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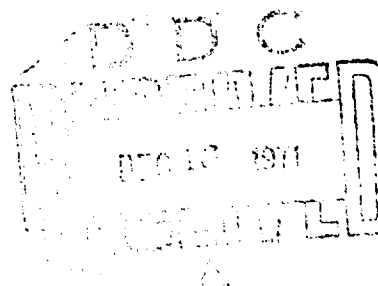
THE EQUATIONS OF INTERIOR BALLISTICS

by

J. P. Vinti

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The Equations of Interior Ballistics

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THE EQUATIONS OF INTERIOR BALLISTICS

Chapter I

Thermodynamic Foundations

In establishing the thermodynamic foundations of interior ballistics we begin for simplicity by assuming for the powder gas an equation of state of the co-volume type. Such an equation is a van der Waals equation containing the "b" term but not the "a" term. Possible modification of the equation of state and the effect on the equations of interior ballistics will be mentioned later. For a gas obeying such an equation of state the specific internal energy depends only on the temperature and not on the density. Even if such an equation holds for each component gas, however, it can hold for the mixture only if all chemical reactions that occur are equivoluminar and if gas imperfection has no appreciable effect on the chemical equilibrium constants. This statement follows from the facts that an equilibrium constant is independent of density only if the latter two assumptions hold and that dependence of equilibrium constants on density would mean dependence of relative concentrations on density as well as temperature, with a resulting dependence of specific internal energy on density as well as temperature.

The assumption of the co-volume type of equation of state thus involves a restriction of the theory to "cool" powders, for with "hot" powders dissociations and other non-equivoluminar reactions come into play. With this same ruling out of non-equivoluminar reactions and of the effect of gas imperfection on equilibrium constants one also has for each powder composition a definite explosion temperature which is independent of the density of loading. Thus the decomposition of one gram of such a powder always liberates the same amount of energy, which then warms the one gram of powder gas to the same temperature, independently of the density of loading. On the other hand, dependence of equilibrium constants on density would involve a dependence on the density of

loading of relative concentrations and thus of specific heat and explosion temperature.

Earlier treatments of interior ballistics involved the assumption that the specific heat of the powder gas is independent of temperature. Modern developments in quantum physics and in heat measurements have shown that such an assumption is untrue. Dederick(1) pointed out that the internal energy of the powder gas at temperature T can be expressed in terms of an average of the specific heat over the whole temperature range from absolute zero to T. By restriction of the theory, however, to cool powders and the consequent existence of a definite explosion temperature, such long range averages may be avoided by the introduction of Kent's concept of the potential of the powder. We proceed next to develop this concept and to obtain the fundamental energy equation of interior ballistics.

Let T_0 be the explosion temperature* and T the temperature of the main body of the gas; the gas is formed at temperature T_0 ** and drops to the lower temperature T because of work done on and heat loss to the surroundings. Let $u(T)$ be the specific internal energy*** of the gas at temperature T and $u(T_0)$ the specific internal energy at temperature T_0 . Then if C is the original mass of the powder and G the fraction burned at time t, in time dt mass C dG of gas is formed at temperature T_0 , so that the gas gains energy C dG $u(T_0)$. This gain is expressible as the sum of a term d [CG u(T)], the gain in internal energy of the main body of gas, and a term δW , composed of work done in imparting translational kinetic energy to the projectile, to the powder and powder gases, and to the gun, in overcoming passive resistance, and in stretching the gun, and of heat loss to the gun. We have thus:

$$C dG u(T_0) = d [CG u(T)] + \delta W \quad (1)$$

Eq. (1) can hold strictly only if all gas reactions are equivoluminar and gas imperfection has no effect, on equilibrium constants. Now, since $u(T_0)$ is constant,

$$C dG u(T_0) - d [CG u(T)] = d [CG u(T_0) - CG u(T)] \quad (2)$$

(1) L.S. Dederick: Certain Considerations in the Thermodynamics of Gases with Applications to Interior Ballistics, R-1-14, May 17, 1928.

*The adiabatic closed chamber explosion temperature is referred to throughout.

**See Addendum to Chapter I.

***exclusive of the macroscopic kinetic energy of flow.

Thus δW is a complete differential dW , where W represents the sum of total work done and heat loss up to time t . We have therefore from Eqs. (1) and (2):

$$CG u(T_0) = CG u(T) + W + k_1, \quad (3)$$

where k_1 is a constant of integration. Now customarily the time origin $t = 0$ is taken as the beginning of motion, but Eq. (3) holds also for negative values of time as far back as the beginning of burning, at which time $G = 0$ and $W = 0$. Thus the integration constant k_1 vanishes, so that:

$$CG u(T_0) = CG u(T) + W \quad (4)$$

The introduction into Eq. (4) of a short range average specific heat $\bar{c}(T_0, T)$, defined by:

$$\bar{c}(T_0, T) \equiv \frac{[u(T_0) - u(T)]}{(T_0 - T)} \quad (5)$$

gives

$$CG\bar{c} T_0 = CG\bar{c} T + W \quad (6)$$

We now introduce some definitions:

Free volume Ω : gas volume - co-volume

p : appropriate space - average pressure in the equation of state; We assume that the proper space - average pressure for use in the rate of burning equation is equal to it.

R_1 : gas constant per unit mass of gas.

$$\text{Potential } \phi \equiv \bar{c} T_0 \quad (7)$$

γ , analogous to the ratio of specific heats in the older theory, defined by:

$$\frac{1}{\gamma-1} \equiv \bar{c}/R_1 \quad (8)$$

The equation of state then takes the form:

$$L p\Omega = CGR_1 T \quad (9)$$

Eqs. (6), (7), (8), and (9) together lead to the fundamental energy equation of interior ballistics:

$$\phi CG = p\Omega/(\gamma-1) + W \quad (10)$$

γ is analogous to the ratio of specific heats in the older theory, \bar{c}_v to the specific energy of the powder, and $p\Omega/(\gamma-1)$ to the internal energy appearing in that theory. The analogy, however, is not to be confused with identity. Thus the term $p\Omega/(\gamma-1) = CGcT = CGT [u(T) - u(T_0)]$, internal energy referred to in the older theory was $CG [u(T) - u(0)]$.

It is apparent from their definitions that \bar{c}_v , \bar{c}_p , and γ are weakly varying functions of temperature. We consider them henceforth to be constant, assuming that they are evaluated for some mean temperature in the range of temperatures actually occurring in the gun. In some coefficients that appear later \bar{c}_v and $\gamma-1$ occur together as the product $\bar{c}_v(\gamma-1)$, which is seen by Eqs. (7) and (8) to be given by:

$$\bar{c}_v(\gamma-1) = R_1 T_0 \quad (11)$$

The quantity $R_1 T_0$, which we shall denote by λ , is sometimes called the "force" of the powder. If R is the universal gas constant, i.e., the gas constant per gram mole, and if Z is the number of gram moles per gram of gas, then

$$R_1 = ZR, \quad (12)$$

$$\text{so that } \lambda = ZRT_0 \quad (13)$$

With the restriction to "cool" powders, all gas reactions that occur are equivoluminar or, speaking more strictly, equimolar. Since displacement of such a reaction will not change the number of moles per gram, Z and thus R_1 and λ are independent of temperature and of density of loading for such "cool" powders.

The quantities T_0 , \bar{c}_v , and λ may be calculated as follows. Let Q_1 be the heat of formation* at 15°C of one gram of the solid powder from the elements and Q_2 the constant volume heat of formation at 15°C of one gram of the powder gases. Then $Q_2 - Q_1$ is the energy liberated when one gram of the solid powder at 15°C decomposes into powder gases at 15°C. If the reaction takes place adiabatically rather than isothermally, then this available energy heats the powder gases to the explosion temperature T_0 . In this final state there will be a definite number of moles $x_i(T_0)$ of the i th gas. Although in the actual process the molal concentrations may change during the heating, the same final state will be reached if we assume the solid powder to decompose directly into the final molal concentrations present at temperature T_0 : this statement follows from the first law of thermodynamics. Thus we interpret $Q_2 - Q_1$ as the energy liberated when one gram of the solid powder decomposes into powder gases at 15°C, the concentrations, however, being as at temperature T_0 .

* in a closed chamber

If we now denote by $U_i(T)$ the molal internal energy of gas i at temperature T we have

$$Q_2 - Q_1 = \sum_i x_i(T_0) [U_i(T_0) - U_i(288)] \quad (14)$$

where the summation is over all the species of gases present. To solve Eq. (14) for T_0 , one assumes a value for T_0 , calculates the equilibrium constant for each reaction, and uses the equilibrium equations and equations expressing the conservation of the number of gram atoms of each element; these latter numbers are known from the composition of the solid powder. There are then as many equations for the x_i 's as there are unknowns; these equations are then solved and the solutions inserted into Eq. (14). This process is continued until Eq. (14) checks. In carrying out this process there appears as a by-product a table of values of the x_i 's as functions of T , and thus of the function $F(T)$, where

$$F(T) = \sum_i x_i(T) [U_i(T) - U_i(288)] \quad (15)$$

Values of $U_i(T) - U_i(288)$ are given in a book by Lewis and von Elbe.² Our specific internal energies (per gram) are now expressible as:

$$u(T) = \sum_i x_i(T) U_i(T) \quad (16)$$

$$u(T_0) = \sum_i x_i(T_0) U_i(T_0) \quad (17)$$

Combination of Eqs. (15), (16), and (17) gives:

$$u(T) - u(T_0) = F(T) - F(T_0) + \sum_i U_i(288) [x_i(T) - x_i(T_0)] \quad (18)$$

Now let
$$x_i(T) - x_i(T_0) = \Delta x_i \quad (19)$$

As the gram of powder gas drops in temperature from T_0 to T the equilibria of the chemical reactions get displaced. In each reaction gas i gets a certain mole increase, positive or negative; let $\Delta x_i(j)$ be the mole increase of gas i in the j th reaction as $T_0 \rightarrow T$, and let N' be the number of reactions taken into account.

Then
$$\Delta x_i = \sum_{j=1}^{N'} \Delta x_i(j) \quad (20)$$

Then from Eqs. (18), (19), and (20):

$$u(T) - u(T_0) = F(T) - F(T_0) + \sum_i U_i(288) \sum_{j=1}^{N'} \Delta x_i(j) \quad (21)$$

$$= F(T) - F(T_0) + \sum_j \sum_i U_i(288) \Delta x_i(j) \quad (22)$$

²Bernard Lewis and Gunther von Elbe;

"Combustion, Flames and Explosions of Gases", Cambridge (1938).

Now the quantity $-\sum_i U_i(288) \Delta x_i(j)$ has a simple interpretation, readily deducible from the first law of thermodynamics. It is interpretable as the heat liberated in the j th reaction taking place at 15°C and at constant volume, when the mole shifts are the same as those that occur in the j th reaction when the whole powder gas drops from equilibrium at T_0 to equilibrium at T . Now the constant volume heat at 15° of the j th reaction can be calculated from the constant volume heats of formation at 15°C , and the mole shifts above referred to are known from the calculations already done to find T_0 . Thus the second term in Eq. (22) can be calculated, so that $u(T) - u(T_0)$ can be calculated:

the calculation of $\bar{c}(T_0, T)$ and of λ follows at once. The evaluation of R , γ , and λ follows as soon as Z has been obtained; we have immediately, however,

$$Z = \sum_i x_i(T_0), \quad (23)$$

so that all quantities so far introduced can now be evaluated.

Addendum to Chapter I

In deriving the fundamental equation (4) it was assumed that gas is actually formed at temperature T_0 and then drops to the temperature T of the main body of gas. It is the purpose of this addendum to point out that Eq. (4) holds independently of such an assumption. It is important to show this fact because on certain plausible theories of burning the temperature T_0 would not exist anywhere within the gas.

As before let W = work done by gas on surroundings + kinetic energy of gas and powder + heat lost by gas to surroundings. Let $u(T)$ be the specific energy of the powder gas at temperature T , $u(0)$ the specific energy of the powder gas at $T = 0$, and u_s the specific energy of the solid powder, all referred to a common zero level, viz the elements at 0° K at vanishingly small density. Referred to this zero level, the internal energy of the powder gas is $CG u(T)$ and the loss of internal energy of the solid powder when mass CdG burns is $CdG u_s$. Then by the first law of thermodynamics:

$$d [CG u(T)] - CdG u_s + \delta W = C \quad (23.1)$$

Now u_s is not a function of T , so that

$$d [CG u_s - CG u(T)] = \delta W \quad (23.2)$$

Integration gives:

$$CG [u_s - u(T)] = W + k_1 \quad (23.3)$$

When $G = 0$, $W = 0$, so that $k_1 = 0$. Thus

$$CG [u_s - u(T)] = W \quad (23.4)$$

Now u_s depends only on the chemical composition of the solid powder and on its temperature, which probably differs very little from its initial temperature because of the low thermal conductivity of solid propellants. There will be a temperature of the powder gas, however, at which the specific energy of the gas will be equal to u_s . Inspection of Eq. (23.4) shows that this temperature is T_0 , the adiabatic closed chamber explosion temperature; this statement follows from the facts that for an adiabatic closed chamber $W = 0$ and $T = T_0$.

Thus

$$u_s = u(T_0) \quad (\text{For cool powders}), \quad (23.5)$$

and

$$CG u(T_0) = CG u(T) + W, \quad (4)$$

this being the equation we wished to rederive.

Chapter II

The Heat Loss Problem

We next consider the evaluation of the various terms of which the quantity W in our fundamental energy equation (10) is composed. The most difficult part of W to handle in any rational manner is the loss of heat by contact of the hot gases with the walls of the gun. Unfortunately this heat loss term is an important one in small arms*, where it may be equal to the kinetic energy acquired by the projectile. This chapter is devoted to outlining the main features of a possible rational attack on the problem and a statement of the empirical procedure that we shall actually adopt.

Notation: t , time from beginning of motion

$-t_1$, time at which burning begins

x , distance from breech to an arbitrary cross-section

L , length of powder chamber

s , travel of projectile with respect to the gun

D_1 , diameter of chamber

D_2 , diameter of bore

A_1 , cross-sectional area of chamber

A_2 , cross-sectional area of bore

T , temperature of main body of gas

T_b , temperature of breech face

$T_w(x,t)$ temperature of walls of chamber and bore

T_p , temperature of base of projectile

h_b , coefficient of heat transfer from hot gas to breech face

h_w , coefficient of heat transfer from hot gas to walls of chamber and bore.

* There is some theoretical indication that the heat loss may be important also in large caliber guns.

h_p , coefficient of heat transfer from hot gas to base of projectile.
 $Q(t)$, total heat loss at time t by contact of gas with gun.

The total loss $Q(t)$ is made up of the parts Q_b to the breech, Q_{cw} to the chamber wall, Q_{bw} to the bore wall, and Q_p to the base of the projectile. The component losses are given by:

$$Q_b = \int_{-t_1}^t h_b A_1 (T - T_b) dt \quad (24)$$

$$Q_{cw} = \int_{-t_1}^t \int_0^L h_w dt \pi D_1 (T - T_w) dx \quad (25)$$

$$Q_{bw} = \int_0^t \int_L^{L+s} h_w dt \pi D_2 (T - T_w) dx \quad (26)$$

$$Q_p = \int_{-t_1}^t h_p A_2 (T - T_p) dt \quad (27)$$

From the beginning of burning at time $-t_1$ to the beginning of motion at $t = 0$ there is no motion of the powder gas as a whole, so that the transfer coefficients in this interval are for a sort of "free convection". After time $t = 0$ there is still no motion of powder gas as a whole with respect to the breech face or the projectile face, so that the transfer coefficients h_b and h_p are still for "free convection". There is after $t = 0$, however, motion of the gas as a whole with respect to the walls of chamber and bore, so we speak of h_w as a transfer coefficient for "forced convection".

To determine h_w , one might use the expressions for forced convection given in McAdams³, although these expressions seem to be for values of the Reynolds number somewhat lower than those that occur in a gun. All such expressions, however, give zero as the answer when the velocity is zero, and it is hard to believe that the heat loss with the gas at rest would not be appreciable.

³W. H. McAdams "Heat Transmission", McGraw-Hill Book Company, (1933) Chapter VII.

Furthermore, the "free convection" transfer coefficients that occur here are for situations that do not occur in ordinary cases of free convection, since free convection in a gun is probably associated with strong turbulence. Then again the wall temperatures are not known. Of course if one knew the transfer coefficients one might calculate the wall temperatures from the geometry, thermal conductivity, and specific heat of the gun and the transfer coefficient from the outside of the gun to the air. Such a calculation, however, would have to be a special one for each gun, and would apparently not be applicable to a general treatment of interior ballistics.

Furthermore, even if one could evaluate the heat loss accurately as a function of time, the resulting complication of the equations with the addition of many extra parameters might make it inadvisable to use the accurate formula. For all these reasons we adopt tentatively the procedure of taking Q to be proportional to the kinetic energy of the projectile. Thus if m is the mass of the projectile and v its velocity with respect to the ground, we have:

$$Q(t) = k \frac{1}{2} mv^2 \quad (28)$$

We evaluate k by using muzzle values of $\frac{1}{2} mv^2$ and $Q(t)$. In small arms muzzle values of Q are best known for machine guns, for which one can measure the total heat developed in a large number of rounds, and divide that value by the number of rounds. It is probably better to take heat loss into account in this way than to neglect it altogether, as has been the custom up to the present.

In the place of any better treatment we may regard any energy lost in stretching the gun as proportional to the kinetic energy of the projectile. Thus in Eq. (28) we regard Q as the sum of heat loss and stretch loss, and k as a constant that takes both into account.

Chapter III

The Mechanical Problems

In this chapter we consider the translational and rotational motions of the projectile and the passive resistance. The motion of the powder and powder gases and the recoil of the gun are omitted here, since their proper consideration requires a treatment of the hydrodynamical problems (Chapter IV).

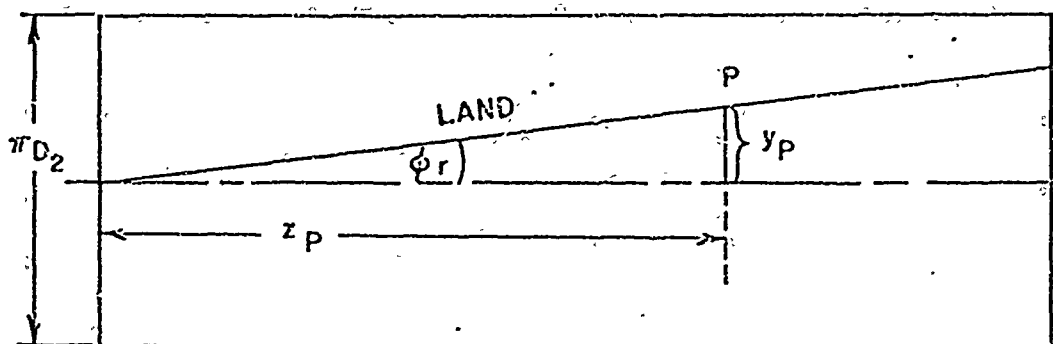
If m is the mass of the projectile, v its velocity with respect to the ground, p_a the pressure required to produce the translational acceleration, W_{trans} the translational kinetic energy of the projectile, and A_2 the cross-sectional area of the bore, then

$$p_a = \frac{m}{A_2} \frac{dv}{dt} \quad (29)$$

$$W_{\text{trans}} = 1/2 mv^2 \quad (30)$$

In treating rotational motion we consider only the case of a constant angle of rifling ϕ_r . With constant rifling a land of the gun is helical in form, so that development of the bore of the gun on a plane results in the situation depicted in Fig. 1.

Fig. 1.
Development of Bore on Plane



For the point P on the rotating band, touching one of the lands,

$$y_p = z_p \tan \phi_r \quad (31)$$

If the angular coordinate (in cylindrical coordinates) of the point P is denoted by θ , and D_2 is the diameter of the bore, then

$$y_p = 1/2 \theta D_2 \quad (32)$$

From Eqs. (31) and (32),

$$\theta = \frac{2}{D_2} z_p \tan \varphi_r \quad (33)$$

If v is the linear velocity of the projectile and ω its angular velocity, then

$$v = \frac{dz_p}{dt} \quad (34)$$

$$\omega = \frac{d\theta}{dt} \quad (35)$$

Eqs. (33), (34), and (35) give:

$$\omega = \frac{2 \tan \varphi_r}{D_2} v \quad (36)$$

Letting W_{rot} denote the rotational kinetic energy of the projectile and R_g its radius of gyration, we have

$$W_{rot} = \frac{1}{2} m R_g^2 \omega^2 \quad (37)$$

Eqs. (36) and (37) then give:

$$W_{rot} = \frac{1}{2} m v^2 f_1 \tan^2 \varphi_r \quad (38)$$

$$\text{where } f_1 \equiv (2R_g/D_2)^2, \quad (39)$$

about 0.6 for most projectiles.

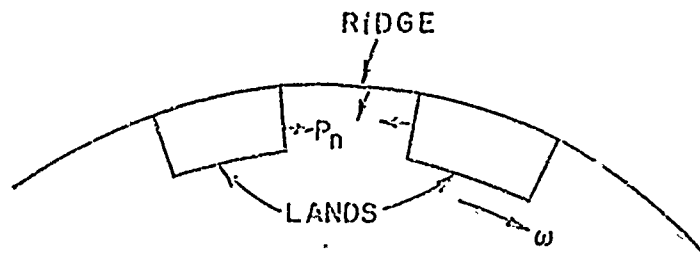
Now let p_{rot} denote the pressure required to produce the rotation. The power going into rotation is

$$A_2 p_{rot} v = \frac{dW_{rot}}{dt} = m v \frac{dv}{dt} f_1 \tan^2 \varphi_r \quad (40)$$

$$\text{so that } p_{rot} = \frac{m}{A_2} \frac{dv}{dt} f_1 \tan^2 \varphi_r \quad (41)$$

We next derive an expression for the energy loss due to passive resistance, i.e., due to friction between the rotating band and the bore of the gun. It is assumed that engraving has been completed. We call the part of the rotating band between two of the

grooves a "ridge" of the band, and of the forces exerted by the bore on the projectile take into account only those forces exerted on the ridges by the driving sides of the lands of the bore. We also assume the law of friction to hold between a ridge and the contiguous driving land. Thus in Fig. 2, of the pressures indicated, only P_n is taken into account:



The forces acting per unit area on the side face of the ridge are P_n , the pressure exerted by the side face of the contiguous driving land, and F , the friction force per unit area. We now introduce the unit vectors l_n along the normal to the ridge in the general direction of rotation and l_t along the tangent to the ridge in the general direction of the breech. In the system of cylindrical coordinates $1/2 D_2$, θ , and z , where θ is positive in the direction of rotation and z is the distance from the breech along the axis of the bore, we introduce the unit vectors l_θ and l_z . Then

$$\vec{P}_n = l_n P_n$$

$$\vec{F} = -l_t f$$

$$\begin{aligned} l_n &= -l_z \sin \phi_r + l_\theta \cos \phi_r \\ l_t &= l_z \cos \phi_r + l_\theta \sin \phi_r \end{aligned} \quad (42)$$

The components of total force per unit area on the side face of the ridge are then:

$$z \text{ component} = - (P_n \sin \phi_r + f \cos \phi_r) \quad (43)$$

$$\theta \text{ component} = P_n \cos \phi_r - f \sin \phi_r \quad (44)$$

Integration of Eq. (43) over the side face of the ridge and multiplication by N , the number of lands or of ridges, gives the total forward force on the projectile due to the bore. Multiplication of Eq. (44) by $1/2 D_2$, integration over the side face of the ridge, and multiplication by N gives the total torque on the projectile. On the assumption that P_n and f are uniform and that the angle of rifling is constant, we get:

$$F_z = - N A_R (P_n \sin \phi_r + f \cos \phi_r) \quad (45)$$

$$\text{Torque} = 1/2 N A_R D_2 (P_n \cos \phi_r - f \sin \phi_r) \quad (46)$$

where A_R is the area of the side face of a ridge.

The law of friction states that f is proportional to P_n , so that

$$f = \mu P_n, \quad (47)$$

where μ is a constant, Eq. (47) may or may not hold accurately for the friction of the lands against the ridges. If it does hold, however, the coefficient of friction μ may not have the value that it has for ordinary cases of the friction of steel and copper. We may now express $N A_R P_n$ in terms of the acceleration $\frac{dv}{dt}$ as follows. Insert Eq. (47) into Eq. (46), equate the torque to $R^2 \frac{d\omega}{dt}$ and use Eq. (36). One readily obtains:

$$N A_R P_n = m f_1 \frac{\tan \alpha_r}{\cos \alpha_r - \mu \sin \alpha_r} \cdot \frac{dv}{dt} \quad (48)$$

where f_1 is given by Eq. (39). If we now let F_B denote the force on the base of the projectile, we have from Eqs. (45) and (47):

$$F_B - N A_R P_n (\sin \alpha_r + \mu \cos \alpha_r) = m \frac{dv}{dt} \quad (49)$$

Eqs. (48) and (49) together give:

$$F_B = m \frac{dv}{dt} \left[1 + \frac{f_1 \tan \alpha_r (\tan \alpha_r + \mu)}{1 - \mu \tan \alpha_r} \right] \quad (50)$$

Letting p_{pass} denote the passive pressure and W_{pass} the energy expended in overcoming passive resistance, and realizing that Eq. (50) gives the total force necessary to produce translational acceleration and rotational acceleration and to overcome passive resistance, we have from Eq. (50):

$$P_{rot} + p_{pass} = \frac{m}{A_2} \frac{dv}{dt} f_1 \frac{\tan \alpha_r (\tan \alpha_r + \mu)}{1 - \mu \tan \alpha_r} \quad (51)$$

$$W_{rot} + W_{pass} = 1/2 m v^2 f_1 \frac{\tan \alpha_r (\tan \alpha_r + \mu)}{1 - \mu \tan \alpha_r} \quad (52)$$

We may expect to have

$$\mu \tan \alpha_r \ll 1 \quad (53)$$

With the approximation (53), comparison of Eq. (51) with Eq. (41) and of Eq. (52) with Eq. (38) gives the equations:

$$p_{pass} = \frac{m}{A_2} \frac{dv}{dt} \mu^2 \quad (54)$$

$$W_{\text{pass}} = 1/2 mv^2 \mu' \quad (55)$$

where $\mu' = \mu f_1 \tan \varphi_r \quad (56)$

In view of the difficulty of obtaining the value of the coefficient of friction under the actual interior ballistic conditions and possible doubts as to the validity of the law of friction, we shall not commit ourselves to the subsequent use of Eq. (56), but shall treat μ' as a quantity to be determined by experiment or by comparison with firings.

In order to take into account the finer details of a trajectory one might use a certain constant value for μ' during the process of engraving and a smaller constant value thereafter. One should also mention here the suggestion that has been made, notably by Dederick⁴, of using a constant value not for μ' but for the passive pressure itself, or of one constant passive pressure during engraving and a smaller one thereafter. Assumption of a constant passive pressure

$$P_{\text{pass}} = P_1 \quad (57)$$

leads to the expression

$$W_{\text{pass}} = P_1 s, \quad (58)$$

where s is the travel.

The pressure p_p on the base of the projectile may now be expressed with the aid of Eqs. (29) and (41) and Eq. (54) or Eq. (57)

Case I. Passive Pressure proportional to Accelerating Pressure:

$$p_p = \frac{m}{A_2} \frac{dv}{dt} (1 + \mu' + f_1 \tan^2 \varphi_r) \quad (59)$$

Case II. Constant Passive Pressure:

$$p_p = \frac{m}{A_2} \frac{dv}{dt} (1 + f_1 \tan^2 \varphi_r) + P_1 \quad (60)$$

If we now let v_1 denote the recoil velocity of the gun, e the ratio v_1/v , (which will be shown later to be a function of travel), and u the velocity of the projectile with respect to the gun, then

$$v = u/(1+e) \quad (61)$$

$$\text{and } p_p = \frac{m}{A_2(1+e)} \left[\frac{du}{dt} - \frac{u}{(1+e)^2} \frac{de}{dt} \right] (1 + \mu' + f_1 \tan^2 \varphi_r) \quad (62)$$

(Case I)

⁴L. S. Dederick: "Project for a New Table for Interior Ballistics", R I 17

or

$$P_p = \frac{m}{A_2(1+e)} \left[\frac{du}{dt} - \frac{u}{(1+e)^2} \frac{de}{dt} \right] (1 + f_1 \tan^2 \phi_r) + F_1 \quad (53)$$

(Case II)

Chapter IV

The Hydrodynamical Problems

1. The Equation of Continuity. We adopt the following notation.

- x , distance from breech face to a variable cross-section in the chamber or the bore.
- X , distance from breech face to base of projectile.
- $a(x)$, cross-sectional area at distance x from breech face.
- $w(x)$, total volume from breech face to cross-section at distance x .
- C , mass of powder charge.
- m , mass of projectile.
- $\epsilon \equiv C/m$
- $\rho(t)$, density of powder and powder gas mixture.
- $u_g(x,t)$, velocity at distance x from breech face of powder gas with respect to the gun.
- $v_g(x,t)$, velocity at distance x from breech face of powder gas with respect to the ground.
- u , velocity of the projectile with respect to the gun.
- γ , defined in Chapter I.

In this chapter we adopt the fundamental hypothesis of Ramsey's⁵ theory of fluid motion in the gun. These are:

- (a) the density of the mixture of powder and powder gas is uniform although variable with time;
- (b) the velocity of flow of this mixture is parallel to the axis of chamber and bore and uniform over a cross-section.

In regard to hypothesis (a) we refer to a paper by Kent⁶, in which he shows that non-uniformity of density can have only a small effect. He shows that non-uniformity of density changes the ratio of breech pressure to projectile base pressure only by the fraction $\epsilon^2/24\gamma$; for $\epsilon = 1/3$ and $\gamma = 1.2$, this figure amounts to $1/260$. His calculation is first carried out on the assumptions that the powder is all burned, that the powder gas is a perfect gas, and that the expansion process is adiabatic; in the latter part of the paper, however, he points out that the result is independent of these assumptions.

Hypothesis (b) is a natural one to make in view of the turbulence that undoubtedly exists in the gun.

⁵H. F. Ramsey, Jr., Ballistic Research Laboratory Report No. 3, (with J. R. Lane) January 25, 1935.

⁶R. H. Kent - Physics 7, 319(1936). See also Love and Pidduck, Phil. Trans. Roy. Soc. 222, 222 (1922)

It has long been known that for viscous flow in a tube the distribution of velocity over a cross-section is parabolic, but that for turbulent flow the velocity is uniform over most of a cross-section. One should realize, however, that there may be violent statistical fluctuations in the velocity, due to the possible existence of large rather than small eddies.

The equation of continuity follows directly from the conservation of mass, with the use of hypothesis (a) and (b). Thus in time dt there flows into a slab of cross-sectional area $a(x)$ and thickness dx the mass $\rho u_g(x)a(x)dt$ and from it there flows the mass $\rho u_g(x+dx)a(x+dx)dt$. The net gain of mass in time dt is thus $-\rho \frac{\partial}{\partial x} (au_g) dx dt$, which expression is to be equated to the product of the volume of the slab and its gain of density in time dt , viz.

$a(x)dx \frac{d\rho}{dt}$. Thus we get the equation of continuity:

$$\rho \frac{\partial}{\partial x} (au_g) = -a \frac{d\rho}{dt} \quad (64)$$

Dividing Equation (64) by ρ , integrating with respect to x from 0 to x , and introducing the definition:

$$= \frac{d}{dt} \ln \rho \equiv \psi(t), \quad (65)$$

we have:

$$au_g \Big|_0^x = \psi(t) \int_0^x a(x) dx \quad (66)$$

Introduction of the boundary condition $u_g(0,t) = 0$, corresponding to the fact that the gas at the breech face has no forward velocity, and use of the fact that

$$\int_0^x a(x) dx = w(x) \quad (67)$$

gives:

$$a(x)u_g(x,t) = \psi(t)w(x) \quad (68)$$

At the base of the projectile $x = X$, and $u_g(x,t) = u(t)$, so that

$$a(X)u(t) = \psi(t)w(X) \quad (69)$$

Elimination of $\psi(t)$ between Equations (68) and (69) gives:

$$u_g(x,t) = \frac{a(X)}{a(x)} \frac{\omega(x)}{\omega(X)} u(t) \quad (70)$$

where $u(t) = \frac{dX}{dt}$, the velocity of the projectile with respect to the gun. Eq. (70) was first derived in a different manner by Bederick**.

2. Recoil of the Gun.

Recoil of the gun is treated in this chapter because its consideration requires the use of Equation (70). We must now make some assumption about the shape of the gun. Actually the greater part of the chamber of a gun is of essentially constant cross-section and the bore is of strictly constant cross-section** somewhat smaller than that of the chamber. The two are ordinarily joined by a portion of a cone. The problem of pressure distribution in the gun would be complicated very seriously by explicit treatment of the cone. We shall therefore use a slightly idealized model for the gun. We assume a constant cross-section A_2 for the bore equal to that required to give the correct volume to the bore when the lands are taken into account, and a constant cross-section A_1 for chamber plus joining cone, A_1 to be so chosen that it gives the correct volume for chamber plus joining cone. We subsequently abbreviate the phrase "chamber plus joining cone" to "chamber" and call its length L . We also assume that when the projectile is seated that its base is flush with the end of the joining cone, so that L is also equal to the distance from the breach face to the base of the seated projectile. We denote the travel of the projectile by s . Then

$$x = L + s \quad (71)$$

$$\omega(x) = A_1 L + A_2 s \quad (72)$$

Abbreviating the ratio A_1/A_2 to σ , we have for

$$x < L : \frac{\omega(x)}{a(x)} = x \quad (73)$$

$$x > L : \frac{\omega(x)}{a(x)} = \frac{A_1 L + A_2(x-L)}{A_2} = x + (\sigma - 1)L \quad (74)$$

We also let v be the velocity of the projectile with respect to the ground, v_1 the velocity of the gun with respect to the ground, and $v_1/v = e$, the recoil ratio. Then

$$u_g(x) = v_g(x) + v_1 \quad (75)$$

$$u = v + v_1 \quad (76)$$

**B.R.L. Report No. X91, Jan. 4, 1933, Addendum to Appendix D.

***We are thus excluding tapered-bore guns from consideration in this report.

Equations (70), (72), (75), and (76) and the relation $a(x) = A_2$ now give:

$$v(x) + v_1 = \frac{A_2}{A_1 L + A_2 s} (v + v_1) \frac{w(x)}{a(x)} = \frac{v + v_1}{s + \sigma L} \frac{w(x)}{a(x)} \quad (77)$$

Equations (77), (73), and (74) now give:

$$x < L: v_g(x) = -v_1 + \left(\frac{v + v_1}{s + \sigma L} \right) x \quad (78)$$

$$x > L: v_g(x) = -v + \frac{v + v_1}{s + \sigma L} \left[x + (\sigma - 1)L \right] \quad (79)$$

We now assume free recoil, so that the total momentum of the system is conserved. The momentum M_3 of powder and powder gas with respect to the ground is:

$$M_3 = \int_0^L \rho v_g(x) A_1 dx + \int_L^{L+s} \rho v_g(x) A_2 dx \quad (80)$$

Using $\rho = C/a(x)$ (81)

and equations (78), (79), (80), and (72), we obtain an integration:

$$M_3 = -Cv_1 + \frac{C(v + v_1)}{2(s + \sigma L)^2} (\sigma L^2 + s^2 + 2\sigma Ls) \quad (82)$$

With the abbreviation $s/L \equiv r$,
Equation (82) becomes:

$$M_3 = -Cv_1 + \frac{C}{2} (v + v_1) \left[1 - \frac{(\sigma - 1)}{(\sigma + r)^2} \right] \quad (83)$$

The momentum of the projectile with respect to the ground is:

$$M_2 = mv \quad (84)$$

and that of the recoiling parts with respect to the ground is:

$$M_1 = m_1 v_1 \quad (85)$$

The conservation of momentum gives:

$$M_2 + M_3 = M_1, \quad (86)$$

so that

$$mv - Cv_1 + \frac{C}{2} (v + v_1) \left[1 - \frac{\sigma(\sigma - 1)}{(\sigma + r)^2} \right] = m_1 v_1 \quad (87)$$

Solution of Eq. (87) gives:

$$e \equiv \frac{v_1}{v} = \frac{m + \frac{C}{2} \left[1 - \frac{\sigma(\sigma - 1)}{(\sigma + r)^2} \right]}{m_1 + \frac{C}{2} \left[1 + \frac{\sigma(\sigma - 1)}{(\sigma + r)^2} \right]} \quad (88)$$

In the special case that $\Lambda_1/\Lambda_2 \equiv \sigma = 1$, Eq. (88) reduces to:

$$e \equiv \frac{m + C/2}{m_1 + C/2} \quad (89)$$

Eq. (89) is the usual expression found in treatments of interior ballistics. Eq. (88) may be somewhat simplified by noting that even in the most extreme case the ratio of $C/2$ to m_1 is only 1 part in 600, so that, to a good accuracy, the denominator of Eq. (88) may be simplified to m_1 . This simplification, together with the use of $C \equiv C/m$, gives

$$e = \frac{m}{m_1} \left[1 + \frac{C}{2} \left\{ 1 - \frac{\sigma(\sigma - 1)}{(\sigma + r)^2} \right\} \right] \quad (90)$$

Actually the recoil is always damped, and the effect of this damping is to reduce e by a very small amount; the effect of a value of $\sigma > 1$ is also such as to reduce e . The two effects thus act in the same direction, so that inclusion of the correction term in σ should improve matters. (If the effects acted in opposite directions, they would partially cancel each other and taking the σ term into account might not be an improvement).

The energy of recoil W_R is given by:

$$W_R = 1/2 m_1 v_1^2 \quad (91)$$

Using $v_1 = ev$, we have:

$$\frac{W_R}{1/2 mv^2} = \frac{m_1}{m} e^2 \quad (92)$$

Applying Eq. (90) and the abbreviation

$$\frac{\sigma(\sigma - 1)}{(\sigma + r)^2} \equiv F_1(r) \quad (93)$$

we get:

$$\frac{W_R}{1/2 mv^2} = \frac{m}{m_1} \left(1 + \frac{C}{2} - \frac{C}{2} F_1 \right)^2 \quad (94)$$

3. Kinetic Energy of Powder and Powder Gases

(a) With Neglect of Recoil.

In this case we take $v_g(x) = u_g(x)$ and $u = v$. The kinetic energy of the powder and powder gases is given by:

$$W_{\text{powd.}} = 1/2 \int_0^L \rho v_g^2(x) A_1 dx + 1/2 \int_L^{L+s} \rho v_g^2(x) A_2 dx \quad (95)$$

With the approximation of $v_g(x)$ by $u_g(x)$ and the use of Eqs. (70), (72), (73), (74), and (81), Eq. (95) gives:

$$W_{\text{powd.}} = 1/2 mv^2 \left[\frac{\epsilon}{3} \left[1 - \frac{\sigma(\sigma^2 - 1)}{(\sigma + r)^3} \right] \right] \quad (96)$$

(b) With Recoil Taken into Account.
Eqs. (95), (78), and (79) give:

$$W_{\text{powd.}} = W_1 + W_2, \text{ where}$$

$$W_1 = 1/2 \rho A_1 \int_0^L \left[-v_1 + \frac{(v + v_1)}{\sigma L + s} x \right]^2 dx \quad (97)$$

$$W_2 = 1/2 \rho A_2 \int_L^{L+s} \left[-v_1 + \frac{(v + v_1)}{\sigma L + s} \left\{ x + (\sigma - 1)L \right\} \right]^2 dx \quad (98)$$

Eqs. (72) and (81) and the relation $v_1 = ev$ give:

$$W_1 = 1/2 mv^2 \frac{\epsilon \sigma}{3(1 + e)} \left[\left(\frac{1 + e}{\sigma + r} - e \right)^3 + e^3 \right] \quad (99)$$

and

$$W_2 = 1/2 mv^2 \frac{\epsilon}{3(1 + e)} \left[1 - \left\{ \frac{\sigma(1 + e)}{\sigma + r} - e \right\}^3 \right] \quad (100)$$

Thus

$$W_{\text{powd.}} = 1/2 mv^2 \frac{\epsilon}{3(1 + e)} \left[1 + \sigma e^3 + \sigma \left(\frac{1 + e}{\sigma + r} - e \right)^3 - \left\{ \frac{\sigma(1 + e)}{\sigma + r} - e \right\}^3 \right] \quad (101)$$

Note that if e is put equal to zero that Eq. (101) reduces to Eq. (96). Eq. (101) may be simplified as follows. Place

$$\frac{1 + e}{\sigma + r} \equiv q \quad (102)$$

Then the expression in brackets $F_2(r)$ is expressible as:

$$F_2(r) = 1 + \sigma e^3 + \sigma (q - e)^3 - (\sigma q - e)^3 \quad (103)$$

Thus

$$F_2 = 1 - \sigma (\sigma^2 - 1)q^3 + 3\sigma (\sigma - 1)q^2 e + e^3 \quad (104)$$

or $F_2 = 1 + e^3 - F_1 \left[\frac{\sigma + 1}{\sigma + r} (1 + e)^3 - 3e(1 + e)^2 \right]$ (105)

In Eq. (105) we may omit, with errors of less than 1 part in 1000, all terms not linear in e . Then

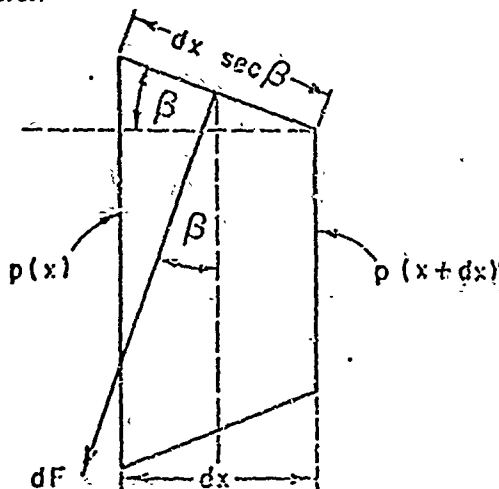
$$F_2 = 1 - F_1 \left[\frac{\sigma + 1}{\sigma + r} (1 + 3e) - 3e \right] \quad (106)$$

or

$$F_2 = 1 - F_1 \left[\frac{\sigma + 1}{\sigma + r} + 3e \left(\frac{\sigma + 1}{\sigma + r} - 1 \right) \right] \quad (107)$$

4. Pressure Distribution without Gas Friction.

Because of the acceleration of the powder and powder gases there is a drop in pressure from the breech to the base of the projectile. The differential equation for this pressure drop may be derived by calculating the net forward force on a slab of gas whose faces are two cross-sections of separation dx , and equating this net force to the product of the mass and the acceleration of the slab. We indicate in Fig. 3 the portion of the gun between two cross-sections. Let $p(x, t)$ be the pressure at time t at distance x from the breech face, R_x the radius of the tube (chamber, joining cone, or bore) at distance x from the breech face, and β the angle between a normal to the tube and a normal to the axis. Then



the force dF exerted on the slab by an element of wall area $dx \sec\beta R_x d\theta$ is:

$$dF = p(x)R_x \sec\beta dx d\theta, \quad (108)$$

and the forward component of this is:

$$-dF \sin \beta = -p(x)R_x \tan\beta dx d\theta \quad (109)$$

Now
$$dR_x = -dx \tan \beta, \quad (110)$$

so that
$$-dF \sin \beta = p(x)R_x dR_x d\theta \quad (111)$$

The total forward force due to the walls is thus

$$dF_w = 2\pi p(x)R_x dR_x = p(x)da(x) \quad (112)$$

To dF_w must be added the net forward force due to the faces of the slab, viz.

$$dF_f = p(x)a(x) - p(x+dx)a(x+dx) = -\frac{\partial}{\partial x} (pa)dx \quad (113)$$

$$= -pda - a \frac{\partial p}{\partial x} dx \quad (114)$$

Thus

$$dF_w + dF_f = -a \frac{\partial p}{\partial x} dx \quad (115)$$

The mass acceleration product of the slab is $\frac{C}{w(X)} \frac{dv_g(x,t)}{dt} adx$,

so that

$$-a \frac{\partial p}{\partial x} dx = \frac{Ca}{w(X)} \frac{dv_g(x,t)}{dt} dx, \quad (116)$$

or

$$\frac{\partial p(x)}{\partial x} = -\frac{C}{w(X)} \frac{dv_g(x,t)}{dt} \quad (117)$$

Eq. (117) is the fundamental law of pressure distribution when gas friction is negligible; $v_g(x,t)$ is the gas velocity with respect to the ground. Now we may express $v_g(x,t)$ by

$$v_g(x,t) = u_g(x,t) - v_1, \quad (75)$$

where $u_g(x,t)$ is the gas velocity with respect to the gun, and v_1 is the velocity of recoil.

Then
$$\frac{dv_g(x,t)}{dt} = \frac{du_g(x,t)}{dt} - \frac{dv_1}{dt} \quad (118)$$

Insertion of Eq. (118) into Eq. (117) gives:

$$\frac{\partial p(x,t)}{\partial x} = - \frac{c}{\omega(X)} \left[\frac{du_g(x,t)}{dt} - \frac{dv_i}{dt} \right] \quad (119)$$

Thus

$$p(x) - p(0) = - \frac{c}{\omega(X)} \int_0^x \frac{du_g(x,t)}{dt} dx + \frac{cx}{\omega(X)} \frac{dv_i}{dt} \quad (120)$$

The derivative $\frac{du_g(x,t)}{dt}$ may be evaluated by means of Eq. (70). This derivation is given in Ramsey's⁵ report, but is summarized here for purposes of completeness and clarity. The acceleration $\frac{du_g(x,t)}{dt}$ is the acceleration of a definite mass of gas and u_g is a function of x and of t , say $f(x,t)$. At time t , u_g has the value $f(x,t)$; at time $t + dt$ the slab has an x -coordinate equal to $x + u_g dt$, so that u_g has the value

$$f(x + u_g dt, t + dt) = f(x,t) + \frac{\partial f}{\partial x} u_g dt + \frac{\partial f}{\partial t} dt \quad (121)$$

The acceleration is thus given by

$$\frac{du_g}{dt} = \frac{\partial f}{\partial x} u_g + \frac{\partial f}{\partial t} \quad (122)$$

Now, from Eq. (70)

$$f(x,t) = f_1(x)f_2(t) \quad (123)$$

where

$$f_1(x) = \frac{\omega(x)}{a(x)} \quad (124)$$

and

$$f_2(t) = \frac{a(X)}{\omega(X)} u(t) \quad (125)$$

In Eq. (125) the factor $\frac{a(X)}{\omega(X)}$ is a function of t , since the x -coordinate X of the base of the projectile is a function of t .

Thus

$$\frac{du_g}{dt} = u_g f_1'(x) f_2(t) + f_1(x) f_2'(t), \quad (126)$$

$$= f_2^2(t) f_1(x) f_1'(x) + f_2'(t) f_1(x), \quad (127)$$

so that
$$\int_0^x \frac{du_g}{dt} dx = 1/2 f_2^2(t) f_1^2(x) \Big|_0^x + f_2^1(t) \int_0^x f_1(x) dx \quad (128)$$

Now $f_1(0) = 0$, so that:

$$\int_0^x \frac{du_g}{dt} dx = 1/2 f_2^2(t) f_1^2(x) + f_2^1(t) \int_0^x \frac{\omega(x)}{a(x)} dx \quad (129)$$

$$= 1/2 u_g^2(x,t) + f_2^1(t) \int_0^x \frac{\omega(x)}{a(x)} dx \quad (130)$$

Insertion of Eq. (130) into Eq. (120) gives:

$$p(x) - p(0) = - \frac{C}{\omega(x)} \left[\frac{1}{2} u_g^2(x,t) + f_2^1(t) \int_0^x \frac{\omega(x)}{a(x)} dx - x \frac{dv_1}{dt} \right] \quad (131)$$

This is the general law of pressure distribution without gas friction.

For our model $\omega(x) = k_1 L + k_2 x$, $a(x) = k_2$, and

$$x < L: \frac{\omega(x)}{a(x)} = x \quad (73)$$

$$x > L: \frac{\omega(x)}{a(x)} = x + (\sigma - 1)L \quad (74)$$

Thus for our model, when

$$x < L: \int_0^x \frac{\omega(x)}{a(x)} dx = 1/2 x^2 \quad (132)$$

$$\begin{aligned} x > L: \int_0^x \frac{\omega(x)}{a(x)} dx &= \int_0^L x dx + \int_L^x [x + (\sigma - 1)L] dx \\ &= 1/2 L^2 + 1/2 \left[\{x + (\sigma - 1)L\}^2 - \sigma^2 L^2 \right] \end{aligned} \quad (133)$$

Also, from Eqs. (123), (124), (125), (72), (73), and (74), we have for:

$$x < L: u_g(x,t) = \frac{xu(t)}{\sigma L + s} \quad (134)$$

and for

$$x > L: u_g(x,t) = \frac{[x + (\sigma - 1)L] u(t)}{\sigma L + s} \quad (135)$$

Also, since $f_2(t) = \frac{u(t)}{\sigma L + s}$ (by Eqs. (125) and (72)),

we have

$$f_2'(t) = \frac{u'(t)}{\sigma L + s} - \frac{u^2(t)}{(\sigma L + s)^2} \quad (136)$$

Insertion of Eq. (136) and Eq. (134) or Eq. (135) into Eq. (131) gives:

$$x < L: p(x) - p(0) = - \frac{C/A_2}{(\sigma L + s)^2} \left[\frac{1}{2} u'(t) x^2 - x(\sigma L + s) \frac{dv_1}{dt} \right] \quad (137)$$

$$x > L: p(x) - p(0) = - \frac{C/A_2}{2(\sigma L + s)^2} \left[\frac{(\sigma^2 - 1)L^2 u^2}{\sigma L + s} + u'(t) \left\{ (x + (\sigma - 1)L)^2 - (\sigma^2 - 1)L^2 \right\} - 2x(\sigma L + s) \frac{dv_1}{dt} \right] \quad (138)$$

Eqs. (137) and (138) give the distribution of gas pressure for our model provided gas friction is neglected. For a gun of uniform cross-section we should have $A_1 = A_2 = A$ and $\sigma = 1$.

For such a case we should have; for all x :

$$p(x) - p(0) = - \frac{C/A}{(L + s)^2} \left[\frac{1}{2} u'(t) x^2 - x(L + s) \frac{dv_1}{dt} \right] \quad (139)$$

$u'(t)$ is the acceleration of the projectile with respect to the gun, and $\frac{dv_1}{dt}$ is the acceleration of the gun with respect to the ground.

5. The Effect of Gas Friction on Pressure Distribution.

It is known that for turbulent flow of fluid in a pipe of diameter D , the pressure drop per unit length is representable by means of the equation:

$$\frac{\partial p}{\partial x} = - \frac{2f_F \rho}{D} u_g^2 \quad (140)$$

where ρ is the mass density of the fluid, u_g its velocity, and f_F is a dimensionless factor, usually called the Fanning friction factor. It is a rather weakly varying function of the dimensionless Reynolds number, the latter being defined as the ratio of the product of diameter, density, and velocity to the viscosity. Curves showing the variation of f_F with Reynolds number will be found on page 110 of McAdams's "Heat Transmission". Rough estimates show that the Reynolds number is of the order 10^7 in a gun; this is somewhat beyond the range of the curves given by McAdams. Furthermore the conditions of turbulence existing in a gun may be so different from those prevailing in steady flow in an ordinary pipe that Eq. (140) may not hold well in a gun. However, Eq. (140) is all that we have to go by in treating gas friction in guns. We shall therefore adopt Eq. (140) as representing the

falling off of pressure due to gas friction, but shall treat f_F as constant throughout the gun, and use some sort of average value for it. For a Reynolds number of 10^7 , McAdams's "rough pipe" curve gives approximately 0.003 as the value of f_F .

Addition of Eqs. (119) and (140), with $\rho = C/\omega(X)$ and $D = D_1$ for $x < L$ and $D = D_2$ for $x > L$, gives as the complete differential equation for pressure distribution:

$$\frac{\partial p(x)}{\partial x} = -\frac{C}{\omega(X)} \left[\frac{du_g(x,t)}{dt} - \frac{dv_1}{dt} + \frac{2f_F}{D} u_g^2(x,t) \right] \quad (141)$$

Our effective diameters are defined by the equations:

$$\frac{\pi}{4} D_1^2 = A_1 \quad (142)$$

$$\frac{\pi}{4} D_2^2 = A_2 \quad (143)$$

where the effective areas A_1 and A_2 have been defined in Section 2 of this chapter. The question may arise as to whether v_g or v_g should be used in Eq. (140) in generalizing from steady flow in a pipe at rest to accelerated flow in a gun in motion. The answer comes at once from the fact that $v_g = 0$ for gas at rest in a tube in motion, in which case, however, there is no pressure drop due to gas friction; use of v_g in place of u_g in Eq. (140) would thus be incorrect.

The gas friction correction to Eq. (137) for $p(x) - p(0)$ when $x < L$ is readily obtained by inserting Eq. (134) into Eq. (141) and integrating. We obtain for

$$x < L: p(x) - p(0) = -\frac{C/A_2}{2(\sigma L + s)^2} \left[u^2(t)x^2 - 2x(\sigma L + s) \frac{dv_1}{dt} + 4/3 \frac{f_F}{D_2 \sigma} \frac{u^2}{(\sigma L + s)} x^3 \right] \quad (144)$$

On inserting Eq. (135) into Eq. (141) and integrating, we obtain for

$$x > L: p(x) - p(0) = -\frac{C/A_2}{2(\sigma L + s)^2} \left[\frac{(\sigma^2 - 1)L^2 u^2}{\sigma L + s} + u^2(t) \left\{ [x + (\sigma - 1)L]^2 - (\sigma^2 - 1)L^2 \right\} - 2x(\sigma L + s) \frac{dv_1}{dt} + 4/3 \frac{f_F}{D_2} \frac{u^2}{(\sigma L + s)} \left\{ [x + (\sigma - 1)L]^3 + L^3(\sigma^{-1/2} - \sigma^3) \right\} \right] \quad (145)$$

To calculate the pressure p_p at the base of the projectile we insert $x = L + s$ into Eq. (145), thus obtaining:

$$\begin{aligned}
 p_p - p(o) = & - \frac{C/A_2}{2(\sigma L + s)^2} \left\{ \frac{(\sigma^2 - 1)L^2 u^2}{\sigma L + s} + u'(t) \left\{ (\sigma L + s)^2 \right. \right. \\
 & \left. \left. - (\sigma^2 - 1)L^2 \right\} - 2(L + s)(\sigma L + s) \frac{dv_1}{dt} + 4/3 \frac{f_F}{D_2(\sigma L + s)} \left\{ (\sigma L + s)^3 \right. \right. \\
 & \left. \left. + L^3 (\sigma^{-1/2} - \sigma^3) \right\} \right\} \quad (146)
 \end{aligned}$$

6. Average Pressure in the Equation of State.

In Chapter I on thermodynamic foundations it was assumed tacitly that the temperature was uniform. Actually it will be a function of x even with our model in which heat loss is treated as in Chapter II. Gas friction, for example, produces a dissipation of mechanical energy into heat energy, and we have seen that its effect depends on x . We may thus expect T to depend on x . It is clear, however, that the $u(T)$ there introduced should be a mass average of $u(T_x)$. Thus, if we let $\rho_g(x)$ be the density of the powder gas (not of powder plus powder gas), our Eq. (1) should be written:

$$Cd\dot{G}u(T_0) = d_t \int u(T_x) \rho_g(x) d\tau + \delta W, \quad (147)$$

The subscript t is appended to the differential sign in front of the integral to denote that the change thought of is a change occurring in time dt , so that the operations d_t and \int are not inverse to each other; $d\tau$ is the gas volume element. Eq. (147) may also be written:

$$Cd\dot{G}u(T_0) = d \left[\overline{CGu(T_x)} \right] + \delta W, \quad (148)$$

where

$$\overline{u(T_x)} = \int u(T_x) \rho_g(x) d\tau / CG, \quad (149)$$

the mass average of $u(T_x)$; the integration is over the volume of chamber and bore unoccupied by solid powder. Comparison of Eq. (148) with Eq. (1) shows that our earlier $u(T)$ must be interpreted as our present $\overline{u(T_x)}$.

Now $u(T_x) = c_v T_x$, where we assume the specific heat c_v at constant volume to be independent of x ; this assumption is equivalent to saying that there is no change in relative gas concentrations with x .

Thus

$$u(T) = \overline{u(T_x)} = c_v \overline{T_x}, \quad (150)$$

where $\overline{T_x}$ is the mass averages:

$$\overline{T_x} \equiv \frac{1}{CG} \int T_x \rho_g(x) dx \quad (151)$$

The integration is again over the volume of powder gas alone. Now if the co-volume of the gas per unit mass is η_1 , then the equation of state may be written as:

$$p(x) \left[\frac{1}{\rho_g(x)} - \eta_1 \right] = R_1 T_x, \quad (152)$$

Where R_1 is the gas constant per unit mass.

Then

$$\rho_g(x) T_x = \frac{1}{R_1} p(x) \left[1 - \eta_1 \rho_g(x) \right], \quad (153)$$

so that

$$\overline{T_x} = \frac{1}{CGR_1} \int p(x) \left[1 - \eta_1 \rho_g(x) \right] dx \quad (154)$$

The factor $1 - \eta_1 \rho_g(x)$ is a correction factor not very different from 1 for all x . Remembering here that Kent found non-uniformity of density to be unimportant, we regard this factor as constant, so that it may be taken outside the sign of integration. The integration is over the volume of powder gas alone, so that:

$$CGR_1 \overline{T_x} = \overline{p} V_g (1 - \eta_1 \rho_g) \quad (155)$$

In Eq. (155), V_g is the volume of powder gas alone, and \overline{p} is a volume average of pressure over the region unoccupied by solid powder. We are naturally led at this point to replace it by a volume average over the whole volume of chamber and bore, since in the Ramsey hydrodynamical theory powder and powder gas are treated as one fluid. Now $V_g(1 - \eta_1 \rho_g)$ is our free volume Ω and $\overline{T_x}$ our earlier T , so that Eq. (155) reduces to our earlier expression of the equation of state Eq. (9) if we use that equation a volume average of the pressure.

Thus the effective pressure p in the equation of state is given by:

$$(A_1 b + A_2 s) p = \int_0^L p(x) A_1 dx + \int_L^{L+s} p(x) A_2 dx \quad (156)$$

Now

$$(A_1 L + A_2 s) p(x) = \int_0^L p(x) A_1 dx + \int_L^{L+s} p(x) A_2 dx \quad (157)$$

Thus

$$(A_1 L + A_2 s) [p - p(x)] = \int_0^L [p(x) - p(x)] A_1 dx + \int_L^{L+s} [p(x) - p(x)] A_2 dx \quad (158)$$

We now insert Eq. (144) into the first integral on the right side of Eq. (158) and Eq. (145) into the second integral, thereby obtaining:

$$\int_0^L [p(x) - p(x)] A_1 dx = \frac{-c\sigma}{2(\sigma L + s)^2} \left[\frac{L^3}{3} u'(t) - L^2(\sigma L + s) \frac{dv_1}{dt} + \frac{f_F L^4 u^2}{3D_2 \sigma L/2 (\sigma L + s)} \right] \quad (159)$$

and

$$\int_L^{L+s} [p(x) - p(x)] A_2 dx = - \frac{c}{2(\sigma L + s)^2} \left[\frac{(\sigma^2 - 1) L^2 s u^2}{\sigma L + s} - (\sigma L + s) \left\{ (L + s)^2 - L^2 \right\} \frac{dv_1}{dt} + \left\{ 1/3 \left[(\sigma L + s)^3 - \sigma^3 L^3 \right] - (\sigma^2 - 1) L^2 s \right\} u'(t) + \frac{4f_F u^2}{3D_2 (L + s)} \left\{ 1/4 \left[(\sigma L + s)^4 - \sigma^4 L^4 \right] + L^3 s (\sigma^{-1/2} - \sigma^3) \right\} \right] \quad (160)$$

Addition of Eqs. (159) and (160) gives $\int_0^{L+s} [p(x) - p(x)] dx$.

On performing this addition and equating the sum to $\omega(x) (p - p_0)$, where where $\omega(x) = A_1 L + A_2 s$, and where p is the desired average pressure, we obtain:

$$p - p(x) = - \frac{c}{2A_2} H_1, \quad (161)$$

where

$$H_1 \equiv 1/3 \left[1 - \frac{(\sigma^2 - 1)(\sigma + 3r)}{(\sigma + r)^3} \right] u'(t) - \left[1 - \frac{(\sigma - 1)(\sigma + 2r)}{(\sigma + r)^2} \right] v_1'(t) + \left[\frac{(\sigma^2 - 1)r/L}{(\sigma + r)^4} + \frac{f_F}{3D_2} \left\{ 1 + \frac{\sigma^{1/2} - \sigma^4 + 4r(\sigma^{-1/2} - \sigma^3)}{(\sigma + r)^4} \right\} \right] u^2 \quad (162)$$

In Eq. (162) $u'(t)$ is the acceleration of the projectile with respect to the gun, u^2 the square of its velocity with respect to the gun, and $v_1'(t)$ is the acceleration of the gun with respect to the ground.

Now Eq. (146) may be put into the form:

$$p_p - p(o) = - \frac{c}{2A_2} H_2, \quad (163)$$

where

$$H_2 = \frac{(\sigma^2 - 1)u^2/L}{(\sigma + r)^3} + \left[1 - \frac{\sigma^2 - 1}{(\sigma + r)^2} \right] u'(t) - \frac{2(1+r)}{\sigma + r} v_1'(t) \\ + H/3 \frac{f_F u^2}{D_2} \left[1 + \frac{\sigma^{-1/2} - \sigma^3}{(\sigma + r)^3} \right] \quad (164)$$

Then

$$p - p_p = - \frac{c}{2A_2} (H_1 - H_2), \quad (165)$$

where

$$H_1 - H_2 = 2/3u'(t) \left[-1 + \frac{\sigma(\sigma^2 - 1)}{(\sigma + r)^3} \right] + v_1'(t) \left[1 - \frac{\sigma(\sigma - 1)}{(\sigma + r)^2} \right] \\ + \left[- \frac{(\sigma^2 - 1)c/L}{(\sigma + r)^4} - \frac{f_F}{D_2} \left[1 + \frac{(\sigma^{-1/2} - \sigma^4)}{(\sigma + r)^4} \right] \right] u^2 \quad (166)$$

In the special case of a uniform cross-section $\sigma = 1$, so that

$$H_1 - H_2 = - 2/3u'(t) + v_1' - \frac{f_F}{D_2} u^2 \quad (167)$$

Now, from Eq. (59), we have:

$$p_p = \frac{m'}{A_2} \frac{dv}{dt} \quad (168)$$

where

$$m' = m(1 + \mu' + f_1 \tan^2 \alpha_p) \quad (169)$$

Inserting Eq. (168) into Eq. (165) and using Eq. (76), we obtain for the average pressure:

$$p = \frac{m'}{A_2} \left[1 + \frac{\epsilon'}{3} \left[1 - \frac{\sigma(\sigma^2 - 1)}{(\sigma + r)^3} \right] \right] u'(t) \\ - \frac{m'}{A_2} \left[1 + \frac{\epsilon'}{2} \left[1 - \frac{\sigma(\sigma - 1)}{(\sigma + r)^2} \right] \right] v_1'(t) \\ + \frac{m'\epsilon'}{2A_2} \left[\frac{(\sigma^2 - 1)\sigma/L}{(\sigma + r)^4} + \frac{f_F}{D_2} \left[1 + \frac{(\sigma^{1/2} - \sigma^4)}{(\sigma + r)^4} \right] \right] u^2, \quad (170)$$

where

$$c' \equiv C/m' \quad (171)$$

It is to be emphasized that in the derivation of Eq. (170) no assumption has been made as to the relative magnitudes of projectile mass and powder mass. Note also that

$$v_1(t) = \frac{e}{1+e} u(t) \quad (172)$$

where e is given by Eq. (90). In Eq. (172) the e in the denominator may be neglected, so that, accurately enough:

$$v_1 = eu \quad (173)$$

and

$$v_1' = eu' + e'u \quad (174)$$

Now

$$e = \frac{m}{m_1} \left[1 + \frac{\epsilon}{2} \left\{ 1 - \frac{(\sigma-1)}{(\sigma+r)^2} \right\} \right], \quad (90)$$

so that

$$e' = -\frac{m}{m_1} \frac{\epsilon}{2} \frac{\sigma(\sigma-1)(-2)}{(\sigma+r)^3} \frac{u}{L} \quad (175)$$

since

$$r' = s'/L = u/L$$

Thus

$$v_1' = \frac{m}{m_1} \left[1 + \frac{\epsilon}{2} \left\{ 1 - \frac{\sigma(\sigma-1)}{(\sigma+r)^2} \right\} \right] u' + \frac{m}{m_1} \epsilon \frac{\sigma(\sigma-1)}{(\sigma+r)^3} \frac{u^2}{L} \quad (176)$$

Now introduce the abbreviations:

$$F_3(r) \equiv 1 + \frac{\epsilon}{2} \left\{ 1 - \frac{\sigma(\sigma-1)}{(\sigma+r)^2} \right\} = 1 + \frac{\epsilon}{2} (1 - F_1) \quad (177)$$

and

$$F_4(r) \equiv 1 + \frac{\epsilon'}{2} \left\{ 1 - \frac{\sigma(\sigma-1)}{(\sigma+r)^2} \right\} = 1 + \frac{\epsilon'}{2} (1 - F_1) \quad (178)$$

and insert Eq. (176) into Eq. (170). We obtain:

$$p = \frac{m'}{A_2} \left[1 + \frac{\epsilon'}{3} \left\{ 1 - \frac{\sigma(\sigma^2-1)}{(\sigma+r)^3} \right\} - \frac{m}{m_1} F_3 F_4 \right] u'(t) + \frac{m'\epsilon'}{2A_2} \left[\frac{(\sigma^2-1)\sigma/L}{(\sigma+r)^4} + \frac{fF}{D_2} \left\{ 1 + \frac{(\sigma^{1/2}-\sigma^2)}{(\sigma+r)^4} \right\} - \frac{2m'}{m_1} \frac{(\sigma-1)\sigma/L}{(\sigma+r)^3} F_4 \right] u^2 \quad (179)$$

It is a matter of some importance to note how the effect of gas friction changes when one changes the caliber of the gun. In order to compare the change in average pressure due to gas friction with the total pressure we may rewrite Eq. (179) as follows (besides the leading term only the gas friction term is retained):

$$\frac{A_2 p}{m'} = u' + \frac{\epsilon' f_F}{2D_2} \left\{ 1 + \frac{(\sigma^{1/2} - \sigma^4)}{(\sigma + r)^4} \right\} u^2 + \dots \quad (180)$$

To obtain some idea of the comparison of the term in u^2 with the term in u' , we approximate u^2 by the Leduc equation, which is known to give a rough representation of an interior ballistic trajectory. The Leduc equation states:

$$u = \frac{c_1 s}{c_2 + s} \quad (181)$$

We easily derive from this:

$$u^2 = u'(t)s \left(1 + \frac{s}{2s_m} \right), \quad *182)$$

where s_m is the value of the travel s for maximum $u'(t)$, which we may take here as equivalent to maximum pressure. Insertion of Eq. (182) into Eq. (180) gives:

$$\frac{A_2 p}{m'} = u' \left[1 + \frac{\epsilon' f_F s}{2D_2} \left\{ 1 + \frac{(\sigma^{1/2} - \sigma^4)}{(\sigma + r)^4} \right\} \left(1 + \frac{s}{2s_m} \right) \right] \quad (183)$$

Now in Eq. (183) let us consider geometrically similar guns, and let s have its muzzle value. Then the values of σ , $\frac{s}{D_2}$, and r are rigorously

equal for the different guns, and those of s/s_m presumably so. ϵ' is about the same for the different guns. Only f_F , the friction factor, remains to be considered. If we compare a 16 inch gun with a caliber .50 gun, the Reynolds numbers will be in the ratio 32 to 1, but inspection of the curves on p. 110 of McAdams's "Heat Transmission" shows that the resulting ratio of friction factors will not be larger than 2 to 1. We have therefore the important result that the fractional change in average pressure due to gas friction is about the same in all guns. Such differences as arise come mainly from differences in shape rather than from differences in caliber.

Chapter V

The Chemical Kinetic Problem

In this short chapter we consider the problem of the rate of burning of the powder. Since this subject is in a state of flux, much of the material of this chapter is necessarily somewhat indefinite and tentative.

The only generally accepted law concerning burning seems to be that known as the "law of burning in parallel layers." According to this law a given layer of a powder grain burns completely before the next layer begins to burn at all*. Stated in another way, at all points on the surface of a powder grain the surface recedes at the same rate, commonly called the rate of regression. It is standard notation to denote the linear distance, of regression in time t by x ; we shall follow this notation, since there seems to be no danger of confusion with the x of Chapter IV. The smallest linear dimension of a powder grain is known as its web, denoted by w . Thus for solid cylinders (cordite), the diameter is the web; for single-perforated powder the difference of the radii is the web; for multiperforated powder the distance from any perforation to the nearest neighboring perforation or to the periphery is the web. The web has the physical significance that when it is burned through the powder is completely burned, except in the case of multiperforated powder, which at that moment falls apart into slivers. This burning through of the web occurs when $x = w/2$; we are thus led naturally to introduce the dimensionless variable z , defined by the equation:

$$z \equiv 2x/w \quad (184)$$

The variable z is the fraction of the web burned through; it has of course the value 1 at the moment when the web is just burned through. For all powder grains of the same shape the fraction G of the powder burned will be a function of z and of z alone; thus $G = G(z)$. E.g. cordite of diameter w and length L_0 ,

$$G(z) = 1 - (1-z)^2 \left(1 - \frac{w}{L_0} z\right) \quad (185)$$

For single perforated powder of length L_0 and difference of radii w ,

$$G(z) = z \left[1 + \frac{w}{L_0} (1-z)\right] \quad (186)$$

For multiperforated powder the formula for $G(z)$ is given in Ordnance Technical Note No. 1 by G. P. St. Clair. Tables of $G(z)$ for multiperforated grains of typical shape are given in Table VIII of Bennett's "Table for Interior Ballistics". For ball powder, just beginning to come into use:

$$G(z) = 1 - (1-z)^3 \quad (187)$$

*Deviations from this law may, however, arise from turbulent flow or flow inside a perforation.

The law of burning of the powder is the expression of the rate of regression, dx/dt , as a function of pressure*. The effect of initial temperature of the powder and of chemical composition, including especially percent moisture and percent volatiles, is ordinarily taken care of through adjustment of the values of parameters occurring in the above function. The particular average pressure that should be used in the law would seem to be an average over the surfaces of the grains⁴. Since, however, the grains are apt to be more or less uniformly distributed throughout the volume behind the projectile, no serious error should be made by treating such an average as identical with the volume average used in the equation of state, viz. that given above in Eq.(179).

The most commonly used form for the law of burning is the pressure index law:

$$\frac{dx}{dt} = bp^n \quad (188)$$

Bennett used $n = 2/3$, Rögglia used $n = 0.7$, and various other ballisticians have used values of n ranging all the way from $1/4$ to 1. The British have customarily used $n = 1$, but Wadley⁷ has recently replaced the first power of the pressure by the first power of the density of the gas. Such a procedure was suggested by the work of Crow and Crimshaw⁸, on the basis of whose kinetic theory the rate should be proportional to the density. An interpretation of their theory by R. H. Kent and the author would indicate, however, that the rate should be proportional on their theory, not to the density of the main body of the gas, but to the density of the freshly formed gas in immediate contact with the grains. Experimental work by R. B. Dow now in progress at this laboratory may lead, however, to results in disagreement with such a conclusion.

Another form for the burning law that has been used to some extent in this laboratory is the general linear function:

$$dz/dt = a + bp \quad (189)$$

This law was first proposed by Mansell⁹ and later used by R. H. Kent on the physical grounds that the law should contain a term independent of pressure to account for the direct effect of radiation on burning.

* Lack of simultaneous ignition may also affect the rate of regression. On the reasonable assumption, however, that ignition is complete before the start of motion, its effect on the trajectory may not be serious.

⁷A.T. Wadley, Research Department Woolwich, R.D. Report No. 72 (1936).

⁸A.D.Crow and W.E.Crimshaw, Phil. Trans. Roy. Soc. London A, 230, 387 (1932).

⁹Mansell, Phil. Trans. A 207 (1907).

A recent analysis by Hirschfelder of the data given in Table V of Crow and Grimshaw's paper on "The Combustion of Colloidal Propellants"⁸ confirms the experimental validity of such a law. Hirschfelder found, moreover, that a is independent of the initial temperature of the powder and that b is inversely proportional to $T_i - T_p$, where T_p is the powder temperature and T_i is the "touchoff temperature" given by Crow and Grimshaw. In some cases a came out to be zero.

We shall consider only the two above laws. In terms of the variable z they become:

$$\text{Pressure Index Law:} \quad dz/dt = \frac{2b}{w} p^n \quad (190)$$

$$\text{General Linear Law:} \quad dz/dt = \frac{2}{w} (a + bp) \quad (191)$$

Questions concerning the dependence of the parameters a and b on the initial temperature of the powder and on powder composition will not be considered in this report.

Chapter VI

The Collected Equations

It is necessary at this point to summarize some of the results already obtained. We have first the general energy equation:

$$c CG(z) = \frac{p \Omega}{\gamma - 1} + W, \quad (10)$$

then the rate-of-burning equation:

$$\frac{dz}{dt} = \frac{2\alpha}{w} p^{1/2} \quad (190)$$

or

$$\frac{dz}{dt} = \frac{2}{w} (a + bp) \quad (191)$$

The energy W in Eq. (10) is given by the sum of Eqs. (28), (30), (38), (55), (94) and (101). On performing this addition and expressing the velocity v with respect to the ground in terms of the velocity u with respect to the gun, so that

$$v = \frac{u}{1 + e}, \quad (192)$$

we obtain:

$$W = 1/2 m u^2 F_5(r), \quad (193)$$

where

$$F_5(r) = \frac{1}{(1+e)^2} \left[1 + u^2 + f_1 \tan^2 \alpha_r + k + \frac{m}{m_1} \left\{ 1 + \frac{\epsilon}{2} - \frac{\epsilon}{2} F_1(r) \right\}^2 + \frac{\epsilon}{3(1+e)} F_2(r) \right] \quad (194)$$

In Eqs. (192) and (194)

$$e = \frac{v_1}{v} = \frac{m}{m_1} \left[1 + \frac{\epsilon}{2} (1 - F_1(r)) \right] \quad (88)$$

Also, with the abbreviations

$$m' = m (1 + u^2 + f_1 \tan^2 \alpha_r) \quad (169)$$

$$\epsilon' = C/m', \quad (171)$$

Eq. (179) for the average pressure p becomes:

$$p = \frac{m}{A_2} \left[F_6(r) u'(t) + F_7(r) u^2 / \dots \right] \quad (195)$$

where

$$F_6(r) = \frac{m'}{m} \left[1 + \frac{\epsilon'}{3} \left(1 - \frac{\sigma(\sigma^2 - 1)}{(\sigma + r)^3} \right) - \frac{m}{m_1} F_3(r)F_4(r) \right] \quad (196)$$

and

$$F_7(r) = \frac{\epsilon}{2} \left[\frac{(\sigma^2 - 1)\sigma}{(\sigma + r)^4} + \frac{f_1 L}{D_2} \left\{ 1 + \frac{(\sigma^{1/2} - \sigma^4)}{(\sigma + r)^4} \right\} - 2 \frac{m'}{m_1} \frac{\sigma(\sigma - 1)}{(\sigma + r)^3} F_4(r) \right] \quad (197)$$

Note that the functions of travel F_1, F_2, \dots, F_7 are all dimensionless. The functions F_5, F_6, F_7 are defined immediately above. The functions F_1, F_2, F_3, F_4 are:

$$F_1(r) \equiv \frac{\sigma(\sigma - 1)}{(\sigma + r)^2} \quad (93)$$

$$F_2(r) \equiv 1 - F_1 \left[\frac{\sigma + 1}{\sigma + r} + 3c \frac{\sigma + 1}{\sigma + r} - 1 \right] \quad (107)$$

$$F_3(r) \equiv 1 + \frac{\epsilon}{2} (1 - F_1) \quad (177)$$

$$F_4(r) \equiv 1 + \frac{\epsilon}{2} (1 - F_1) \quad (178)$$

It is desirable to list the values of the functions F_1, F_2, \dots, F_7 , and of c for the case of uniform cross-section, so that $\sigma = 1$. In this special case they are all constants with the values given in the following table.

Table I

Function	Value
F_1	0
F_2	1
c	$\frac{m}{m_1} (1 + \frac{\epsilon}{2}) \equiv e_0$ for short
F_3	$1 + \frac{\epsilon}{2}$
F_4	$1 + \frac{\epsilon'}{2}$
F_5	$\frac{1}{(1 + e_0)^2} \left[1 + \mu' + \epsilon_1 \tan^2 \alpha_r + k + \frac{m}{m_1} (1 + \frac{\epsilon}{2})^2 + \frac{\epsilon}{3(1 + e_0)} \right]$

Table 1 (Cont'd)

Function	Value
F_6	$\frac{m'}{m} \left[1 + \frac{\epsilon'}{3} - \frac{m}{m_1} \left(1 + \frac{\epsilon'}{2} \right) \left(1 + \frac{\epsilon'}{2} \right) \right]$
F_7	$\frac{\epsilon'}{2} f_T$

If the further specializing assumption is made that gas friction is negligible, then F_7 is zero.

If Eqs. (193) and (195) are inserted into Eq. (10), the latter becomes:

$$QCG(z) = \frac{\Omega}{\gamma-1} \frac{m}{A_2} \left[F_6(r)u'(t) + F_7(r)u'^2 \right] + 1/2 \mu u^2 F_5(r) \quad (198)$$

Eqs. (198) and (190) or (191), together with Eq. (195), constituted the fundamental equations of interior ballistics of which the simultaneous solution is desired. In previous treatments of interior ballistics the effects of gas friction and non-uniformity of cross-section on pressure distribution have been neglected. These neglects have permitted the very great simplification of taking F_7 to be zero and F_5 and F_6 to be constants. With such neglects one can absorb the constants F_5 and F_6 into the mass, defining a "momentum" mass

$$m_m \equiv m F_6 \quad (199)$$

and an "energy mass"

$$m_E \equiv m F_5 \quad (200)$$

so that

$$p = \frac{m_m}{A_2} u'(t) \quad (201)$$

and

$$QCG(z) = \frac{\Omega}{\gamma-1} \frac{m_m}{A_2} u'(t) + 1/2 m_E u^2 \quad (202)$$

Suppose one attempts now to generalize to the case in which non-uniformity of cross-section and gas friction are taken into account. Then, in so far as the energy equation alone is concerned, one can group the term in F_7 with the μ term, so that one could introduce variable masses as follows:

$$m_m = m F_6(r) \quad (203)$$

$$m_p = m \left[F_5(r) + \frac{2\theta}{(\gamma - 1)A_2} \right] \quad (204)$$

Now, however, in the pressure equation it is no longer possible to introduce an effective variable mass $m_p(r)$ such that:

$$p = \frac{m_p(r)}{A_2} u'(r)$$

The problem of succeeding chapters is to put the equations of interior ballistics into dimensionless forms containing as few parameters as possible. One might for this purpose begin with the equations (195) and (198), perhaps later finding that the functions F_5 , F_6 , and F_7 could be replaced by some sort of average without causing too much error. With such a purpose in mind, however, it seems desirable if possible first to transform the equations (195) and (198) into such forms that variable effective masses can be used to correct for gas friction and non-uniformity of cross section. The equations will then bear a greater resemblance to previous interior ballistics equations.

In Eq. (195) the term in u^2 is a correction term which would vanish for uniform cross-section and no gas friction. Since it is only a correction term we may try to express it by some approximation as the product of the acceleration $u'(t)$ and some function of the travel s . The Leduc theory enables us to do this. We have, in fact, already obtained the necessary equation, viz:

$$u^2 = u'(t)s \left(1 + \frac{s}{2s_m} \right), \quad (182)$$

where s_m is the value of the travel s for maximum $u'(t)$. Denoting as before s/L by r and s_m/L by r_m , we then have:

$$u^2/L = u'(t)r \left(1 + \frac{r}{2r_m} \right) \quad (205)$$

Then

$$\frac{u^2}{L} F_7(r) = u'(t)F_8(r), \quad (206)$$

where $F_8(r) \equiv r \left(1 + \frac{r}{2r_m} \right) F_7(r)$ (207)

so that $p = \frac{m}{A_2} u'(t) \left[F_6(r) + F_8(r) \right]$ (208)

The ratio r_m may be looked on as a parameter to be determined by comparison with firings, or it may be given an approximate value as follows.

The quantity s is the value of s for maximum $u'(t)$. Now let V denote the total volume behind the base of the projectile, i.e., our $w(X)$ of Chapter IV. If V_0 denotes the initial value of V , i.e., the volume of the chamber, then the total volume expansion ratio ξ is given by:

$$\xi = V/V_0 \quad (209)$$

The variable ξ is identical with the travel variable occurring in Bennett's "Tables for Interior Ballistics"³. Now we have

$$\xi = \frac{A_1 b + A_2 s}{A_1 b} = 1 + \frac{s}{\sigma b} = 1 + \frac{r}{\sigma} \quad (210)$$

In Bennett's Tables the value of ξ for maximum acceleration is seen to range from about 1.2 to 1.8 with the most important cases around 1.5. If we adopt the value 1.5, we obtain $r_m = \sigma/2$.

Now in the energy equation it is of course still possible to move the u^2 correction term occurring in the $p\Omega$ term over into the W term and thereby obtain the effective masses of Eqs. (203) and (204). In the interest of consistency, however, it seems desirable to approximate the u^2 term in $p\Omega$ just as we did in the pressure equation. Then

$$\lambda CG(z) = \frac{m}{\gamma-1} \cdot \frac{m}{A_2} u'(t) \left[F_6(r) + F_8(r) \right] + 1/2 m u^2 F_5(r) \quad (211)$$

Eqs. (208) and (211) are formally the same as Eqs. (201) and (202) for the case of uniform cross-section and no gas friction, where now the "momentum mass"

$$m_m(r) \equiv m \left[F_6(r) + F_8(r) \right] \equiv m F_9(r), \quad (212)$$

$$\text{with } F_9(r) = F_6(r) + F_8(r), \quad (213)$$

$$\text{and the "energy mass" } m_E(r) \equiv m F_5(r), \quad (214)$$

are, however, functions of travel.

Suppose we multiply the energy equation by $\gamma-1$ and use the fact that $\tau(\gamma-1) = \lambda$, the "force" of the powder $R_1 T_0$.

$$\text{Then } \lambda CG(z) = \frac{m_m(r)}{A_2} u'(\tau) + 1/2 m_E(r) (\gamma-1) u^2 \quad (215)$$

In the event that $m_m(r)$ and $m_E(r)$ may be replaced by constant values we may take into account the difference of the effective masses m_m and m_E through use of an effective γ suggested by R. H. Kent. We simply introduce an effective γ_e defined by the equation

*Ordnance Department Document No. 2039.

$$m_p(\gamma-1) = m_m(\gamma_e - 1), \quad (216)$$

so that

$$\gamma_e = 1 + \frac{m_p}{m_m} (\gamma-1) \quad (217)$$

Then

$$\lambda CG(z) = \frac{\Omega m_m}{A_2} u'(t) + 1/2 m_m (\gamma_e - 1) u^2 \quad (218)$$

The equations which will form the basis for the succeeding chapters are the energy equation (215) together with the definitions (212) and (214), the rate-of-burning equation (190) or (191), and the pressure equation (208), which we may write as:

$$p = \frac{m_m(r)}{A_2} u'(t) \quad (208')$$

Chapter VII

Dimensionless Travel Variables

1. The Total Volume Expansion Ratio ξ

The total volume expansion ratio ξ has already been defined in equation (209) in Chapter VI. It is identical with Bennett's dimensionless travel variable. In order to express our previous equations in terms of ξ we must specify free volume, travel, velocity, and acceleration in terms of ξ . Suppose we denote the mass density of the solid powder by δ ; then the total volume of the powder packed in the gun is C/δ and the volume burned at time t is $CG(z)/\delta$. If V is the total volume from breech to projectile base, then the gas volume at time t is $V - (C/\delta)(1 - G(z))$. The free volume Ω is obtained from the latter by subtraction of the co-volume. Let us now denote the co-volume per unit volume of solid powder by $1 + \alpha$. Then the total co-volume is $(1 + \alpha)CG(z)/\delta$, so that the free volume Ω is given by:

$$\begin{aligned}\Omega &= V - \frac{C}{\delta} [1 - G(z)] - \frac{(1 + \alpha)C}{\delta} CG(z) \\ &= V - \frac{C}{\delta} [1 + \alpha G(z)]\end{aligned}\quad (219)$$

If V_0 is the initial total volume from breech to projectile base, then $V = V_0 \xi$. If we introduce ξ in this way into Eq. (219) and introduce the density of loading

$$\Delta_1 \equiv C/V_0, \quad (220)$$

Then

$$\Omega = V_0 \left[\xi - \frac{\Delta_1}{\delta} (1 + \alpha G) \right] \quad (221)$$

It is to be especially noted that our density of loading Δ_1 has dimensions, so that its value agrees numerically with Bennett's dimensionless density of loading Δ only if C is expressed in grams and V_0 in cubic centimeters. Of course the ratio Δ_1/δ is dimensionless and thus always agrees in numerical value with Bennett's Δ/δ . In the energy equation (215), however, the term $\lambda CG(z)$ gives rise to a factor Δ_1 which does not get divided by δ . If we wish therefore to introduce the usual dimensionless Δ , we must express Δ_1 as:

$$\Delta_1 = \Delta \rho_0, \quad (222)$$

where ρ_0 represents a density of 1 gram per cubic centimeter.

In the British Engineering system ρ_0 has the value 1.940 slugs per cubic foot.

Bennett used $\alpha = 0.5$ for his interior ballistic tables and Röggl, with his assumption that the co-volume is equal to the volume of the powder burned, used $\alpha = 0$. Crow and Grimshaw obtained $\alpha = 0.49$ by analysis of their closed chamber records on nitrocellulose powders. With the use of improved cooling corrections given in B.R.L. Report No. 281 by R. H. Kent and the author, the author's analysis of Crow and Grimshaw's data gave $\alpha = 0.70$ (B.R.L. Report No. 282).

Now if A_2 is the cross-sectional area of the bore, s the travel with respect to the gun, and V_0 the initial volume from base to projectile base, we have

$$A_2 ds = V_0 d\xi \quad (223)$$

If we use a superscript dot to denote differentiation with respect to the time t , the velocity $u(t)$ is given by

$$u(t) = \frac{V_0}{A_2} \dot{\xi} \quad (224)$$

and the acceleration $u'(t)$ is given by

$$u'(t) = \frac{V_0}{A_2} \ddot{\xi} \quad (225)$$

It is to be emphasized that V_0 in the above equation is the actual total volume behind the projectile as it sits in the gun before being fired. It will not be exactly equal to the actual chamber volume in those cases where the base of the projectile initially juts out into the powder chamber or in those cases of separate loading in an old gun where the projectile is rammed in until the rotating band comes in contact with the origin of rifling. Now, however, we must express the functions $F_5(r)$ and $F_9(r)$ in terms of ξ . Departure of $F_5(r)$ from constancy arises from non-uniformity of cross-section and departure of $F_9(r)$ from constancy arises from non-uniformity of cross-section and gas friction. We have treated these effects in Chapter IV in terms of a definite model for which $V_0 = A_1 L$ and $V = A_1 L + A_2 s$. Thus for such a model

$$\xi = \frac{A_1 L + A_2 s}{A_1 L} = 1 + \frac{s}{L\sigma} = 1 + \frac{r}{\sigma} \quad (210)$$

and

$$r = \sigma(\xi - 1) \quad (226)$$

For the actual gun Eq. (226) is thus an approximation giving the ratio of travel to length of chamber in terms of the expansion ratio ξ , thus enabling us to express the functions $F_9(r)$ and $F_5(r)$ in terms of ξ .

Insertion of Eqs. (220), (221), (224), (225), and (226) into the energy equation (215), the pressure equation (208') and the rate-of-burning equation (190) or (191) furnishes the following system of equations.

$$\frac{\lambda_1 \mu_1^2 (s)}{R V_0} = F_9(r) \left[\xi - \frac{A_2}{\delta} (1 + \sigma C) \right] \ddot{\xi} + 1/2 F_5(r) (\gamma - 1) \dot{\xi}^2 \quad (227)$$

$$p = \frac{m V_0 F_9(r)}{A_2^2} \ddot{\xi} \quad (228)$$

$$\frac{dz}{dt} = \frac{2}{1} \left\{ b \left(\frac{m V_0 F_9(r)}{A_2^2} \ddot{\xi} \right)^n \right. \quad (229.1)$$

$$\left. \begin{array}{l} \text{or} \\ a + \frac{b m V_0 F_9(r)}{A_2^2} \ddot{\xi} \end{array} \right\} \quad (229.2)$$

where

$$r = \sigma(\xi - 1) \quad (226)$$

It may be possible to find, by experience with actual calculations, a method of choosing constant values of the functions $F_9(r)$ and $F_5(r)$ that will give good results. Let us denote the best possible constant values of these functions by \bar{F}_9 and \bar{F}_5 , and define the functions $B_1(r)$ and $B_2(r)$ as

$$B_1(r) \equiv F_9(r) / \bar{F}_9 \quad (230)$$

$$B_2(r) \equiv F_5(r) / \bar{F}_5 \quad (231)$$

*Eq. (225) is sufficient for hydrodynamical corrections. Accurate translation from tabular values of ξ to travel s requires the use of

$$\xi = 1 + \frac{A_2}{V_0} s, \quad (226')$$

where V_0 is as in Eq. (225).

If we now express $F_9(r)$ and $F_5(r)$ in Eq. (227) in terms of $B_1(r)$ and $B_2(r)$, divide through by \bar{F}_9 , and define an effective γ_e by the equation:

$$\bar{F}_5/\bar{F}_9 (\gamma - 1) \equiv \gamma_e - 1 \quad (232)$$

we obtain for the energy equation:

$$\frac{\lambda A_1 A_2^2 G(z)}{m V_0 \bar{F}_9} = B_1(r) \left[\xi - \frac{A_1}{\delta} (1 + \alpha G) \right]'' + 1/2 B_2(r) (\gamma_e - 1) \xi'' \quad (233)$$

The pressure equation becomes

$$p = \frac{m V_0 \bar{F}_9}{A_2^2} B_1(r) \xi'' \quad (234)$$

The rate-of-burning equation becomes:

$$\frac{dz}{dt} = \frac{2}{w} \left\{ \begin{array}{l} b \left(\frac{m V_0 \bar{F}_9}{A_2^2} \right)^n B_1(r) \xi'' \quad (235.1) \\ \text{or} \\ a + \frac{b m V_0 \bar{F}_9}{A_2^2} B_1(r) \xi'' \quad (235.2) \end{array} \right.$$

The choice of constant values for $F_9(r)$ and $F_5(r)$ leads to the value unity for $B_1(r)$ and $B_2(r)$. Replacement of the functions $F_9(r)$ and $F_5(r)$ by the functions $B_1(r)$ and $B_2(r)$ should make the effect of the parameter occurring therein somewhat weaker, i.e., less effective in governing the behavior of the equations.

2. The Free Volume Expansion Ratio

Röggla in his system of interior ballistics used a dimensionless travel variable η , defined as the ratio of the free volume at time t to the initial free volume. On Röggla's assumption that the co-volume is equal to the volume of the powder burned, the free volume remains constant from the beginning of burning to the beginning of motion. Thus Röggla could take his initial free volume, i.e., his free volume at $t = 0$, the beginning of motion, simply as $V_0 - C/\delta$.

As soon as one attempts to improve Röggl's co-volume assumption, however, one runs into complications with the use of the free volume expansion ratio as a travel variable. Thus the free volume does not remain constant from the beginning of burning to the beginning of motion. We may of course use as "initial" value of Ω the value of Ω at the beginning of burning, viz.

$$\Omega_i = V_0 - C/\delta \quad (236)$$

The free volume expansion ratio η will then be given by:

$$\eta = \Omega/\Omega_i \quad (237)$$

The difficulty then arises that at $t = 0$, the value of η is not known accurately. It will be somewhere near the value unity, but will vary from case to case. Furthermore, the equations become much more complicated than those above. In order, therefore, to preserve some of the simplicity of Röggl's treatment without using his incorrect co-volume assumption, we proceed to the introduction of a new dimensionless travel variable ζ .

3. The Dimensionless Travel Variable ζ

Let \bar{G} be the value of $G(z)$ at some point in the trajectory. Also, let

$$g \equiv 1 - \frac{\Delta_1}{\delta} (1 + \alpha\bar{G}) \quad (238)$$

(We do not have to save the letter g for the acceleration of gravity, since the latter could enter into interior ballistics only when effects due to elevation of gun are considered, and such effects are always negligibly small in interior ballistics). Then with $\xi \equiv V/V_0$, the total volume expansion ratio, we define ζ as:

$$\zeta \equiv \frac{1}{g} \left[\xi - \frac{\Delta_1}{\delta} (1 + \alpha\bar{G}) \right] \equiv 1 + 1/g (\xi - 1) \quad (239)$$

Then the time derivative is given by:

$$\dot{\zeta} = \dot{\xi}/g \quad (240)$$

Initially $\xi = 1$ and $\dot{\xi} = 0$, so that the initial values of ζ and $\dot{\zeta}$ are:

$$\zeta_0 = 1 \quad (241)$$

$$\dot{\zeta}_0 = 0 \quad (242)$$

If we were to assume that $\alpha = 0$, we should have:

$$\zeta_{\alpha=0} = \frac{\xi - \Delta_1 / \delta}{1 - \Delta_1 / \delta} = \frac{V - \bar{C} / \delta}{V_0 - \bar{C} / \delta} \quad (243)$$

Thus ζ is a linear function of the total volume expansion ratio ξ , satisfying the initial conditions $\zeta_0 = 1$ and $\dot{\zeta}_0 = 0$, and reducing to the free volume expansion ratio η if Röggl's assumption $\alpha = 0$ is made.

To obtain the equations in terms of ζ , note that Eq. (239) can be solved for ξ :

$$\xi = 1 + g (\zeta - 1) \quad (244)$$

On inserting Eq. (244) into Eqs. (233), (234), and (235), one obtains:

$$\frac{\lambda \Delta_1 A_2^2 G(z)}{m V_0 g^2 \bar{F}_g} = B_1(r) \left[\zeta - h(G - \bar{G}) \right] \ddot{\zeta} + 1/2 B_2(r) (\gamma_e - 1) \dot{\zeta}^2 \quad (245)$$

where

$$h = \frac{\Delta_1}{\delta} \frac{\alpha}{\bar{g}} \quad (246)$$

$$p = \frac{m V_0 \bar{g} \bar{F}_g}{A_2^2} B_1(r) \ddot{\zeta} \quad (247)$$

$$\frac{dz}{dt} = \frac{2}{w} \left\{ b \left(\frac{m V_0 \bar{g} \bar{F}_g B_1(r)}{A_2^2} \ddot{\zeta} \right)^n \right. \quad (248.1)$$

$$\left. \left[a + \frac{b m V_0 \bar{g} \bar{F}_g B_1(r)}{A_2^2} \ddot{\zeta} \right] \right\} \quad (248.2)$$

Note that if α vanishes that h also vanishes. In such a case the entire right side of (245) is free of terms in z . When α is correctly treated as non-vanishing, then h remains in the energy equation and there is some explicit dependence of the right hand side on z . The factor of h , however, viz. $G - \bar{G}$, changes sign during the motion, so that it is reasonable to believe that a proper choice of G will minimize the effect of h in influencing the solution of the equations. In fact the term $h(G - \bar{G})$ become small compared to ζ not far beyond the pressure maximum, and is appreciable compared to ζ only on the initial side of the pressure maximum. It appears then that a value of G at some point between the start of the motion and the occurrence of maximum pressure should be chosen for \bar{G} .

Such a value of \bar{G} will be somewhat less than $1/2$, perhaps $1/4$. It should be mentioned here that Debye¹¹ chose a travel variable that gave rise to a free-volume term resembling ours in containing $G = 1/2$.

In order to express $B_1(r)$ and $B_2(r)$ as functions of ζ we need also the equation connecting r and ζ . From Eqs. (226) and (244) we obtain:

$$r = g\sigma(\zeta - 1) \quad (249)$$

Chapter VIII

Dimensionless Time Variables

We now consider various means of introducing a dimensionless time variable into the fundamental interior ballistic equations. To do so, let us first define v to be some constant quantity having the dimensions of the reciprocal of a time. Then we define τ by

$$\tau \equiv vt \quad (250)$$

We continue to indicate derivatives of dimensionless space variables with respect to t by superscript dots, and now introduce the new convention that derivatives of such quantities with respect to τ shall be denoted by primes. We have also

$$\frac{d}{dt} = v \frac{d}{d\tau} \quad (251)$$

Thus

$$\dot{\xi} = v\xi' \quad (252.1)$$

$$\ddot{\xi} = v^2\xi'' \quad (252.2)$$

$$\dot{\zeta} = v\zeta' \quad (253.1)$$

$$\ddot{\zeta} = v^2\zeta'' \quad (253.2)$$

Then Eqs. (234), (235), and (236) involving the total volume expansion ratio ξ become:

$$EG(z) = B_1(r) \left[\xi = \frac{\Delta_1}{\delta} (1 + \alpha G) \right] \xi'' + 1/2 B_2(r)(\gamma_e - 1)\xi'^2 \quad (254)$$

$$p = PB_1(r)\xi'' \quad (255)$$

$$\frac{dz}{d\tau} = \left\{ \begin{array}{l} QB_1^n(r)\xi'' \\ \text{or} \\ Q_0 + QB_1(r)\xi' \end{array} \right\} \quad (256.1)$$

$$(256.2)$$

where the "energy parameter"

$$E \equiv \frac{\lambda \Delta_1 \Lambda^2}{mV_0 \Gamma_9 v^2} = \frac{\lambda \Delta_1}{P} \quad (257)$$

the "pressure parameter"

$$P \equiv \frac{mV_0 \Gamma_9 v^2}{\Lambda^2} \quad (258)$$

and the "quicknesses"

$$Q \equiv \frac{2b}{wv} P^n \quad (259.1)$$

$$Q_0 \equiv \frac{2a}{wv} \quad (259.2)$$

The parameters E , Q_0 , and Q are dimensionless, but P is a pressure. For the second rate-of-burning law the exponent n in Eq. (259.1) is to be taken equal to unity.

On introduction of τ the equations in ζ become:

$$EG(z) = B_1(r) \left[\zeta - h(G - \bar{G}) \right] \zeta'^2 + 1/2 B_2(r)(\gamma_0 - 1)\zeta'^2 \quad (260)$$

$$p = PB_1(r)\zeta'' \quad (261)$$

$\frac{dz}{dt} =$ or

$$\left\{ \begin{array}{l} QB_1^n(r)\zeta'^{2n} \\ Q + QB_1(r)\zeta'' \end{array} \right. \quad (262.1)$$

$$(262.2)$$

where

$$E \equiv \frac{\lambda \Delta_1 A_2^2}{mV_0 g^2 \bar{F}_9 v^2} = \frac{\lambda \Delta_1}{gP} \quad (263)$$

$$P \equiv \frac{mV_0 g \bar{F}_9 v^2}{A_2^2} \quad (264)$$

$$Q \equiv \frac{2b}{wv} P^n \quad (259.1)$$

$$Q_0 \equiv \frac{2a}{wv} \quad (259.2)$$

It is instructive at this point to make comparisons with Bennett. With Bennett the functions $B_1(r)$ and $B_2(r)$ are unity, γ_0 is fixed at 1.30, and the first of the rate-of-burning laws (256) is used. His values of F_9 and F_5 also correspond to $f_p = 0$, $\sigma = 1$, and $k = 0$, i.e., to neglect of gas friction, non-uniformity of cross-section, and of heat loss. His procedure amounts to using the ξ equations, choosing v in such a way that

$$P = R_0 \lambda / \lambda_s \quad (265)$$

where P_0 denotes unit pressure in a standard system of units (lb/in²) and λ_s denotes a standard value of the "force". From Eq. (11) $\lambda = \xi(\gamma - 1)$, and since he always takes $\gamma = 1.3$, he has

$$\frac{\lambda}{\lambda_s} = \frac{\xi}{\xi_s} \quad (266)$$

where ξ_s is a standard value of the potential ξ , which corresponds formally to his specific energy n' . In this connection note the remarks following Eq. (10). Then from Eqs. (267) and (265),

$$E = \frac{\lambda_s \Delta_1}{P_0} \quad (267)$$

The energy equation then contains only one parameter Δ_1 , which together with the quickness Q in the rate-of-burning equation, gives a differential system with two strong parameters (Δ_1 and Q).

In order to get rid of a weak parameter that might be brought in by initial conditions, he assumes that the initial pressure is universal, except for proportionality to the "force". Thus by assuming that the initial pressure

$$P_i = 2500 P_0 \lambda / \lambda_s, \quad (268)$$

he obtains a universal initial value for ξ , viz. 2500.

By a "strong" parameter we mean one the variation of which will produce large changes in the solution and by a "weak" parameter one whose variation will produce only small changes in the solution. Now the parameter P does not occur in the equations that have to be integrated, but only in the equation connecting actual pressure p with "tabular pressure" ξ . One might thus imagine that it would pay to choose v in a different manner, viz. by setting equal to unity one of the parameters E or Q . With the ξ equations, however, nothing would be gained by such a procedure, since the parameter Δ_1 would still be left as a factor of $1 + \alpha G$, and would presumably have a strong effect on the solution. Thus two strong parameters, viz. Δ_1 and either E or Q would still remain, and they would be independent, since the relation (267) would no longer hold, because Eq. (265) no longer holds.

The situation is different with the ζ equations. In order to compare with Bennett we again assume that the functions $B_1(r)$ and $B_2(r)$ are unity, that γ_c is universal, and that the first rate-of-burning equation is used. Instead of having a strong parameter Δ_1 multiplying $1 + \alpha G$ we now have a weak parameter h multiplying $G - G_0$. Choice of P equal to P_0 or to $P_0 \lambda / \lambda_s$ would still leave two strong parameters E and Q .

Now, however, if we choose v in such a way as to make $E = 1$ or $Q = 1$, we reduce the number of strong parameters by one. Thus the ζ equations should be a great improvement over the ξ equations.

There is still another way of reducing by one the number of strong parameters. This is a method which under certain special conditions reduces to Röggl's. To obtain the Röggl-type equations, differentiate the energy equation (260) with respect to τ . Then,

$$EG'(z) \frac{dz}{d\tau} = \frac{d}{d\tau} J(\zeta, z) \quad (269)$$

where

$$G'(z) \equiv \frac{d}{dz} G(z) \quad (270)$$

$$\text{and } J(\zeta, z) \equiv B_1(r) \left[\zeta - h(G-\bar{G}) \right] \zeta'' + 1/2 B_2'(r) (\gamma_e - 1) \zeta'^2 \quad (271)$$

Now insert Eq. (262) for $\frac{dz}{d\tau}$ into Eq. (269) and call

$$\frac{G'(z)}{G'(0)} \equiv \alpha_R(z) \quad (272)$$

Then

$$EG'(0)\alpha_R(z) \cdot \left\{ \begin{array}{l} QB_1^n(r)\zeta''^n \\ \text{or} \\ Q_0 + QB_1(r)\zeta'' \end{array} \right\} = \frac{d}{d\tau} J(\zeta, z) \quad (273)$$

Now choose v in such a way that

$$EQG'(0) = 1 \quad (274)$$

Then

$$\alpha_R(z) \cdot \left\{ \begin{array}{l} B_1^n(r)\zeta''^n \\ \text{or} \\ \frac{Q_0}{Q} + B_1(r)\zeta'' \end{array} \right\} = \frac{d}{d\tau} J(\zeta, z) \quad (275)$$

In the above equations $G'(0)$ is the value of $G'(z)$ for $z=0$, i.e., at the beginning of burning. Moreover $\alpha_R(z)$ is equal to the ratio of the powder surface 0 at time t to the powder surface 0_0 at the beginning of burning. To see this, express the mass of powder burned in time t in terms of both $G'(z)$ and the surface 0 . Thus the mass burned is

$$CdG = CG'(z) dz \quad (276)$$

$$\text{or } 0\delta dx = 0\delta \frac{W}{2} dz, \quad (277)$$

where δ is the mass density of the solid powder and dx is the linear distance burned. Thus

$$CG'(z) = 0\delta \frac{W}{2} \quad (278)$$

and

$$CG'(0) = 0_0\delta \frac{W}{2} \quad (279)$$

On dividing Eq. (278) by Eq. (279) and using the definition (272) one obtains:

$$\alpha_R(z) = 0/0_0 \quad (280)$$

Thus for powder of constant burning surface $\alpha_R(z)$ remains equal to unity. Moreover, $\alpha_R(z)$ corresponds to Röggl's α which he takes to be a function of expansion ratio. It is seen that such an assumption is false; α_R is a function of z . In the event that one can use constant values for $F_3(r)$ and $F_5(r)$, the functions $B_1(r)$ and $B_2(r)$ reduce to unity, so that

$$\alpha_R(z) \left\{ \begin{array}{l} \zeta''' \\ \text{or} \\ \frac{Q_0}{Q} + \zeta'' \end{array} \right\} \left[\zeta - h(G-\bar{G}) \right] \left[\zeta'' + 1/2(\gamma_e - 1)\zeta'^2 \right] \quad (281)$$

If one chooses the first rate-of-burning law, assumes constant burning surface so that $\alpha_R(z) = 1$, and makes Röggl's co-volume assumption $\alpha = 0$ so that $h = 0$, then

$$\zeta''' = \frac{d}{dt} \left[\zeta\zeta'' + 1/2(\gamma_e - 1)\zeta'^2 \right] = \zeta\zeta''' + \gamma_e \zeta'\zeta'' \quad (282)$$

Eq. (282) is the Röggl equation in the universal form due to Kent¹⁰ Eq. (281) or the more general Eq. (275) gives the form for use when the co-volume is properly treated. If the surface is not constant, one must use also the rate-of-burning equation:

$$\frac{dz}{dt} = \left\{ \begin{array}{l} QB_1^n(r)\zeta''' \\ \text{or} \\ Q_0 + QB_1(r)\zeta'' \end{array} \right\} \quad (256.1)$$

$$\quad \quad \quad (256.2)$$

¹⁰R. H. Kent, B.R.L. Report No. 48 "Röggl's Equation and its Application to Interior Ballistic Problems (Revised)", July 3, 1942.

so that one then has a pair of simultaneous differential equations rather than a single equation. The quantity α affects answers to problems in two ways. First, in the differential equation, it governs the value of \dot{z} , which has a weak effect on the solution. Second, it enters in Eq. (239) for transition from total volume expansion ratio ξ to tabular expansion ratio ζ . In this second place it clearly has a strong effect. Some improvement over Röggl's co-volume assumption may thus be made by neglecting α in the differential equation but retaining it in the equation for tabular entry. Such a procedure amounts to neglecting α in the variable η as our variable ζ . Such a procedure might be useful in the laboratory before new interior ballistic tables have been available. The main importance of these remarks however, lies in the realization of the fact that construction of interior ballistic tables need not wait upon an accurate equation of state. The principal effect of improvements in the equation of state will be improvements in the equation for entering the table.

According to the above treatment it will be noted, on inspection of Eq. (281), that even for constant burning surface the improved treatment of co-volume necessitates the integration of two simultaneous equations. This state of affairs follows from the fact that the correction coefficient α is still defined by $\alpha = \xi$, a function of z . There is one but unfortunate, due to Zaroodny, which reduces the problem to one differential equation when the surface is strictly constant. For variable burning surface, however, the Zaroodny equation appears to be much more complicated. In view of the fact that there is no powder in use, except possibly flake powder, which approximates at all closely to constant burning surface, it seems highly improbable that Zaroodny's equation offers any practical improvements. This statement is further strengthened by the fact that, as pointed out above, the co-volume correction can be largely taken into account merely by the equation for tabular entry without any explicit correction term in the system of differential equations. The latter then reduce to a single equation if constant burning surface is assumed. It hardly seems worth while to treat co-volume exactly without taking into account the variability of burning surface. As soon as one decides to treat the burning surface properly, one sees that the above system of equations is far more practicable. For completeness, however, Zaroodny's equation will now be presented. To obtain his equation, begin with Eq. (260), choosing $G = 0$, and solve for $EG(z)$. One obtains:

$$EG(z) = \frac{B_1(r)\zeta'' + 1/2 B_2(r)(\gamma_e - 1)\zeta'''}{1 + \int B_1(r)\zeta''} \quad (283)$$

Next differentiate Eq. (283) with respect to z , place $G'(z) = G'(0)\alpha_R(z)$, insert the rate-of-burning equation (256), and make the Röggl choice for v , i.e., choose v so that $EG'(0) = 1$. Then

$$\alpha_R(z) \left\{ \begin{array}{l} B_1^n(r) \zeta^{n-1} \\ \text{or} \\ \frac{Q_0}{Q} + B_1(r) \zeta^n \end{array} \right. = \frac{d}{d\tau} \left[\frac{B_1(r) \zeta^n + 1/2B_1(r)(\gamma_c - 1)\zeta^{1/2}}{1 + \frac{h}{E} B_1(r) \zeta^n} \right] \quad (284)$$

For variable burning surfaces the factor $\alpha_R(r)$ on the left hand side necessitates the simultaneous use of the rate-of-burning equation; for constant burning surface one sees that only ζ is involved, so that one has only a single differential equation. Actually Zaretsky did not choose v in the above manner, but left B as a factor of ζ^{n-1} , thus having a strong parameter P in the equation instead of the weaker parameter h/E . Eq. (284) is thus Zaretsky's equation as modified by the author. The complications above referred to arise from the required differentiation of a fraction.

The following table gives the values of the parameters v , P , E , and Q for choices of v which lead to $\beta = 1$, $\beta = 1$, or $P(\zeta) = 1$. If the second rate-of-burning equation is used, we have

$$\frac{Q_0}{Q} = \frac{a}{bP} \quad (285)$$

where a and b are the constants in the law $\frac{dx}{dt} = a + bp$, giving linear rate-of-burning as a function of pressure. Also, when the second rate-of-burning equation is used, we must put the exponent $n = 1$. The formulas for $P = P_0$ or $P = P_0 \lambda / \lambda_S$ are not included, since such a choice leads to an additional strong parameter. In writing down the formulas for the Röggl case $EQG'(0) = 1$, we use Eq. (279) to express $G'(0)$ in terms of the initial burning surface 0_0 .

TABLE II

	$E = 1$	$Q = 1$	$W = 1$
V	$\left(\frac{\lambda \Delta_1 \Delta_2}{mV_0 \delta F_9} \right)^{1/2}$	$\frac{1}{2b} \left(\frac{A_2}{mV_0 \delta F_9} \right)^{2n-1}$	$\frac{1}{V_0 \delta} \left(\frac{\lambda \Delta_0 \delta b}{V_0 \delta} \right)^{2n-1} \left(\frac{A_2}{mV_0 \delta F_9} \right)^{2n}$
P	$\frac{\lambda \Delta_1}{E}$	$\frac{2}{2b} \left(\frac{A_2}{mV_0 \delta F_9} \right)^{2n-1}$	$\frac{2}{V_0 \delta} \left(\frac{\lambda \Delta_0 \delta b}{V_0 \delta} \right)^{2n-1} \left(\frac{mV_0 \delta F_9}{A_2} \right)^{2n}$
E	1	$\frac{2}{2b} \left(\frac{\lambda \Delta_1 / 2b}{W} \right)^{2n-1} \left(\frac{mV_0 \delta F_9}{A_2} \right)^{2n-1}$	$\frac{2}{V_0 \delta} \left(\frac{\lambda \Delta_1 / V_0 \delta}{E} \right)^{2n-1} \left(\frac{A_2}{mV_0 \delta F_9} \right)^{2n}$
Q	$\frac{2b}{W} \left(\frac{mV_0 \delta F_9}{\lambda \Delta_1 A_2} \right)^{1/2} \left(\frac{\lambda \Delta_1}{E} \right)^n$	1	$\frac{2b}{W} \left(\frac{\lambda \Delta_0 \delta b}{V_0 \delta} \right)^{2n-1} \left(\frac{mV_0 \delta F_9}{A_2} \right)^{2n}$

Chapter IX

Strong and Weak Parameters and Initial Conditions

Let us examine first the effect of the parameter γ_c by Eq. (232)

$$\gamma_c - 1 = (\gamma - 1)(\bar{F}_5/\bar{F}_9),$$

where \bar{F}_5 and \bar{F}_9 are the first initial values of the function $F_5(r)$ and $F_9(r)$. So far we have assumed a constant velocity of cross-section. Then $17(r)$ vanishes and $F_9(r)$, by Eqs. (207) and (213), reduces to $F_6(r)$. Then from table I we have the values:

$$\bar{F}_5 = \frac{1}{(1 + \epsilon_0)^2} \left[\mu' + f_1 \tan^2 \alpha_r + k + \frac{m}{m_1} \left(1 + \frac{\epsilon}{2}\right)^2 + \frac{\epsilon}{3(1 + \epsilon_0)} \right]$$

$$\bar{F}_9 = \bar{F}_6 = \frac{m'}{m} \left[1 + \frac{\epsilon'}{3} - \frac{m}{m_1} \left(1 + \frac{\epsilon}{2}\right) \left(1 + \frac{\epsilon'}{2}\right) \right],$$

where $\epsilon_0 = \frac{m}{m_1} \left(1 + \frac{\epsilon}{2}\right)$. On neglecting recoil and using the equations

$$\frac{m'}{m} = 1 + \mu' + f_1 \tan^2 \alpha_r \quad (169)$$

$$\epsilon' = C/m' \quad (171)$$

we have

$$\bar{F}_5 \approx 1 + \mu' + f_1 \tan^2 \alpha_r + k + \frac{\epsilon}{3} \quad (286)$$

$$\bar{F}_9 = 1 + \mu' + f_1 \tan^2 \alpha_r + \frac{\epsilon}{3} \quad (287)$$

Thus
$$\frac{\bar{F}_5}{\bar{F}_9} \approx 1 + \frac{k}{1 + \mu' + f_1 \tan^2 \alpha_r + \frac{\epsilon}{3}} \quad (288)$$

$$\approx 1 + k \left[1 - \mu' - f_1 \tan^2 \alpha_r - \frac{\epsilon}{3} + \text{higher order terms} \right] \quad (289)$$

In general the passive resistance term μ' and the rotational term $f_1 \tan^2 \alpha_r$ will be small compared to 1, so that approximately:

$$\frac{\bar{F}_5}{\bar{F}_9} \approx 1 + k \left[1 - \frac{\epsilon}{3} + \frac{\epsilon^2}{9} - \dots \right] \quad (290)$$

For large guns the heat loss coefficient k may be small compared to 1; for small arms it may reach the value 1. Ordinarily ϵ will not exceed 1/3, but recent high velocity guns in some cases use a greater mass of powder than of projectile, so that ϵ may reach or exceed 1 in these extreme cases. In any case, either from Eq. (288) or from

Eq. (20), one sees that the departure of \bar{F}_5/\bar{F}_9 from unity depends entirely upon heat loss in case of gas friction and non-uniformity of chamber diameter. It appears then that variation of γ_c from the value γ will depend largely on the heat loss coefficient k . In general the variation in γ is small, the variation for PM powders ordinarily being from about 1.24 to 1.28. Thus for large guns, (small k), one may expect γ_c to vary only over a small range, so that probably two values of k or γ_c would be sufficient in such a table. For small arms, the variation in k will be considerable, so the variation in γ_c will be large. Values of γ_c from 1.2 to 1.6 at intervals of 0.05 may be necessary. k or γ_c is thus essentially a strong parameter which has to be introduced to improve the treatment of heat loss.

The functions $B_1(r)$ and $B_2(r)$, are, respectively, the ratios of $F_9(r)$ and $F_5(r)$ to their best constant values. Now from Chapter VI the function $F_5(r)$ is seen to depend not only on k , but also on the parameters $\mu' + f_1 \tan^2 \alpha = \mu''$, m/m_1 , ϵ , and σ . The function $F_9(r)$ does not depend on k , but does depend not only on μ'' , m/m_1 , ϵ , and σ , but also on $f_F/2$ and r_m . We shall assume that the best constant values \bar{F}_5 and \bar{F}_9 can be found in terms of these same parameters plus one other parameter, viz the ratio of the total travel S to the chamber length L or to the bore diameter D_2 ; S/D_2 is the length of the bore in calibers. Altogether then the functions $B_1(r)$ and $B_2(r)$ depend on the eight parameters k , μ'' , m/m_1 , ϵ , σ , $f_F L/D_2$, r_m , and S/D_2 . The facts that $B_1(r)$ is the ratio of $F_9(r)$ to its best constant value and that $B_2(r)$ is the ratio of $F_5(r)$ to its best constant value should serve to make the variations in $B_1(r)$ and $B_2(r)$ small. Thus each of these eight parameters should have only a small effect on the solution; i.e., each one should be a weak parameter as regards its influence on the solution through $B_1(r)$ or $B_2(r)$. k is not a new parameter, since it occurred also in γ_c . The functions B_1 and B_2 thus introduce only seven new parameters, and this number will be reduced to six if we adopt the value $r_m = \sigma/2$, suggested in Chapter VI. In the event that B_1 and B_2 may be chosen equal to 1, all these parameters disappear.

The coefficient $h = \frac{\Delta_1}{g} \frac{\alpha}{g}$ of $G - \bar{G}$ introduces only one new weak parameter, viz. Δ_1 , since g is a function of Δ_1 .

The choice $E = 1$ leaves Q as the only strong parameter (besides γ_c).

The choice $Q = 1$ leaves E as the only strong parameter (besides γ_c). The choice $KQ^{-1}(c) = 1$ leaves Q as a strong parameter in the case of variable burning surface. For constant Q and constant burning surface it leaves Q as a weak parameter.

In general the initial conditions will introduce an additional weak parameter, the initial pressure p_i or the ratio p_i/f^0 of initial pressure to the pressure parameter F . From Eq. (261) we have:

$$p_i = B_1(\alpha) \zeta_0'' \quad (291)$$

$B_1(\alpha)$ is the initial value of $B_1(r)$ and contains only weak parameters already considered. ζ_0'' is the initial value of ζ'' .

In considering the possible choices of initial pressure it will be useful to consider the situation in the special Rögglä case of a uniform cross-section with neglect of gas friction and with the co-volume taken equal to the volume of powder burned. If the first rate-of-burning law is used, there results in such a case the equation:

$$g'''' = \zeta_0'''' + \gamma_c \zeta_0'' \quad (292)$$

With fixed values of α and γ_c this is a universal equation, so that use of a universal value for ζ_0'' leads to a universal maximum value ζ_{max}'' . In such a case the use of a universal value for ζ_0'' is equivalent to the assumption that initial pressure is proportional to maximum pressure.

Without the above assumptions of the simple Rögglä theory one does not get a universal equation, so that a universal ζ_0'' will not lead to a universal ζ_{max}'' . In general then, the assumption of a universal ζ_0'' is not equivalent to the assumption of proportionality of initial pressure to maximum pressure. Since the assumption by Rögglä and Kent of such a proportionality, however, was made solely to secure a universal ζ_0'' , it would seem pointless in the general case to preserve this proportionality and thus lose the universality of ζ_0'' . A priori the latter assumption, and furthermore it has the merit of reducing by one the number of weak parameters.

The assumption of a universal value for ζ_0'' is probably incorrect, since it leads in Rögglä's special case to the above mentioned proportionality of initial pressure to maximum pressure, and such a result appears to be non-physical. Although Rögglä's success indicates that initial pressure is not critical, we may perhaps improve the treatment of it as follows. Physically more reasonable than the proportionality above mentioned would be the assumption of a universal initial pressure. A little less reasonable assumption

would be that initial pressure is proportional to the "force" but otherwise universal, i.e., always the same for powder of given force λ . This latter assumption has the virtue of reducing the number of work parameters by one, and it may be recalled that Bennett made such an assumption. Thus, letting λ_0 be a standard value of λ and p_0 a standard value of initial pressure, we assume that the initial pressure

$$P_i = p_0 \frac{\lambda}{\lambda_0} \quad (292)$$

We also have

$$P_i = EP_1(\sigma) \zeta_0'' \quad (291)$$

and

$$EP = \frac{\lambda \Delta_1}{E} \quad (293)$$

From these three equations we readily obtain:

$$\zeta_0'' = \frac{p_0}{\lambda_0 B_1(\sigma)} \frac{E}{\Delta_1 / \lambda} \quad (293)$$

Thus the second assumption about initial pressure leads to the result that ζ_0'' is a function only of parameters already introduced, so that it reduces by one the number of work parameters. The assumption of strict universality of initial pressure would not lead to such a result.

Chapter X

Use of an Improved Equation of State

So far we have restricted ourselves to an equation of state of the type

$$p\Omega = \lambda CG, \quad (294)$$

where

$$\Omega = v_g \left[1 - \eta_1 \frac{CG}{v_g} \right], \quad (295)$$

v_g being the volume of powder and CG/v_g its density. In the Ballistic Research Laboratory Report No. 288 by the author it was found that Crew and Grimshaw's closed chamber data for their nitrocellulose powder K(1) could be represented better by taking Ω to be of the form:

$$\Omega' = v_g \left[1 - \eta_1 \frac{CG}{v_g} + \eta_2 \left(\frac{CG}{v_g} \right)^2 \right] \quad (296)$$

Weighted averages for the two sizes of chamber used by Crew and Grimshaw give for numerical values:

Eq. (295): $\eta_1 = 1.674 \pm 0.021 \text{ cm}^3/\text{gm}$

Eq. (296): $\eta_1 = 1.50 \pm 0.05 \text{ cm}^3/\text{gm}$

$$\eta_2 = 1.51 \pm 0.21 \text{ cm}^6/\text{gm}^2$$

For the Ω of Eq. (295) we have written η_1 as $(1 + \alpha)/\delta$, so that the values $\delta = 1.58 \text{ gm/cm}^3$ and $\eta_1 = 1.674 \text{ cm}^3/\text{gm}$ lead to $\alpha = 0.70$.

If we now represent the η_1 occurring in the Ω' of Eq. (296) by

$$\eta_1 = \frac{1 + \alpha'}{\delta}, \quad (297)$$

we find $\alpha' = 1.37$ for $\eta_1 = 1.50 \text{ cm}^3/\text{gm}$ and $\delta = 1.58 \text{ gm/cm}^3$. Let us also define:

$$g' = 1 - \frac{\Delta_1}{\delta} (1 + \alpha' \bar{G}) \quad (298)$$

$$h' = \frac{\Delta_1}{\delta} \frac{\alpha'}{g'} \quad (299)$$

corresponding to the g and h of Eqs. (238) and (246), the difference being that we now use the value α' instead of α . As before we let V denote the total volume from the breech to the base of the projectile.

We now consider the changes that have to be made in the interior ballistic equations when Eq. (295) is replaced by Eq. (296). Only the energy equation is affected, and it is affected only through the term containing Ω . We have:

$$v_g = v - \frac{C}{\delta} (1 + G) \quad (300)$$

Combination of Eqs. (300), (296), and (297) gives

$$\Omega' = V_0 \frac{C}{\xi} (1 + \alpha' \xi) + \frac{\eta_2 \Delta_1^2 G^2}{V_0 \frac{C}{\xi} (1 - C)} \quad (301)$$

$$= V_0 \left[\xi - \frac{\Delta_1}{\delta} (1 + \alpha' \xi) + \frac{\eta_2 \Delta_1^2 G^2}{\xi - \frac{\Delta_1}{\delta} (1 - C)} \right] \quad (302)$$

where V_0 is the chamber volume and ξ is Bennett's total volume expansion ratio V/V_0 . Therefore

$$\xi = \frac{1}{\xi'} \left[\xi - \frac{\Delta_1}{\delta} (1 + \alpha' \xi) \right] \quad (302')$$

gives

$$\Omega' = V_0 \xi' \left[\xi - h'(G - \bar{G}) + \frac{\eta_2 \Delta_1^2 G^2}{g'^2 (\xi + h' \bar{G} + \frac{\Delta_1}{\delta} \frac{G}{g'})} \right] \quad (303)$$

Besides the replacement of h by h' and of ρ by ρ' the only change that has to be made is the addition of a term:

$$" \Delta \zeta " = \frac{\eta_2 \Delta_1^2 G^2}{g'^2 (\xi + h' \bar{G} + \frac{\Delta_1}{\delta} \frac{G}{g'})} \quad (304)$$

to the term $\xi - h'(G - \bar{G})$ in the energy equation.

We now estimate the relative values of ζ , $h'(G - \bar{G})$, and " $\Delta \zeta$ " in about as unfavorable a case as can be found. To do so we make use of Bennett's Table II, taking a density of loading $\Delta_1 = 0.800 \text{ gm/cm}^3$ and Bennett's quickness $q = 0.0750$. His Table II gives the value $\xi = 1.94$ for maximum pressure. Interpolation in his Table III gives the corresponding values at maximum pressure: $\xi' = 190.1$ and $\xi'' = 47,610$. Insertion of these values into his energy equation on page (x) gives $G = 0.3515$. Using $C = 1/4$, we then find:

$$\zeta - h'(G - \bar{G}) + " \Delta \zeta " = 2.374 - 0.212 + 0.335 = 2.497 \quad (305)$$

" $\Delta \zeta$ " thus turns out to be 14% of ζ in this unfavorable case.

Neglect of the term " $\Delta \zeta$ ", however, would not be so serious as such a calculation would indicate, for the reason that when we use the simpler Equation (295) we largely compensate for omission of the term containing η_2 by use of a different value of η_1 . This fact is easily seen by calculation of Ω and Ω' . Thus since

$$\Omega' = g' \left[\zeta - h'(G - \bar{G}) + " \Delta \zeta " \right] V_0, \quad (305')$$

$$\text{where } g^* = 1 - \frac{A_2}{A_1} (1 + a^* \bar{G}) \quad (205)$$

$$= 1 - \frac{0.806}{1.58} \left[1 + (1.27)(0.70) \right]$$

$$= 0.3203,$$

we have

$$\Omega' = (0.3203)(2.087) \frac{V}{V_0} = 0.669 \frac{V}{V_0} \quad (206)$$

$$\text{Now } \beta = g \left[\frac{V}{V_0} - h (G - \bar{G}) \right] \frac{V}{V_0}.$$

Using $a = 0.70$, we have

$$\Omega = 0.4051 \left[2.086 - 0.089 \right] \frac{V}{V_0} = 0.809 \frac{V}{V_0}.$$

Use of the less accurate equation of state leads therefore to only 1% error in the free volume even at maximum pressure with a high density of loading. It does not therefore appear that use of the more accurate equation of state would be advisable.

John P. Vinti

Glossary of Symbols

(The order is that in which they appear)

Chapter I

- T_0 adiabatic closed chamber explosion temperature of the powder gas. ($^{\circ}\text{K}$)
- $u(T)$ specific internal energy of the powder gas at absolute temperature T .
- C initial mass of powder
- G fraction of powder burned at time t
- W work done by powder gas at time t plus heat loss to gun
- k_1 integration constant that vanishes
- $\bar{c}(T_0, T) \equiv \frac{u(T_0) - u(T)}{T_0 - T}$, mean constant volume specific heat over temperature range T_0 to T
- V free volume of the powder gas
- R_1 gas constant per unit mass
- $\bar{Q} \equiv \bar{c}T_0$, mean potential of the powder over the temperature range T_0 to T
- $\gamma \equiv 1 + R_1/\bar{c}$, analogous to ratio of specific heats in older theories of interior ballistics
- Z number of gram moles per gram of powder gas, or pound moles per pound of powder gas, etc.
- R universal gas constant (gas constant per mole)
- $\lambda \equiv ZRT_0$, "force" of the powder
- Q_1 closed chamber heat of formation at 15°C of unit mass of the solid powder from the elements at 15°C
- Q_2 constant volume heat of formation at 15°C of unit mass of the powder gas, the relative concentrations being as at chemical equilibrium at absolute temperature T_0
- $x_i(T)$ Number of moles of i th gas per unit mass at absolute temperature T

$U_i(T)$ molar internal energy of gas i at absolute temperature T

$$E(T) = \sum_{i=1}^n x_i(T) \left[U_i(T) - U_i(298.0) \right]$$

$$\Delta x_i = x_i(T) - x_i(T_0)$$

$\Delta x_i(j)$ mole increase of gas i in j th reaction as $T_0 \rightarrow T$

n' number of reactions taken into account

Addendum to Chapter I

u_s specific energy of the solid powder, referred to the elements $^{\circ}\text{K}$ at zero density as the zero level

Chapter II

t time from beginning of motion

$-t_1$ time at which burning begins

x distance from breech to an arbitrary cross-section of chamber or bore

L length of powder chamber (distance from breech face to beginning of bore)

s travel of projectile with respect to the gun

A_1 mean cross-sectional area of chamber

A_2 cross-sectional area of bore

D_1 diameter of chamber, first accurately defined in Chapter IV by relation $\frac{\pi}{4} D_1^2 = A_1$

D_2 diameter of bore, first accurately defined in Chapter IV by relation $\frac{\pi}{4} D_2^2 = A_2$

T (absolute) temperature of main body of gas

T_b temperature of breech face

$T_w(x,t)$ temperature of walls of chamber and bore

- T_p temperature of base of projectile
 h_b coefficient of heat transfer from hot gas to breech face
 h_w coefficient of heat transfer from hot gas to walls of chamber and bore
 h_p coefficient of heat transfer from hot gas to base of projectile
 $Q(t)$ total heat loss at time t by contact of hot gas with gun
 $Q(t) = Q_b + Q_{cw} + Q_{bw} + Q_p$
 Q_b loss to breech
 Q_{cw} loss to chamber wall
 Q_{bw} loss to bore wall
 Q_p loss to base of projectile
 k ratio of heat loss $Q(t)$ to translational energy of projectile, this ratio being assumed constant

Chapter III

- m mass of projectile
 v velocity of projectile with respect to the ground
 p_a pressure required to produce translational acceleration
 $W_{trans} \equiv 1/2mv^2$
 α_r angle of rifling
 y_p, z_p coordinates of point P on rotating band (on developed bore)
 θ angular coordinate of point P
 $\omega = \frac{d\theta}{dt}$ angular velocity of projectile
 W_{rot} rotational energy of projectile
 R_g radius of gyration of projectile

$$f_1 \approx (2R_g/D_2)^2$$

P_{rot} pressure required to produce rotational acceleration

\rightarrow
 P_n pressure of driving side of land on ridge of projectile

\rightarrow
 f friction force per unit area

l_n unit vector along normal to ridge

l_t unit vector along normal to tangent

N number of lands or of ridges

F_z total frictional force on rotating band in direction of motion

A_R area of side face of a ridge

μ coefficient of friction between land and ridge

F_B total force on the base of the projectile

P_{pass} forcing resistance per unit cross-sectional area of projectile ("passive" pressure)

W_{pass} energy lost in overcoming passive pressure

μ' ratio of W_{pass} to $1/2 mv^2$, this ratio being assumed constant, = $\mu f_1 \tan \alpha_r$ if the law of friction holds

P_1 passive pressure (if taken to be constant)

v_1 recoil velocity of gun

e recoil ratio v_1/v

Chapter IV

X distance from breech face to base of projectile

$a(x)$ cross-sectional area at distance x from breech

$w(x)$ total volume from breech face to cross-section at distance x

$c \equiv C/m$

ρ density of powder plus powder gas mixture

$v_p(x,t)$ velocity with respect to the gun of powder gas mixture at distance x from breech face

$v_g(x,t)$ velocity with respect to the ground of powder gas mixture at distance x from breech face

u velocity of the projectile with respect to the gun

$$\dot{V}(t) = - \frac{d}{dt} \ln \rho$$

$$\sigma \equiv A_1/A_2$$

M_1 momentum with respect to the ground of the recoiling parts

M_2 momentum with respect to the ground of the projectile

Γ_3 moment of momentum of the powder gas mixture

W_R energy of recoil

$$r \equiv s/L$$

$$q \equiv \frac{1+\epsilon}{\sigma+r}$$

$$m' \equiv m(1 + \mu + f_1 \tan^2 \phi_p)$$

$$\epsilon' \equiv c/m'$$

$$F_1(r) \equiv \frac{(\sigma-1)}{(\sigma+r)^2}$$

$$F_2(r) = 1 - F_1 \left[\frac{\sigma+1}{\sigma+r} + 3 \frac{\sigma+1}{\sigma+r} - 1 \right]$$

$$F_3(r) \equiv 1 + \frac{\epsilon}{2} (1 - F_1)$$

$$F_4(r) \equiv 1 + \frac{\epsilon'}{2} (1 - F_1)$$

W_1 kinetic energy of powder gas mixture in chamber

W_2 kinetic energy of powder gas mixture in bore

β angle defined in Fig. 3

$p(x,t)$ pressure at time t at distance x from breech

$p(0,t)$ pressure at time t at breech

$$f_1(x) \equiv \frac{u(x)}{a(x)}$$

$$f_2(t) \equiv \frac{a(X)}{u(X)} u(t)$$

f_P Fanning friction factor for gas friction

$\rho_g(x)$ density of powder gas alone

$u(T_x)$ specific internal energy of powder gas at distance x from breech

$\overline{u(T_x)}$ mass average of $u(T_x)$, interpreted as $u(T)$.

n_j co-volume of powder gas per unit x

\bar{p} volume average of pressure

$H_1(r)$ see Eq. (162) on page 31

$H_2(r)$ see Eq. (164) on page 32

c_1, c_2 , constants occurring in the Leduc equation for velocity as a function of travel.

s_m travel at maximum acceleration

Chapter V

x , regression of any surface of the powder grain

w , web thickness

$$z \equiv 2x/w$$

L_0 , length of a single perforated grain

n exponent in rate-of-burning law

b , coefficient of p^n in rate-of-burning law $\frac{dx}{dt} = bp^n$

or coefficient of p in law $\frac{dx}{dt} = a + bp$

- a , constant term in rate-of-burning law
 T_p , initial temperature of solid powder
 T_i , "touch off" temperature of solid powder

Chapter VI

- $F_5(r)$ see Eq. (194) on page 38
 $F_6(r)$ see Eq. (196) on page 39
 $F_7(r)$ see Eq. (197) on page 39
 $F_8(r)$ see Eq. (207) on page 41
 $F_9(r) \equiv F_6(r) + F_8(r)$
 $m_m(r)$ "momentum mass"; see Eq. (212) on page 42
 $E_L(r)$ "energy mass"; see Eq. (214) on page 42
 $K_m \equiv s_m/L$
 V , total volume from breech to base of projectile = $w(x)$
 V_0 , initial value of V
 $\xi \equiv V/V_0$
 γ_G , defined for constant m_m and m_g in Eq. (217) on page 43

Chapter VII

- l +a co-volume of powder gas unit volume of solid powder
 $\Delta_1 \equiv C/V_0$, density of loading
 \bar{F}_5 best constant value of $F_5(r)$
 $\Delta \equiv$ ratio of Δ_1 to a density of 1 gm per cm^3
 \bar{F}_9 best constant value of $F_9(r)$
 $B_1(r) \equiv F_9(r)/\bar{F}_9$

$$B_2(r) \equiv \Gamma_5(r)/\bar{\Gamma}_5$$

$$\gamma_e \equiv 1 + \frac{\bar{\Gamma}_5}{\Gamma_9} (\gamma-1)$$

n , Röggl's free volume expansion ratio

\bar{G} , arbitrary value of G at some point in trajectory (probably to be taken as 1/4)

$$g \equiv 1 - \frac{\Delta_1}{\delta} (1 + \alpha \bar{G})$$

$\zeta \equiv 1 + \frac{1}{g} (\xi - 1)$, modified expansion ratio

$$h \equiv \frac{\Delta_1}{\delta} \frac{e}{g}$$

Chapter VIII

τ , dimensionless time variable

ν , proportionality constant connecting physical time t and dimensionless time τ by the relation $t = \nu\tau$

E , dimensionless energy parameter, coefficient of G^2z in dimensionless energy equation

P , pressure parameter, proportionality constant occurring in equation connecting physical pressure and dimensionless pressure

Q
 Q_0 } quickness, parameters occurring in dimensionless rate-of-burning equation

P_0 , unit pressure in standard system of units

λ_s , a standard value of the "force" λ

ϕ_s , a standard value of the potential

P_i , the initial pressure

$J(\zeta z)$, see Eq. (271) on page 54

$$g_R(z) \equiv G'(z)/G'(0)$$

0 surface of solid powder at time t

0_0 , initial surface of solid powder

Superscript dots or primes over dimensionless travel variables denote derivatives with respect to t or τ respectively.

Chapter IX

$\mu'' = \mu' + \eta_1 \tan^2 \phi_p$, a coefficient that takes both forcing resistance and rotation into account

Chapter X

Ω' , a more accurate expression for the free volume

η_1 and η_2 , coefficients in the more accurate expression for the free volume

σ_1' , σ_2' , μ_1' , coefficients analogous to the σ , μ , μ' of Chapter VII but formed from the η_1 of the more accurate expression for the free volume.

" $\Delta\zeta$ ", defined in Eq. (304).