATHEMATICS

CLEMSON UNIVERSITY
Clemson, South Carolina



NATIONAL TECHNICAL
INFORMATION SERVICE



14

UNCLASSIFIED
Security Classification

1. <u>ORIGINATING ACTIVITY:</u> Dept. of Mathematical Sciences Clemson University Clemson, S. C. 29631	2a. <u>REPORT SECURITY CLASSIFICATION:</u> Unclassified 2b. <u>GROUP:</u>
---	---

3. REPORT TITLE: On the distribution of a product of two random variables.

4. DESCRIPTIVE NOTES:

5. AUTHOR(S): Khursheed Alam

6. <u>REPORT DATE:</u> Sept. 15, 1971	7a. <u>TOTAL NO. OF PAGES</u> 11	7b. <u>NO. OF REFERENCES</u> 1
--	-------------------------------------	-----------------------------------

8a. <u>CONTRACT OR GRANT NO.</u> NR00014-71-A-0339-0002	9a. <u>ORIGINATOR'S REPORT NUMBER(S)</u> : 76
b. <u>PROJECT NO.</u>	9b. <u>OTHER REPORT NO(S)</u> :
c. <u>TASK NO.</u> NR 042-271	

10. AVAILABILITY/LIMITATION NOTICES:
Unlimited

11. <u>SUPPLEMENTARY NOTES:</u>	12. <u>SPONSORING MILITARY ACTIVITY:</u> Statistics and Probability Program Office of Naval Research Arlington, Virginia
---------------------------------	---

13. ABSTRACT: Let $W=UV$ where U and V are independent random variables. It is shown that if V is distributed according to the non-central Chi-square distribution, then W is distributed according to the Chi-square distribution if and only if $U=1$ with probability 1. If V is normally distributed, it is shown that W is normally distributed if and only if the distribution of U is a two-point distribution.

14. <u>KEY WORDS:</u>	Link A		Link B		Link C	
	Role	Wt.	Role	Wt.	Role	Wt.
1. Chi-square distribution						
2. Non-centrality parameter						

ON THE DISTRIBUTION OF A PRODUCT
OF TWO RANDOM VARIABLES

KHURSHEED ALAM

Department of Mathematical Sciences
Clemson University
(Tr 76)

REPORT N15

Research Supported by

THE OFFICE OF NAVAL RESEARCH

Task NR 042-271 Contract N00014-71-A-0339-0002



Reproduction in whole or part is permitted for any purposes of
the U. S. Government. Distribution of this document is unlimited.

On The Distribution Of A Product
Of Two Random Variables

Khursheed Alam
Clemson University

Summary. Let U and V be two independent random variables, and let $W = UV$. It is shown that if V is distributed as $\chi^2_{2m, \delta}$, a non-central Chi-square with $2m$ degrees of freedom and non-centrality parameter δ , then W is distributed as a Chi-square random variable if, and only if $U = 1$ with probability 1. If V is normally distributed with mean μ , say, it is shown that W is normally distributed if, and only if, the distribution of U is degenerate at a point distinct from the origin, when $\mu \neq 0$, or the distribution has probability mass at two points equidistant from the origin. This result is generalized for the case in which V is a random vector, distributed according to the multivariate normal distribution and U is an independent random matrix.

1. Main results. Let U and V be independent random variables, and let U be distributed as $\chi^2_{2m, \delta}$, that is, non-central Chi-square with $2m$ degrees of freedom and non-centrality parameter δ . Let $W = UV$. Theorem 1.1 below, gives a necessary and sufficient condition that W is distributed as a Chi-square random variable.

Theorem 1.1. W is distributed as a Chi-square random variable if, and only if, $U = 1$ with probability 1.

Proof: The sufficiency part of the theorem is trivial. We consider the necessary condition of the theorem. Let W be distributed as $\chi^2_{2m', \delta'}$, where $m' > 0$ and $\delta' \geq 0$. Clearly, $U > 0$ with probability 1. Suppose that $P\{U \geq 1+\zeta\} = \alpha$, where α and ζ are positive numbers. Let $C = \frac{1}{2}(1+\zeta)^{-1}$. Then

$$(1.1) \quad E e^{CW} = E(e^{CUV}) \\ \geq \alpha E(e^{\frac{1}{2}V}).$$

where E denotes expectation. From the distribution of Chi-square it is seen that the left hand side of (1.1) is finite, whereas the right hand side is infinite. Therefore, $P\{U > 1\} = 0$.

The density function of the distribution of $\chi_{2m,\delta}^2$ is given by

$$(1.2) \quad g(x) = \sum_{r=0}^{\infty} e^{-\delta} \frac{\delta^r}{r!} f_{2m+2r}(x); \quad x > 0$$

where

$$(1.3) \quad f_{2m}(x) = x^{m-1} e^{-x/2} / (2^m \Gamma(m))$$

denotes the density function of the distribution of χ_{2m}^2 , a central Chi-square with $2m$ degrees of freedom. The Laplace transform of the distribution of $\chi_{2m,\delta}^2$ is given by

$$\begin{aligned} (1.4) \quad \phi(\lambda) &= \int_0^{\infty} e^{-\lambda x} g(x) dx \\ &= \sum_{r=0}^{\infty} e^{-\delta} \frac{\delta^r}{r!} (1+2\lambda)^{-m-r} \\ &= (1+2\lambda)^{-m} \exp(-2\lambda\delta/(1+2\lambda)) \end{aligned}$$

where $\lambda > -\frac{1}{2}$. Therefore, the Laplace transform of the distribution of $W = UV$ is given by

$$(1.5) \quad \psi(\lambda) = E((1+2\lambda U)^{-m} \exp(-2\lambda\delta U/(1+2\lambda U))).$$

As W is distributed as $\chi^2_{2m', \delta'}$, we have from (1.4) that

$$(1.6) \quad \psi(\lambda) = (1+2\lambda)^{-m'} \exp(-2\lambda\delta'/(1+2\lambda)).$$

We shall show that $m = m'$ and $\delta = \delta'$. Let $H(u)$ denote the distribution function of U , and let

(1.7)

$$h(u, \lambda) = (1+2\lambda u)^{-m} \exp(-2\lambda\delta u/(1+2\lambda u)) / \int_0^1 (1+2\lambda u)^{-m} \exp(-2\lambda\delta u/(1+2\lambda u)) dH(u)$$

$$0 < u \leq 1.$$

Let $\lambda > 0$. It is seen that $h(u, \lambda)$ represents a density function with monotone likelihood ratio property in λ , that is,

$$\int_0^1 h(u, \lambda) dH(u) = 1$$

and

$$(1.8) \quad h(u, \lambda)h(u', \lambda') - h(u', \lambda)h(u, \lambda') \geq 0$$

for $u' \geq u$ and $\lambda' \geq \lambda$. From the monotone likelihood ratio property, given by (1.8), it follows that

$$H'(x; \lambda) = \int_0^x h(u, \lambda) dH(u), \quad 0 < x \leq 1$$

is non-increasing in λ for each x . Therefore, for any non-decreasing (non-increasing) integrable function $\zeta(u)$, we have that

$$\eta(\lambda) = \int_0^1 \zeta(u) dH'(u; \lambda)$$

is non-decreasing (non-increasing) in λ . Thus

$$(1.9) \quad \eta(\lambda) \leq \int_0^1 \zeta(u) dH(u)$$

if $\zeta(u)$ is non-increasing in u .

From (1.5) we have

$$-\delta \log \psi(\lambda) / \delta \lambda = \int_0^1 \left(\frac{2mu}{1+2\lambda u} + \frac{2\delta u}{(1+2\lambda u)^2} \right) dH'(u; \lambda).$$

or

$$(1.10) \quad -\lambda \partial \log \psi(\lambda) / \partial \lambda = m + \int_0^1 \left(\frac{\delta - m}{1+2\lambda u} - \frac{\delta}{(1+2\lambda u)^2} \right) dH'(u; \lambda).$$

On the other hand, from (1.6) we have

$$(1.11) \quad -\lambda \partial \log \psi(\lambda) / \partial \lambda = m' + \frac{\delta' - m'}{1+2\lambda} - \frac{\delta'}{(1+2\lambda)^2}$$

From (1.9) we have that

$$\int_0^1 (1+2\lambda u)^{-1} dH'(u; \lambda) \leq \int_0^1 (1+2\lambda u)^{-1} dH(u)$$

for $\lambda > 0$. Therefore, the integral on the right hand side of (1.10) tends to zero as $\lambda \rightarrow \infty$. It follows from (1.10) and (1.11), letting $\lambda \rightarrow \infty$, that $m = m'$, and also $\delta - m = \delta' - m'$. Thus, $\delta = \delta'$.

As V and W have the same Chi-square distribution, it follows that $U = 1$ with probability 1. The Theorem is proved.

Let U and V be independent random variables as in the preceding. From Theorem 1.1 we have the following corollary.

Corollary 1.1. If V is normally distributed with mean μ then W is normally distributed if, and only if, the distribution of U is degenerate at a point distinct from the origin, when $\mu \neq 0$, or the distribution has probability mass at two points, equidistant from the origin.

Proof: The sufficiency part of the corollary is trivial. We consider the necessary condition of the corollary. It is given that V and W are normally distributed. Let v denote the mean of W . Without loss of generality we may assume that $\text{Var}(V) = \text{Var}(W) = 1$. Then V^2 and W^2 are distributed as Chi-square with 1 degree of freedom. As $W^2 = U^2 V^2$, from Theorem 1.1 we have $\mu^2 = v^2$, and

$$(1.12) \quad P\{U^2 = 1\} = 1.$$

As $v = E(W) = \mu E(U)$, we have that $E(U) = 1$ or -1 if $\mu \neq 0$. Therefore, $P\{U = 1\} = 1$ or $P\{U = -1\} = 1$, if $\mu \neq 0$. Thus the distribution of U is degenerate if $\mu \neq 0$. This completes the proof of the corollary.

Next we generalize the results of Corollary 1.1. Let A be a $p \times p$ random matrix which is non-singular with probability 1, and let Y be a p -component random vector, distributed according to the multivariate normal distribution with mean ζ and covariance Σ , where Σ is a diagonal matrix. Let ζ_i and α_i denote the i th component and the i th vector, respectively, of ζ and A . We shall say that the distribution of α_i has property π if $P\{\alpha_i = \pm \alpha\} = 1$ for some non-null vector α . Suppose that $\alpha_1, \dots, \alpha_p$ and Y are mutually independent. Let $Z = AY$. We have the following result.

Corollary 1.2. The distribution of Z is multivariate normal if, and only if, for each $i = 1, \dots, k$, the distribution of α_i is either degenerate at a point distinct from the origin if $\zeta_i \neq 0$, or if the distribution of α_i has property π .

Proof: It is known that a p -component vector X is multivariate normal if and only if the distribution of $\lambda'X$ is univariate normal for all non-null p -component row vector λ' . Therefore Z is normally distributed if and only if

$\lambda'Z$, given by

$$(1.13) \quad \lambda'Z = \sum_{i=1}^p (\lambda'\alpha_i)Y_i$$

is normally distributed for all non-null vectors λ' , where Y_i denotes the i th component of Y . The sufficiency part of the corollary is shown easily. We shall prove the necessary condition of the corollary.

Suppose that $\lambda'Z$ is normally distributed for all non-null vectors λ' . As the components of Y and the columns of A are independent, the terms of the summation on the right hand side of (1.13), are independently distributed. By the reproductive property of the normal distribution it follows that $(\lambda'\alpha_i)Y_i$ is normally distributed for each $i = 1, \dots, p$. From Corollary 1.1 it follows that the distribution of $\lambda'\alpha_i$ is discrete which is either degenerate at a point distinct from the origin, or it has saltus at two points equidistant from the origin, and that the distribution is degenerate if $\zeta_i \neq 0$. This result, which holds for all non-null vector λ' , implies the conclusion of the corollary.

The following result, due to Kingman and Graybill [1], is related to Corollary 1.2. Let Y be a p -component random vector, and let the components of Y be independently and

identically distributed. Let $A = (a_{ij})$ be a $p \times p$ random matrix which is orthogonal with probability 1, and

$E\left(\sum_{j=1}^p a_{ij}\right) \neq 0$ for some i . Let $Z = AY$. From Theorem 3.1

of [1] it follows that $Z \stackrel{d}{\sim} N(0, I_p)$ if, and only if, $Y \stackrel{d}{\sim} N(0, I_p)$ where I_p denotes the $p \times p$ identity matrix.

A plausible result supplementing Theorem 1.1 is given below as a conjecture. Let X and Y be independent random variables, and let $Z = XY$.

Conjecture I: If $E(Z) \neq 0$ then Z is normally distributed if and only if either X or Y is normally distributed.

Suppose that Conjecture I is true. From Corollary 1.1 we have the following proposition characterizing the normal distribution.

Proposition: If $E(Z) \neq 0$ then Z is normally distributed if, and only if, between the two random variables X and Y one of them is normally distributed and the other is degenerate at a point distinct from the origin.

The following example shows that the assumption $E(Z) \neq 0$ is necessary for Conjecture I to be true. Let X' be a random variable distributed as $(\chi_{2m}^2)^{1/4}$, that is, 4th root of Chi-square with $2m$ degrees of freedom, and let U be an independent random variable taking values $+1$ and -1 with equal probability $\frac{1}{2}$. Set $X = UX'$. Let Y be distributed as $(4\chi_{2m+1}^2)^{1/4}$ independent of U and X' . Clearly, the distribution of $Z = XY$ is symmetrical about the origin.

Let $\phi(t)$ denote the characteristic function of $\text{Log} Z^2$. As the characteristic function of $\log \chi_{2m}^2$ is equal to $2^{it} \Gamma(m+it)/\Gamma(m)$, and $Z^2 = (X')^2 Y^2$, we have

$$\begin{aligned} (1.14) \quad \phi(t) &= 2^{2it} \Gamma(m+\frac{it}{2}) \Gamma(m+\frac{it}{2} + \frac{1}{2}) / \Gamma(m) \Gamma(m+1/2) \\ &= 2^{it} \Gamma(2m+it) / \Gamma(2m). \end{aligned}$$

The second line on the right hand side of (1.14) is obtained from Legendre's duplication formula for the gamma function. Let $m = \frac{1}{4}$ then (1.14) shows that Z^2 is distributed as χ_1^2 . As the distribution of Z is symmetrical about the origin, it follows that Z is normally distributed with mean zero and variance 1, whereas neither X nor Y is normally distributed.

References

- [1] Kingman, A. and Graybill, F. A. (1970). A non-linear characterization of the normal distribution. Ann. Math. Statist. (41) 1889-1895.