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A RESOURCE ALLOCATION MODEL WITH DEVELOPMENT COSTS AND PRIOR INVESTMENTS

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Kenneth D. Shere

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U.S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

 1 November 1971

A Resource Allocation Model with Development Costs and Prior Investments

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Captain, USN
Commander

J.W. Enig for
E. K. RITTER
By direction

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I. INTRODUCTION

The theory of max-min is extended to a class of functions having direct application to the allocation of resources among strategic retaliatory systems. The model presented below permits the inclusion of both committed resources and "prices of admission."

The price of admission includes development costs, and these terms are used interchangeably; however, preliminary research costs necessary for the serious consideration of any product or system are excluded. By its nature, research allocation must be considered separately. Committed resources includes the value of existing systems and resources previously committed for a variety of (possibly irrational) reasons.

The problems considered describe "zero-sum" competition among two players. In the absence of development costs and committed resources the model becomes a zero-sum, two-person continuous game with a continuous payoff function. Specifically, we consider a payoff function for the x-player

$$F(x,y) = \sum_{i=1}^{n} f_i(x_i,y_i) \tag{1.1}$$

where $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$ and

$$f_i(x_i,y_i) \equiv \begin{cases} 0 & : \ 0 \le x_i \le q_i; \\ h_i(x_i) & : \ x_i \ge q_i \ , \ 0 \le y_i < r_i; \\ g_i(x_i,y_i) & : \ x_i \ge q_i \ , \ y_i \ge r_i. \end{cases} \tag{1.2}$$

Call $f_i(x_i,y_i)$ the value of the ith system; q_i denotes the x-player's price of admission for the ith system and r_i denotes the y-player's price of admission. Since a competitive situation is being described $g_i(x_i,y_i)$ is an increasing (decreasing), concave (convex) function of $x_i(y_i)$. For each i define

$$h_i(x_i) \equiv g_i(x_i, r_i)$$

and

$$h_i(q_i) \equiv 0.$$

Further assume that $g_i(x_i, y_i) > 0$ whenever $x_i > q_i$. The allocation of resources to the ith system is denoted x_i or y_i.

Allowing R_x and R_y to be the total resources (excluding research funds) available to the x-player and the y-player respectively, the allocation is constrained by

(1.3) $$x_1 + \ldots + x_n = R_x \quad ; \quad y_1 + \ldots + y_n = R_y.$$

The allocation is additionally constrained by

(1.4) $$x_i \geq s_i \geq 0 \quad ; \quad y_i \geq t_i \geq 0 \quad (i = 1,2,\ldots,n)$$

where s_i and t_i represent the amount of previously committed resources to the ith system. We assume for convenience of presentation that $s_i > 0$ $(t_i > 0)$ implies $s_i > q_i$ $(t_i > r_i)$. For physical reasons, we also assume $R_x > \max_i q_i$, $R_y > \max_i r_i$ and that $t_i = 0$ whenever $s_i = 0$.

Since we are interested in applications where the x-player moves first, the purpose of the problem is to determine the "residual value" V and "optimal strategies" $u = (u_1,\ldots,u_n)$ and $w = (w_1,\ldots,w_n)$ such that

(1.5) $$V = F(u,w) = \text{Max}_x\, P(x)$$

where

$$P(x) \equiv \text{Min}_y\, F(x,y)$$

P(x) is called the security function.

A particular example is determined by

(1.6) $$g_i(x_i, y_i) = v_i(x_i - q_i) \exp\,[-a_i(y_i - r_i)]$$

where v_i and a_i are constants. The zero development cost, zero committed resources game corresponding to this example has been solved by Danskin [1]. Shere and Cohen have extended Danskin's max-min theory to solve this example with committed resources but no development costs [2] and with development costs but no committed resources [3].

In the following section (1.1)-(1.5) is solved in a constructive manner; additional constraints are imposed as needed on $g_i(x_i,y_i)$. The final section consists of observations on problems related to (1.1)-(1.5) and conclusions.

II. MATHEMATICAL ANALYSIS

This section is organized in two parts. The first part consists of nine lemmas and theorems which describe the nature of the solution of (1.1)-(1.5). It is shown by adding hypotheses related to the differentiability of $g_i(x_i,y_i)$ that the optimal strategies (u,w) are also optimal strategies of a certain game. A scheme for determining the game and consequently (u,w) is subsequently provided.

The following four lemmas are trivial extensions of the corresponding lemmas of [3].

Lemma 1. Let $x = (x_i)$ and $\eta(x) = (\eta_i(x))$. If $P(x) = F(x,\eta(x)) > 0$, then $x_k > q_k$ and $\eta_k > r_k$ for some $i = k$.

Lemma 2. If $P(x) = F(x,\eta(x)) > 0$, then $x_i \le q_i$ implies $\eta_i(x) = 0$.

Lemma 3 (Modified Gibbs' Lemma). Let $f_i(x_i)$ be continuous with right- and left-derivatives. Let $z = (z_1,\dots,z_n)$ maximize $\Sigma_i f_i(x_i)$ constrained by $\Sigma_i x_i = R_x > 0$ and $x_i \ge s_i \ge 0$. Then there exists a number λ such that

$$f_i'(z_i^+) \le \lambda \quad \text{for all } i;$$

$$f_i'(z_i^-) \ge \lambda \quad \text{for all } i \text{ with } z_i > s_i.$$

Furthermore, λ is unique if $f_i(x_i)$ is differentiable at $x_i = z_i > s_i$ for some i.

Lemma 4. Either $w_i = 0$ or $w_i > r_i$.

This lemma says that payment of a fraction f, $0 < f < 1$, of a price of admission for the ith system is nonoptimal. The y-player's ith system cannot exist unless the price of admission, r_i, has been fully paid. Therefore an intermediate payment deminishes the y-player's resources without effecting the x-player. The corresponding result is obtained for the x-player as a corollary to the following theorem.

Theorem 5. Let us suppose, for each i, that $g_i(x_i,y_i)$ is a strictly increasing function of x_i over the domain $0 \leq x_i \leq D$. The residual value V of (1.1)-(1.5) is a strictly increasing function of R_x on $(q^*,D]$ where $q^* \equiv \min_i q_i$.

Proof. Suppose that $R_x^* < R_x$, $F(u,w) = V$ and $F(u^*,w^*) = V^*$. Assume that $V(R_x^*) \geq V(R_x)$. Select ξ as any point such that $\Sigma_i \xi_i = R_x$ and $\xi_i > u_i^*$ for each i. Defining η by:

$$F(\xi,\eta) = \mathrm{Min}_y F(\xi,y) = P(\xi)$$

it is noted that

$$F(\xi,\eta) \leq F(u,w) \leq F(u^*,w^*).$$

From Lemma 1, $u_i^* > q_i$ and consequently $\xi_i > q_i$ for some i. By the increasing nature of $g_i(x_i,y_i)$ for every i,

$$F(u^*,\eta) < F(\xi,\eta) \leq F(u^*,w^*)$$

contrary to the definition of w^*. Hence $V(R_x) > V(R_x^*)$.

Corollary 6. Under the hypotheses of Theorem 5, either $u_i = 0$ or $u_i > q_i$.

Proof. Define $I \equiv \{i : 0 < u_i \leq q_i\}$. If I is nonempty then $z \equiv \Sigma_I u_i > 0$. Define u^* by $u_i^* = u_i (i \notin I)$ and $u_i^* = 0 (i \in I)$. Then

$$V(R_x) = P(u) = P(u^*) \leq V(R_x - z).$$

Since $R_x > q^*$, $0 < V(R_x)$. If $R_x - z > q^*$ we have a contradiction to Theorem 5 and if $R_x - z \leq q^*$, $V(R_x - z) = 0$ so again we have a contradiction.

After introducing some notation two lemmas are presented; these lemmas are used to reduce the solution of (1.1)-(1.5) to the solution of a zero-sum, two-person continuous game (with pure strategy).

Notation. Let (u,w) be a solution of (1.1)-(1.5) and define:

$$A \equiv \{i : u_i > s_i\} \cap \{i : w_i > t_i\};$$

$$B \equiv \{i : u_i = s_i\} \cap \{i : w_i = t_i\};$$

$$C \equiv \{i : u_i > s_i\} \cap \{i : w_i = t_i\};$$

$$D \equiv \{i : u_i = s_i > 0\} \cap \{i : w_i > t_i\};$$

$$X \equiv \{x = (x_1,\ldots,x_n) : x_i \ge \max(s_i,q_i) \text{ for } i \in A\cup C,$$
$$x_i = s_i \text{ for } i \in B\cup D \text{ and } \Sigma_{A\cup C}\, x_i = R_x - \Sigma_{B\cup D}\, s_i\};$$

$$Y \equiv \{y = (y_1,\ldots,y_n) : y_i \ge \max(t_i,r_i) \text{ for } i \in A\cup D,$$
$$y_i = t_i \text{ for } i \in B\cup C \text{ and } \Sigma_{A\cup D}\, y_i = R_y - \Sigma_{B\cup C}\, t_i\ .$$

Define the game:

(2.1) Given: $G_{XY}(x,y) = \Sigma_A\, g_i(x_i,y_i) + \Sigma_B\, f_i(s_i,t_i) + \Sigma_C\, g_i(x_i,t_i) + \Sigma_D\, g_i(s_i,y_i)$

(2.2) Constrained by: $x \in X \quad , \quad y \in Y$

(2.3) Determine: $V_{XY} = \mathrm{Max}_X\, P_{XY}(x) = \mathrm{Max}_X\, \mathrm{Min}_Y\, G_{XY}(x,y)$.

In (2.1), replace $g_i(x_i,t_i)$ by $h_i(x_i)$ whenever $t_i = 0$.

Lemma 7. $P_{XY}(x)$ is a concave function of x in X.

This lemma is given in greater generality by Shiffman [4]; because of the inaccessibility of that reference the proof is given below.

Proof. Let x, $x^* \in X$ and $0 \le \alpha \le 1$. Then $P_{XY}[\alpha x + (1-\alpha)x^*] = \mathrm{Min}_Y\, G_{XY}[\alpha x+(1-\alpha)x^*,y]$
$\ge \mathrm{Min}_Y[\alpha\, G_{XY}(x,y)+(1-\alpha)G_{XY}(x^*,y)] \ge \alpha P_{XY}(x,y) + (1-\alpha)P_{XY}(x^*,y)$.

Hitherto there has been no restrictions related to the differentiability of $g_i(x_i,y_i)$ and $h_i(x_i)$. For the remainder of this section we assume for each i that $h_i'(x_i)$ exists and is continuous, and we assume that $\partial g_i(x_i,y_i)/\partial x_i$ exists and is continuous in its variables taken together. The following lemma can be proved by appealing to the definition of the directional derivative and modifying slightly the proof of Danskin's corresponding theorem [1, p. 19-22].

Lemma 8. Let Γ be a hypercurve in X and assume that X is not a single point. Let D_Γ denote the directional derivative along Γ. For each $x \in \Gamma$, $D_\Gamma P(x)$ and $D_\Gamma P_{XY}(x)$ exist.

Theorem 9. The strategies (u,w) are optimal strategies for the game (2.1)-(2.3).

Proof. Suppose to the contrary that (u,w) is not optimal for (2.1)-(2.3) and let (u^*,w^*) be a solution of (2.1)-(2.3). Define Γ as the hyperline segment formed by the intersection of X and the hyperline passing through u and u^*. Either X consists of one point, in which case $u = u^*$, or u lies in the interior of Γ. Since P(x) is maximized at x = u and $D_\Gamma[P(u)]$ exists, $D_\Gamma[P(u)] = 0$. Since $P_{XY}(x)$ is increasing as Γ is traversed from u to u^*, $D_\Gamma[P_{XY}(u)] > 0$. This means that $P_{XY}(x') < P(x')$ for some point x' such that $u \in (x', u^*)$, as illustrated in Figure 1. This conclusion contradicts the fact that P(x') is minimum over a space which includes Y. Hence $u = u^*$. Define $\eta^*(x)$ by $F(x,\eta^*(x)) = P_{XY}(x)$. Since $\eta^*(x)$ is well-defined on Y [cf. 4], $w^* = w$.

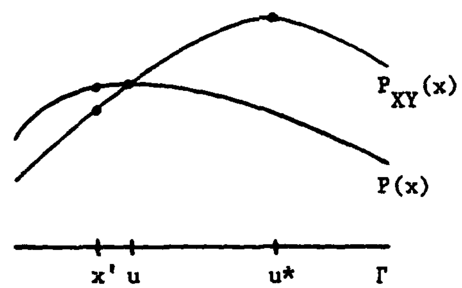

Figure 1. Typical Security Functions

Although Theorem 9 reduces the solution of (1.1)-(1.5) to the solution of a game (2.1)-(2.3), the spaces X and Y (or equivalently the sets A, B, C, D) are not *a priori* known. After establishing further notation a method for determining X and Y is provided.

Let $\Pi = (A', B', C', D')$ be a partition of $\{1,2,\ldots,n\}$ constrained by "$i \in D'$ implies $s_i > 0$." Define X' and Y' with respect to Π in the same manner as X and Y are defined with respect to A, B, C, and D. Let Φ be the set of all such pairs (X',Y'). Finally define $\Psi(X') \equiv \{Y' : (X',Y') \in \Phi\}$.

Lemma 10. For each $(X',Y') \varepsilon \Phi$ let $V_{X'Y'}$ be the value of the game (2.1)-(2.3) with (X,Y) replaced by (X',Y'). Define

$$V_{X'} \equiv \text{Min}_{Y' \varepsilon \Psi(X')} V_{X'Y'}.$$

Then for some $(X,Y) \varepsilon \Phi$,

$$V = V_{XY} = V_X.$$

Proof. By Theorem 9, $V = V_{XY}$ for some $(X,Y) \varepsilon \Phi$. Suppose that $V_{XY} > V_X = V_{XY'}$. Then

$$P(u) = P_{XY}(u) > P_{XY'}(u') \geq P_{XY'}(u) \tag{2.4}$$

where u' is the x-player's optimal strategy for the (X,Y')-game. The inequality (2.4) is inconsistent with the definition of P(u). Hence $V_{XY} = V_X = V$.

For the case of no committed resources it is shown in [3] for the example (1.6) that X = Y. Unfortunately, there does not appear to be a similar result for the committed resources problem. The following crude procedure shows how (1.1)-(1.5) can be solved.

Algorithm 11. (i) For each $(X',Y') \varepsilon \Phi$ determine $V_{X'Y'} = P_{X'Y'}(u_{X'Y'})$.

(ii) Find $V_{X'}$ for each X'.

(iii) Define $U \equiv \{u_{X'Y*} : P_{X'Y*}(u_{X'Y*}) = V_{X'} = P(u_{X'Y*})\}$.

(iv) Find $P(u) = \text{Max}\ \{P(u_{X'Y*}) : u_{X'Y*} \varepsilon U\}$.

(v) Determine $u = u_{X'Y*}$, $w = w_{X'Y*}$ and $V = V_{X'}$ from step (iv).

It is noted that U is nonempty and whenever $P_{X'Y*}(u_{X'Y*}) \neq P(u_{X'Y*})$, this choice of X' could not have been optimal.

III. OBSERVATIONS AND CONCLUSIONS

There has been no discussion in the preceding sections on how $V_{X'Y'}$, $u_{X'Y'}$ and $w_{X'Y'}$ are determined. By imposing the additional hypothesis that $\partial g_i(x_i,y_i)/\partial y_i$ exists and is decreasing in x_i for each fixed y_i, the "Gibbs' lemma approach" of Danskin [1] can be used. The necessary generlizations for the application of this technique are straightforward. A class of functions which satisfy all of preceding hypotheses is:

$$F(x,y) = \sum_{i=1}^{n} \left[\sum_{j=1}^{m_i} \alpha_{ij}(x_i)\beta_{ij}(y_j) \right]$$

where α_{ij} and β_{ij} are respectively concave-increasing and convex-decreasing for $x_i > q_i$ and $y_j > r_j$. Set $\alpha_{ij}(x_i) = 0$ whenever $0 \le x_i \le q_i$ and $\beta_{ij}(y_j) = 0$ whenever $0 \le y_j \le r_j$. An example of a concave-increasing, convex-decreasing function which does not satisfy the above hypothesis is:

$$g(x,y) = (xy)^{-1} + 2 \ln x.$$

If the x-player has previously committed resources of ν of n systems, at most $3^{n-\nu} 4^{\nu}$ games must be solved. Although this bound grows quite rapidly, in practice n is small. For example, if $n = 9$ and $\nu = 5$ there are 82944 games to be solved compared to 19683 games for the corresponding problem with no committed resources ($\nu = 0$). At the rate of 0.05 seconds of computer time per game, an upper bound of over an hour of computer time per choice of parameters is obtained. This illustrates the need for both careful programming and for a consideration of additional special properties a particular problem may possess. To consider a large number of systems there is a need for a more direct method of solving (1.1)-(1.5).

There are several other problems related to (1.1)-(1.5); for example, suppose that some of the $g_i(x_i,y_i)$ were convex-convex. This additional complication may be treated by showing that additional investment should be made in at most one of the convex-convex systems. The analysis can then proceed by using the approach of Danskin [1, p. 52+]. Functions $g_i(x_i,y_i)$ which are concave-convex for some uncomplicated regions of x_i and convex-convex elsewhere can also be considered without excessive difficulty. Of course, the additional complexities increase the computer time.

Additional improvements in mathematical modeling are also needed. We end this work by posing two open questions. How can operational costs be separated from procurement costs? How does the time phasing of procurements affect the result?

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