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THE CALCULATION OF ELECTRICAL CAPACITANCE

by

Yu. Ya. Iossel', E. S. Kochanov,
and M. G. Strunskiy

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This book is devoted to presentation of methods of calculation of electrical capacitance, ~~and~~ contains calculation formulas, tables, and graphs necessary for determination of the capacitance of various forms of conductors. It is intended for engineers and scientists engaged in electromagnetic calculations; it can be useful also to students and to graduate students of electrical specialties.

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1969

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
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Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
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Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
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П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}
rot	curl
lg	log

The book is dedicated to the presentation of methods of calculation of electrical capacitance, and it contains a summary of calculation formulas, tables, and graphs necessary for the determination of the capacitance of conductors of various form.

The book is intended for engineers and scientists engaged in electromagnetic calculations; it can be useful also to students and to graduate students of electrical specialities.

PREFACE

The necessity for the calculation of capacitance (or parameters analogous to it - electrical magnetic, and thermal conductivity) appears with the designing of various electroautomatic and radio engineering devices, the calculation of telephonic, telegraphic, and television cables, of transmission lines and communication lines, separate elements of television, telemetering and electrometric apparatus, calculation of grounding electrodes, of various magnetic systems, and with the solution of a whole series of other problems which must be encountered by engineers and scientific workers of various specialties.

Because of this the problems of calculation of capacitance and parameters analogous to it have for several decades been considered in physical, radio engineering, and electrical literature, and the bibliography of works dedicated to this problem published at the present time is vast.

Unfortunately, the vast majority of these works are devoted to giving an account of only individual special problems of calculation of electrical capacitance. As for the very few works in which attempts were made to give a systematic account of the problems of calculation of capacitance, they are either too antiquated,[1] or

---

[1]Orlich E., Kapazität und Induktivität; 1909.

they concern (similar to the book of R. Brüderlink[1]) only conductors of a certain type.

In connection with this there has long been a need for publication of a reference book on the calculation of capacity reflecting the contemporary state of this problem and containing both the fundamental methods of calculation of capacity and ready formulas, tables, and curves which refer to the most important particular cases. This book, proposed for the readers' attention, is dedicated to the solution of this problem.

In developing the plan of the book, the authors in many respects likened it to the plan of the known reference book of P. L. Kalantarov and L. A. Tseytlin on calculation of inductance, published by the State Scientific and Technical Power-Engineering Publishing House in 1955. The authors feel that this will not only be convenient for the readers of this or other books, but also will create prerequisites for a uniform account in many respects of connected problems of calculation of capacitance and inductance in the future.

Following such a plan, the authors broke up the fundamental material of the book into two parts, in the first of which an account is given of the methods of the calculation of capacitance, and in the second of which are given formulas, tables, graphs necessary for calculation of capacitance in various cases.

One of the things concerning problems on calculation of electrical capacitance is that strict methods of their solution are essentially inseparable from methods of calculation of the electrostatic field of the system of charged bodies being considered. Along with this during the calculation of capacitance approximation methods are used, not requiring knowledge of the electrostatic field in the space surrounding the conductors, also auxiliary methods which allow converting the system of conductors considered to a form more convenient for calculation.

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[1]Brüderlink R., Induktivität und Kapäzrität der Starkstromfreileitungen; 1954.

Taking into account that the methods of calculation of electrostatic fields in the majority are well illuminated in electrical engineering and physicomathematical literature,[1] in the first part of the book only the less known approximation and auxiliary methods used in calculating capacitance are stated.

The account of each of the methods of calculation of capacitance is accompanied by illustrations which should help the reader master not only the idea of the method, but also the characteristics of its application to the solution of concrete practical problems.

In the second, reference, part of the book, the authors strove as fully as possible to present the data necessary for calculation of capacitance of conductors of the most typical form, without facing the problem of summarizing all results published up to the present time (within the confines of one book this would be, apparently, generally impossible). The application of reference data is illustrated by illustrations of a calculation reduced to numerical results.

In conclusion the authors express sincere gratitude to the reviewer, Doctor of Technical Sciences L. A. Tseytlin and the scientific editor, Candidate of Technical Sciences R. A. Pavlovskiy, the participation of whom in the consideration and preparation of the present book went far beyond the scope of their formal responsibilities.

The authors hope that this book will be useful to a wide circle of engineers and scientific workers engaged in electromagnetic calculations.

Comments and remarks on the content of the book should be sent to: Leningrad, USSR, Leningrad, D-41, Marsovo pole, d. 1, Leningradskoye otdeleniye izdatel'stva "Energiya."

Authors

[1]See, for example, V. Smayt, Elektrostatika i elektrodinamika (Electrostatics and Electrodynamics), IL, 1954, N. N. Mirolyubov et al., Metody rascheta elektrostaticheskikh poley (Methods of Calculation of Electrostatic Fields), Vysshaya shkola, 1963.

INTRODUCTION

V-1. Basic Definitions

Between charges and potentials in any system of conductors that create an electrostatic field, a one-to-one linear relation exists, for the expression of which the concept of electrical capacitance or simply capacitance is introduced.[1]

Depending on the type of system of conductors considered, the capacitance of a solitary conductor, the capacitance between two conductors and the capacitance in a system of many conductors are distinguished.

The capacitance of a solitary conductor is a scalar quantity characterizing the ability of the conductor to accumulate an electrical charge and is equal to the ratio of the charge of the conductor to its potential on the assumption that all other loaded conductors are an infinite distance away.

If the charge of a solitary conductor is designated Q, and its potential V, then in accordance with the given definition, the

[1]Here and subsequently, if nothing is said to the contrary, it is assumed that the specific inductive capacitance of the medium surrounding the conductors does not depend on electrostatic field strength, all the conductors being considered are in a finite region of space, and that the potential at an infinitely distant point is equal to zero.

capacitance of this conductor will be expressed by the formula

$$C_0 = \frac{Q}{V}. \tag{V-1}$$

The capacitance between two conductors is a scalar quantity equal to the absolute value of the ratio of the electrical charge of one of the conductors to the difference in their potentials on condition that these conductors have charges identical in amount, but opposite in sign and that all other loaded conductors are infinitely far away.

If the charges of the conductors are equal to $\pm Q$, and their potentials have quantity V_1 and V_2, then in accordance with the given definition, the capacitance between these conductors can be expressed by the formula

$$C = \left|\frac{Q}{V_1 - V_2}\right|. \tag{V-2}$$

An arrangement of two conductors separated by a dielectric (plates) intended for utilization of capacitance between them is called a capacitor; therefore, the capacitance between two conductors is sometimes called also *capacitor capacitance*.

The generalization of introduced concepts in the case of a system with a random finite number of conductors is a concept about intrinsic and mutual partial capacitances.

The conductor's *intrinsic partial capacitance* that enters the system of many bodies is a scalar quantity equal to the ratio of the charge of this conductor to its potential on the assumption that all the conductors of the system (including the one being considered) have identical potential.

Mutual partial capacitance between two conductors that enter the system of many bodies is a scalar quantity equal to the ratio of the charge of one of the conductors being considered to the potential of another on the assumption that all conductors, except the latter, have potential equal to zero.

In accordance with the introduced definitions the relation between charges and potentials in a system of n conductors is expressed by the following equations:

$$\begin{aligned} Q_1 &= C_{11}V_1 + C_{12}(V_1 - V_2) + \ldots + C_{1n}(V_1 - V_n); \\ Q_2 &= C_{21}(V_2 - V_1) + C_{22}V_2 + \ldots + C_{2n}(V_2 - V_n); \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Q_n &= C_{n1}(V_n - V_1) + C_{n2}(V_n - V_2) + \ldots + C_{nn}V_n, \end{aligned} \qquad (V\text{-}3)$$

where Q_i and V_i are the charge and potential of the i-th conductor ($i = 1, 2, \ldots, n$); C_{ii} is the intrinsic partial capacitance of the i-th conductor ($i = 1, 2, \ldots, n$); C_{ik} is the mutual partial capacitance between the i-th and k-th conductors ($i, k = 1, 2, \ldots, n$; $i \neq k$), in this case it is possible to show that $C_{ik} = C_{ki}$.

The distribution of concepts of intrinsic and mutual partial capacitances is to a considerable extent arbitrary in nature. Really any system of n conductors which occupies a finite volume can be conditionally considered a system of $n + 1$ conductors, where $(n + 1)$-th conductor is a sphere of infinite radium having zero potential. In a new system the intrinsic partial capacitance of any conductor [except the $(n + 1)$-th] can be interpreted as the mutual partial capacitance between this conductor and the sphere.

In the particular case when the algebraic sum of the charges of all conductors of a system is equal to zero (such a system is called electroneutral), the system of equations (V-3) can be converted to the form:

$$\begin{aligned} Q_1 &= C'_{12}(V_1 - V_2) + \ldots + C'_{1n}(V_1 - V_n), \\ Q_2 &= C'_{21}(V_2 - V_1) + \ldots + C'_{2n}(V_2 - V_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Q_n &= C'_{n1}(V_n - V_1) + \ldots + C'_{n,\,n-1}(V_n - V_{n-1}). \end{aligned} \qquad (V\text{-}4)$$

where C'_{ik} is the mutual partial capacitance between the i-th and k-th electrodes in an electroneutral system ($C'_{ik} = C'_{ki}$).

The quantities C'_{ik} can be defined in the same way as C_{ik}, on the assumption that all conductors, except one, have about the same (but not necessarily equal to zero) potential. In general the quantities C'_{ik} are not equal to the quantities C_{ik}, but can be expressed through them.

Equations (V-3) or (V-4) can be converted, grouping on their right sides terms having a factor value V_k. In this case the system of equations connecting charges and potentials of conductors takes the form:

$$\begin{aligned} Q_1 &= \beta_{11}V_1 + \beta_{12}V_2 + \ldots + \beta_{1n}V_n \\ Q_2 &= \beta_{21}V_1 + \beta_{22}V_2 + \ldots + \beta_{2n}V_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ Q_n &= \beta_{n1}V + \beta_{n2}V_2 + \ldots + \beta_{nn}V_n \end{aligned} \qquad \text{(V-5)}$$

The quantities entering these equations β_{ik} are called *coefficients of electrostatic induction* (intrinsic when $i = k$ and mutual when $i \neq k$), and, as can be shown,

$$\beta_{kk} > 0, \quad \beta_{ik} = \beta_{ki} < 0.$$

Another form of recording of relationships (V-5) is:

$$\begin{aligned} V_1 &= \alpha_{11}Q_1 + \alpha_{12}Q_2 + \ldots + \alpha_{1n}Q_n \\ V_2 &= \alpha_{21}Q_1 + \alpha_{22}Q_2 + \ldots + \alpha_{2n}Q_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ V_n &= \alpha_{n1}Q_1 + \alpha_{n2}Q_2 + \ldots + \alpha_{nn}Q_n \end{aligned} \qquad \text{(V-6)}$$

The quantities α_{ik} entering (V-6) are called *potential coefficients* (intrinsic when $i = k$ and mutual when $i \neq k$), $\alpha_{kk} > 0$, $\alpha_{ik} > 0$, $\alpha_{ik} = \alpha_{ki} < \alpha_{kk}$.

The systems of equations (V-3)-(V-6) are various forms of the expression of one and the same interrelationship between charges and potentials of conductors in a system of many bodies. Therefore, the coefficients which enter the equations are also interconnected. Thus

$$C_{ik} = -\beta_{ik},$$
$$C_{kk} = \beta_{1k} + \beta_{2k} + \ldots \beta_{kk} + \ldots + \beta_{nk},$$
$$\beta_{kk} = C_{1k} + C_{2k} + \ldots + C_{kk} + \ldots + C_{nk}.$$

When a system consists of one conductor ($n = 1$), the concept of intrinsic partial capacitance coincides with the concept of the capacitance of a solitary conductor: $C_0 = C_{11}$.

When a system consists of two conductors ($n = 2$) and is electroneutral, the concept of mutual partial capacitance coincides with the concept of the capacitance between two conductors: $C = C'_{12}$. In this case the following relationships are also valid:

$$C = \frac{C_{11}C_{12} + C_{12}C_{22} + C_{11}C_{22}}{C_{11} + C_{22}},$$
$$C = \frac{\beta_{11}\beta_{22} - \beta_{12}^2}{\beta_{11} + \beta_{22} + 2\beta_{12}},$$
$$C = \frac{1}{\alpha_{11} + \alpha_{22} - 2\alpha_{12}}.$$

As follows from the definitions given above, the values of the capacitance of solitary conductors, of the capacitance between two conductors and of the capacitances in a system of many conductors are substantially positive and are defined only by the geometric parameters of conductors and by the specific inductive capacitance of the environment. From these determinations it is evident also that the quantities C_0, C, C_{ik}, C_{kk}, β_{ik}, β_{kk} and C'_{ik} are quantities of the same dimension and can be united under the name of capacitive coefficients (unlike potential coefficients having reverse dimension).

V-2. General Features of Capacitance and Classification of Conductors

A. Formulated below are some general positions expressing the dependence of the capacitance of conductors upon their geometric parameters and the specific inductive capacitance of the environment.

1. *At a constant value of specific inductive capacitance* the relationships of the capacitances in two geometrically similar systems of conductors are equal to the relationship of the characteristic sizes of these systems:

$$\frac{C_0^{\mathrm{I}}}{C_0^{\mathrm{II}}}=\frac{a^{\mathrm{I}}}{a^{\mathrm{II}}};\quad \frac{C^{\mathrm{I}}}{C^{\mathrm{II}}}=\frac{a^{\mathrm{I}}}{a^{\mathrm{II}}};\quad \frac{C_{ik}^{\mathrm{I}}}{C_{ik}^{\mathrm{II}}}=\frac{a^{\mathrm{I}}}{a^{\mathrm{II}}}, \tag{V-7}$$

where a^{I} and a^{II} are the characteristic sizes of systems I and II.

When the form of conductors is such that the electrostatic fields being induced by them can be considered plane-parallel,[1] the capacitances (per unit of length of conductors) in geometrically similar electroneutral systems of two or more bodies[2] equal between themselves:

$$C_l^{\mathrm{I}}=C_l^{\mathrm{II}};\quad C_{ik,l}^{\mathrm{I}}=C_{ik,l}^{\mathrm{II}}, \tag{V-8}$$

where $C_l^m=\frac{C^m}{l^m}$, $C_{ik,l}^m=\frac{C_{ik}^m}{l^m}$, $m=\mathrm{I},\ \mathrm{II}$, l^m is the length of the conductors (in the direction of their axis).

2. *At identical geometric parameters of two systems* of conductors in uniform media with various specific inductive capacitances, the relationships of similar quantities characterizing capacitance in these systems are equal to the ratio of specific inductive capacitances:

$$\frac{C_0^{\mathrm{I}}}{C_0^{\mathrm{II}}}=\frac{\varepsilon^{\mathrm{I}}}{\varepsilon^{\mathrm{II}}};\quad \frac{C^{\mathrm{I}}}{C^{\mathrm{II}}}=\frac{\varepsilon^{\mathrm{I}}}{\varepsilon^{\mathrm{II}}};\quad \frac{C_{ik}^{\mathrm{I}}}{C_{ik}^{\mathrm{II}}}=\frac{\varepsilon^{\mathrm{I}}}{\varepsilon^{\mathrm{II}}}, \tag{V-9}$$

---

[1] Such systems of conductors will subsequently be called plane-parallel.

[2] The concept of the capacitance of a solitary conductor in this instance makes no sense physically.

where ε^{I} and ε^{II} are the specific inductive capacitances of the media in systems I and II.[1]

This feature is valid also for the case of heterogeneous media on condition that the spatial distributions of specific inductive capacitance in systems I and II are similar. Together with those given can be shown a number of features of capacitance which are valid only for individual types of systems united by any general criteria. One of such criteria is the presence of a boundary of division of two uniform media with various specific inductive capacitances.

Let the conductors being considered be located in a medium with specific inductive capacitance ε_1 near the boundary of division of media with specific inductive capacitances ε_1 and ε_2. If $\varepsilon_1 \ll \varepsilon_2$, then the boundary of division of media can be considered equipotential, i.e., it can be considered a surface of an ideal conductor. If $\varepsilon_1 \gg \varepsilon_2$, then the boundary of division can be considered impenetrable for power lines of an electrostatic field and therefore it can be considered the surface of a certain conditional medium with zero specific inductive capacitance. Such a boundary will be subsequently called *impenetrable*.

For the capacitance of conductors near an infinitely extended flat ideally conducting or impenetrable surface, the following basic relationships are valid.

1. The capacitance between any solitary conductor and an infinite ideally conducting surface (Fig. V-1a) is equal to the doubled value of the capacitance between this conductor and its mirror reflection relative to the plane (Fig. V-1b).

2. The capacitance of any solitary conductor 1 near an infinitely extended flat impenetrable boundary (Fig. V-2a), is equal to the half of the capacitance of the solitary conductor formed by the union of

[1]Analogous equations are valid even for all remaining capacitive coefficients while for potential coefficients the opposite relationships are fulfilled.

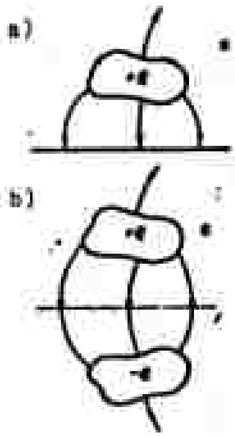

Fig. V-1.

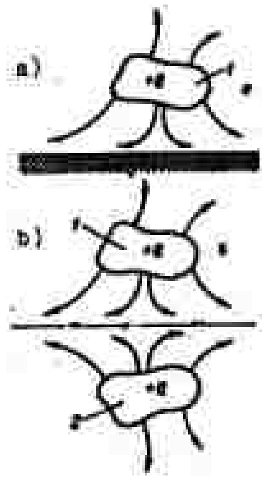

Fig. V-2.

conductor 1 with its mirror reflection 2 relative to the plane (Fig. V-2b).

B. Subsequently we will subdivide conductors according to their geometric form into *wires, flat plates, open and closed shells.* The latter in an electrostatic sense is equivalent to the solid conductors of the same form, with the exception of those cases when other charged conductors are inside the shells. In considering wires we will assume that their sections are constant in length and the linear dimensions of the section are considerably less than the length of wire and the distances to other conductors. In considering flat plates and shells we will consider that their thickness at every point of surface is constant and in all cases when nothing is said to the contrary is infinitesimal.

With the assumptions made the following extremum properties of capacitance are valid.

1. Of all solitary straight wires of assigned length and area of transverse section, the one with the least capacitance is the wire of circular section.

2. Of all flat plates of assigned area the one with least capacitance is the circular disc.

3. Of all triangular flat plates of assigned area, the one with least capacitance is a plate in the form of an equilateral triangle.

4. Of all rectangular flat plates of assigned area, the one having least capacitance is the square plate.

5. Of all bodies of an assigned volume the one having the least capacitance is the sphere.

6. Of all right cylinders of assigned altitude and area of transverse section, the one having the least capacitance is the right circular cylinder.

7. Of all systems in the form of two circular infinitely long cylinders with parallel axes, one of which envelopes the other, the one with least capacity per unit of length is the system in the form of coaxial cylinders.

Very characteristic features are possessed also by the capacitance of the system shown in Fig. V-3. Let curve $OAO'A'$ represent the section of an infinitely long cylinder, symmetrical with respect to line OO'. Considering the surface of a cylinder an impenetrable boundary of a medium with specific inductive capacitance ε, filling the inside of the cylinder, we assume that OA' and $O'A$ are sections of infinitely long conductors *1* and *2*, and points A and A' are symmetric relative to plane OO'.

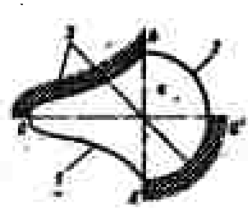

Fig. V-3.

In the conditions shown the capacitance between conductors *1* and *2* (per unit of length) is numerically equal to ε.[1]

An analogous feature can be formulated also for a system which differs from the one shown in Fig. V-3 only by the fact that it contains not two, but four divided infinitesimally thin gaps of conductor *1*, *2*, *3* and *4*, the sections of which coincide with lines *OA*, *AO'*, *O'A'* and *A'O*, respectively (Fig. V-4). For this system the mutual partial capacitance between any two crosswise lying conductors per unit of length ($C_{13,l}$ or $C_{24,l}$) is equal to ε ln 2, whatever the form and dimensions of the section of the cylinder.[2]

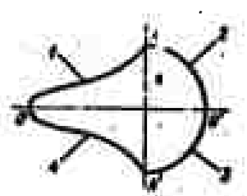

Fig. V-4.

V-3. Units of Measurement of Capacitance

The unit of measurement of capacitance in the system SI [International System] is the farad (F). Furthermore, fractional units are used: microfarad (μF) and picofarad (micromicrofarad) (pF):

$$1\ \mu F = 10^{-6}\ F,$$

$$1\ pF = 10^{-12}\ F.$$

To find capacitance in farads it is necessary to multiply its value in another system of units by the appropriate conversion factor.

[1]The feature shown was noted for the first time for a particular case in the work of Lees C. H., Proc. Manch. Lit. and Phil. Soc. 1899, 1-3; in general form it was formulated by F. Bowman (Bowman, F. Proc. of the Lond. Math. Soc. 1935, ser 2, V. 39, p. 3, 205-213), and then was again considered by A. V. Netushil ("Elektrichestvo" 1951, No. 3).

[2]See, for example, Lampard D. G., Proc. IEE, 1957, C. 104, N 6, 271-280.

The conversion factors of values of capacitance from other systems of units to the SI system have the following values:

System of units	Conversion factor
SGSE	$10^5/c^2$
SGSM	10^9
SGS	$10^5/c^2$
MKSA	1

[SGSE - Centimeter-gram-second electrostatic system; SGSM - Cgs electromagnetic system; SGS - Centimeter-gram-second; MKSA - meter-kilogram-second-ampere].

where c is the number value of the velocity of the propagation of electromagnetic waves in free space (in m/s), equal to $2.997925 \cdot 10^8$.

V-4. Analogy Between Capacitance and Other Physical Quantities

Because of the mathematical analogy of potential fields of different physical nature, for each of them it is possible to show the analog of electrical capacitance. Thus, for instance, for stationary electrical, magnetic, and thermal fields such analogs are electrical, magnetic, and thermal conductivities, respectively. At assigned geometric parameters of the system of bodies, the value-analogs of electrical capacitance are proportional to it, and the coefficients of proportionality are the relationships of the appropriate physical parameters of a medium to specific inductive capacitance. Specifically, for two bodies

$$G = \frac{\gamma}{\varepsilon} C; \tag{V-10}$$

$$G_{м} = \frac{\mu}{\varepsilon} C; \tag{V-11}$$

$$G_{т} = \frac{\lambda}{\varepsilon} C, \tag{V-12}$$

where G is the electrical conductivity between the bodies being considered in a uniform medium with specific electrical conductivity γ; G_M is magnetic conductivity between bodies in a uniform medium with permeability μ; G_T is the thermal conductivity between bodies in a uniform medium with thermal conductivity coefficient λ; C is the capacitance between bodies in a uniform medium with specific inductive capacitance ε.

The same relationships connect partial conductivities and partial capacitances in the system of many bodies.

Apart from the one indicated there is also an approximate analogy between electrical capacitance and certain parameters of high-frequency electromagnetic systems.[1] At assigned geometric layout of the system of conductors, at high frequency especially, the following approximate relationships are valid:

$$W \simeq \frac{\sqrt{\varepsilon\mu}}{C}, \tag{V-13}$$

where W is the wave resistance of a system of two conductors in a uniform medium with specific inductive capacitance ε and permeability μ, C is the capacitance between these conductors;

$$L_l \simeq \frac{\varepsilon\mu}{C_l}, \tag{V-14}$$

where L_l is the inductance per unit of length of a two-wire line in a uniform medium with permeability μ; C_l is the capacitance between these conductors (per unit of their length) in a uniform medium with specific inductive capacitance ε.

For rectilinear wires the following relationships can also be shown:

$$L_{kk} \simeq \varepsilon\mu l_k^2 \alpha_{kk}, \tag{V-15}$$

[1]In this case frequency is assumed to be so high that the lines of the magnetic field can be considered outside the sections of conductors.

where l_{kk} is the inductance of a wire l_k long in a homogeneous medium with permeability μ; α_{kk} is the intrinsic potential coefficient of a wire in a homogeneous medium with specific inductive capacitance ε, calculated by the method of mean potentials (see § 1-2);

$$M_{ik} \propto \varepsilon\mu l_i l_k \cos\varphi_{ik}\alpha_{ik}, \tag{V-16}$$

where M_{ik} is the mutual inductance of two wires l_i and l_k long at an angle ϕ_{ik} to each other in a homogeneous medium with permeability μ; α_{ik} is the mutual potential coefficient of the same wires in a homogeneous medium with specific inductive capacitance ε, calculated by the method of mean potentials.

The examined analogy makes the calculation of capacitance equivalent to the calculation of a number of other physical parameters, specifically:

a) magnetic conductivity of various magnetic circuits;

b) resistance of spreading out of electrodes connecting electrical circuits with conducting medium (for example, grounds);

c) wave resistance of wave guides, strip lines, antennas, and other transmitting and radiating systems;

d) thermal conductivity between various heated bodies.

V-5. Means of Calculation of Capacitance

Formulas (V-1)-(V-6) cannot be directly used for calculation of capacitance (or quantities connected with it) because usually only geometric parameters of the system of conductors and the specific inductive capacitance of the surrounding medium are known. Therefore, to determine capacitance it is necessary either to design charges of conductors, having been assigned by their potentials, or, on the contrary, to find the potentials of conductors, having been assigned by the quantity of charges.

Both these problems can be strictly solved on the basis of calculation of the electrostatic field of the system of conductors being considered. Really, knowing the distribution of electrostatic field potential (u) in the space surrounding the conductors, it is possible to find the charges of each of them with the aid of the relationship:

$$Q_i = -\int_{S_i} \varepsilon \frac{\partial u}{\partial n} ds, \qquad \text{(V-17)}$$

where Q_i is the charge of the i-th conductor; S_i is the surface of the i-th conductor; n is the external normal to the surface of the conductor.

When the electrostatic field cannot be calculated, special methods of calculating capacitance are used which are based either on directly establishing the connection of the charge of the conductor with the potential of its surface (methods of direct determination of capacitance), or upon simplification of problems of calculation of electrostatic field (auxiliary methods).

Formulation of problems of calculation of capacitance depends upon the selection of initial quantities (charges or potentials), which, in turn, is determined by the form of the system of conductors considered.

In calculating the capacitance of a conductor, its potential or charge can be assigned at random. If it is supposed that potential is equal to one, then the charge of a solitary conductor will be numerically equal to its capacitance. In calculating the capacitance between conductors, as a rule, it is possible to define only their charges, and the condition $Q_2 = -Q_1$ must be observed.

The potentials of both conductors in general cannot be selected at random since they are connected by the relationship

$$\frac{V_1}{V_2} = -\frac{C_{22}}{C_{11}}. \qquad \text{(V-18)}$$

following from (V-3) when $n = 2$, $Q_1 = -Q_2$.

The assignment of potentials as initial quantities is possible only in certain special cases, for example, the following.

1. The system of two conductors is symmetrical relative to a certain plane. In this case $C_{11} = C_{22}$, and at $Q_1 = -Q_2$ $V_1 = -V_2 = A$, where A is a random quantity.

2. The dimensions of one of the conductors (for example, the first) are incommensurably great in comparison with the dimensions of the other. Here $C_{11} >> C_{22}$, $C_{22}/C_{11} \approx 0$, i.e., $V_1 \approx 0$, $V_2 = A$, where A is a random quantity.

With the calculation of partial capacitances in a system, initial quantities can be in general only their potentials.

Thus, with calculation of intrinsic partial capacitance, the potentials of all conductors of the system must be taken equal to one and the same random constant, and in calculating the mutual partial capacitance between the i-th and k-th conductor, the potential of one of them can be selected at random, and the potentials of all the remaining conductors must be taken equal to zero.

As already noted in the preface, methods of calculation of electrostatic fields are covered in sufficient detail in literature; therefore, in the last two chapters of this book, only special methods of calculating capacitance are considered.

PART ONE

SPECIAL METHODS OF CALCULATING CAPACITANCE

CHAPTER 1

METHODS OF DIRECT DETERMINATION OF CAPACITANCE

1-1. General Remarks

Methods of direct determination of capacitance are applicable when conductors are in homogeneous media. These methods are based on replacement of each of the conductors considered with a dielectric body having the same form as the conductor, and the same specific inductive capacitance as the surrounding medium. Instead of an unknown true (equilibrium) distribution of charge over the surface of the conductor, a certain fictitious distribution of charge over the surface of the body $\sigma(S)$ or in its volume $\rho(v)$ is assigned. Methods of assignment of functions $\sigma(S)$ or $\rho(v)$ depend on the features of concrete methods of direct determination of capacitance; however, in any selected form of these functions, the value of the total charge of the body is found from the formulas

$$Q_i = \int_{S_i} \sigma(S)\, dS \tag{1-1}$$

or

$$Q_i = \int_{v_i} \rho(v)\, dv, \tag{1-2}$$

and the potential at the random point (P_k) of the surface of the body from the formulas

$$V(P_k) = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{n} \int_{S_i} \frac{\sigma(S)}{r(P_k, P_i)}\, dS \tag{1-3}$$

$$V(P_k) = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{n} \int_{v_i} \frac{\rho(v)}{r(P_k; P_i)} dv, \qquad (1\text{-}4)$$

where P_i is a point either on the surface of the i-th body (1-3), or in its volume (1-4); $r(P_k; P_i)$ is the distance between points P_k and P_i; n is the number of conductors in the system.

For a plane-parallel system of conductors, instead of (1-3) and (1-4) one ought to use the formulas:

$$V(P_k) = \frac{1}{2\pi\varepsilon} \sum_{i=1}^{n} \int_{L_i} \tau(L) \cdot \ln \frac{1}{r(P_k; P_i)} dL \qquad (1\text{-}3')$$

or

$$V(P_k) = \frac{1}{2\pi\varepsilon} \sum_{i=1}^{n} \int_{S_i} \sigma(S) \ln \frac{1}{r(P_k; P_i)} dS, \qquad (1\text{-}4')$$

where L_i is the contour of the section of the i-th body; $\tau(L)$ is the linear density of a charge on the contour of the section of the i-th body; $\sigma(S)$ is the surface density of the charge in a section of the i-th body; P_k is a point of the contour of the section of the k-th body; P_i is a point either on the contour of the section of the i-th body (1-3') or inside its section (1-4'), $r(P_k; P_i)$ is the distance between points P_k and P_i, lying in the section.

In general the surface of the body considered is not equipotential, whereas the surface of any conductor is equipotential. To get rid of this discrepancy of the whole surface of the body, a certain constant potential V_i is conditionally added, the value of which is determined by this or that method according to the distribution of the potential found from (1-3) or (1-4).

Disposing Q_i and V_i for each of the conductors of the system ($i = 1, 2, \ldots, n$), the capacitances of this system can be found approximately using formulas (V-1)-(V-4).

1-2. The Method of Mean Potentials

The method of mean potentials is based on assignment of a fictional distribution of charge over the surface of or in the volume of bodies replacing conductors. In this case the surface of each of the bodies is ascribed a constant potential equal to the arithmetic mean of values of potential in all points of the surface of the body ($V = V_{cp}$). This quantity (V_{cp}) is called the mean potential of the surface or the mean potential of the conductor.

When the method is used to determine V, the law of fictional distribution of charge has comparatively little effect on accuracy of determination of capacitance (inasmuch as capacitance is an integral characteristic of electrostatic field) and is usually selected only from conditions of simplicity of calculations. The most widespread assumption is that the charge is uniformly distributed over the surface of the body. A method of calculation of capacitance based on this was proposed by G. Howe [1-1] and bears his name.

Other methods besides this were proposed for assigning the law of surface distribution of charge, using the method of mean potentials. Thus, in [1-2] it is proposed to select this law in the form

$$\sigma(S) = A\left[\int_S \frac{dS}{r}\right]^{-1}. \qquad (1\text{-}5)$$

where S is the surface of the conductor; A is a random quantity; r is the distance between two points of the surface S, one of which is a running point and the other of which is a fixed point.

Formula (1-5) in a number of cases gives a better approximation than in the Howe method to equilibrium distribution of charge; however, the calculation formulas obtained are usually more complex.

Below we shall limit ourselves mainly to consideration of the Howe method, which is the most widespread method of direct determination of capacitance.

For a solitary conductor mean potential can be determined according to the formula

$$V_{cp} = \frac{1}{S}\int_S V(p)\,dS = \frac{Q}{4\pi\epsilon S^2}\int_S dS'\int_S \frac{dS}{r}, \qquad (1\text{-}6)$$

where S is the surface of the conductor considered (and also the area of this surface); $V(p)$ is the potential at point p of surface S, determined by formula (1-3); Q is the total charge of the conductor; r is the distance between the points of the surface of the conductor.

Calculation of mean potential by formula (1-6) in a number of cases can be simplified, having broken the surface of the conductor into individual sections and consecutively calculating the mean potential of each of them as a solitary body. Into these cases the mean potential of the whole conductor is determined by the formula

$$V_{cp} = \sum_{k=1}^{n} V_{cp\,k}\cdot\frac{S_k}{S}, \qquad (1\text{-}7)$$

where S_k is the area of the surface of the k-th section; S is the total area of the surface of the conductor; V_{cpk} is the mean potential of the k-th section; n is the number of sections into which the surface of the conductor is divided.

For wire the ratio S_k/S in this form can be replaced by the ratio l_k/l, where l_k is the length of the k-th segment of wire, and l is the total length of wire.

From formulas (1-6) and (V-1) it follows that the capacitance of a solitary conductor calculated by the Howe method is determined by the expression

$$C_0 \simeq 4\pi\epsilon S^2\left[\int_{S_1} dS'\int_S \frac{dS}{r}\right]^{-1}. \qquad (1\text{-}8)$$

With calculation of the capacitance between two conductors, the mean potential of each of them is found from the formulas

$$V_{cp1} = \frac{Q}{4\pi\varepsilon S_1} \int\limits_{S_1} \left(\frac{1}{S_1} \int\limits_{S_1} \frac{dS}{r_{11}} - \frac{1}{S_2} \int\limits_{S_2} \frac{dS}{r_{12}} \right) dS';$$

$$V_{cp2} = -\frac{Q}{4\pi\varepsilon S_2} \int\limits_{S_2} \left(-\frac{1}{S_1} \int\limits_{S_1} \frac{dS}{r_{21}} + \frac{1}{S_2} \int\limits_{S_2} \frac{dS}{r_{22}} \right) dS', \qquad (1\text{-}9)$$

where S_1 and S_2 are the surfaces of each of the conductors considered (and also the areas of these surfaces); r_{11} and r_{22} are the distances between two points of one and the same conductor (of the first and second, respectively); $r_{12} = r_{21}$ is the distance between two points, one of which lies on the surface of the first conductor and the second of which lies on the surface of the second; Q is the total charge of one conductor.

As in the previous case, calculation of mean potentials of conductors can be simplified, having divided the surfaces of each or of one of them into separate sections or segments (in the case of a wire) and having used formulas (1-7).

In calculation of the difference of mean potentials between two conductors, use can also be made of the principle of mutuality of mean potentials, which consists of the following.

The mean potential of conductor A induced by charge Q, is evenly distributed on conductor B, and is equal in absolute value to the mean potential of conductor B induced by a charge $-Q$, uniformly distributed on conductor A.

Use of formulas (1-9), taking the mutuality principle into account, leads to the expression

$$V_{cp1} - V_{cp2} = \frac{Q}{4\pi\varepsilon} \left(\frac{1}{S_1^2} \int\limits_{S_1} dS' \int\limits_{S_1} \frac{dS}{r_{11}} - \frac{2}{S_1 \cdot S_2} \int\limits_{S_1} dS' \int\limits_{S_2} \frac{dS}{r_{12}} + \frac{1}{S_2^2} \int\limits_{S_2} dS' \int\limits_{S_2} \frac{dS}{r_{22}} \right). \qquad (1\text{-}10)$$

From formulas (1-10) and (V-2) it follows that the capacitance between two conductors calculated by the Howe method is determined by the expression

$$C \simeq 4\pi\varepsilon\left(\frac{1}{S_1^2}\int\limits_{S_1} dS'\int\limits_{S_1}\frac{dS}{r_{11}}-\frac{2}{S_1S_2}\int\limits_{S_1} dS'\int\limits_{S_2}\frac{dS}{r_{12}}+\frac{1}{S_2^2}\int\limits_{S_2} dS'\int\limits_{S_2}\frac{dS}{r_{22}}\right)^{-1}. \qquad (1\text{-}11)$$

When a system of two conductors is plane-parallel instead of (1-11) the next formula, analogous to it, for capacitance per unit of length of conductors should be used:

$$C_l \simeq 2\pi\varepsilon\left(\frac{1}{L_1^2}\int\limits_{L_1} dL'\int\limits_{L_1}\ln\frac{1}{r_{11}}\,dL-\frac{2}{L_1L_2}\int\limits_{L_1} dL'\int\limits_{L_2}\ln\frac{1}{r_{12}}\,dL+\frac{1}{L_2^2}\int\limits_{L_2} dL'\int\limits_{L_2}\ln\frac{1}{r_{22}}\,dL\right)^{-1}, \qquad (1\text{-}12)$$

where L_1 and L_2 are the contours of the sections of conductors considered (and also the perimeters of these sections); r_{11}, r_{12}, and r_{22} are the distances between the corresponding points on the contours of the sections (see designations to formula (1-9)).

In calculation of partial capacitances in a system of many bodies, direct use of the method of mean potentials is difficult since it usually leads to bulky calculations. Therefore, in the given cases the method of mean potentials is used, as a rule, to calculate potential coefficients with subsequent conversion to partial capacitances on the basis of the relationships given in V-1.

Calculation of mean potentials in a system of n conductors is based on utilization of the formula

$$V_{\text{cp}\,i} = \frac{1}{4\pi\varepsilon S_i}\int\limits_{S_i}\left(\sum_{k=1}^{n}\frac{Q_k}{S_k}\int\limits_{S_k}\frac{dS}{r_{ik}}\right)dS', \qquad (1\text{-}13)$$

where V_{cpi} is the mean potential of the i-th conductor; S_k is the surface of the k-th conductor ($k = 1, 2, \ldots, n$) and also the area of this surface; Q_k is the full charge of the k-th conductor; r_{ik} is the distance between two points on the surface of different conductors ($k \neq i$) or one conductor ($k = i$). In this case between the quantities of the mean potentials of any two conductors (A and B) the relationship is satisfied[1]

$$\frac{V_{Acp}}{V_{Bcp}} = \frac{Q_B}{Q_A}. \tag{1-14}$$

where V_{Acp} is the mean potential of the conductor A, created by charge Q_B, uniformly distributed on conductor B; V_{Bcp} is the mean potential of conductor B created by charge Q_A, uniformly distributed on conductor A.

In determination of partial capacitances the charges of all the conductors of the system must be taken as different from zero, and calculations made using formula (1-13), even allowing for relationship (1-14), become very lengthy. Upon finding potential coefficients (when only one of the conductors must be considered charged) formula (1-13) is strongly simplified and coincides in form with (1-6). This leads to the following expressions for intrinsic and mutual potential coefficients calculated according to the Howe method:

$$\alpha_{ii} \approx \frac{1}{4\pi\varepsilon S_i^2} \int_{S_i} dS' \int_{S_i} \frac{dS}{r_{ii}}, \tag{1-15}$$

$$\alpha_{ik} \approx \frac{1}{4\pi\varepsilon S_i S_k} \int_{S_k} dS' \int_{S_i} \frac{dS}{r_{ik}}. \tag{1-16}$$

For a plane-parallel system of n conductors, analogous formulas take the form:

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[1]When $|Q_A| = |Q_B|$, this formula expresses the principle of mutuality of mean potentials formulated above.

$$\alpha_{ii,l} = \frac{1}{2\pi\varepsilon L_i^2} \int_{L_i} dL' \int_{L_i} \ln \frac{1}{r_{ii}} \, dL_i \tag{1-17}$$

$$\alpha_{ik,l} = \frac{1}{2\pi\varepsilon L_i L_k} \int_{L_i} dL' \int_{L_k} \ln \frac{1}{r_{ik}} \, dL_k \tag{1-18}$$

where $\alpha_{ii,l}$ and $\alpha_{ik,l}$ are intrinsic and mutual potential coefficients per unit of length of conductors; L_l, L_k are contours of sections of the i-th and k-th conductors, and also the perimeters of these circuits; r_{ii} and r_{ik} are the distance from any fixed point on the contour of the section of the i-th conductor up to a random point of this contour (r_{ii}) or the contour of the section of the k-th conductor (r_{ik}).

All the above formulas for the calculation of capacitance by the Howe method are approximation methods.

1. The values of capacitance of a solitary converter calculated by the Howe method [formula (1-8)] and of the capacitance between two conductors [formula (1-11)] do not exceed the accurate values of these quantities.

For a solitary conductor this affirmation follows directly from the variation principle of Gauss [1-3], which is expressed in the form

$$C_0 < \frac{\int_S V(S) \cdot \sigma(S) \, dS}{Q^2}. \tag{1-19}$$

where C_0 is the true value of capacitance of a solitary conductor limited by surface S; $\sigma(S)$ is any assigned distribution of charge Q over surface S; $V(S)$ is the potential at a random point of surface S at assigned distribution of charge.

Supposing in (1-19) that $\sigma(S) = \frac{Q}{S}$, where S also designates the area of the surface of the conductor being considered, and using formula (1-3), we obtain in the right side of the inequality the quantity being determined by formula (1-8).

For the capacitance between two conductors the proof is conducted analogously.

2. The values of intrinsic and mutual potential coefficients [formulas (1-15), (1-16)] are greater than the true values of these quantities.

3. For conductors of one and the same (or close) layout the error of the method of mean potentials is less, the more uniform the equilibrium distribution of charge on these conductors. Specifically:

the relative error of calculation of capacity of any straight solitary wire (or cylindrical conductor) with the assigned form of cross section is less the greater the ratio of its length to maximum dimension of cross section, its lowest value is reached when the section is round;

the relative error of calculation of capacitance of a flat rectilinear plate of assigned area is less the greater the ratio of dimensions of the plate;

the relative error of calculation of capacitance of solitary conductors in the form of right polyhedrons inscribed in a certain sphere or described relative to it is less the greater the number of sides;

the relative error of calculation of capacitance between two plates of the same form and dimensions in one plane is less the greater the ratio of distance between plates to any dimension of them.

Errors of calculation of capacitances are numerically evaluated by the Howe method taking into account the affirmations, by means of comparison of the corresponding approximation expressions with accurate ones (see Example 1-2).

Example 1-1. Let us determine the capacitance of a conductor in the form of an $a \times b$ rectangular plate. Using formula (1-8) let us precalculate $\int_S \frac{dS}{r}$. For this let us introduce a rectangular system of coordinates the origin of which is compatible with one of the peaks of the rectangular contour of the plate and direct the axes along the sides of this contour. Then the value of $\int_S \frac{dS}{r}$ at a certain point with coordinates x_1; y_1 $(0 \le x_1 \le a;\ 0 \le y_1 \le b)$ will be determined by the expression

$$\int_S \frac{dS}{r} = \int_0^a dx \int_0^b \frac{dy}{\sqrt{(x-x_1)^2+(y-y_1)^2}} =$$
$$= (a-x_1)\left(\operatorname{Arsh}\frac{b-y_1}{a-x_1} + \operatorname{Arsh}\frac{y_1}{a-x_1}\right) +$$
$$+ (b-y_1)\left(\operatorname{Arsh}\frac{a-x_1}{b-y_1} + \operatorname{Arsh}\frac{x_1}{b-y_1}\right) + y_1\left(\operatorname{Arsh}\frac{a-x_1}{y_1} + \operatorname{Arsh}\frac{x_1}{y_1}\right) +$$
$$+ x_1\left(\operatorname{Arsh}\frac{b-y_1}{x_1} + \operatorname{Arsh}\frac{y_1}{x_1}\right) = f(x_1;\ y_1).$$

Repeatedly integrating,[1] after the corresponding conversions we obtain

$$\int_S dS' \int_S \frac{dS}{r} = \int_0^a dx_1 \int_0^b f(x_1;\ y_1)\, dy_1 =$$
$$= 2\left[a^2 b \operatorname{Arsh}\frac{b}{a} + b^2 a \operatorname{Arsh}\frac{a}{b} + \frac{1}{3}(a^3+b^3) - \frac{1}{3}(a^2+b^2)^{\frac{3}{2}}\right].$$

Substituting the obtained expression into formula (1-8), we obtain the following approximation expression for the capacitance

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[1] Using the symmetry of expression $f(x_1, y_1)$ relative to the quantities entering it and also the obvious relation $\int_0^a \varphi(a-x)\,dx = \int_0^a \varphi(x)\,dx$, where ϕ is a random function, it is sufficient to carry out integration of only one of the components entering $f(x_1, y_1)$.

of the plate considered:

$$C_0 \approx 2\pi\varepsilon \frac{a^2b^2}{a^2b \operatorname{Arsh}\frac{b}{a} + ab^2 \operatorname{Arsh}\frac{a}{b} + \frac{1}{3}(a^3+b^3) - \frac{1}{3}(a^2+b^2)^{\frac{3}{2}}}.$$

Example 1-2. Using the Howe method, let us determine the value of the capacitance of a conductor in the form of a solitary circular disc of radius R.

Using formula (1-8) again, let us precalculate

$$\int_S \frac{dS}{r} = \int_0^R r\,dr \int_0^{2\pi} \frac{d\theta}{\sqrt{r^2 + r_1^2 - 2rr_1\cos\theta}} = 4R\mathrm{E}\left(\frac{r_1}{R}\right),$$

where E is a complete elliptical integral of the 2nd kind with modulus $k = r_1/R$.

Then

$$\int_S dS' \int_S \frac{dS}{r} = 4R\int_0^{2\pi} d\theta \int_0^R r_1\mathrm{E}\left(\frac{r_1}{R}\right) dr_1 = 8\pi R^3 \int_0^1 k\mathrm{E}(k)\,dk = \frac{16\pi R^3}{3}.$$

Substituting the obtained expression in formula (1-8), we find that the capacitance of a circular disc calculated by the Howe method is equal to

$$C_0 \approx \frac{3}{4}\pi^2\varepsilon R \approx 7{,}40\varepsilon R.$$

The accurate value of the capacitance of the disc is equal to $8\varepsilon R$. Thus, the relative inaccuracy of the calculation of the capacitance of a solidary disc by the Howe method is about 7.5%.

In calculation of capacitance of closed shells, a fictitious charge can be considered distributed not only over the surface, but also in the volume of the bodies replacing these conductors.

In this case the general scheme of using the method of mean potentials remains constant; however, the features of its application depend on the character of distribution of charge in the volume of bodies.

With continuous distribution of charge with assigned volume density $\rho(v)$, the course of calculation differs only by the fact that to determine potential at points of the surface of the body instead of formulas (1-3) it is necessary to use formula (1-4). This does not usually lead to simplification of calculations since instead of surface integrals entering (1-3), it is necessary to reckon integrals in terms of volume.

With continuous distribution of charge along certain lines in a volume of bodies (it is expedient to use such distribution in calculating the capacitance of conductors of drawn out or axisymmetric form) in formula (1-4) the volume density of a charge must be replaced by linear density, the volume integral must be replaced by a curvilinear integral, and calculations are simplified. Thus, for a solitary axisymmetrical shell

$$C_0 = 4\pi\varepsilon \cdot S \cdot L \left[\int_S dS \int_L \frac{dL}{r} \right]^{-1},$$

where L is the segment of the axis of symmetry inside the conductor (and also the length of this segment); S is the surface of the conductor (and also its area); r is the distance from the fixed point of the surface S to the running point of the axis L.

With discrete distribution of charge in the volume of bodies, the potential at every point of surface of the body is calculated as the sum of potentials of point charges.

1-3. Method of Grounds

During the calculation of capacitance by the method of grounds the surface of each of the bodies replacing the conductors is divided into a number of grounds the simplest possible form of which is selected and the dimensions of which are so small that the fictional distribution of charge in the limits of each ground can be considered uniform.[1]

The surface of each ground is ascribed a fixed potential $\tilde{V}_k$ equal to the potential at any one (characteristic) point of this ground.

At sufficiently small dimensions of grounds, the method of location of characteristic points on their surface has comparatively little effect on the results of calculation. Therefore, it is usually selected only from conditions of simplicity of calculations.[2]

The potential at the characteristic point of each ground can be determined with the aid of formula (1-3) and with the accepted law of fictional distribution of charge

$$V_k = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{n}\sigma_i \cdot a_{ki}, \tag{1-20}$$

where V_k is the potential at the characteristic point of the k-th ground; n is the number of grounds; σ_i is the density of a charge on the surface of the i-th ground; S_i is the surface of the i-th ground; r_{ki} is the distance from the characteristic point of the k-th ground to a random point on the surface of the i-th ground; $a_{ki} = \int_{S_i}\frac{dS}{r_{ki}}$.

The value of coefficients a_{ki} are determined only by geometric parameters of grounds and their mutual location. When the distance between any two grounds considerably exceeds the dimensions of at least one of them (for example, the i-th), the quantity a_{ki} can be determined with sufficient accuracy as the ratio of the area of the

[1]See examples of use of the method of grounds in works [1-5, 1-6].

[2]The location of characteristic points on the surface of identical grounds are usually selected identical, and on the surface of geometrically similar grounds similar.

i-th ground to the distance between the characteristic points of the i-th and k-th grounds.

The values of potentials of grounds ($\tilde{V}_k = V_k$) found from (1-20) in the limits of the surface of one conductor are equated with one and the same constant.

For a solitary conductor this value (A) can be selected at random. This leads to the following system of linear algebraic equations relative to unknown values of density of charge on the surface of each ground[1]

$$\left.\begin{array}{l} a_{11}\sigma_1 + a_{12}\sigma_2 + \ldots + a_{1n}\sigma_n = 4\pi\varepsilon A, \\ a_{21}\sigma_1 + a_{22}\sigma_2 + \ldots + a_{2n}\sigma_n = 4\pi\varepsilon A, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}\sigma_1 + a_{n2}\sigma_2 + \ldots + a_{nn}\sigma_n = 4\pi\varepsilon A. \end{array}\right\} \quad (1\text{-}21)$$

Hence the charge density on the surface of the k-th ground is

$$\sigma_k = 4\pi\varepsilon \cdot A \frac{\Delta_k}{\Delta}, \quad (1\text{-}22)$$

where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_n \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix},$$

and Δ_k is the determinant formed from Δ by replacement of all the elements of the k-th column with ones.

At the found values of σ_k the total charge of the conductor in general (at random separation of surface of conductor into grounds)

[1]In a number of cases from conditions of symmetry it is possible knowingly to show certain grounds with the same charge density. In these cases the number of unknowns in (1-21) is reduced.

is determined by the formula

$$Q = 4\pi\varepsilon \cdot A \cdot \frac{1}{\Delta} \sum_{k=1}^{n} S_k \cdot \Delta_k, \tag{1-23}$$

where S_k is the area of the k-th ground.

If all the grounds are identical, then

$$Q = 4\pi\varepsilon A \frac{S}{n\Delta} \sum_{k=1}^{n} \Delta_k, \tag{1-23a}$$

where S is the total area of the surface of the conductor.

The obtained expressions for total charge lead directly to the following approximation expressions for the capacitance of a solitary conductor:

a) in general

$$C_0 \simeq 4\pi\varepsilon \frac{1}{\Delta} \sum_{k=1}^{n} S_k \cdot \Delta_k; \tag{1-24}$$

b) for identical grounds

$$C_0 \simeq 4\pi\varepsilon \frac{S}{n\Delta} \sum_{k=1}^{n} \Delta_k. \tag{1-24a}$$

During the calculation of capacitances in a system of two and more conductors, direct utilization of the method of grounds is difficult since it leads to lengthy computations. Therefore, in these cases the method of grounds is used, as a rule, to calculate coefficients of electrostatic induction with subsequent conversion to values of capacitance on the basis of the relationships given in § V-1.

Let the number of conductors in the system be equal to N, and the number of grounds into which the surface of the p-th conductor is divided n_p ($p = 1, 2, \ldots, N$). Then the potential at the characteristic point of each ground can be found from formula (1-20) when $n = \sum_{p=1}^{N} n_p$. Then the potentials of all platforms on the surface of each conductor should be equated with one and the same constant A_p ($p = 1, 2, \ldots, N$); however, the values of the constants A_p can no longer be assigned at random (as in the case of a solitary conductor), but must be selected taking into account the conditions indicated in V-1 and V-5. These conditions are the simplest in the calculation of coefficients of electrostatic induction since the potentials of all conductors, except one, should be taken equal to zero.

Let us assume, for example, that it is necessary to determine the intrinsic coefficient of electrostatic induction for the p-th conductor and the mutual coefficient of electrostatic induction for the p-th and q-th conductors ($p, q = 1, 2, \ldots, N, p \neq q$). Without losing generality one can assume that areas with numbers $1, 2, \ldots, n_p$ belong to the surface of the p-th conductor, and areas with numbers $n_p + 1; n_p + 2, \ldots, n_p + n_q$ belong to the area of the q-th conductor. Furthermore, let us assume that the potential of the p-th conductor is equal to a certain constant A.

Then the system of equations for determination of unknown values of charge density on the surface of the grounds takes the form:

$$\sum_{l=1}^{n} a_{kl}\sigma_l = 4\pi\varepsilon A \quad \text{with } k = 1, 2, \ldots, n_p,$$

$$\sum_{l=1}^{n} a_{kl}\sigma_l = 0 \text{ with } \quad k = n_p + 1, \ldots, n_p + n_q \qquad (1\text{-}25)$$

The solution of this system again leads to formula (1-22), where this time Δ_k is formed from Δ by replacement of the first n_p elements of the k-th column with ones, and all the rest of the elements of this column with zeroes.

The found values of charge density allow directly determining the quantity of total charge of the p-th and q-th conductors, and thereby the sought values of intrinsic and mutual coefficients of electrostatic induction. The formulas for the determination of these coefficients have the form:[1]

$$\beta_{pp} \approx 4\pi\varepsilon \frac{1}{\Delta} \sum_{k=1}^{n_p} S_k \Delta_{kp} \tag{1-26}$$

$$\beta_{pq} \approx 4\pi\varepsilon \frac{1}{\Delta} \sum_{k=n_p+1}^{n_p+n_q} S_k \Delta_{kp} \tag{1-27}$$

The given formulas (1-24), (1-26) and (1-27) are approximation formulas, and, as can be shown, give underestimated values of the capacitance of a solitary conductor and of coefficients of electrostatic induction in a system of two and more conductors. From the essence of the given method it is clear that the inaccuracy of calculation from these formulas is less the smaller the grounds into which the surface of the conductors is divided. At rather small sizes of grounds the accuracy of calculation of capacitance by the method being considered can be brought to any required limits and, in particular, can be higher than when using the method of mean potentials.

Example 1-3. Using the method of grounds, let us consider the same problem as in Example 1-1, having assumed that the surface of the plate is divided into 4 identical grounds, numbered as shown in Fig. 1-1.

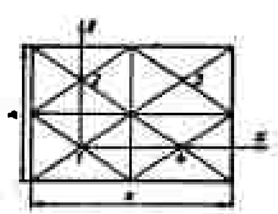

Fig. 1-1. Rectangular plate divided into 4 grounds.

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[1]With separation of the surfaces of the conductors into identical grounds formulas (1-26) and (1-27) can be simplified similar to formula (1-24).

Using a rectangular system of coordinates (Fig. 1-1) and selecting as characteristic points the points of intersection of diagonals of each ground, in accordance with formula (1-20) we find:

$$a_{11} = \int_{S_1} \frac{dS}{r_{11}} = 4\int_0^{a/4} dx \int_0^{b/4} \frac{dy}{\sqrt{x^2+y^2}} = a \operatorname{Arsh}\frac{b}{a} + b \operatorname{Arsh}\frac{a}{b},$$

$$a_{12} = \int_{S_2} \frac{dS}{r_{12}} = 2\int_0^{a/4} dx \int_{b/4}^{3b/4} \frac{dy}{\sqrt{x^2+y^2}} = -\frac{b}{2}\operatorname{Arsh}\frac{a}{b} + \frac{3b}{2}\operatorname{Arsh}\frac{a}{3b} + \frac{a}{2}\operatorname{Arsh}\frac{3b}{a} - \frac{a}{2}\operatorname{Arsh}\frac{b}{a},$$

$$a_{13} = \int_{S_3} \frac{dS}{r_{13}} = \int_{a/4}^{3a/4} dx \int_{b/4}^{3b/4} \frac{dy}{\sqrt{x^2+y^2}} = b \operatorname{Arsh}\frac{a}{b} + a \operatorname{Arsh}\frac{b}{a} - \frac{3}{4} b \operatorname{Arsh}\frac{a}{3b} - \frac{a}{4}\operatorname{Arsh}\frac{3b}{a} - \frac{b}{4}\operatorname{Arsh}\frac{3a}{b} - \frac{3a}{4}\operatorname{Arsh}\frac{b}{3a},$$

$$a_{14} = \int_{S_4} \frac{dS}{r_{14}} = 2\int_0^{b/4} dy \int_{a/4}^{3a/4} \frac{dx}{\sqrt{x^2+y^2}} = -\frac{b}{2}\operatorname{Arsh}\frac{a}{b} - \frac{a}{2}\operatorname{Arsh}\frac{b}{a} + \frac{b}{2}\operatorname{Arsh}\frac{3a}{b} + \frac{3a}{2}\operatorname{Arsh}\frac{b}{3a}.$$

Because of the symmetry of the location of grounds $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma_0$; furthermore, with the accepted division of surface into grounds $a_{11} = a_{22} = a_{33} = a_{44}$; $a_{12} = a_{34}$; $a_{13} = a_{24}$. Therefore, in system (1-21) it is sufficient to keep only one equation, whence

$$\sigma_0 = 4\pi\varepsilon \frac{A}{\sum_{k=1}^{4} a_{1k}} = 4\pi\varepsilon A\left(a \operatorname{Arsh}\frac{b}{a} + b\operatorname{Arsh}\frac{a}{b} + \frac{3b}{4}\operatorname{Arsh}\frac{a}{3b} + \frac{a}{4}\operatorname{Arsh}\frac{3b}{a} + \frac{b}{4}\operatorname{Arsh}\frac{3a}{b} + \frac{3a}{4}\operatorname{Arsh}\frac{b}{3a}\right)^{-1}.$$

Using then formulas (1-23a) and (1-24a), we find that with the means shown of division of plate into grounds, its capacitance is

$$C_0 \simeq 4\pi\varepsilon \frac{ab}{a\operatorname{Arsh}\frac{b}{a} + b\operatorname{Arsh}\frac{a}{b} + \frac{3b}{4}\operatorname{Arsh}\frac{a}{3b} + \frac{a}{4}\operatorname{Arsh}\frac{3b}{a} + \frac{b}{4}\operatorname{Arsh}\frac{3a}{b} + \frac{3a}{4}\operatorname{Arsh}\frac{b}{3a}}.$$

1-4. The Method of Equivalent Charges

The method being considered consists of determination of the distribution of charges in the volume of bodies replacing conductors in the form of closed shells at which the surface of these bodies is equipotential.[1] If such distribution of charges is found, then the values of capacitance in the system of conductors can be determined according to the formulas shown in § V-1, substituting in place of potentials of conductors potentials of surfaces of bodies, and instead of charges of conductors values of the total charge in the volume of each body.

There is no general means of determining the distribution of charges creating electrostatic fields with assigned configuration of equipotential surfaces in existence at present. Therefore, in determination of capacitance from the method of equivalent charges, the reverse method is usually employed: assigning this or that concrete distribution of charges, the form of equipotential surfaces of electrostatic field is determined for each of them, and thereby a certain "set" of distributions of charges which create known fields is obtained. Using it, it is possible in a number of cases to find such a distribution of charges for which the form of equipotential surfaces coincides (or closely enough) with the form of surfaces of the conductors considered.

Sometimes the required distribution of charges can be found also directly from the assigned form of the surface of conductors.

Thus, during calculation of capacitance in a system of conductors bounded by surfaces of spherical form, the required distribution of charge can be found directly by means of utilization of the following known features of electrostatic field of point charges.

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[1]At the shown distribution of charge the electrostatic field outside the surface of the bodies coincides with the electrostatic field of the system of conductors being considered. In this sense the charges concentrated in the volume of bodies are equivalent to the charges distributed over the surface of the conductors.

The method considered is also sometimes called the method of "consolidation" or "congelation" of equipotential surfaces.

1. In the field of point charge q any spherical surface with center at the point of location of charge is equipotential. If the potential of this surface is equal to A, and the radius is equal to a, then the charge located at the center of the sphere is $q = 4\pi\varepsilon a A$.

2. In a field of unlike point charges q_1 and q_2 separated by a distance of d, there is a surface of zero potential having the form of a sphere, the center of which is the line passing through the points of location of charges, and the radius of the sphere R and the location of its center are determined from the relationships:

$$R^2 = h_1 \cdot h_2; \quad h_1 = -\frac{q_1}{q_2} R; \quad |h_1 - h_2| = d, \tag{1-28}$$

where h_1 and h_2 are the distances between the points of location of the charges q_1 and q_2 and the center of the sphere.

At assigned radius of sphere R, and quantity and location of one of the charges (for example, charge q_1), relationships (1-28) can be used to determine the quantity and location of the second charge q_2, which is called the reflection of charge q_1 relative to the sphere or simply the reflected charge.

The application of these features allows showing the means of determination of distribution of equivalent charge in a volume of bodies bounded by spherical surfaces. In general form this method consists of the fact that, locating in the center of each sphere a charge of appropriate quantity, its influence on the potentials of the remaining spheres is compensated with the aid of a definitely selected system of reflections.

Example 1-4. Let us determine the capacitance of a solitary conductor formed by two spheres of radii a and b, which intersect at an angle of $\pi/2$; $a > b$, and the distance between the centers of spheres $l > a - b$ (Fig. 1-2).

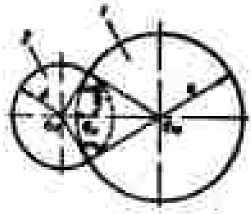

Fig. 1-2. A solitary conductor formed by two spheres with radii a and b ($a > b$), intersecting at right angles.

Taking the potential of the conductor to be equal to the constant A, we locate in the center of a sphere of radius a (sphere 1) a point charge $q_{10} = 4\pi\varepsilon a \cdot A$.

In the field of this charge the surface of the sphere 1 acquires a potential A, but the potential of sphere 2 is inconstant. Reflecting charge q_{10} relative to sphere 2, we find that the reflected charge is

$$q_{11} = -\frac{b}{\sqrt{a^2+b^2}} q_{10} = -4\pi\varepsilon \frac{ab}{\sqrt{a^2+b^2}} A$$

and is at a distance of

$$h_{11} = \frac{b^2}{\sqrt{a^2+b^2}}$$

from the center of sphere 2, i.e., at a distance of

$$h'_{11} = \sqrt{a^2+b^2} - \frac{b^2}{\sqrt{a^2+b^2}} = \frac{a^2}{\sqrt{a^2+b^2}}$$

from the center of sphere 1.

In the field of charges q_{10} and q_{11} the potential of sphere 2 is equal to zero, and sphere 1 is not equipotential. To restore constancy of potential of sphere 1 we reflect relative to it charge q_{11}. The reflected charge is

$$q_{12} = -\frac{a\sqrt{a^2+b^2}}{a^2} q_{11} = 4\pi\varepsilon b A$$

and is at a distance of $h_{12} = \frac{a^2\sqrt{a^2+b^2}}{a^2} = \sqrt{a^2+b^2}$ from the center of sphere 1, i.e., at the center of sphere 2.

From the means of selection of quantity and location of charges q_{10}, q_{11} and q_{12}, it is clear that in an electrostatic field induced by them the potential of each of the spheres considered is constant and equal to A.

Summarizing the found quantities q_{10}, q_{11} and q_{12}, we find that the equivalent charge is

$$Q = q_{10} + q_{11} + q_{12} = 4\pi\varepsilon A\left(a + b - \frac{ab}{\sqrt{a^2 + b^2}}\right).$$

Therefore, the capacitance of the conductor being considered is

$$C_0 = 4\pi\varepsilon a\left(1 + \frac{b}{a} - \frac{b}{\sqrt{a^2 + b^2}}\right).$$

<u>Example</u> 1-5. Let us determine the capacitance of a solitary conductor formed by two adjoining spheres of equal radii (Fig. 1-3).

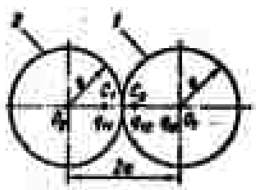

Fig. 1-3. Conductor formed by two tangent spheres of equal radii.

Assuming again the potential of the conductor considered equal to A, let us first pick the distribution of equivalent charge at which one of the spheres (sphere 1 for sure) has potential equal to A, and the other has potential equal to zero.

The required value of the potential on sphere 1 is obtained, as before, placing in its center a point charge $q_{10} = 4\pi\varepsilon aA$, where a is the radius of the spheres. However, the potential of sphere 2 in this case is not equal to zero. To achieve zero potential on sphere 2 we reflect charge q_{10} relative to this sphere. In this case we obtain the reflected charge q_{11}, the quantity and location of which is shown in Table 1-1. In the field of charges q_{10} and q_{11}

Table 1-1. Quantity and location of initial and reflected equivalent charges for the conductor shown in Fig. 1-3.[1]

k	$\frac{q_{1k}}{4\pi\varepsilon A a}$	O_2C_k	O_1C_k
0	1	$2a$	0
1	$-\frac{1}{2}$	$\frac{a}{2}$	$\frac{3a}{2}$
2	$\frac{1}{3}$	$\frac{4a}{3}$	$\frac{2a}{3}$
3	$-\frac{1}{4}$	$\frac{3a}{4}$	$\frac{5}{4}a$
4	$\frac{1}{5}$	$\frac{6}{5}a$	$\frac{4}{5}a$
5	$-\frac{1}{6}$	$\frac{5}{6}a$	$\frac{7}{6}a$
k	$\frac{(-1)^k}{k+1}$	$\left[1-\frac{(-1)^{k+1}}{k+1}\right]a$	$\left[1-\frac{(-1)^k}{k+1}\right]a$

[1] O_1C_k and O_2C_k are the distance of the point of location of charge q_{1k} from the centers of spheres 1 and 2, respectively.

the potential of sphere 2 is equal to zero, but the potential of sphere 1 is inconstant. To restore the constancy of this potential we reflect charge q_{11} relative to sphere 1, finding the charge shown in Table 1-1 q_{12}. Continuing this process (see Table 1-1), we see that the required values of potentials of spheres can be achieved only with an infinite number of reflections; the k-th reflected charge is

$$q_{1k} = \frac{(-1)^k}{k+1} q_{10}.$$

In completely analogous manner a distribution of equivalent charges q_{2k}, can be found with which the potential of sphere 2 is equal to A, and the potential of sphere 1 is equal to zero. In this case it is obvious that $q_{1k} = q_{2k}$.

In the total field of all charges found in this way the potential of each sphere is equal to one and the same constant A. Therefore, in this case the equivalent charge is

$$Q = 2\sum_{k=0}^{\infty} q_{1k} = 8\pi\varepsilon a\cdot A\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1} = 8\pi\varepsilon aA\cdot\ln 2.$$

Thus, the capacitance of the conductor considered is

$$C_0 = 8\pi\varepsilon a\cdot\ln 2.$$

The scheme of application of the method of equivalent charges for calculation of capacitance between two conductors is analogous to the scheme of calculation of capacitance of solitary conductors. An example is the calculation of capacitance between two spheres given in [1-4].

CHAPTER 2

AUXILIARY METHODS IN THE DETERMINATION OF CAPACITANCE

2-1. General Remarks

The methods considered in the present chapter pursue the objective of bringing the problems of determination of capacitance to a form permissible for calculations or to a form simplifying them. Such methods will subsequently be called auxiliary methods.

The majority of auxiliary methods consist of geometric conversions of systems of conductors and are based on the fact that in some of these conversions the values of capacitance remain constant or vary in a known manner. If such conversion is carried out, then the problem boils down to calculation of capacitance in the converted system, which can be done either by methods of direct determination of capacitance or by means of calculation of electrostatic field.

Some of the auxiliary methods are based on simplification of the problems of calculation of electrical field (and thereby of capacitance) with constant geometric parameters of the system of conductors. Such methods consist in introduction of the relief functions, which are connected in a known way with the potential of the electrostatic field, but satisfy simpler boundary conditions. If the introduced auxiliary functions satisfy the Laplace equation, then the problem of their calculation turns out to be simpler than calculation of the electrostatic field.

2-2. Method of Conformal Conversions

The method of conformal conversions is used to calculate capacitance in plane-parallel systems consisting of two or more conductors. The basis of the method is the feature of capacitance to remain constant during conformal conversions of shown systems (the invariance of the capacitance relative to conformal conversion).

Let us recall that conformal conversion is geometrical conversion in which the angles between any two intersecting lines remain constant, and the length of all infinitesimal segments passing through the given point of the plane changes the same number of times. Conformal conversion is described by the analytical function of a complex variable on condition that this function is unambiguous, and its derivative in the reflected area nowhere turns into zero. The analyticity of the function of the complex variable $W(z) = \phi(x, y) + i\psi(x, y)$ is checked with the aid of the conditions of Cauchy-Riemann:

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y};\quad \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}.$$ [1]

The invariance of capacitance relative to conformal conversion permits replacing the problem of determination of capacitance of any plane-parallel system of conductors by calculating the capacitance of another system obtained from the initial system by means of one or several repeated conformal conversions. If, especially, the initial system can be reduced to any system with known capacitance, then it thereby ceases to be necessary to calculate capacitance.

With practical utilization of the method considered, the section of the plane-parallel system of conductors is taken as the plane of the complex variable z, and a conformal conversion $f(z)$ is selected as a result of which the system takes a simpler form permissible for

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[1]The reader will find more detailed information on conformal conversions in numerous works on the theory of functions of complex variable, for example, in works [2-1 to 2-4].

calculations. Expressions for the functions which realize conformal conversion of some simple areas to upper semiplane are given in Table 2-1.

Table 2-1. Conformal conversions of very simple areas to upper semiplane.

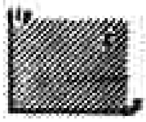
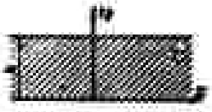
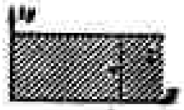
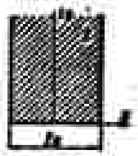
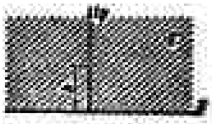
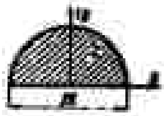
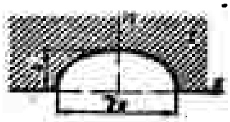
Form of initial area in plane z	The function which realizes the conformal reflection of the area in plane z on the upper half-plane of plane ζ
	$\zeta = \frac{1}{a_0}$ [illegible]
	$\zeta =$ [illegible] $\left(\frac{[illegible]}{2} \cdot \frac{[illegible]}{h}\right)$
	$\zeta =$ [illegible] $\left(\frac{[illegible]}{2} \cdot \frac{[illegible]}{h}\right)$
	$\zeta =$ [illegible] $\left(\frac{[illegible]}{2} \cdot \frac{[illegible]}{h}\right)$
	$\zeta = \sqrt{z^2 + h^2}$

Table 2-1 (Continued).

Form of initial area in plane z	The function which realizes the conformal reflection of the area in plane z on the upper half-plane of plane ζ
	$\zeta = R\cdot\left(\frac{z+R}{z-R}\right)^2$
	$\zeta = \frac{a}{a^2-b^2}\left[az - b\cdot\sqrt{z^2-(a^2-b^2)}\right]$ when $a \neq b$, $\zeta = \frac{z^2+a^2}{2z}$ when $a = b$

Note. a_0 - the dimensional coefficient of length, numerically equal to one.

In a number of problems encountered in practice the geometry of systems proves to be so complex that it is impossible to carry out its conformal conversion to a form permissible for calculations. In these cases use is sometimes made of methods of approximation conformal conversions (see, for example, [2-4]).

Example 2-1. Let us determine the capacitance (per unit of length) between an infinitely long elliptical cylinder and an infinite band, the sections of which are shown in Fig. 2-1a.

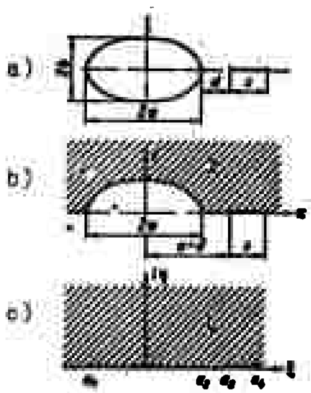


Fig. 2-1. Elliptical cylinder and infinite band in boundless homogeneous medium: a) initial system; b) auxiliary system obtained by cutting the initial system with a plane of its symmetry; c) reflected system in plane ζ.

The sought capacitance is equal to twice the capacitance between the conductors presented in Fig. 2-1b. To calculate the capacitance of this auxiliary system we use the method of conformal conversions, taking plane xOy as the plane of the complex variable z. According to Table 2-1, the function conformally reflecting the area considered on the half-plane of the new complex variable ζ (Fig. 2-1c), has the form

$$\zeta = \frac{a}{a^2 - b^2}\left(az - b\sqrt{z^2 - a^2 + b^2}\right).$$

With the aid of this expression we find the coordinates of the edges of the plates of the reflected system:

$$a_3 = \frac{a}{a^2 - b^2}\left[a(a+d) - b\sqrt{(a+d)^2 - (a^2 - b^2)}\right],$$

$$a_4 = \frac{a}{a^2 - b^2}\left[a(a+d+c) - b\sqrt{(a+d+c)^2 - (a^2 - b^2)}\right].$$

Inasmuch as the geometric parameters of the converted system are now known, it is possible to consider the problem of determination of its capacitance in the normal way (presentation of the plane of the section of this system as the plane of a complex variable was only an auxiliary method necessary for construction of the converted system).

Using, especially, the method of direct determination of field strength (see § 2-6), it is possible to obtain that the capacitance of the converted system is

$$C_U = 2\varepsilon\frac{K'(k)}{K(k)}, \qquad (2\text{-}1)$$

where $K(k)$ and $K'(k) = K(\sqrt{1 - k^2})$ are complete elliptical integrals of kind I with modules being determined by formula (2-24).[1] Specifically, when $a = b$ (circular cylinder), the expression for the

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[1]The basic concepts relating to elliptic integrals are given in Appendix 1.

module of elliptical integrals takes the form:

$$k = \sqrt{\frac{(d-a)(3a+c+d)}{(3a+d)(c+d-a)}}.$$

Inasmuch as capacitance is invariant relative to conformal conversion, formula (2-1) determines the capacitance of the system depicted in Fig. 2-1b. Thus, the sought capacitance of the initial system (Fig. 2-1a) is determined by the expression

$$C_l = 4\epsilon \frac{K'}{K}.$$

<u>Example</u> 2-2. Let us determine the capacitance (per unit of length) between two conductors, each of which is formed by the joining of two infinitely long bands depicted in Fig. 2-2a. Using the general features of capacitance (§ V-2), it can be established that the sought capacitance C_l is four times as great as the capacitance C_{1l} of the auxiliary system shown in Fig. 2-2b,

$$C_l = 4C_{1l}.$$

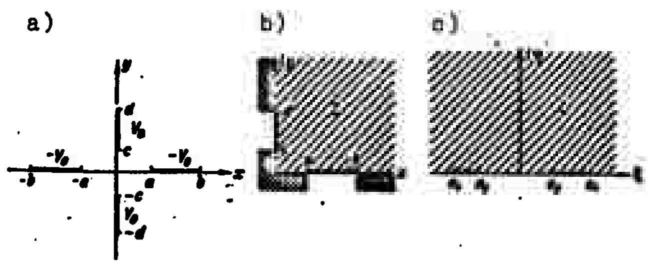


Fig. 2-2. System of two conductors, each of which is formed by the joining of two symmetrically located infinite bands: a) initial system; b) auxiliary system obtained by means of cutting the initial system with the plane of its symmetry; c) reflected system in plane ζ.

To detect capacitance C_{1l} we use the method of conformal conversions. Taking the plane of section of the auxiliary system as the plane of the complex variable z, let us select the reflecting function in the form of $\zeta = \frac{1}{a_0} z^2$, which, as is evident from Table 2-1, converts the auxiliary system considered into a system of two plates lying in one plane (Fig. 2-2c). The capacitance (per unit of length) between these plates is determined by the above formula (2-1) if the modulus of elliptical integrals is assumed to be in it

$$k = \sqrt{\frac{(a^2 + c^2)(b^2 + d^2)}{(a^2 + d^2)(b^2 + c^2)}}. \tag{2-2}$$

Thus the sought value of capacitance is

$$C_l = 8\varepsilon \frac{K'}{K},$$

where the value of the module of integrals is determined by expression (2-2).

2-3. The Method of Spatial Inversion

The method of spatial inversion[1] is applicable during calculation of capacitance of solitary conductors in a homogeneous medium and is based on the use of geometrical conversion of the surface of these converters by their reflection reflective to the sphere.

Reflection (or inversion) relative to a certain sphere of radius R_0 (the radius of inversion) is geometrical conversion in which any point with spherical coordinates r; θ; ϕ becomes another (inverted) point with coordinates R_0^2/r; θ; ϕ. The locus of inverted points of a certain surface forms an inverted surface, which in a number of cases has a simpler form than the original. The determination of inverted surfaces is carried out either according to an assigned equation of initial surfaces (by replacement in it of the coordinate r with the coordinate $r_1 = R_0^2/r$) or by means of direct construction.

[1]Do not confuse with the method of plane inversion (reflection relative to a circle), which is a special case of the method of conformal conversions.

The latter is substantially facilitated by the fact that spatial inversion keeps constant the angles between any two intersecting lines.

In using the method of inversion one ought to have in view that a reflection relative to a sphere is reversible; therefore, any of the surfaces that correspond to each other can be considered both original and inverted.

A number of very simple examples of inversions are given in Table 2-2. The capacitance of a solitary conductor the surface of which is converted by means of reflection relative to a sphere can be determined according to formula [2-5]

$$C_0 = 4\pi\varepsilon \cdot R_0^2 \cdot V_0, \qquad (2-3)$$

where ε is the specific inductive capacitance of the medium; V_0 is so-called normalized potential in an inverted system.

To determine potential V_0 it is necessary:

1. Considering an inverted surface a surface of a grounded conductor ($V = 0$), to dispose in the center of inversion a point charge q_0 numerically equal to $-4\pi\varepsilon$.

2. Having calculated the electrostatic field of the shown point charge inside the grounded inverted surface,[1] to find the density of the charges induced on this surface from the relationship

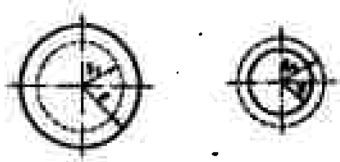
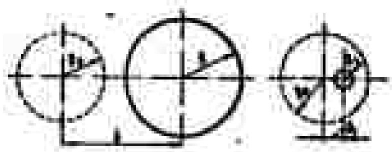
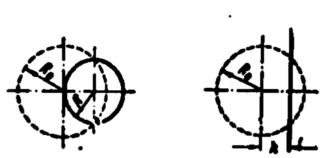
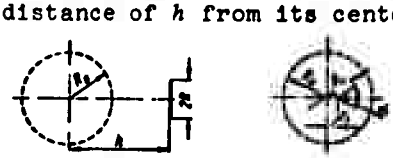
$$\sigma = -\varepsilon \frac{\partial u}{\partial n}\Big|_s,$$

where u is the potential of the found electrostatic field, and n is the internal normal to the inverted surface s.

3. Using formula (1-3), to find V_0 as the potential in the center of inversion being induced by induced charges.

[1]This problem coincides with the determination of the Green function for an inverted surface (see § 2-6).

Table 2-2. Spatial inversion of certain very simple surfaces.

Initial surface	Inverted surface
Sphere of radius R encompassing sphere of inversion and concentric with it 	A sphere of radius $R_1 = \frac{R_0^2}{R}$, encompassed by a sphere of inversion and concentric with it
A sphere of radius R the center of which is at a distance of $b (b > R + R_0)$ from the center of inversion 	A sphere of radius $R_1 = R_0^2 \frac{R}{b^2 - R^2}$, the center of which is at a distance of $b_1 = \frac{R_0^2 \cdot b}{b^2 - R^2}$ from the center of inversion
A sphere of radius R passing through the center of inversion 	A plane passing at a distance of $h = \frac{R_0^2}{2R}$ from the center of inversion
Circular disc of radius R perpendicular to the radius of a sphere of inversion at a distance of h from its center 	Part of the surface of a sphere of radius $R_1 = \frac{R_0^2}{h}$, cut by a right circular cone the peak of which is at point $\left(r_1 = \frac{R_0^2}{2h}\right)$ the angle at the peak is $\alpha = 2 \operatorname{arctg} \frac{R}{h}$

With the simple form of inverted surface the calculation of electrostatic field necessary for the determination of V_0 is simpler than the initial problem of calculation of capacitance. Specifically, when an inverted surface if formed by the intersection of several planes, the electrostatic field of a charge q_0 inside a grounded inverted surface can be calculated using the principle of mirror reflections, and the potential is found simply as the sum of the potentials of reflected charges.

Example 2-3. Using the method of spatial inversion, let us determine the capacitance of the same conductor as in Example 1-4 (two spheres intersecting at right angles).

Considering the meridional section of this conductor (Fig. 2-3a), let us dispose the center of inversion at one of the points of intersection of circumferences, for example, at point A, and let us take the radius of inversion equal to the diameter of one of these circumferences, for example, $R_0 = 2a$.

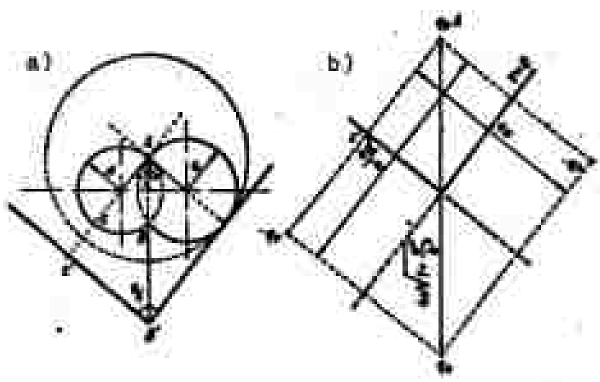


Fig. 2-3. The conductor formed by two spheres intersecting at right angles with radii a and b ($a > b$): a) is the section of the initial and inverted surface; b) is the system of mirror reflections of charge $q_0 = -4\pi\varepsilon$, located at the center of inversion relative to the inverted surface.

As is evident from Table 2-2 or from direct construction, the inverted surface in the given case is formed by two semi-infinite planes intersecting at an angle of $\pi/2$ at point B', which is the initial surface inverted for point B.

Placing further in the center of inversion a negative point charge $q_0 = -4\pi\varepsilon$, we take the potential of the inverted surface equal to zero. Constructing then the system of mirror reflections of this charge relative to the shown half-planes (Fig. 2-3b), we find that

$$V_0 = \frac{1}{4a} + \frac{b}{4a^2} - \frac{b}{4a\sqrt{a^2+b^2}}.$$

Substituting the value of V_0 into formula (2-3), we have

$$C_0 = 4\pi\varepsilon 4a^2 \frac{1}{4a}\left(1 + \frac{b}{a} - \frac{b}{\sqrt{a^2+b^2}}\right) = 4\pi\varepsilon a\left(1 + \frac{b}{a} - \frac{b}{\sqrt{a^2+b^2}}\right),$$

which coincides with the formula obtained in Example 1-4 by the method of equivalent charges.

2-4. The Method of Symmetrization of Conductors

The method of symmetrization is used in lower estimation of the values of the capacitance of solitary conductors in a uniform medium, and is based on utilization of geometrical conversion called symmetrization.

In general symmetrization can be defined as geometrical conversion of a spatial or planar body which permits reducing it to a form symmetrical relative to a certain plane or axis.

The symmetrization of the spatial body relative to a plane (so-called spatial symmetrization of Steiner) is carried out in the following manner.

Let there be a certain spatial body A and any plane P (plane of symmetrization). Drawing through every point of the surface of body A straight lines perpendicular to P, plotted on these straight lines symmetrically relative to P are segments equal to the total lengths of the chords being cut on the straight line being considered by body A. The locus of the ends of such segments forms the surface of a new body symmetric relative to plane P. Thus, for instance, a hemisphere of radius a with symmetrization relative to any plane parallel to its base becomes a condensed spheroid with axes $2a$ and a.

Completely analogously carried out is symmetrization of the flat body relatively to any straight line in its plane. One of the examples of such symmetrization is given in Fig. 2-4.

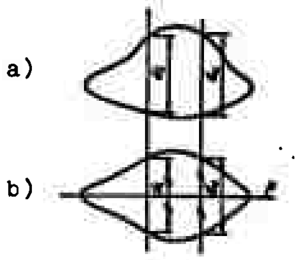


Fig. 2-4. The symmetrization of an arbitrary flat plate: a) initial; b) symmetrized plate.

Symmetrization of spatial body relative to an axis (Schwary symmetrization) consists in the following.

Given a certain spatial body A and any straight line L (axis of symmetrization). Drawing through the points of the surface A planes perpendicular to L, plotted at each of them is a circle with center at L equal in area to the section of body A by the plane being considered. The locus of such circumferences forms a surface of new, axisymmetric body. Thus, for instance, a cube with side a with this means of symmetrization relative to the axis parallel to one of its ribs becomes a right cylinder with altitude a and radius $a\sqrt{\pi}$.

Apart from this there are other, less widespread means of symmetrization.

Use of the method of symmetrization when evaluating capacitance is based on the fact that capacitance by any means of symmetrized solitary conductors never exceeds the capacitance of these conductors in their original form [1-3], i.e., $C_{0\text{сим}} < C_0$. Therefore, having determined in one way or another the capacitance of a symmetrized conductor, the lower boundary of capacitance of the conductor of initial form can be determined in the same way.

If after a single symmetrization the form of the conductor still remains so complex that the capacitance of the symmetrized conductor cannot be found, then symmetrization is carried out repeatedly, until the form of the symmetrized conductor is simple enough.[1] Thus, the method of symmetrization permits determining the boundary for the capacitance of a solitary conductor of a form no matter how complex.

Example 2-4. Let us find the lower boundary of the values of the capacitance of a flat plate in the form of a semicircle of radius a.

The capacitance of a conductor of the form considered cannot be accurately calculated by existing methods. Therefore, we deform the conductor in advance by means of planar symmetrization relative to a straight line parallel to the base of the semicircle. The form of a conductor thus symmetrized can be determined in the following manner.

Let us introduce rectangular coordinates (x, y) with origin at the center of the semicircle, having combined the Ox axis with its base. Then the connection between the coordinates of points on the contour of the initial (x, y) and symmetrized (x_1, y_1) plates will be determined by the equations

$$x_1 = x; \quad y_1 = \pm\frac{1}{2}y = \pm\frac{1}{2}\sqrt{a^2 - x^2} = \pm\frac{1}{2}\sqrt{a^2 - x_1^2}.$$

---

[1]With an infinite number of symmetrizations the surface of any conductor of spatial form is converted into a sphere.

Hence $\frac{x^2}{a^2}+\frac{y^2}{\left(\frac{a}{2}\right)^2}=1$, i.e., the symmetrized conductor has the form of a planar elliptical disc with axes $2a$ and a. The expression for the capacitance of an elliptical disc is known [see formula (4-3)]. Using it, we find that the capacitance of a plate in the form of a semicircle of radius a satisfies the inequality

$$C_0 > \frac{4\pi\varepsilon a}{K\left(\frac{\sqrt{3}}{2}\right)} = 8\varepsilon a \cdot 0{,}729.$$

2-5. The Method of Small Strains

The method of small strains is based on replacement of conductors of assigned (complex) form with other conductors of close but simpler form, which permits calculating electrostatic field or directly determining capacitance.

The strain of the surface of a conductor (as of any other body) is commonly called small, if the displacement of the points with respect to the normal to the surface of this conductor (h) is considerably less than its characteristic dimensions and is a continuous function of surface. Under these conditions the potential and strength of the electrostatic field of the electrodes, just as their capacitance, can be presented in the form of an exponential series of h, the zero term of which characterizes the electrostatic field (or, the capacitance, respectively) of an unstrained electrode. Limiting ourselves to this or that finite number of terms of this series, it is possible to obtain approximation expressions for an electrostatic field or the capacitance of the considered electrodes of complex form. The characteristics of utilization of the method of small strains depend on the number and form of conductors entering the system considered.

Let us consider for example, the problem of determination of the capacitance of a solitary conductor of "almost spherical" form

[2-6], i.e., of a conductor the surface of which S can be determined by an equation of the form

$$r = R_0[1 + \delta(\theta; \varphi)], \qquad (2\text{-}4)$$

where r; θ; ϕ are spherical coordinates of the points of the surface of the conductor; R_0 is the radius of a certain sphere close to the surface of the conductor considered ("reference" surface); $\delta(\theta; \phi)$ is the comparative amount of the normal displacement of the points of the surface of the conductor from the surface of the sphere:

$$\delta(\theta; \varphi) = \frac{h(\theta; \varphi)}{R_0}; \quad |\delta(\theta; \varphi)| < 1.$$

The quantity $\delta(\theta; \phi)$ can be presented in the form of $\delta(\theta; \phi) = \delta_0 \cdot F(\theta; \phi)$ where δ_0 is the comparative normal displacement at any fixed point of the surface of the conductor; $F(\theta; \phi)$ is the function which characterizes the distribution of normal displacements with respect to the surface of the conductor, and

$$F(\theta; \varphi) = F(\theta; \varphi + 2\pi).$$

Assigning the fixed quantity of the potential of the surface

$$V|_S = A, \qquad (2\text{-}5)$$

we will search for the potential of its electrostatic field in the form

$$V = \frac{D}{r} + \delta_0 V_1(r; \theta; \varphi), \qquad (2\text{-}6)$$

where D and $V_1(r; \theta; \phi)$ are the constant and function to be determined, respectively.[1]

Substituting this expression into the boundary condition (2-5), we find that

$$A = \frac{D}{R_0[1 + \delta_0 F(\theta; \varphi)]} + \delta_0 V_1 \{R_0[1 + \delta_0 F(\theta; \varphi)]; \theta; \varphi\}. \qquad (2\text{-}7)$$

[1]Let us note, that the given means can be generalized to the case when the "reference" surface selected is any (not only spherical) surface the form of which admits the solution of the external problem of Dirikhle for the Laplace equation.

Expanding the right side of this equation into an exponential series of small parameter δ_0 and retaining the terms of this series containing δ_0 to a power not above the first, we obtain

$$A = \frac{D}{R_0} + \delta_0\left[-F(\theta;\varphi)\frac{D}{R_0} + V_1(R_0;\theta;\varphi)\right]. \tag{2-8}$$

Inasmuch as the left side of equation (2-8) does not depend on θ and ϕ, the right side of it also should not depend upon these quantities. This is executed, especially, on condition that

$$-F(\theta;\varphi)\cdot\frac{D}{R_0} + V_1(R_0;\theta;\varphi) = 0. \tag{2-9}$$

From (2-8) and (2-9) it follows that

$$\begin{gathered} D = A\cdot R_0; \\ V_1(R_0;\theta;\varphi) = AF(\theta;\varphi). \end{gathered} \tag{2-10}$$

The last of the given equations can be considered the boundary condition for determination of a harmonic function $V_1(r;\ \theta;\ \phi)$ at any point of an area outside a sphere of radius R_0. Thereby the boundary surface of the problem was deformed into a sphere. Determination of $V_1(r;\ \theta;\ \phi)$ with such a form of boundary surface can be carried out with any assigned type of function $F(\theta;\ \phi)$ by the method of distribution of variables (see, for example [2-7]).

Substituting the expression found for $V_1(r;\ \theta;\ \phi)$ in (2-6), it is possible to obtain the approximation formula for the potential of the electrostatic field of the conductor considered, and then, using the general expressions (V-18), (V-1), approximately to find its capacitance. The approximation formula obtained by such a method for the determination of the capacitance of a conductor of "almost spherical" form has the form

$$C_0 \approx 4\pi\varepsilon R_0(1 + \delta_0 M), \tag{2-11}$$

where M is the coefficient with the first term of the expansion of function $V_1(r;\ \theta;\ \phi)$ into an exponential series of $1/r$. In general this coefficient is determined [2-7] by the formula

$$M = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} F(\theta;\ \varphi)\sin\theta\, d\theta. \qquad (2\text{-}12)$$

If the conductor is axisymmetric, then

$$M = \frac{1}{2}\int_0^{\pi} F(\theta)\sin\theta\, d\theta. \qquad (2\text{-}12a)$$

If more accurate formulas must be obtained, the equation of the surface of the conductor can be assigned in the form

$$r = R_0\left[1 + \sum_{k=1}^{n} \varepsilon_0^k F_k(\theta;\ \varphi)\right],$$

where

$$\left|\sum_1^n \varepsilon_0^k F_k(\theta;\ \varphi)\right| < 1, \quad F_k(\theta;\ \varphi) = F_k(\theta;\ \varphi + 2\pi).$$

In this case, finding the potential of the electrostatic field in the form

$$V(r;\ \theta;\ \varphi) = \frac{D}{r}\sum_{k=1}^{n} \varepsilon_0^k V_k(r;\ \theta;\ \varphi)$$

and using the given method, it is possible to obtain the following approximation formula for the capacitance of a conductor of "almost spherical" form

$$C_0 = 4\pi\varepsilon R_0\left(1 + \varepsilon_0 M_1 + \varepsilon_0^2 M_2 + \dots + \varepsilon_0^n M_n\right).$$

where M_n is the coefficient at the k-th term of the expansion of function $V(r;\ 0;\ \phi)$ into an exponential series of $1/r$.

Example 2-5. Let us determine the capacitance of a solitary conductor of axisymmetric form the section of which is described by the equation

$$r = R_0[1 + \varepsilon_0(\cos^2\theta - \cos\theta)], \quad |\varepsilon_0| < 1.$$

With the assigned type of equation of the surface of a conductor its capacitance can be determined only as a first approximation: $F(\theta) = \cos^2\theta - \cos\theta$. Substituting this expression $F(\theta)$ into formula (2-12a) and integrating, we find that $M = 1/3$. Then in accordance with (2-11)

$$C_0 \approx 4\pi\varepsilon R_0\left(1 + \frac{\varepsilon_0}{3}\right).$$

In using the method of small strains to calculate capacitance in a system of two conductors, it is possible to make use of the fact that the relative change in capacitance between any two conductors with any strain of them is expressed [2-8] by the formula[1]

$$\frac{\Delta C}{C} = \frac{\int\limits_{\Delta v} \vec{E}_1\vec{E}_2\,dv}{\int\limits_{(v)} |\vec{E}_1|^2\,dv}, \tag{2-13}$$

where ΔC is the absolute value of the change in capacitance; C is the initial capacitance between conductors; $\vec{E}_1$ and $\vec{E}_2$ are the strength of an electrostatic field before and after strain, respectively; v is the volume of space in which the electrostatic field being considered exists (if one conductor wholly encompasses another, then v is the volume limited by the surfaces of the electrodes); Δv is the change in v as a result of the strain of the electrodes.

If the strain of the conductors is small, then $\vec{E}_1 \approx \vec{E}_2 \approx \vec{E}$ and

$$\frac{\Delta C}{C} \approx \frac{\int\limits_{S} E^2 h\,dS}{\int\limits_{(v)} E^2\,dv}, \tag{2-14}$$

where S is the initial surface of the conductors; h is the quantity of the normal mixing of the points of an initial surface during strain.

[1]The given formula can be generalized also to the case when the media filling the space between electrodes is heterogeneous. In this instance the specific inductive capacitance of the medium should be introduced by a factor into the subintegral functions of the numerator and denominator.

Formula (2-14) allows approximately calculating the capacitance of any little strained system of two conductors, if only the electrostatic field of this system in its initial state is known or can be found.[1]

Example 2-6. Using the method of small strains, let us find an approximation expression for capacitance (per unit of length) between two noncoaxial cylinders, one of which encompasses the other (Fig. 2-5).

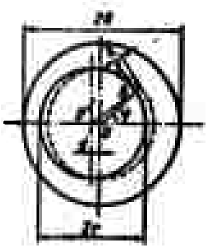


Fig. 2-5. System of two infinitely long cylinders with parallel axes displaced a distance of d from one another.

If $d < r < R$, then the system being considered can be presented as the result of the small strain of a system of two coaxial cylinders with radii R and r. Introducing polar coordinates ρ, ϕ with center at point 0, we find that for any value of ϕ the amount of the normal displacement of the surface of an interior cylinder is

$$h = r - OB = r\left[1 - \sqrt{1 - \left(\frac{d}{r}\right)^2 \sin^2\varphi} + \frac{d}{r}\cos\varphi\right].$$

The strength of the electrostatic field in the space between the coaxial cylinders is determined by the known formula

$$E = -A\frac{Rr}{R-r}\cdot\frac{1}{\rho^2},$$

where A is the difference in potentials between cylinders.

---

[1]It is understandable, that any of the surfaces considered can be taken as the initial and strained surfaces.

Taking into account that in the case being considered $dS = r \cdot d\phi$; $dv = \rho d\rho d\phi$ and substituting the obtained values for h and E in formula (2-14) we find

$$\frac{\Delta C_l}{C_l} \approx \frac{A^2\left(\frac{R}{r}\right)^2 \cdot \frac{r^2}{(R-r)^2}\int_0^{2\pi}\left[1-\sqrt{1-\left(\frac{d}{r}\right)^2\sin^2\varphi}+\frac{d}{r}\cos\varphi\right]d\varphi}{A^2\frac{R^2r^2}{(R-r)^2}\int_0^{2\pi}d\varphi\int_r^R\frac{d\rho}{\rho^3}} = \frac{2}{\pi}\cdot\frac{R^2}{R^2-r^2}\left[\pi-2E\left(\frac{d}{r}\right)\right],$$

where $E(d/r)$ is the complete elliptical integral of kind II.

From the general features of capacitance shown in § V-2, it follows that the increase in capacitance induced by the strain considered is positive, on volume $C_{l1} = C_l + \Delta C$, where C_l and C_{l1} are the capacitances between coaxial and noncoaxial cylinders, respectively. Using the known expression for C_l, we obtain the following approximation formula for capacitance (per unit of length) between two noncoaxial infinitely long cylinders:

$$C_{l1} = \frac{2\pi\varepsilon}{\ln\frac{R}{r}}\left\{1+\frac{2}{\pi}\cdot\frac{1}{1-\left(\frac{r}{R}\right)^2}\left[\pi-2E\left(\frac{d}{r}\right)\right]\right\}. \qquad (2\text{-}15)$$

To evaluate the inaccuracy of this formula let us compare it with the known accurate expression for capacitance between noncoaxial cylinders, which has (see § 5-4) the form

$$C_{l1} = \frac{2\pi\varepsilon}{\operatorname{Arch}\frac{R^2+r^2-d^2}{2Rr}}. \qquad (2\text{-}16)$$

If we obtain, especially, $r/R = 0.5$, $d/r = 0.3$, then using formula (2-15) $(C_{l1}/2\pi\varepsilon) \approx 1.53$ while the accurate value of this quantity calculated from formula (2-16) is equal to $(C_l/2\pi\varepsilon) = 1.48$. Thus, the comparative inaccuracy of the calculation from formula (2-15) in a given case is 3.4%.

2-6. Methods of Auxiliary Functions

Methods of auxiliary functions consist in simplification of problems of calculation of electrostatic field (and respectively of capacitance) of conductors with their constant geometric form.

These methods are: a) the method of function of source (the method of Green); b) the method of direct determination of field intensity; c) and the method of consecutive approximations.

The first of the shown methods allows reducing uniform[1] boundary conditions assigned on any surface to zero conditions; the second method makes it possible to replace compound boundary conditions on the surface of some plane-parallel systems with uniform boundary conditions; the third of the enumerated methods allows simplifying compound boundary conditions on the surface of some typical systems.

The method of function of source is based on use of the formula:

$$V_N = \frac{1}{4\pi} \oint V \cdot \frac{\partial G}{\partial n} dS, \qquad (2\text{-}17)$$

where V_N is the potential at a certain point N inside the closed surface S; n is an interior normal to this surface, and G is a Green function of kind I determined in the following manner:

a) at any point inside surface S function $G = \frac{1}{r} + f$, where r is the distance from point N to a random point lying inside surface S or on this surface itself; f is a random harmonic function (hence it follows that function G is also everywhere harmonic, except point $r = 0$, where it has the feature of type $1/r$);

[1]Let us recall that uniform boundary conditions are boundary conditions in which the values of one and the same function are assigned on the entire boundary, and compound boundary conditions are boundary conditions in which the values of various functions are assigned in individual sections of the boundary surface (for example, in one section of the boundary surface potential is assigned, and in another its normal derivative is assigned). The solution of the boundary problems under compound boundary conditions, as a rule, is considerably more complex than under uniform boundary conditions.

b) at all points of surface S function $G = 0$.

Formula (2-17) makes it possible to calculate the electrostatic field inside surface S, if the values of the potentials on this surface are assigned and the Green function G is found.

From the given determination of the Green function, it is evident that it coincides with the potential of the electrostatic field of a point charge numerically equal to $4\pi\varepsilon$ and in a volume bounded by a grounded metal surface S, i.e., inside a surface with zero boundary conditions. Determination of this auxiliary function in a number of cases is simpler than solution of an initial problem with nonzero boundary conditions. Therefore, the method considered is widely used both in the calculation of electrostatic fields, and in the direct calculation of the capacitance of a number of conductors.[1]

The method of direct determination of field strength is used to calculate a certain class of flat electrostatic fields with compound boundary conditions. This method is based on the preliminary determination of auxiliary function $\gamma(x, y)$ expressing the size of the angle formed by the vector of electrostatic field strength at any point of the area considered with one of the axes of the Cartesian system of coordinates. Function $\gamma(x, y)$ is harmonic [2-9]: it satisfies the two-dimensional Laplace equation. Boundary conditions for this function can be established from conditions of orthogonality of power and equipotential lines of field and, as is seen from the illustration given in Fig. 2-6, can be uniform even when potential is assigned in one part of the boundary surface, and its normal derivative is assigned on the other.

In connection with this the problem of calculation of function $\gamma(x, y)$ proves to be considerably simpler than the initial problem of calculation of potential under compound boundary conditions.

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[1]Thus, calculation of capacitance by the method of spatial inversion actually boils down to computation of Green function at the center of inversion (compare with § 2-3).

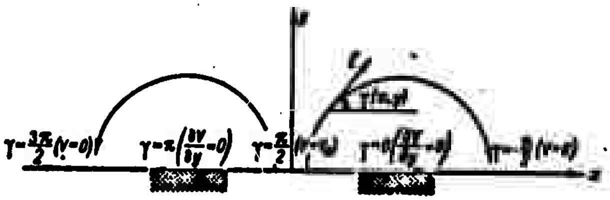


Fig. 2-6. Boundary conditions for a potential and an angle $\gamma(x, y)$ in the case of a system of plates lying in one plane.

Having found this auxiliary function, it is possible then directly (passing the stage of determination of potential) to find the modulus of the strength of the electrostatic field of the system of conductors being considered from the relationships:

$$\left.\begin{aligned} \frac{\partial\gamma}{\partial x} &= \frac{\partial(\ln|E|)}{\partial y}; \\ \frac{\partial\gamma}{\partial y} &= -\frac{\partial(\ln|E|)}{\partial x}. \end{aligned}\right\} \tag{2-18}$$

From (2-18) it is possible, especially, directly to determine the modulus of the strength of a planar electrostatic field created by a system of any number of charged infinitely long plates lying in one plane.

In the points of this plane ($y = 0$) the modulus of the strength of the electrostatic field of the system considered is determined by the formula

$$|E|_{y=0} = \frac{B\sqrt{\prod\limits_{i=1}^{m}\left[(x-x_{0i})^2+y_{0i}^2\right]}}{\sqrt{\prod\limits_{k=1}^{n}|x-a_k|}}, \tag{2-19}$$

where x_{0i}; y_{0i} are coordinates of the special points of the field, i.e., of points in which $E = 0$; m is the number of special points; a_k is the coordinate of the edges of plates; n is the number of plates; B is a constant determined (along with the constants x_{0i}

and y_{0i}) from assigned charges or potentials of conductors. In particular for an electroneutral system consisting of two plates (Fig. 2-7),

$$|\mathbf{E}|_{y=0}=\frac{B}{\sqrt{|(x-a_1)(x-a_2)(x-a_3)(x-a_4)|}}. \tag{2-20}$$

Using formula (2-20), it is possible to find that the difference in potentials between the plates is

$$V_1-V_2=\int_{a_2}^{a_3}|\mathbf{E}|_{y=0}dx=B\int_{a_2}^{a_3}\frac{dx}{\sqrt{(x-a_1)(x-a_2)(a_3-x)(a_4-x)}}, \tag{2-21}$$

and the charge per unit of length of each plate is

$$\tau=2\pi\int_{a_1}^{a_2}|\mathbf{E}|_{y=0}dx=2\pi B\int_{a_1}^{a_2}\frac{dx}{\sqrt{(x-a_1)(a_2-x)(a_3-x)(a_4-x)}}. \tag{2-22}$$

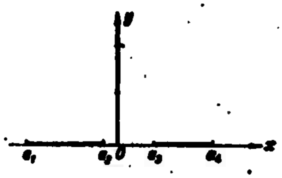


Fig. 2-7. Two infinitely long plates lying in one plane.

Calculating the integrals entering expressions (2-21) and (2-22), we find that the capacitance (per unit of length) between the plates considered is

$$C_l=\frac{\tau}{V_1-V_2}=2\pi\frac{\mathrm{K}'(k)}{\mathrm{K}(k)}, \tag{2-23}$$

where $K(k)$ is the complete elliptical integral of kind I with modulus

$$k=\sqrt{\frac{(a_3-a_2)(a_4-a_1)}{(a_4-a_2)(a_3-a_1)}};\qquad \mathrm{K}'(k)=\mathrm{K}\left(\sqrt{1-k^2}\right). \tag{2-24}$$

The method of direct determination of field strength is especially effective in conjunction with the above (§ 2-2) method of conformal conversions. Thus, because of invariance of capacitance during conformal conversion, formula (2-23) determines the capacitance between any two infinitely long conductors, which as a result of this or that conformal conversion can be reduced to the form shown in Fig. 2-7. In this instance the coordinates of the edges of plates are determined from assigned parameters of the initial system with the aid of an appropriate reflecting function (see Examples 2-1 and 2-2).

Example 2-7. Let us determine the capacitance per unit of length between the conductors presented in Fig. 2-8.

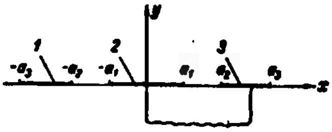


Fig. 2-8. Three infinitely long plates lying in one plane; plates 2 and 3 are interconnected.

From the type of system considered it follows that in the electrical field induced by it, only one special point can exist which is located on plane $y = 0$; $a_1 < x_0 < a_2$. Therefore, assuming in formula (2-19) $y_{0i} = 0$, and $m = 1$, we obtain that the modulus of the strength of the electrostatic field on plane $y = 0$ is

$$|E|_{y=0} = B\frac{|x-x_0|}{\sqrt{|(x^2-a_1^2)(x^2-a_2^2)(x^2-a_3^2)|}}.$$

Taking into account that the difference in potentials between electrodes 2 and 3 is equal to zero, we find

$$\int_{a_1}^{a_2}\frac{(x-x_0)\,dx}{\sqrt{(x^2-a_1^2)(a_2^2-x^2)(a_3^2-x^2)}} = 0.$$

whence

$$x_0 = \frac{\int\limits_{a_1}^{a_2} \frac{x \cdot dx}{\sqrt{(x^2-a_1^2)(a_2^2-x^2)(a_3^2-x^2)}}}{\int\limits_{a_1}^{a_2} \frac{dx}{\sqrt{(x^2-a_1^2)(a_2^2-x^2)(a_3^2-x^2)}}} = a_2 \frac{K(k)}{K(k_1)},$$

i.e.,

$$k = \sqrt{\frac{a_2^2-a_1^2}{a_3^2-a_1^2}}, \quad k_1 = \frac{a_3}{a_2} \sqrt{\frac{a_2^2-a_1^2}{a_3^2-a_1^2}}.$$

Using the expression found for field strength, we find, that the charge per unit of length of every conductor is

$$\tau = 2B\varepsilon \int\limits_{-a_2}^{-a_1} \frac{(x-x_0)\,dx}{\sqrt{(x^2-a_1^2)(x^2-a_2^2)(a_3^2-x^2)}} =$$

$$= 2B\varepsilon \frac{1}{\sqrt{a_3^2-a_1^2}} \left[K(k') + K(k) \frac{K(k_1')}{K(k_1)} \right],$$

where $k' = \sqrt{1-k^2}$; $k_1' = \sqrt{1-k_1^2}$.

The difference in potentials between the conductors considered is

$$V_1 - V_2 = V_1 - V_3 = B \int\limits_{-a_2}^{-a_1} \frac{(x_0+x)\,dx}{\sqrt{(x^2-a_1^2)(a_2^2-x^2)(a_3^2-x^2)}} = \frac{2BK(k)}{\sqrt{a_3^2-a_1^2}}.$$

Hence we obtain the following expression for capacitance per unit of length of conductors considered:

$$C_l = \frac{\tau}{V_1 - V_2} = \varepsilon \left[\frac{K(k')}{K(k)} + \frac{K(k_1')}{K(k_1)} \right].$$

The method of successive approximations of boundary conditions allows reducing the solution of certain problems on calculation of an electrostatic field with complex boundary conditions to the solution of a succession of simpler problems. No general method of creating this succession exists at the present time: selection of the initial approximation and method of construction of successive approximations depend upon the type of this or that concrete system. Let us limit ourselves therefore to illustration of the method of successive approximations in the example of the calculation of the capacitance of a flat circular ring [2-10].

The problem of determination of the capacitance of a flat circular ring under a strict posing requires the calculation of electrostatic field under compound boundary conditions, and the plane on which boundary conditions of various type are assigned has 2 boundaries[1] ($r = a$ and $r = b$). Solution of such problems is very difficult, while the procedure for solving compound problems with one circular boundary of boundary conditions is developed to a considerably greater extent; therefore, we replace solution of the initial problem with solution of a succession of compound problems with one boundary of boundary conditions.

In the first approximation we replace the ring considered (Fig. 2-9a) with a circular disc of radius $r = a$ (Fig. 2-9b). Using the fact that the capacitance of any conductor is greater than the capacitance of any part of it, we arrive at the inequality

$$0 < C_{\text{кольца}} < C_{\text{диска}}.$$

which gives a rough estimate of the upper and lower limits of the capacitance of the ring.

To get a more accurate estimate we go to the second approximation, which we construct in the following manner:

a) having assigned the potential of the disc (A) and having calculated the field of the system in Fig. 2-9b, let us find the

[1]Subsequently we will call such lines boundaries of boundary conditions.

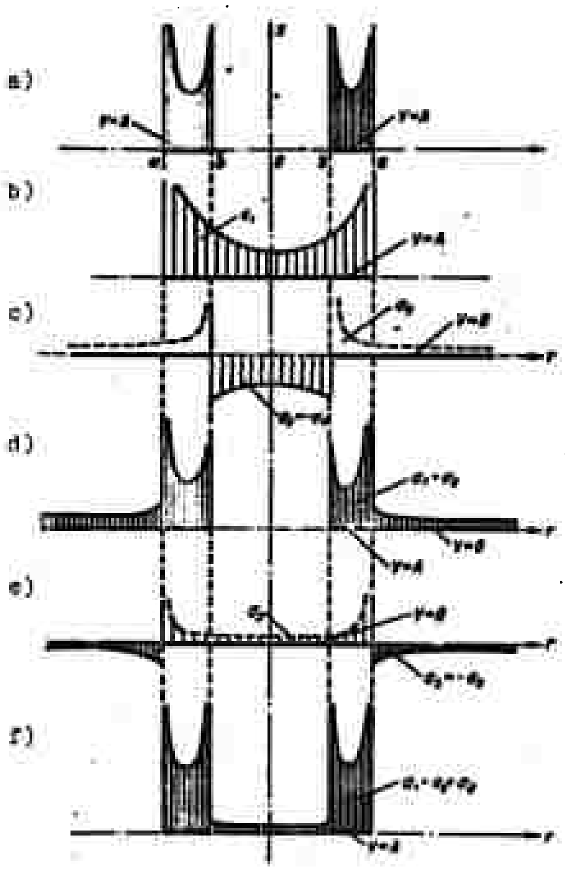


Fig. 2-9. To the creation of a system of successive approaches for calculation of capacitance of a plane circular ring: a) initial system - a ring with radii a and b; b) 1st approximation - circular disc with radius a; c) 1st auxiliary system; d) 2nd approximation; e) 2nd auxiliary system; f) 3rd approximation.

charge on the surface of the disc $\sigma_1(r)$ and the potential in its plane $V_1(r)$ at $r > a$;

b) let us build an auxiliary system (Fig. 2-9c) in the form of an infinitely extended plane, in part of which $r \leq b$ charge distribution is assigned $\sigma_2(r) = -\sigma_1(r)$; and in the remaining part of which potential is equal to zero;

c) having calculated the field of the system in Fig. 2-9c, let us find the charge density and distribution of potential on the boundary surface;

d) superimposing the systems depicted in Fig. 2-9b and 2-9c, we obtain the system boundary conditions for which is shown in Fig. 2-9d.

The built system differs from the initial system in that in it the ring $z = 0$, $a < r < b$ is in the field of positive charge distributed with density $\sigma_2(r)$, but retains the same potential (A) as in the initial system. Hence it follows that the complete charge of the ring in the system of Fig. 2-9d is less than the true charge, and

$$\frac{Q_1 + Q_2}{A} < C_{[illegible]} < C_{[illegible]}$$

where Q_1 is the complete charge of the surface $a < r < b$ in the system in Fig. 2-9b; Q_2 is the complete charge of the surface $a < r < b$ in the system in Fig. 2-9c.

The method of construction of the third approach is analogous to the one considered: it is based on solution of the auxiliary problem of finding the distribution of charge induced on a grounded flat disc of radius a with negative charge distributed on part of plane $r > a$ with density $\sigma_3(r) = -\sigma_2(r)$ (Fig. 2-9e), and on the subsequent superposition of the systems shown in Fig. 2-9d and 2-9e.

In a new auxiliary system thus built (Fig. 2-9f) a ring with potential A is located in the field of positive charge distributed over the surface $r < b$ with a density of $\sigma_3(r)$. The complete charge of the ring in this system is greater than in the second approximation, but as before it is less than the true charge; therefore, we obtain a more accurate inequality for the capacitance of the ring in the form

$$\frac{Q_1 + Q_2 + Q_3}{A} < C_{\text{кольца}} < C_{\text{диска}}.$$

where Q_3 is the complete charge on the surface $b < r < a$ in the system of Fig. 2-9e.

All the subsequent approximations of even order are built in the same way as the second, and those of odd order are built in the same way as the third approximation. As a result we arrive at a convergent series of boundary conditions. In every subsequent approximation the complete charge of the ring is increased, remaining less than the true value, while the potential of the ring remains constant; therefore, the capacitance of the ring is determined with ever greater accuracy.

Detailed computations of the capacitance of the ring by the method of successive approximations are given in [2-10].

PART TWO

CALCULATION FORMULAS, TABLES AND GRAPHS

1. The material of this part is divided into three chapters. In Chapter 3 the data are given on the capacitance of wires, in Chapter 4 data on the capacitance of flat plates, and in Chapter 5 data on the capacitance of wires in the form of open and closed shells.

In all these chapters it is assumed that the medium surrounding the conductors is either uniform in infinite, or is bounded by one flat impenetrable boundary. In the latter case capacitance is calculated by means of analysis of the auxiliary systems of conductors located in an infinite uniform medium and obtained by means of a single mirror reflection of the initial system (see § V-2).

2. At the beginning of every chapter general remarks are given, in which the geometric forms of the conductors considered are briefly scanned, and the general characteristic of the data given on their capacitance is given.

3. The material of each chapter is arranged in increasing order of the number of conductors that form this or that system.

One ought to take account of the fact that the system formed by the union of several conductors is considered one conductor; in this case the effect of the connecting conductors on the capacitance of a system is assumed to be negligible.

4. For the majority of the systems of conductors considered

both accurate, and approximation formulas are given with indication of the limits of their applicability and accuracy. The latter is characterized by relative error

$$\delta = \frac{C_{\text{точн}} - C_{\text{прибл}}}{C_{\text{точн}}} 100\%,$$

where $C_{\text{точн}}$ and $C_{\text{прибл}}$ are the accurate and approximation values of capacitance, respectively.

5. References to operations used in obtaining individual formulas, as a rule, are not given. However, for some typical systems the basic results obtained by various authors are briefly compared.

C H A P T E R 3

CAPACITANCE OF WIRES

3-1. General Remarks

1. In this chapter formulas are given for calculation of the capacitance of wires, i.e., of conductors the form of which satisfies the conditions shown in § V-2. In all cases when nothing is said to the contrary, it is assumed that the form of the section of wire is circular.

2. In all the formulas below the distance between wires is understood to be the distance between their axis.

3. All formulas given in this chapter are approximation formulas, and a majority of them are obtained by the method of mean potentials.

4. The limits of applicability of the given approximation formulas depend upon the relationship of the sizes of a wire and of the form of its axis; in most cases accuracy of formulas is evaluated by solving numerical examples.

3-2. The Capacitance of Solitary Conductors Formed by Wires Arranged in Infinite Space

1. *The rectilinear wire of finite length* (Fig. 3-1).

$$C_0 \simeq \frac{2\pi\varepsilon l}{\ln\frac{l}{a} - 0{,}3069 - \frac{0{,}1776}{\ln\frac{l}{a}} - \frac{0{,}8519}{\left(\ln\frac{l}{a}\right)^2}} \qquad (3\text{-}1)$$

$[|\delta| < 1{,}0\%$ when $l/a > 10]$.

Fig. 3-1. A rectilinear wire of finite length.

When greater inaccuracy is tolerable, the following less accurate formulas can also be used:

$$C_0 \simeq \frac{2\pi\varepsilon l}{\operatorname{Arsh}\frac{l}{a} + \frac{a}{l} - \sqrt{1 + \frac{a^2}{l^2}}}; \qquad (3\text{-}2)$$

$$C_0 \simeq \frac{2\pi\varepsilon l}{\ln\frac{2l}{a} - 1}. \qquad (3\text{-}3)$$

Example 3-1. To determine the capacitance of a rectilinear wire in air l = 0.5 m long and a = 0.025 m in radius.

To determine capacitance let us make use of formulas (3-1)-(3-3). Taking into account that for air $\varepsilon = \varepsilon_0 = \frac{1}{36\pi}10^{-9}$ F/m, and using the formula (3-1), we find

$$C_0 = \frac{2\pi\varepsilon_0 \cdot 0{,}5}{\ln\frac{0{,}5}{0{,}025} - 0{,}307 - \frac{0{,}177}{\ln\frac{0{,}5}{0{,}025}} - \frac{0{,}552}{\left[\ln\frac{0{,}5}{0{,}025}\right]^2}} =$$

$$= \frac{2\pi}{36\pi}10^{-9}\frac{0{,}5}{2{,}995 - 0{,}307 - 0{,}059 - 0{,}0614} = \frac{10^{-9}}{18}\cdot\frac{0{,}5}{2{,}568} =$$

$$= 10{,}8\cdot 10^{-12}\text{ F} = 10{,}8 \text{ pF};$$

At calculation from formula (3-2) analogously we obtain

$$C_0 = \frac{2\pi}{36\pi}\cdot 10^{-9}\frac{0{,}5}{\operatorname{Arsh}\frac{0{,}5}{0{,}025} + \frac{0{,}025}{0{,}5} - \sqrt{1 + \left(\frac{0{,}025}{0{,}5}\right)^2}} =$$

$$= \frac{10^{-9}}{18}\cdot\frac{0{,}5}{2{,}737} = 10{,}15\cdot 10^{-12}\text{ F} = 10{,}15 \text{ pF}.$$

Finally, using formula (3-3) we find

$$C_0 = \frac{2\pi}{36\pi}\cdot 10^{-9}\frac{0{,}5}{\ln 40 - 1} = 10{,}3\cdot 10^{-12}\text{ F} = 10{,}3 \text{ pF}.$$

With respect to the result obtained using formula (3-1), the inaccuracy of calculation from formulas (3-2) and (3-3) in the case considered is respectively 6 and 4.6%.

2. *A wire, bent along the arc of a circumference* (Fig. 3-2).

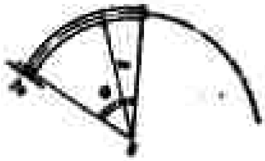


Fig. 3-2. Wire bent along the arc of a circumference.

$$C_0 \simeq \frac{2\pi\varepsilon l}{\ln\frac{8R}{a} - \frac{4}{\theta}I} \qquad (3\text{-}4)$$

$$|\delta| < 2{,}0\% \text{ when } R/a > 10]$$

where θ is the central angle of the arc (in radians); $l = \theta R$ is the length of the wire; I is a parameter the numerical values of which are given in Table 3-1.

Table 3-1. Value of parameter I, which enters formula (3-4).

θ, deg	I	θ, deg	θ, deg	I	θ, deg
0	0,0000	360	90	0,7529	270
5	0,1052	355	95	0,7715	265
10	0,1803	350	100	0,7887	260
15	0,2439	345	105	0,8047	255
20	0,3000	340	110	0,8196	250
25	0,3506	335	115	0,8332	245
30	0,3968	330	120	0,8456	240
35	0,4393	325	125	0,8572	235
40	0,4786	320	130	0,8676	230
45	0,5151	315	135	0,8774	225
50	0,5492	310	140	0,8852	220
55	0,5809	305	145	0,8925	215
60	0,6107	300	150	0,8968	210
65	0,6385	295	155	0,9041	205
70	0,6645	290	160	0,9083	200
75	0,6889	285	165	0,9117	195
80	0,7117	280	170	0,9141	190
85	0,7330	275	175	0,9155	185
90	0,7529	270	180	0,9160	180

3. *A wire in the form of a circular ring* (Fig. 3-3).

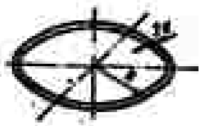


Fig. 3-3. A wire in the form of a circular ring.

$$C_0 \approx \frac{4\pi^2\varepsilon R}{\ln\frac{8R}{a}} \qquad (3\text{-}5)$$

$[|\delta| < 2{,}0\%$ when $R/a > 10]$.

Example 3-2. To determine the capacitance of a circular ring and semiring in air and having a radius R = 0.1 m, the diameter of a section is 2 a = 0.01 m $= 0{,}01$ m $\left(\varepsilon = \varepsilon_0 = \frac{1}{36\pi}\cdot 10^{-9}\right.$ F/m).

Using formula (3-5) for the capacitance of the ring we obtain

$$C_0 \approx \frac{4\pi^2\cdot 0{,}1\cdot 10^{-9}}{36\pi \ln\frac{8\cdot 0{,}1}{0{,}005}} = 6{,}87\cdot 10^{-12}\ \text{F} = 6{,}87\ \text{pF}.$$

To determine the capacitance of a semiring we preliminarily find from Table 3-1 the value of parameter I. When $\theta = \pi$ parameter $I = 0.916$. Thus, for the capacitance of the semiring considered we have

$$C_0 \approx \frac{2\pi\cdot 10^{-9}}{36\pi}\cdot\frac{0{,}1}{\ln\frac{8\cdot 0{,}1}{0{,}005} - \frac{4}{\pi}0{,}916} = 3{,}74\cdot 10^{-12}\ \text{F} = 3{,}74\ \text{pF}.$$

The ratio of the found values of the capacitance a semiring and a ring is 0.544, i.e., the capacitance of a semiring is somewhat greater than half of the capacitance of a ring of the same radius. With decrease in R/a this difference is increased.

4. *Two interconnected parallel rectilinear wires of finite length* (Fig. 3-4).

Let us examine several cases:

a) $b = 0$ (Fig. 3-5), then

$$C_0 = \frac{\alpha_{22} + \alpha_{11} - 2\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}^2}, \tag{3-6}$$

where

$$\alpha_{11} \simeq \frac{1}{2\pi\varepsilon l_1}\left\{\mathrm{Arsh}\,\frac{l_1}{a_1} + \frac{a_1}{l_1} - \sqrt{\left(\frac{a_1}{l_1}\right)^2 + 1}\right\},$$

$$\alpha_{22} \simeq \frac{1}{2\pi\varepsilon l_2}\left\{\mathrm{Arsh}\,\frac{l_2}{a_2} + \frac{a_2}{l_2} - \sqrt{\left(\frac{a_2}{l_2}\right)^2 + 1}\right\},$$

$$\alpha_{12} \simeq \frac{1}{4\pi\varepsilon l_2}\left[\mathrm{Arsh}\,\frac{l_1}{d} + \frac{l_2}{l_1}\,\mathrm{Arsh}\,\frac{l_2}{d} - \left(\frac{l_2}{l_1} - 1\right)\mathrm{Arsh}\,\frac{l_2 - l_1}{d} + \frac{d}{l_1} + \right.$$
$$\left. + \sqrt{\left(\frac{d}{l_1}\right)^2 + \left(\frac{l_2}{l_1} - 1\right)^2} - \sqrt{\left(\frac{d}{l_1}\right)^2 + 1} - \sqrt{\left(\frac{d}{l_1}\right)^2 + \left(\frac{l_2}{l_1}\right)^2}\right].$$

b) $b = 0;\ l_1 = l_2 = l;\ a_1 = a_2 = a$

$$C_0 \simeq \frac{4\pi\varepsilon l}{\ln\left[\frac{l}{a} + \sqrt{1 + \left(\frac{l}{a}\right)^2}\right] + \ln\left[\frac{l}{d} + \sqrt{1 + \left(\frac{l}{d}\right)^2}\right] + N}, \tag{3-7}$$

where $N = \frac{a}{l} + \frac{d}{l} - \sqrt{1 + \left(\frac{a}{l}\right)^2} - \sqrt{1 + \left(\frac{d}{l}\right)^2}.$

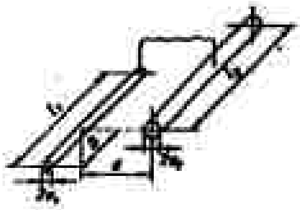


Fig. 3-4. The general case of a solitary conductor, formed by the union of two parallel straight wires.

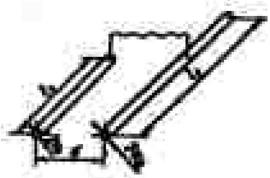


Fig. 3-5. The conductor shown in Fig. 3-4, when $b = 0$.

When $2d/l \gg 1$

$$C_0 \simeq \frac{4\pi\varepsilon l}{\ln\frac{l}{a} - \frac{l}{2d} - 0{,}307}. \tag{3-8}$$

When $2d/l \ll 1$

$$C_0 \simeq \frac{4\pi\varepsilon l}{\ln\frac{l}{a} + \ln\frac{l}{d} - 0{,}614}. \qquad (3\text{-}9)$$

c) $b = l_1 + 2h;\ a_1 = a_2 = a$ (Fig. 3-6)

$$C \simeq 4\pi\varepsilon\left\{\frac{l_1^2}{2l_1\left\{\ln\frac{l_1}{a} - 0{,}307\right\} + F} + \frac{l_2^2}{2l_2\left\{\ln\frac{l_2}{a} - 0{,}307\right\} + F}\right\}. \qquad (3\text{-}10)$$

where

$$F = 2h\ln\frac{\left[2h + l_1 + l_2 + \sqrt{d^2 + (2h + l_1 + l_2)}\right]\left[2h + \sqrt{d^2 + 4h^2}\right]}{\left[2h + l_1 + \sqrt{d^2 + (2h + l_1)^2}\right]\left[2h + l_2 + \sqrt{d^2 + (2h + l_2)^2}\right]} +$$
$$+ l_1\ln\frac{2h + l_2 + l_1 + \sqrt{d^2 + (2h + l_1 + l_2)^2}}{2h + l_1 + \sqrt{d^2 + (2h + l_1)^2}} +$$
$$+ l_2\ln\frac{2h + l_1 + l_2 + \sqrt{d^2 + (2h + l_2 + l_1)^2}}{2h + l_2 + \sqrt{d^2 + (2h + l_2)^2}} +$$
$$+ \sqrt{d^2 + (2h + l_1)^2} + \sqrt{d^2 + (2h + l_2)^2} -$$
$$- \sqrt{d^2 + (2h + l_1 + l_2)^2} - \sqrt{d^2 + 4h^2};$$

d) $d = 0;\ l_1 = l_2 = l;\ a_1 = a_2 = a;\ b = l + 2h$ (Fig. 3-7).

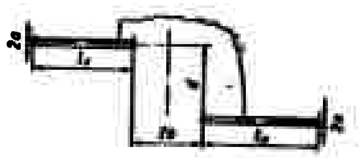


Fig. 3-6. The conductor shown in Fig. 3-4, when $b = l_1 + 2h$.

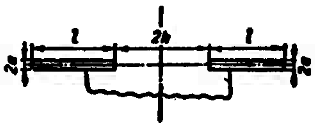


Fig. 3-7. Two joint identical wires, arranged on one straight line.

When $h > l/4$

$$C_0 \simeq \frac{4\pi\varepsilon l}{\ln\frac{l}{a} + \ln\frac{2h + 2l}{2h + l}}. \qquad (3\text{-}11)$$

5. *Two interconnected intersecting or crossing rectilinear wires of finite length.*

a) General case (Fig. 3-8).

$$C_0 = \frac{a_{11} + a_{22} - 2a_{12}}{a_{11}a_{22} - a_{12}^2}, \qquad (3\text{-}12)$$

where

$$a_{11} \simeq \frac{1}{2\pi\varepsilon(x_2 - x_1)}\left\{\ln\left[\frac{x_2 - x_1}{a} + \sqrt{1 + \left(\frac{x_2 - x_1}{a}\right)^2}\right] + \right.$$

$$\left. + \frac{a}{x_2 - x_1} - \sqrt{1 + \frac{a^2}{(x_2 - x_1)^2}}\right\};$$

$$a_{22} \simeq \frac{1}{2\pi\varepsilon(y_2 - y_1)}\left\{\ln\left[\frac{y_2 - y_1}{a} + \sqrt{1 + \left(\frac{y_2 - y_1}{a}\right)^2}\right] + \right.$$

$$\left. + \frac{a}{y_2 - y_1} - \sqrt{1 + \frac{a^2}{(y_2 - y_1)^2}}\right\};$$

$$a_{12} = \frac{F_{11} + F_{22} + F_{12} - F_{21}}{4\pi\varepsilon(x_2 - x_1)(y_2 - y_1)};$$

$$F_{pq} = x_p \ln|y_q - x_p\cos\varphi + D_{pq}| + y_p \ln|x_p - y_q\cos\varphi + D_{pq}| +$$

$$+ \frac{2d}{\sin\varphi}\,\mathrm{arctg}\left(\frac{x_p + y_q - D_{pq}}{d}\,\mathrm{tg}\,\frac{\varphi}{2}\right);$$

$$D_{pq} = \sqrt{x_p^2 + y_q^2 - 2x_p y_q \cos\varphi + d^2}, \quad p = 1, 2;\; q = 1, 2.$$

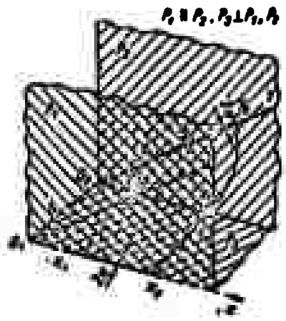


Fig. 3-8. Conductor formed by two intersecting or crossing rectilinear wires P_1 and P_2 are parallel planes passing through wires 1 and 2, respectively; P_3 is a plane perpendicular to P_1 and P_2; d is the distance between planes P_1 and P_2; ϕ is the angle between wire 2 and the projection of wire 1 on plane P_2 (or, what amounts to the same thing, between wire 1 and the projection of wire 2 on plane P_1); x_1, x_2, and y_1, y_2 are the coordinates of the ends of each of the wires reckoned along the line of their location from points O_1 and O_2, respectively.

b) Perpendicular wires of equal length are located in one plane: $d = 0$; $\phi = \pi/2$; $x_1 = y_1 = h$; $x_2 - x_1 = y_2 - y_1 = l$ (Fig. 3-9)

$$C_0 \simeq \frac{4\pi\varepsilon l}{\ln\left[\frac{l}{a}+\sqrt{1+\left(\frac{l}{a}\right)^2}\right]+\frac{a}{l}-\sqrt{1+\left(\frac{a}{l}\right)^2}+N}, \qquad (3\text{-}13)$$

$$N = \frac{h}{l}\ln\frac{2{,}41h}{l+h+\sqrt{h^2+(l+h)^2}}+\left(1+\frac{h}{l}\right)\ln\frac{2{,}41\,(l+h)}{h+\sqrt{h^2+(l+h)^2}}$$

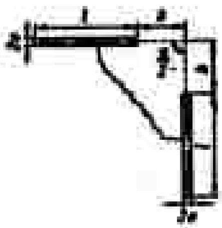


Fig. 3-9. Two perpendicular lines.

When $h = 0$

$$C_0 \simeq \frac{4\pi\varepsilon l}{\ln\left\{2\left[\frac{l}{a}+\sqrt{1+\left(\frac{l}{a}\right)^2}\right]\right\}+\frac{a}{l}-\sqrt{1+\left(\frac{a}{l}\right)^2}}. \qquad (3\text{-}14)$$

6. *Several (n) interconnected identical parallel rectilinear wires.*

a) the wires are located in one plane at equal distance from one another (Fig. 3-10).

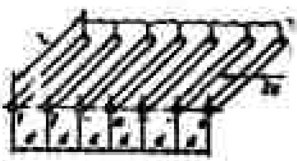


Fig. 3-10. A system of n identical interconnected parallel wires in one plane at equal distance from one another.

When $d/l \ll 1$

$$C_0 \simeq \frac{2\pi\varepsilon n l}{n\left(\ln\frac{l}{d}-0{,}307\right)+\ln\frac{d}{a}+B}, \qquad (3\text{-}15)$$

where

$$B = \frac{1}{n}\sum_{m=1}^{m=n} \ln[(m-1)!\,(n-m)!].$$

The values of coefficient B for n = 2-12 are given in Table 3-2.

Table 3-2. Values of the parameter B which enters formula (3-15).

n	2	3	4	5	6	7	8	9	10	11	12
B	0	0,46	1,24	2,26	3,48	4,85	6,40	8,06	9,8	11,65	13,58

b) The wires are located evenly on the surface of a circular cylinder (Fig. 3-11).

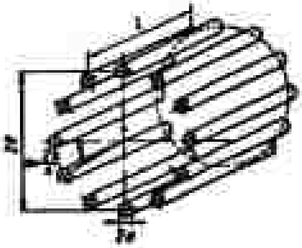


Fig. 3-11. A system of n identical interconnected parallel wires arranged over the surface of a circular cylinder.

$$C_0 \approx \frac{2\pi \varepsilon n l}{\ln\frac{l}{a} + (n-1)\ln\frac{l}{R} + \ln 2 - n - \ln\left[\sin\frac{\alpha}{2}\cdot\sin\frac{2\alpha}{2}\cdots\sin\frac{(n-1)\alpha}{2}\right]},$$

$$\alpha = 2\pi/n. \qquad (3\text{-}16)$$

When n = 2 this formula coincides with (3-9), and when n = 2-8 it leads to the formulas shown in Table 3-3.

c) the wires are located on the parallel edges of a rectangular parallelepiped (Fig. 3-12).

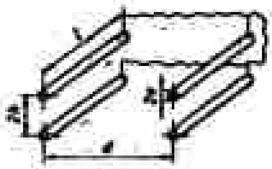
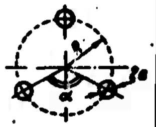
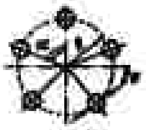
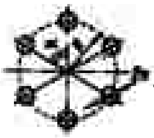
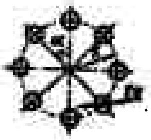


Fig. 3-12. Four identical rectilinear wires arranged on the parallel edges of a parallelepiped.

Table 3-3. Capacitance of conductor shown in Fig. 3-11 at various values of n.

No. in order	Calculation system	n	α	Capacitance
1		3	$\frac{2\pi}{3}$	$C_0 \simeq \frac{6\pi\varepsilon l}{\ln \frac{l}{a} + 2 \ln \frac{l}{R} - 2{,}023}$
2		4	$\frac{\pi}{2}$	$C_0 \simeq \frac{8\pi\varepsilon l}{\ln \frac{l}{a} + 3 \ln \frac{l}{R} - 2{,}615}$
3		5	$\frac{2\pi}{5}$	$C_0 \simeq \frac{10\pi\varepsilon l}{\ln \frac{l}{a} + 4 \ln \frac{l}{R} - 3{,}138}$
4		6	$\frac{\pi}{3}$	$C_0 \simeq \frac{12\pi\varepsilon l}{\ln \frac{l}{a} + 5 \ln \frac{l}{R} - 3{,}540}$
5		8	$\frac{\pi}{4}$	$C_0 \simeq \frac{16\pi\varepsilon l}{\ln \frac{l}{a} + 7 \ln \frac{l}{R} - 4{,}540}$

When $d \geq 5h$

$$C_0 \simeq \frac{8\pi\varepsilon l}{\ln \frac{l^4}{2hd^2a}}. \qquad (3\text{-}17)$$

Example 3-3. To calculate the capacitance of the conductor shown in Fig. 3-11, when l = 2.0 m; 2 a = 0.04 m; R = 0.5 m, at various values of n.

Using the formulas given in Table 3-3, we directly find

for $n = 3$

$$C_0 \approx \frac{6\pi\varepsilon_0 \cdot 2{,}0}{\ln\frac{2{,}0}{0{,}02} + 2\ln\frac{2{,}0}{0{,}5} - 2{,}023} = 62{,}5 \cdot 10^{-12}\ \text{F} = 62{,}5\ \text{pF}.$$

for $n = 4$

$$C_0 \approx \frac{8\pi\varepsilon_0 \cdot 2{,}0}{\ln\frac{2{,}0}{0{,}02} + 3\ln\frac{2{,}0}{0{,}5} - 2{,}615} = 72{,}8 \cdot 10^{-12}\ \text{F} = 72{,}8\ \text{pF}.$$

for $n = 5$

$$C_0 \approx \frac{10\pi\varepsilon_0 \cdot 2{,}0}{\ln\frac{2{,}0}{0{,}02} + 4\ln\frac{2{,}0}{0{,}5} - 3{,}138} = 79{,}5 \cdot 10^{-12}\ \text{F} = 79{,}5\ \text{pF}.$$

for $n = 6$

$$C_0 \approx \frac{12\pi\varepsilon_0 \cdot 2{,}0}{\ln\frac{2{,}0}{0{,}02} + 5{,}0\ln\frac{2{,}0}{0{,}5} - 3{,}54} = 83{,}5 \cdot 10^{-12}\ \text{F} = 83{,}5\ \text{pF}.$$

for $n = 8$

$$C_0 \approx \frac{16\pi\varepsilon_0 \cdot 2{,}0}{\ln\frac{2{,}0}{0{,}02} + 7 \cdot \ln\frac{2{,}0}{0{,}5} - 4{,}54} = 91 \cdot 10^{-12}\ \text{F} = 91\ \text{pF}.$$

Hence it is apparent that with increase in the number of wires from 3 to 6 the capacitance of the conductor being considered is increased 34%, and with increase from 4 to 6 only 15%. This evidences significant mutual effect of wires.

7. *Rectilinear wires connected in the form of a polygon.*

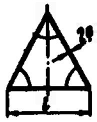
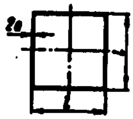
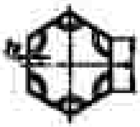
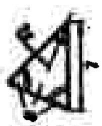
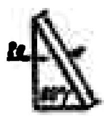
Approximation formulas for the calculation of capacitance of conductors in the form of polygons of various type are given in Table 3-4, and give an inaccuracy of not more than 10%.

Example 3-4. Disregarding the effect of earth, to determine the capacitance of a frame antenna in the form of a square with side $l = 10.0$ m with a wire 6 mm in diameter.

Using the formula of Table 3-4, we find that the capacitance of the antenna considered is

$$C_0 \approx \frac{8\pi\varepsilon_0 \cdot 10{,}0}{\ln\frac{10{,}0}{0{,}003} + 1{,}91} \approx 0{,}22 \cdot 10^{-9}\ (\text{F}) = 220\ \text{pF}.$$

Table 3-4. Formulas for calculation of capacitance of solitary conductors in the form of polygons formed by rectilinear wires.

No. in order	Form of circuit	Calculation system.	Capacitance
1	Equilateral triangles		$C_0 \approx \frac{6\pi\varepsilon l}{\ln\frac{l}{a} + 1{,}80}$
2	Square		$C_0 \approx \frac{8\pi\varepsilon l}{\ln\frac{l}{a} + 1{,}91}$
3	Regular hexagon		$C_0 \approx \frac{12\pi\varepsilon l}{\ln\frac{l}{a} + 2{,}176}$
4	Isosceles triangle $\alpha = 40°$		$C_0 \approx 4\pi\varepsilon l\left\{\frac{1}{\ln\frac{l}{a} + 2{,}18} + \frac{0{,}767}{\ln\frac{l}{a} + 2{,}31}\right\}$
5	Right triangle		$C_0 \approx 4\pi\varepsilon l\left\{\frac{0{,}433}{\ln\frac{l}{a} + 1{,}88} + \frac{0{,}25}{\ln\frac{l}{a} + 1{,}59} + \frac{0{,}5}{\ln\frac{l}{a} + 1{,}9}\right\}$

8. *Wires connected in the form of spatial bodies.*

a) The wires are located on the edges of a cube (Fig. 3-13):

$$C_0 \approx \frac{24\pi\varepsilon l}{\ln\frac{l}{a} + 6{,}36}. \tag{3-18}$$

b) Wires are located along the directrixes of a right circular cylinder and along four of the generatrixes, lying in two naturally perpendicular planes (Fig. 3-14).

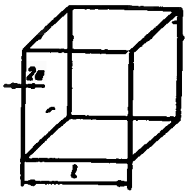


Fig. 3-13. Conductor formed by wires on edges of cube.

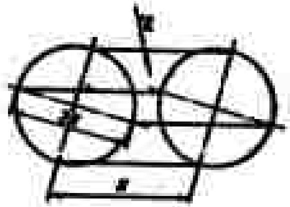


Fig. 3-14. Conductor formed by the wires arranged along the directrix and four generatrixes of a right circular cylinder.

When $H/4R < 1$

$$C_0 \simeq \frac{4\pi^2\varepsilon_0(\pi R + H)}{\frac{\pi}{2}\ln\frac{8R}{a} + \frac{\pi R}{\sqrt{H^2 + 4R^2}}K + \frac{H}{R}\left(\ln\frac{16R}{H} + 1\right)}, \qquad (3\text{-}19)$$

where K is a complete elliptical integral of the first kind (see Appendix 1) with modulus $k = \sqrt{\frac{4R^2}{4R^2 + H^2}}$.

3-3. The Capacitance of Solitary Conductors, Formed by Wires Arranged Near an Infinite Flat Impenetrable Boundary

The formulas given in the present paragraph were obtained by the method of mirror reflection of the conductors being considered relative to a planar impenetrable boundary. Some of the auxiliary systems obtained in this way coincide with those considered in the previous paragraph. In these cases calculation of capacitance boils down to utilization of the formulas of appropriate sections § 3-2. The numerical examples given in the present paragraph concern basically the determination of the resistance of grounds on the basis of an analogy between conductivity and capacitance (see § V-4).

1. *Rectilinear wire of finite length.*

a) The wire is parallel to the boundary plane (Fig. 3-15):

$$C_0 = \frac{C_0'}{2}. \qquad (3\text{-}20)$$

where C_0' is determined from formulas (3-7)-(3-9).

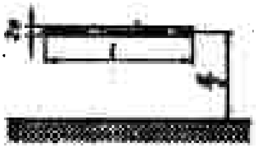


Fig. 3-15. Rectilinear wire of finite length parallel to a flat impenetrable boundary.

b) The wire is perpendicular to the boundary plane (Fig. 3-16):

$$C_0 = \frac{C_0'}{2}, \tag{3-21}$$

where C_0' when $h = 0$ is determined from formulas (3-1)-(3-3), and when $h \neq 0$ from the formula (3-11).

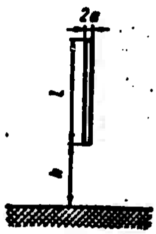


Fig. 3-16. Rectilinear wire of finite length perpendicular to a flat impenetrable boundary.

Example 3-5. To find the resistance R of horizontal and vertical grounds with radius $a = 0.1$ m and length $l = 1.0$ m in ground with electrical conductivity $\gamma = 2.0 \cdot 10^{-2}$ 1/Ω·m, arranged on depth: horizontal ground – $d/2 = 1.0$ m, vertical ground – $h = 0.5$ m (see Figs. 3-15 and 3-16).

Using the relation between R and C_0 (see § V-4), we find for a horizontal ground [formulas (3-20) and (3-7)]

$$R = \frac{1}{G} = \frac{\varepsilon}{\gamma C_0} = \frac{2\pi}{\gamma C_0'} = \frac{1}{2\pi \cdot 2 \cdot 10^{-2}} \left\{ \ln\left[\frac{1}{0,1} + \sqrt{1 + \left(\frac{1}{0,1}\right)^2}\right] + \ln\left[\frac{1}{2,0} + \sqrt{1+\left(\frac{1}{2}\right)^2}\right] + \frac{0,1}{1,0} + \frac{2,0}{1,0} - \sqrt{1+\left(\frac{0,1}{1,0}\right)^2} - \sqrt{1+\left(\frac{2}{1,0}\right)^2} \right\} = 17,8 \ \Omega,$$

for a vertical ground [formulas (3-21) and (3-11)]

$$R = \frac{1}{G} = \frac{\varepsilon}{\gamma C_0} = \frac{2\varepsilon}{\gamma C_0'} \approx \frac{1}{\pi \cdot 2 \cdot 10^{-2}}\left[\ln\frac{1,0}{0,1} + \ln\frac{2 \cdot 0,5 + 2 \cdot 1,0}{2 \cdot 0,5 + 1,0}\right] = 22,5 \;\;\Omega .$$

2. *A wire in the form of a circular ring arranged in a plane parallel to boundary* (Fig. 3-17).

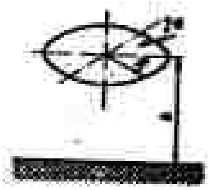


Fig. 3-17. A wire in the form of a circular ring arranged in a plane parallel to an impenetrable boundary.

When $h \ll R$

$$C_0 \simeq \frac{4\pi^2\varepsilon R}{\ln\frac{8\pi R^2}{ah}}. \qquad (3\text{-}22)$$

When $h \gg R$

$$C_0 \simeq \frac{4\pi^2\varepsilon R}{\frac{\pi R}{h} + \ln\frac{8R}{a}}. \qquad (3\text{-}23)$$

3. *Two identical rectilinear wires perpendicular to a boundary plane* (Fig. 3-18).

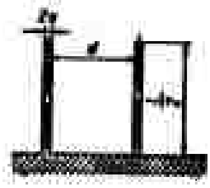


Fig. 3-18. Two identical rectilinear wires perpendicular to an impenetrable boundary.

$$C_0 \simeq \frac{C_0'}{2}, \qquad (3\text{-}24)$$

where C_0' is determined from formulas (3-7)-(3-9).

4. *Several (n) identical rectilinear wires perpendicular to a boundary plane.*

a) Wires are located in one plane at an equal distance from one another (Fig. 3-19):

$$C_0 = \frac{C_0'}{2}, \qquad (3\text{-}25)$$

where C_0' determined by formula (3-15).

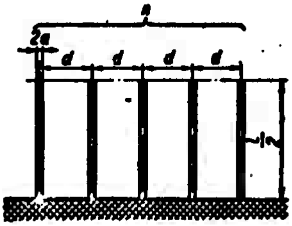


Fig. 3-19. *n* identical wires perpendicular to an impenetrable boundary and arranged in one plane at an equal distance from one another.

b) The wires are located uniformly on the surface of a circular cylinder (Fig. 3-20):

$$C_0 = \frac{C_0'}{2}, \qquad (3\text{-}26)$$

where C_0' is determined by formula (3-16).

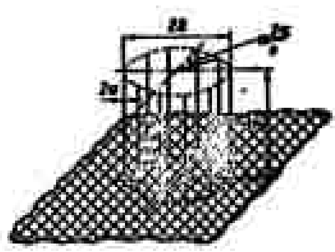


Fig. 3-20. *n* identical wires perpendicular to an impenetrable boundary and arranged along the generatrix of a circular cylinder.

c) The wires are located on the parallel edges of a parallelepiped (Fig. 3-21):

$$C_0 = \frac{C_0'}{2}, \qquad (3\text{-}27)$$

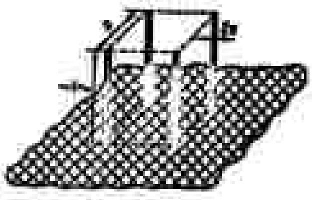


Fig. 3-21. Four identical wires perpendicular to an impenetrable boundary and arranged along the edges of a parallelepiped.

where C_0' is determined by formula (3-17) when $d = h/2 = h_1$.

5. *Wires connected in the form of a rectangle parallel to a boundary* (Fig. 3-22).

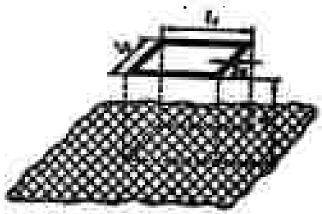


Fig. 3-22. A conductor in the form of a rectangle parallel to an impenetrable boundary.

$$C_0 \simeq \frac{2\pi\epsilon L}{\ln \frac{kL^2}{2ah}}, \tag{3-28}$$

where $L = 2\,(l_1 + l_2)$, and the values of the coefficient k, depending on the ratio l_1/l_2 are given below:

l_1/l_2	1,5	2,0	3,0	4,0
k	3,81	6,42	8,17	10,4

When $l_1 = l_2 = l$

$$C_0 \simeq \frac{2\pi\epsilon L}{\ln \frac{5{,}53L^2}{2ah}}, \tag{3-29}$$

where $L = 4l$.

6. *Horizontal rectangular grating parallel to a boundary* (Fig. 3-23).

$$C_0 \simeq \frac{2\pi\epsilon L}{\ln \frac{L^2}{2ah} + D}, \tag{3-30}$$

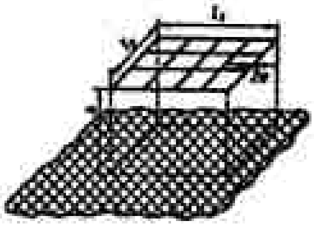
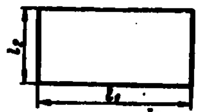
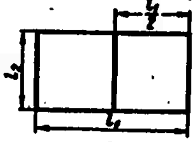
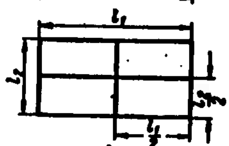
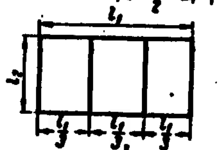


Fig. 3-23. A conductor in the form of a rectangle parallel to an impenetrable boundary.

where L is the total length of all conductors that form a grating; D is a coefficient depending on the ratio of the dimensions of the grating and the number of its cells.

The values of the coefficient D for some types of rectangular lattices are given in Table 3-5.

Table 3-5. Values of the coefficient D which enters formula (3-30).

Nr. in order	[illegible] of [illegible] / [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1		1,71	[illegible]	[illegible]	2,10	[illegible]
2		3,67	3,41	3,31	3,29	3,35
3		4,95	5,16	5,44	6,00	6,52
4		4,33	4,43	4,73	5,04	5,61
5		8,55	8,94	9,40	10,30	11,11

Example 3-6. To find the resistance of a horizontal ground in the form of a rectangular grating of tubes with 4 2.0 × 1.0 m cells with a diameter of tubes of 0.02 m. The grating is placed into ground with electrical conductivity of $\gamma = 10^{-2}\frac{1}{\Omega\cdot m}$ to a depth of $h = 2.0$ m.

Using an analogy between conductivity and capacitance, let us use for calculation the formula (3-30).

The total length of the conductors of the ground being considered is $L = 3\ (l_1 + l_2) = 3\ (4.0 + 2.0) = 18.0$ m.

The coefficient D which enters (3-30) is determined according to an assigned ratio $l_1/l_2 = 2.0$ from Table 3-6, with the aid of which we find that $D = 5.44$.

Table 3-6. The values of coefficient D_1, depending on l/d.

$\frac{l}{d}$	D_1	$\frac{l}{d}$	D_1	$\frac{l}{d}$	D_1
0,0	0,0	0,90	0,364	0,45	0,576
10	0,042	0,85	0,379	0,40	0,617
5	0,082	0,80	0,396	0,35	0,664
2,5	0,157	0,75	0,414	0,30	0,721
2,0	0,191	0,70	0,435	0,25	0,790
1,25	0,288	0,65	0,457	0,20	0,874
1,11	0,310	0,60	0,482	0,15	0,990
1,00	0,336	0,55	0,510	0,10	1,155
0,95	0,350	0,50	0,541	0,05	1,445

Thus,

$$R = \frac{1}{G} = \frac{\varepsilon}{\gamma C_0} \approx \frac{\ln\frac{18^2}{2\cdot 0,01\cdot 2,0} + 5,44}{2\pi\cdot 18,0\cdot 10^{-2}} = 12,7$$

7. *Flat n-ray stars parallel to a boundary,*[1] *when $h/l < 1$* (Figs. 3-24 thru 3-28).

a) 2-ray star (Γ-shaped wire) (Fig. 3-24):

$$C_0 \simeq \frac{4\pi\varepsilon l}{\ln\frac{2l}{a} + \ln\frac{2l}{h} - 0,2373 + 0,2146\frac{h}{l} + 0,1035\frac{h^2}{l^2} - 0,0494\frac{h^4}{l^4}} \qquad (3-31)$$

[1] *n*-ray star will be the name given the conductor formed by *n* rectilinear wires intersecting at one point.

$[|\delta| < 1{,}0\%$ when $h/l < 0{,}8]$.

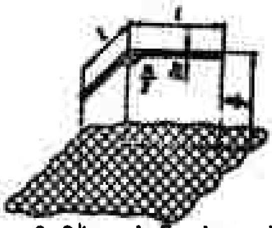


Fig. 3-24. A Γ-shaped wire, parallel to an impenetrable boundary.

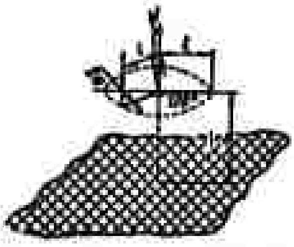


Fig. 3-25. A three-ray star, parallel to an impenetrable boundary.

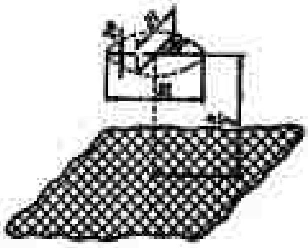


Fig. 3-26. Four-ray star parallel to impenetrable boundary.

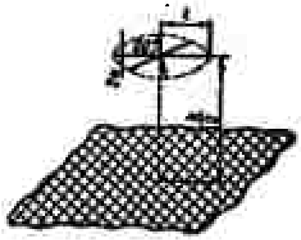


Fig. 3-27. Six-ray star parallel to impenetrable boundary.

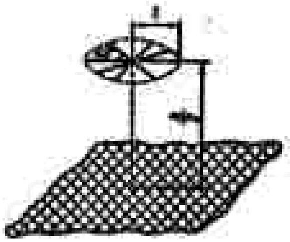


Fig. 3-28. Eight-ray star parallel to impenetrable boundary.

Furthermore, a less accurate formula can be used

$$C_0 \simeq \frac{2\pi\varepsilon L}{\ln \frac{1,46L^2}{ah}}, \qquad (3\text{-}32)$$

where $L = 2l$, $|\delta| < 10\%$ when $h/l < 0,8$.

b) 3-ray star (Fig. 3-25):

$$C_0 \simeq \frac{6\pi\varepsilon l}{\ln \frac{2l}{a} + \ln \frac{2l}{h} + 1,071 - 0,209 \frac{h}{l} + 0,238 \frac{h^2}{l^2} - 0,054 \frac{h^4}{l^4}} \qquad (3\text{-}33)$$

$[|\delta| < 1,0\%$ when $h/l < 0,8]$

or

$$C_0 \simeq \frac{2\pi\varepsilon L}{\ln \frac{2,38L^2}{a \cdot h}}, \qquad (3\text{-}34)$$

where $L = 3l$, $|\delta| < 10\%$ when $h/l < 0,8$.

c) 4-ray star (Fig. 3-26):

$$C_0 \simeq \frac{8\pi\varepsilon l}{\ln \frac{2l}{a} + \ln \frac{2l}{h} + 2,912 - 1,071 \frac{h}{l} + 0,645 \frac{h^2}{l^2} - 0,145 \frac{h^4}{l^4}} \qquad (3\text{-}35)$$

$[|\delta| < 1,0\%$ when $h/l < 0,8]$

or

$$C_0 \simeq \frac{2\pi\varepsilon L}{\ln \frac{8,45L^2}{ah}}, \qquad (3\text{-}36)$$

where $L = 4l$, $|\delta| < 10\%$ when $h/l < 0,8$.

d) 6-ray star (Fig. 3-27):

$$C_0 \simeq \frac{12\pi\varepsilon l}{\ln \frac{2l}{a} + \ln \frac{2l}{h} + 6,851 - 3,128 \frac{h}{l} + 1,758 \frac{h^2}{l^2} - 0,490 \frac{h^4}{l^4}} \qquad (3\text{-}37)$$

$[|\delta| < 1,0\%$ when $h/l < 0,8]$

or

$$C_0 \simeq \frac{2\pi\epsilon L}{\ln\frac{19{,}2L^2}{a\cdot h}}, \qquad (3\text{-}38)$$

where $L = 6l$, $|\delta| < 10\%$ when $h/l < 0{,}8$.

e) 8-ray star (Fig. 3-28):

$$C_0 \simeq \frac{16\pi\epsilon l}{\ln\frac{2l}{a} + \ln\frac{2l}{h} + 10{,}98 - 5{,}51\frac{h}{l} + 3{,}26\frac{h^2}{l^2} - 1{,}17\frac{h^4}{l^4}} \qquad (3\text{-}39)$$

$[|\delta| < 1{,}0\%$ when $h/l < 0{,}8]$.

3-4. Capacitor Capacitance of Systems of Wires

In the present paragraph formulas are given for the calculation of capacitance between two conductors, each of which is formed either by a single wire, or by the combination of several wires.[1]

1. *Two parallel wires of circular section* (Fig. 3-29)

a) $b = 0$:

$$C = \frac{1}{\alpha_{11} + \alpha_{22} + 2\alpha_{12}}, \qquad (3\text{-}40)$$

where

$$\alpha_{11} \simeq \frac{1}{2\pi\epsilon l_1}\left\{\operatorname{Arsh}\frac{l_1}{a_1} + \frac{a_1}{l_1} - \sqrt{1 + \left(\frac{a_1}{l_1}\right)^2}\right\},$$

$$\alpha_{22} \simeq \frac{1}{2\pi\epsilon l_2}\left\{\operatorname{Arsh}\frac{l_2}{a_2} + \frac{a_2}{l_2} - \sqrt{1 + \left(\frac{a_2}{l_2}\right)^2}\right\},$$

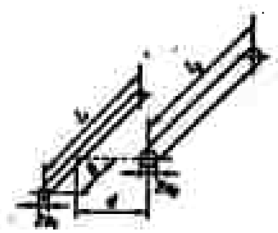


Fig. 3-29. A system of two rectilinear wires (general case).

[1]The basis of the majority of the data of the present and following paragraphs is the results of work [3-1].

$$a_{12} = \frac{1}{4\pi\varepsilon l_2}\left\{\mathrm{Arsh}\,\frac{l_1}{d} + \frac{l_2}{l_1}\,\mathrm{Arsh}\,\frac{l_2}{d} - \left(\frac{l_2}{l_1} - 1\right)\mathrm{Arsh}\,\frac{l_2 - l_1}{d} + \right.$$

$$+ \frac{d}{l_1} + \sqrt{\left(\frac{d}{l_1}\right)^2 + \left(\frac{l_2}{l_1} - 1\right)^2} - \sqrt{\left(\frac{d}{l_1}\right)^2 + 1} -$$

$$\left. - \sqrt{\left(\frac{d}{l_1}\right)^2 + \left(\frac{l_2}{l_1}\right)^2}\right\};$$

b) $b = 0,\ l_1 = l_2 = l,\ a_1 = a_2 = a$:

$$C = \frac{\pi\varepsilon l}{\ln\frac{d}{a} - 2{,}303 D_1}, \qquad (3\text{-}41)$$

where coefficient D_1 is determined by expressions:

at $l/d > 1$

$$D_1 = \frac{\frac{d}{l} - \left[\sqrt{1 + \left(\frac{d}{l}\right)^2} - 1\right]}{2{,}303} + \lg\frac{1 + \sqrt{1 + \left(\frac{d}{l}\right)^2}}{2};$$

at $l/d \leq 1$

$$D_1 = \frac{0{,}307 - \frac{d}{l}\left[\sqrt{1 + \left(\frac{l}{d}\right)^2} - 1\right]}{2{,}303} + \lg\frac{\frac{l}{d} + \sqrt{1 + \left(\frac{l}{d}\right)^2}}{\frac{l}{d}}.$$

The values of the coefficient D_1 depending on Table 3-6.

At $\varepsilon = \varepsilon_0$

$$C(\mathrm{pF}) = \frac{27{,}8 l}{\ln\frac{d}{a} - 2{,}303 D_1}. \qquad (3\text{-}41a)$$

c) $a_1 = a_2 = a$, $b = l_1 + 2m$ (Fig. 3-30):

$$C = \frac{1}{a_{11} + a_{22} - 2a_{12}}, \qquad (3\text{-}42)$$

where

$$a_{11} \simeq \frac{1}{2\pi\varepsilon l_1}\left\{\ln\left[\frac{l_1}{a} + \sqrt{1 + \left(\frac{l_1}{a}\right)^2}\right] + \frac{a}{l_1} - \sqrt{1 + \left(\frac{a}{l_1}\right)^2}\right\};$$

$$a_{22} \simeq \frac{1}{2\pi\varepsilon l_2}\left\{\ln\left[\frac{l_2}{a} + \sqrt{1 + \left(\frac{l_2}{a}\right)^2}\right] + \frac{a}{l_2} - \sqrt{1 + \left(\frac{a}{l_2}\right)^2}\right\}.$$

$$a_{12} \simeq \frac{1}{4\pi\varepsilon l_1 l_2}\left\{h_1 \ln\frac{h_1 + h_2 + \sqrt{d^2 + (h_1 + h_2)^2}}{h_1 + m + \sqrt{d^2 + (h_1 + m)^2}} + \right.$$

$$+ h_2 \ln\frac{h_1 + h_2 + \sqrt{d^2 + (h_1 + h_2)^2}}{h_2 + m + \sqrt{d^2 + (h_2 + m)^2}} +$$

$$+ m\ln\frac{\left(2m+\sqrt{d^2+4m^2}\right)^2}{\left[h_1+m+\sqrt{d^2+(h_1+m)^2}\right]\left[h_2+m+\sqrt{d^2+(h_2+m)^2}\right]} +$$

$$+\sqrt{d^2+(h_1+m)^2}+\sqrt{d^2+(h_2+m)^2}-$$

$$-\sqrt{d^2+(h_1+h_2)^2}-\sqrt{d^2+4m^2}\Big\}.$$

d) $d=0,\ l_1=l_2=l,\ a_1=a_2=a,\ b=l+2m$ (Fig. 3-31)

$$C=\frac{\pi\varepsilon l}{\ln\frac{l}{a}-2{,}303D_2}, \quad (3\text{-}43)$$

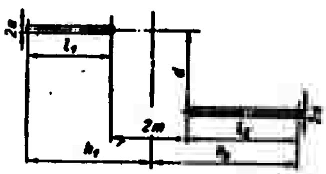


Fig. 3-30. Two parallel wire at $b = l_1 + 2m$.

where coefficient D_2 depends upon the ratio m/l and is determined by formulas:

at $m/l < 1$

$$D_2 = 0{,}434+\frac{m}{l}\lg\left(\frac{4m}{l}\right)+\left(1+\frac{m}{l}\right)\lg\left(1+\frac{m}{l}\right)-$$

$$-\left(1+\frac{2m}{l}\right)\lg\left(1+\frac{2m}{l}\right);$$

at $m/l > 1$

$$D_2 = 0{,}133+\frac{m}{l}\left(1+\frac{l}{m}\right)\lg\left(1+\frac{l}{m}\right)-\frac{2m}{l}\left(1+\frac{l}{2m}\right)\lg\left(1+\frac{l}{2m}\right).$$

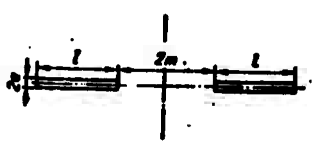


Fig. 3-31. Two identical wires arranged on one straight line.

The values of coefficient D_2 are given in Table 3-7.

Table 3-7. Values of coefficient D_2, depending on m/l.

$\frac{m}{l}$	D_2	$\frac{m}{l}$	D_2	$\frac{m}{l}$	D_2
0,02	0,403	0,30	0,280	1,0	0,207
0,04	0,384	0,40	0,261	1,11	0,202
0,06	0,369	0,50	0,247	1,25	0,196
0,08	0,356	0,60	0,236	2,0	0,177
0,10	0,345	0,70	0,227	2,5	0,170
0,15	0,323	0,80	0,219	5,0	0,153
0,20	0,305	0,90	0,2125	10,0	0,144
0,25	0,291				

When $\varepsilon = \varepsilon_0$

$$C(\text{pf}) = \frac{27{,}8 l}{\ln \frac{l}{a} - 2{,}303\, D_2}. \qquad (3\text{-}43a)$$

e) $b = 0$, $l_1 = l_2 = l \gg d$ (a plane-parallel system, Fig. 3-32).

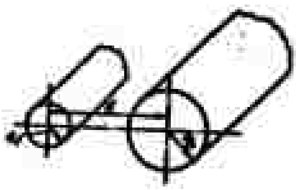


Fig. 3-32. Two parallel infinitely long wires of different diameter.

Capacitance per unit of length of system is determined by formulas: at random a and d

$$C_l = \frac{2\pi\varepsilon}{\operatorname{Arch} \frac{d^2 - a_1^2 - a_2^2}{2a_1 a_2}}; \qquad (3\text{-}44)$$

at $a_1 = a_2 = a$

$$C_l = \frac{\pi\varepsilon}{\operatorname{Arch} \frac{d}{2a}}; \qquad (3\text{-}45)$$

at $a_1 = a_2 = a$, $d \gg a$

$$C_l \approx \frac{\pi\varepsilon}{\ln \frac{d}{a}}. \qquad (3\text{-}46)$$

Example 3-7. To determine capacitance per unit of length of a two-wire located in air and consisting of wires $2a = 4$ mm in diameter and $d = 10$ cm apart.

Since in this case $d/a = 50 >> 1$, it is possible to use formula (3-46), with the aid of which we find (when $\varepsilon = \varepsilon_0$)

$$C_l = \frac{\pi\varepsilon_0}{\ln\frac{d}{a}} = \frac{\pi\cdot 10^{-9}}{36\pi\cdot\ln 50} = 7,1\cdot 10^{-12}\,\mathrm{F/m} = 7,1\ \mathrm{pF/m}.$$

2. *Two infinitely long wires of rectangular section.*

Capacitance per unit of length of system is determined by the formulas:

a) in general (Fig. 3-33)

$$C_l \simeq \frac{2\pi\varepsilon}{\ln\left[k\nu^2 - \nu\left(\frac{k^2}{k\nu - 1} + \frac{1}{k(\nu - 1)}\right)\right]} \qquad (3\text{-}47)$$

$$[|\delta| < 2,0\% \text{ when } \nu > 10],$$

or

$$C_l \simeq \frac{2\pi\varepsilon}{\ln(k\nu^2)} \qquad (3\text{-}48)$$

$$[|\delta| < 5,0\% \text{ when } \nu > 7],$$

where

$$k = \frac{a_1}{a_2}\cdot\frac{0,4\gamma_1^2 - 1,76\gamma_1 - 1}{0,4\gamma_2^2 - 1,76\gamma_2 - 1},$$

$$\nu = \frac{d}{a_1\left(-0,1\gamma_1^2 + 0,44\gamma_1 + 0,28\right)},$$

$$\gamma_1 = \frac{b_1}{a_1}, \quad \gamma_2 = \frac{b_2}{a_2};$$

b) in the case of a symmetric system (Fig. 3-34) $a_1 = b_1 = a_2 = b_2 = a$;

$$C_l \simeq \frac{2\pi\varepsilon}{\ln\left(\nu^2 - \frac{2\nu}{\nu - 1}\right)}, \qquad (3\text{-}49)$$

where $\nu = 1,695\frac{d}{a}$ and

$$|\delta| < 3,0\% \text{ when } \nu > 7,\ d > 4a,$$

or

$$C_l \simeq \frac{\pi\varepsilon}{\ln\frac{1,72d}{a}} \qquad (3\text{-}50)$$

$$[|\delta| < 4,0\% \text{ when } \nu > 10,\ d > 6a].$$

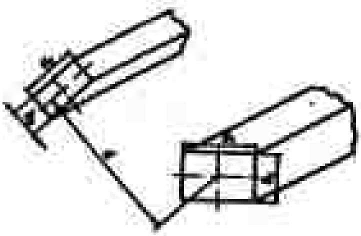


Fig. 3-33. Two infinitely long rectilinear parallel wires of rectangular section.

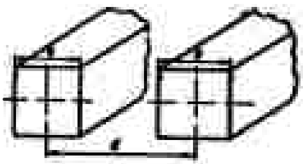


Fig. 3-34. Two identical infinitely long rectilinear parallel wires of square section.

Example 3-8. To find capacitance per unit of length of a two-wire line formed by wires of square section with side $a = 1$ cm, arranged in air at a distance of $d = 5$ cm from one another.

At the assigned dimensions of a line parameter ν, entering formula (3-49), $\nu = 1{,}695\frac{d}{a} = 8{,}48$.

Then, using formula (3-49), we find that when $\varepsilon = \varepsilon_0$

$$C_l \approx \frac{2\pi\cdot 10^{-9}}{36\pi \ln\left[8{,}48^2 - \frac{2\cdot 8{,}48}{8{,}48-1}\right]} = 13{,}25\cdot 10^{-12}\ \text{F/m} = 13{,}25\ \text{pF/m}.$$

If we make use of the simplified formula (3-50), then

$$C_l \approx \frac{\pi\cdot 10^{-9}}{36\pi\cdot(1{,}72\cdot 5)} = 12{,}9\cdot 10^{-12}\ \text{F/m} = 12{,}9\ \text{pF/m},$$

i.e., the relative difference in the results of calculations from the formulas (3-49) and (3-50) for the given value of ν is 3.6%.

3. *Two intersecting or crossing rectilinear wires of finite length:*

a) the general case (Fig. 3-35):

$$C = \frac{1}{\alpha_{11} + \alpha_{22} - 2\alpha_{12}}, \qquad (3\text{-}51)$$

where

$$\alpha_{11} = \frac{1}{2\pi\varepsilon(x_2 - x_1)}\left\{\ln\left[\frac{x_2 - x_1}{a} + \sqrt{1 + \left(\frac{x_2 - x_1}{a}\right)^2}\right] + \right.$$

$$\left. + \frac{a}{x_2 - x_1} - \sqrt{1 + \frac{a^2}{(x_2 - x_1)^2}}\right\},$$

$$\alpha_{22} = \frac{1}{2\pi\varepsilon(y_2 - y_1)}\left\{\ln\left[\frac{y_2 - y_1}{a} + \sqrt{1 + \left(\frac{y_2 - y_1}{a}\right)^2}\right] + \right.$$

$$\left. + \frac{a}{y_2 - y_1} - \sqrt{1 + \frac{a^2}{(y_2 - y_1)^2}}\right\},$$

$$\alpha_{12} = \frac{F_{11} + F_{22} + F_{12} - F_{21}}{4\pi\varepsilon(x_2 - x_1)(y_2 - y_1)},$$

$$F_{pq} = x_p \ln\left[y_q - x_p\cos\varphi + D_{pq}\right] + y_q \ln\left[x_p - y_q\cos\varphi + D_{pq}\right] +$$

$$+ \frac{2d}{\sin\varphi}\operatorname{arctg}\left(\frac{x_p + y_q + D_{pq}}{d}\operatorname{tg}\frac{\varphi}{2}\right),$$

$$D_{pq} = \sqrt{x_p^2 + y_q^2 - 2x_p y_q\cos\varphi + d^2},$$

$$p = 1, 2; \quad q = 1, 2;$$

b) perpendicular wires of equal length are located in one plane; $d = 0$; $\varphi = \frac{\pi}{2}$; $x_1 = y_1 = h$; $x_2 - x_1 = y_2 - y_1 = l$ (Fig. 3-36):

$$C \simeq \frac{\pi\varepsilon l}{\ln\left[\frac{l}{a} + \sqrt{1 + \left(\frac{l}{a}\right)^2}\right] + \frac{a}{l} - \sqrt{1 - \left(\frac{a}{l}\right)^2} - N}, \qquad (3\text{-}52)$$

$$N = \frac{h}{l}\ln\frac{2{,}41h}{h + l + \sqrt{h^2 + (h + l)^2}} +$$

$$+ \left(1 + \frac{h}{l}\right)\ln\frac{2{,}41(h + l)}{h + \sqrt{h^2 + (h + l)^2}}$$

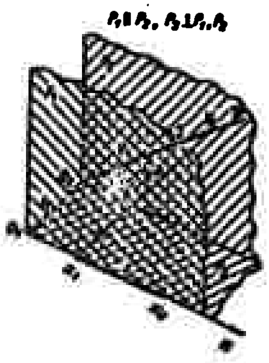


Fig. 3-35. Two intersecting wires 1 and 2. P_1 and P_2 are parallel planes passing through wires 1 and 2, respectively; P_3 is a plane perpendicular to P_1 and P_2. d is the distance between planes P_1 and P_2; ϕ is the angle between wire 2 and the projection of wire 1 on plane P_2 (or between wire 1 and the projection of wire 2 on plane P_1). x_1, x_2 and y_1, y_2 are the coordinates of the ends of wires reckoned along the line of their location from points O_1 and O_2, respectively.

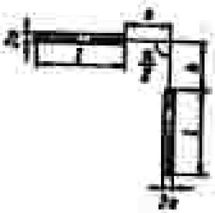


Fig. 3-36. Two straight wires of finite length arranged at right angles.

When $\frac{l}{a} \gg 1$; $\frac{h}{l} \gg 1$ a simpler formula can also be used

$$C \simeq \frac{\pi \varepsilon l}{\ln \frac{l}{a} - 1{,}307}. \qquad (3\text{-}53)$$

4. *Two identical wires in the form of a circular ring arranged symmetrically in parallel planes* (Fig. 3-37).

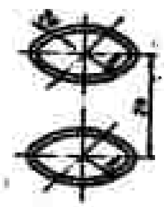


Fig. 3-37. Two identical circular rings lying in parallel planes.

$$C \simeq \frac{4\pi^2 \varepsilon R}{\ln \frac{8R}{a} - \frac{R}{\sqrt{R^2 + h^2}} K}, \qquad (3\text{-}54)$$

where K is the complete elliptic integral of kind I with modulus $k^2 = \frac{R^2}{R^2 + h^2}$, (see Appendix 1).

<u>Example</u> 3-9. To determine the capacitance between the conductors shown in Fig. 3-37, considering that they are located in air, the radius of every ring is equal to 5 cm, the distance between them is 10 cm, and the diameter of the wire is 0.1 cm.

Calculating the modulus of an elliptic integral, we find that

$$k = \sqrt{\frac{R^2}{R^2 + h^2}} = \sqrt{\frac{25}{25 + 25}} = 0{,}707.$$

Then from the tables of elliptic integrals we find that $K = 1.854$.

Substituting this value into formula (3-54), we find that the capacitance between the conductors considered when $\varepsilon = \varepsilon_0$ is

$$C \approx \frac{4\pi^2 \cdot 10^{-9} \cdot 5 \cdot 10^{-2}}{36\pi\left(\ln\frac{8 \cdot 5}{0{,}05} - 0{,}708 \cdot 1{,}854\right)} = 1{,}04 \cdot 10^{-12}\ \text{F} = 1{,}04\ \text{pF}.$$

5. *An infinitely long straight wire and the coaxial circular ring enveloping it* (Fig. 3-38).

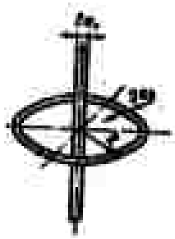


Fig. 3-38. A circular ring and rectilinear wire coaxial with it.

$$C \approx \frac{2\pi\varepsilon l}{\ln\frac{2R}{a}}. \qquad (3\text{-}55)$$

6. *A straight wire of finite length passing through circular cut of the plane* (Fig. 3-39).

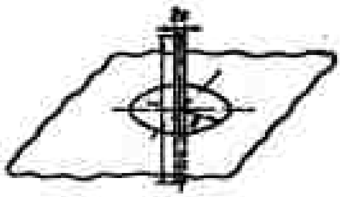


Fig. 3-39. A straight wire of finite length passing through a circular cut in conducting plane.

When $a \ll R$ and $2R \approx l$

$$C \approx \frac{2\pi\varepsilon l}{\ln\left[\frac{2lR}{a(2R+l)}\right]}; \qquad (3\text{-}56)$$

when $2R \ll l$

$$C \approx \frac{4\pi^2\varepsilon a_0}{\ln\frac{2R}{a_0}}. \qquad (3\text{-}57)$$

Example 3-10. To determine the capacitance between a wire 2 mm in diameter and 40 mm long and the metal panel of a voltmeter, if the wire passes through an opening in the panel 10 mm in diameter. Disregard the influence of insulation.

Using formula (3-56), when $\varepsilon = \varepsilon_0$ we find

$$C = \frac{2\pi\cdot 10^{-9}\cdot 40\cdot 10^{-3}}{36\pi\cdot \ln\frac{2\cdot 40\cdot 5}{40+10}} = 1{,}05\cdot 10^{-12}\ \mathrm{F} = 1{,}05\ \mathrm{pF}.$$

From a less accurate formula (3-57) we have

$$C = \frac{2\pi\cdot 10^{-9}\cdot 40\cdot 10^{-3}}{36\pi\cdot \ln\frac{2\cdot 5}{1}} = 0{,}96\ \mathrm{pF}.$$

Comparison of these results shows when $l/2R = 4$ formula (3-57) gives significant error (≈10%).

When $l/2R \geq 10$ the difference in quantities calculated from formulas (3-56) and (3-57) does not exceed 6.5%, when $l/2R \geq 20$-0.7%.

7. *2 n identical wires in two parallel planes, in each of which the wires are interconnected* (Fig. 3-40).

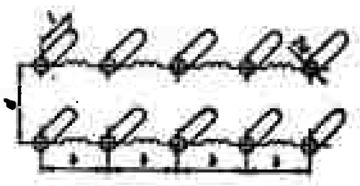


Fig. 3-40. 2 n wires arranged in two parallel planes.

When $(n - 1)\, b \leq l$

$$C \simeq \frac{n\pi\varepsilon l}{\ln\frac{d}{a} + (n-1)\ln\frac{d}{b} - 2{,}303n\,(D_1 + B_n)}, \tag{3-58}$$

where D_1 is found from Table 3-6, and coefficient B_n is determined by the formula

$$B_n = \frac{2}{n^2}\,[\lg(n-1) + 2\lg \times (n-2) + 3\lg(n-3) + \ldots + (n-2)\lg 2].$$

The values of coefficient B_n are given in Table 3-8.

Table 3-8. Values of the coefficient B_n entering formuls (3-58), depending on the number of wires.

n	B_n	n	B_n	n	B_n	n	B_n
2	0,0	8	0,347	14	0,550	20	0,688
3	0,067	9	0,388	15	0,576	30	0,847
4	0,135	10	0,425	16	0,601	40	0,970
5	0,197	11	0,460	17	0,625	50	1,063
6	0,252	12	0,492	18	0,647	100	1,367
7	0,302	13	0,522	19	0,668		

When $\varepsilon = \varepsilon_0$

$$C(\text{pF}) \simeq \frac{27{,}84\, nl}{\ln\frac{l}{a} + (n-1)\ln\frac{l}{d} - 2{,}303\,(D_1 + B_n)\,n}. \qquad (3\text{-}58a)$$

8. *2 n identical rectilinear wires of finite length arranged in one plane and connected in accordance with Fig. 3-41.*

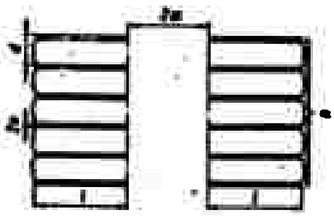


Fig. 3-41. 2 n identical rectilinear wires of finite length arranged in one plane.

When $(n-1)\, d \leq m$

$$C \simeq \frac{\pi\varepsilon n l}{\ln\frac{l}{a} + (n-1)\ln\frac{l}{d} - 2{,}303\cdot n\,(D_2 + B_n)}, \qquad (3\text{-}59)$$

where D_2 is determined from Table 3-7, and B_n from Table 3-8.

When $\varepsilon = \varepsilon_0$

$$C(\text{pF}) \simeq \frac{27{,}84\, nl}{\ln\frac{l}{a} + (n-1)\ln\frac{l}{d} - n\,(D_2 + B_n)}. \qquad (3\text{-}59a)$$

9. *Conductors formed by the union of infinitely long parallel wires.*

a) Three wires in one plane, the extreme of which are united (Fig. 3-42),

$$C_l \simeq \frac{2\pi\varepsilon}{\ln\left[\frac{d}{a_1}\left(\frac{d}{a_2}\right)^{1/2}\right]}. \qquad (3\text{-}60)$$

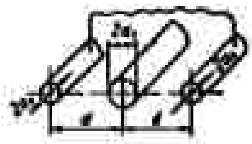


Fig. 3-42. Three infinitely long wires lying in one plane.

b) 2 *n* wires of alternating polarity lying in one plane (Fig. 3-43),

$$C_l \simeq \frac{2\pi\varepsilon n}{\ln\frac{d}{\pi a}}; \qquad (3\text{-}61)$$

c) 2 *n* wires of alternating polarity arranged evenly on the surface of a circular cylinder (Fig. 3-44).

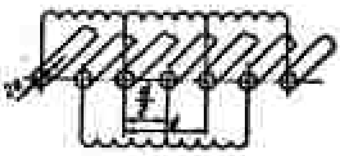


Fig. 3-43. 2 *n* loaded wires of alternating polarity lying in one plane.

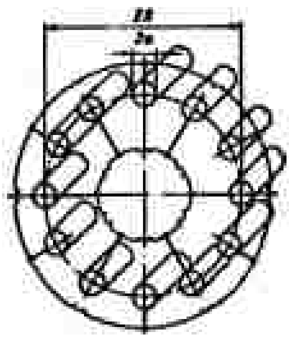


Fig. 3-44. *n* charged wires of alternating polarity lying on the surface of circular cylinder.

When 0.4 < α < 0.65 and $a \ll \frac{2\pi R}{n}$,
where $\alpha = \frac{2na}{\pi R}$:

$$C_l \simeq \frac{\pi\varepsilon}{2\ln\left(x + \sqrt{x^2 - 1}\right)}, \tag{3-62}$$

where $x = \frac{0.64 + 1.57\alpha^2}{\alpha(1 + 1.85\alpha)}$.

When α < 0.4

$$C \simeq \frac{\pi\varepsilon}{2\ln\frac{2R}{na}}. \tag{3-63}$$

10. *Various combinations of infinitely long wires and plates (planes).*

The formulas for determining the capacitance of the systems are given in Table 3-9.

11. *An infinitely long wire and two butting planes* (Fig. 3-45).

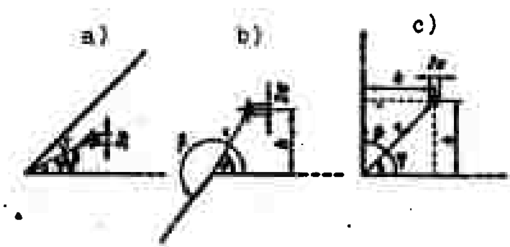


Fig. 3-45. An infinitely long wire in a sector.

When β ≤ π (Fig. 3-45a)

$$C_l \simeq \frac{2\pi\varepsilon}{\ln\left[\frac{\beta r}{\pi a}\sqrt{4\sin^2\left(\frac{\pi\varphi}{\beta}\right) + \left(\frac{\pi a}{r\beta}\right)^2}\right]}. \tag{3-64}$$

When β > π (Fig. 3-45b)

$$C_l \simeq \frac{2\pi\varepsilon}{\ln\left\{\frac{2h}{a}\sqrt{\left[\frac{\beta}{\pi}\frac{\sin\left(\frac{\pi\varphi}{\beta}\right)}{\sin\varphi}\right]^2 + \left(\frac{a}{2h}\right)^2}\right\}}. \tag{3-65}$$

Table 3-9. Formula for determining capacitance between an infinitely long wire and a plate or plane.

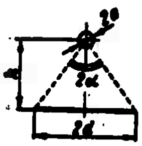
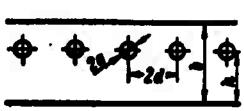
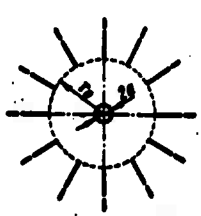
No. in order	Name of system	Calculated model	Calculation formula	Note
1	Linear wire over half-plane		$C_l = \dfrac{2\pi\varepsilon}{\ln\left[\dfrac{\sin\frac{\gamma}{2}}{1-\sqrt{1-\frac{a}{R}}}+N\right]}$, where $N=\sqrt{\left(\dfrac{\sin\frac{\gamma}{2}}{1-\sqrt{1-\frac{a}{R}}}\right)^2-1}$,	when $a \ll R$
			$C_l = \dfrac{2\pi\varepsilon}{\ln\dfrac{4R\sin\frac{\gamma}{2}}{a}}$	when $a \ll R$, $\dfrac{2R\sin\gamma/2}{a} \gg 1$
2	Linear wire over a plate		$C_l = \dfrac{2\pi\varepsilon}{\ln\left[\dfrac{2(d^2+h^2)}{d\cdot a}\cdot\cos^2\alpha\right]}$, where $\alpha = \operatorname{arctg}\dfrac{d}{h}$	when $a \ll d$, $a \ll h$
3	Linear wire over plane with cut		$C_l = \dfrac{2\pi\varepsilon}{\ln\left[\dfrac{2(d^2+h^2)}{d\cdot a}\sin^2\alpha\right]}$, where $\alpha = \operatorname{arctg}\dfrac{d}{h}$	when $a \ll d$, $a \ll h$

Table 3-9 continued.

No. in order	Name of system	Calculated model	Calculation formula	Note
4	Linear wire over plane with out		$C_l = \frac{2\pi\varepsilon d}{\sqrt{d^2-a^2}\ln\left[\frac{d}{a}+\sqrt{\left(\frac{d}{a}\right)^2-1}\right]}$, $C_l = \frac{2\pi\varepsilon}{\ln\frac{2d}{a}}$	when $a < d$ when $a \ll d$
5	Linear wire between two planes		$C_l = \frac{2\pi\varepsilon}{\ln\left(\frac{2b}{\pi a}\cdot\sin\frac{\pi h}{b}\right)}$, $C_l = \frac{2\pi\varepsilon}{\ln\left(1,27+\frac{b}{2a}\right)}$	when $a \ll h$ when $\frac{a}{b} < 0,5$ $h = b/2$
6	System of linear wire between two planes		$C_l = \frac{2\pi\varepsilon n}{\ln\left[\frac{b}{\pi a}\vartheta_1\left(\frac{h}{b};\frac{id}{b}\right)e^{\frac{\pi d}{4b}}\right]}$ where ϑ_1 — theta-function n — the number of wires for which capacitance is determined $C_l = \frac{2\varepsilon\pi d}{b-\frac{d}{\pi}\sin^2\frac{\pi a}{2d}\,\mathrm{th}\left(\frac{\pi b}{2d}\right)}$	 when $h = \frac{b}{2}$
7	System of linear wire and semi-infinite plates evenly arranged along the radii		$C_l = \frac{\pi\varepsilon n}{\left[\ln\left(\frac{2r_0}{a}\right)\right]^{\frac{n}{2}}}$	when $a \ll r_0$

When $\beta = \pi/2$; $\phi = \pi/4$ (Fig. 3-45c)

$$C_l \simeq \frac{2\pi\varepsilon}{\ln\left(1{,}41\frac{h}{d}\right)}. \tag{3-66}$$

<u>Example</u> 3-11. To determine capacitance per unit of length between a linear wire and a plane, in the cut of which it is located. The diameter of the wire is $2a = 2$ mm, and the width of the cut is $2d = 10$ mm.

The sought capacitance is determined from the formulas of paragraph 4 of Table 3-9.

From the first formula when $\varepsilon = \varepsilon_0$ we obtain

$$C_l \simeq \frac{2\pi\cdot 5\cdot 10^{-9}}{36\pi\cdot\sqrt{25-1}\ln\left[5+\sqrt{25-1}\right]} = 24{,}7\cdot 10^{-12}\ \mathrm{F/m} = 24{,}7\ \mathrm{pF/m}$$

From the second

$$C_l \simeq \frac{2\pi\cdot 10^{-9}}{36\pi\ln 10} = 24{,}2\cdot 10^{-12}\ \mathrm{F/m} = 24{,}2\ \mathrm{pF/m}.$$

As it appears, even when $a/d = 0.2$ the difference in determining capacitance from the given formulas does not exceed 2.5%.

12. *An infinitely long wire in the center of a shell of square section* (Fig. 3-46).

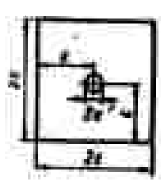


Fig. 3-46. An infinitely long wire inside a shell of square section.

$$C_l \simeq \frac{2\pi\varepsilon}{\ln\left[1{,}08\frac{c}{a}\right]}. \tag{3-67}$$

<u>Example</u> 3-12. In the center of a copper tube of square section with side $c = 20$ mm, there is a linear conductor 2 mm in radius.

To determine the mutual inductance of a wire and tube at high frequency (per unit of length).

Using formulas § V-4, we find that

$$L_1 = \frac{\varepsilon\mu_0}{C_1}.$$

The capacitance of the system considered is determined by formula (3-67), therefore,

$$L_1 = \frac{\mu_0}{2\pi}\ln\left[1{,}08\frac{c}{a}\right] = \frac{4\pi\cdot10^{-7}}{2\pi}\cdot\ln\left[1{,}08\frac{20}{2}\right] = 4{,}76\cdot10^{-7}\ \text{H/m}$$

13. *System of touching infinitely long wires arranged on a circumference, and the shell of circular section eveloping it* (Fig. 3-47).

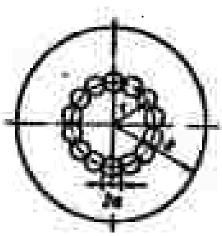


Fig. 3-47. System of touching infinitely long wires arranged along the circumference inside a shell of circular section.

$$C_1 \simeq \frac{2\pi\varepsilon}{\ln\frac{2R}{2r+a}}. \qquad (3\text{-}68)$$

14. *System of touching infinitely long wires arranged on a circumference, and circular shell inside it* (Fig. 3-48).

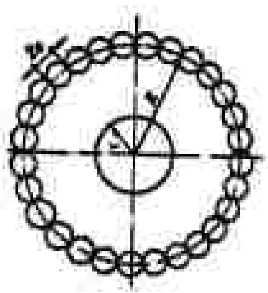


Fig. 3-48. System of touching infinitely long wires arranged along the circumference outside a shell of circular section.

$$C_l \doteq \frac{2\pi\varepsilon}{\ln\frac{2R-a}{2r}}. \qquad (3\text{-}69)$$

3-5. Capacitance Between Systems of Wires and Infinite Conducting Plane

The formulas given in this paragraph were obtained by the method of mirror reflection of conductors considered relative to a flat conducting boundary. Some of the auxiliary systems thus obtained coincide with those considered in the preceding paragraph. In these cases the calculation of capacitance between conductors and conducting plane boils down to the use of formulas of appropriate sections § 3-4. The numerical illustrations given in this paragraph mainly concern determination of the capacitance of antennas in air $\left(\varepsilon = \varepsilon_0 = \frac{1}{36\pi}\cdot 10^{-9}\right.$ F/m.

1. *Rectilinear wires parallel to a boundary plane and each other.*

a) A wire of finite length (Fig. 3-49):

$$C = 2C', \qquad (3\text{-}70)$$

where C' is determined from formulas (3-41) and (3-41a) when $d = 2h$.

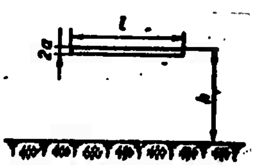


Fig. 3-49. A striaght wire of finite length parallel to a conducting plane.

Example 3-13. To determine the capacitance between grounds and a horizontal wire 30 m long and 6 mm in diameter arranged parallel to the surface of the earth at an altitude of 15 m.

In this case the quantity $\frac{d}{l} = \frac{2h}{l} = \frac{30}{30} = 1$. At this value of d/l the quantity D_1 in Table 3-6 is equal to 0.336.

With the aid of formulas (3-70) and (3-41a) we find

$$C \approx \frac{27,84\cdot 30}{\ln\frac{30}{0,003} - 2,303\cdot 0,336} = 198 \text{ pF}.$$

b) An infinitely long wire (Fig. 3-50):

$$C_l = 2C_l', \tag{3-71}$$

where C_l' is determined from formulas (3-45) and (3-46) when $d = 2h$.

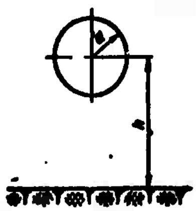


Fig. 3-50. An infinitely long straight wire of circular section parallel to a conducting plane.

c) Infinitely long wire of square section (Fig. 3-51):

$$C_l = 2C_l', \tag{3-72}$$

where C_l' is determined from formulas (3-49), (3-50) when $d = 2h$.

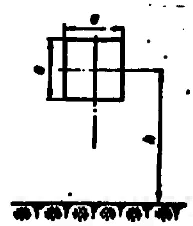


Fig. 3-51. An infinitely long straight wire of square cross section parallel to a conducting plane.

d) n identical parallel wires of finite length lying in a plane parallel to the boundary plane (Fig. 3-52):

$$C = 2C', \tag{3-73}$$

where C' is determined from formulas (3-58) and (3-58a) when $d = 2h$.

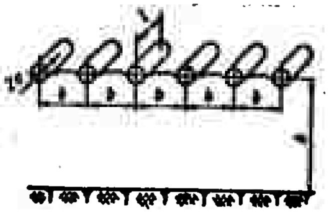


Fig. 3-52. n infinitely long rectilinear wires lying in a plane parallel to the boundary plane.

Example 3-14. To determine the capacitance to the ground of a horizontal antenna placed at an altitude of $h = 15$ m and consisting of 6 parallel wires $l = 30$ m long and 6 mm in diameter if the distance between the wires is $b = 0.6$ m.

In the case considered $\frac{d}{l}=\frac{2h}{l}=1$. At this value of d/l the coefficient in Table 3-6 is $D_1 = 0.336$. The coefficient B_n in Table 3-8 when $n = 6$ is equal to 0.252. Therefore, using formulas (3-73) and (3-58a), we find that

$$C \approx \frac{27{,}84\cdot 30\cdot 6}{\ln\frac{2\cdot 15}{0{,}003}+5\ln\frac{30}{0{,}6}-2{,}303\,(0{,}336+0{,}252)\cdot 6}=488 \text{ pF}.$$

e) n identical wires of finite length parallel to the boundary plane (Fig. 3-53).

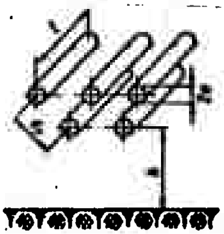


Fig. 3-53. n identical rectilinear wires of finite length parallel to a boundary plane.

If the distance between any wires d_r ($r = 1, 2, \ldots, n - 1$) is significantly shorter than their mean distance from a boundary ($d_r \ll h$), then

$$C \simeq \frac{2\pi\varepsilon n l}{2{,}303\cdot F_2}, \qquad (3\text{-}74)$$

where

$$F_1 = \lg \frac{2h}{a} + \sum_{r=1}^{r=n-1} \left(\lg \frac{2h}{d_r} + 0{,}434 \frac{d_r}{l} \right) - nD_1,$$

and D_1 is determined from Table 3-6 when $d = 2h$.

When $\varepsilon = \varepsilon_0$

$$C_{(pF)} \simeq \frac{24{,}16\, nl}{F_1}. \tag{3-75}$$

When the wires are located on the surface of a circular cylinder (Fig. 3-54),

$$d_r = 2R \sin r \frac{\pi}{n} \quad (r = 1, 2, \ldots, n-1),$$

where n is the number of wires.

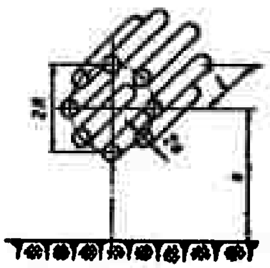


Fig. 3-54. n identical wires parallel to a boundary plane and arranged on the surface of a circular cylinder.

Example 3-15. To determine the capacitance between grounds and a horizontal antenna consisting of 6 wires 30 m long and 6 mm in diameter arranged over the surface of circular cylinder $2R = 1.5$ m in diameter, the axis of which is 15 m from the surface of the earth.

The distance between the wires which enter a system are equal to

$$d_1 = d_5 = 0{,}75 \text{ m}, \; d_2 = d_4 = \frac{\sqrt{3}}{2} R = 1{,}29 \text{ m}; \; d_3 = 1{,}5 \text{ m}.$$

The coefficient D_1 in Table 3-6 when $\frac{d}{l} = \frac{2h}{l} = \frac{2 \cdot 15}{30} = 1$ is equal to 0.336, and $nD_1 = 6 \cdot 0.336 = 2.016$. The coefficient which enters formula (3-75) is $F_1 = 4.0 + 2\,(1.602 + 0.011) + 2\,(1.364 + 0.018) + (1.301 + 0.022) - 2.016 = 0.297$.

Using formula (3-75), we find that

$$C = \frac{24{,}16 \cdot 30 \cdot 6}{9{,}297} = 472 \text{ (pF)}.$$

2. *Rectilinear wires of infinte length perpendicular to boundary plane.*

a) One wire (Fig. 3-55):

$$C = 2C', \tag{3-76}$$

where C' is determined from formulas (3-43) and (3-43a) when $m = h$.

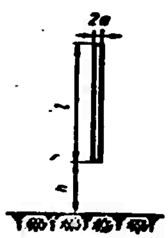


Fig. 3-55. Straight wire of finite length perpendicular to a plane.

<u>Example</u> 3-16. To determine the capacitance to the ground of a vertical l = 12 m long and 6 mm in diameter, the lower end of which is at a distance of 3 m from the surface of the earth.

In this case $m/l = 0.25$, therefore, the value of D_2 in Table 3-7 is equal to 0.291. Using then formulas (3-76) and (3-43a), we find

$$C \approx \frac{2 \cdot 27{,}84 \cdot 12}{8{,}28 - 2{,}303 \cdot 0{,}291} \approx 85 \text{ pF}.$$

b) n identical wires lying in one plane (Fig. 3-56):

$$C = 2C', \tag{3-77}$$

where C' is determined from formulas (3-59), (3-59a) when $m = h$.

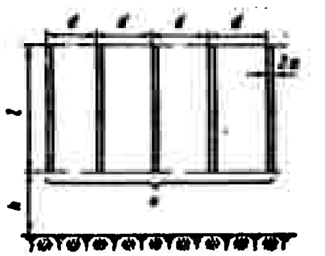


Fig. 3-56. n identical wires perpendicular to a boundary and lying in one plane.

Example 3-17. To determine the capacitance between grounds and vertical antenna formed by 6 rectilinear wires $l = 12$ m and 6 mm in diameter, if the distance between neighboring wires is $d = 0.6$ m, and the distance of the lower end of each wire up to the ground is $h = 3.0$ m.

Using Tables 3-7 and 3-8, we find that at assigned dimensions and number of wires of the system $D_2 = 0.291$, but $B_n = 0.252$.

Using then formulas (3-77) and (3-59a), we find that

$$C \simeq \frac{2 \cdot 27{,}84 \cdot 12 \cdot 6}{8{,}28 + 5 \cdot 2{,}995 - 2{,}303\,(0{,}291 + 0{,}252)} = 256 \quad \text{pF}.$$

c) n identical wires arranged on the surface of a circular cylinder (Fig. 3-57):

$$C \simeq \frac{2\pi\varepsilon n l}{2{,}303 F_2}, \qquad (3\text{-}78)$$

where

$$F_2 = \lg\frac{l}{a} + \sum_{r=1}^{r=n-1}\left(\lg\frac{l}{d_r} + 0{,}434\frac{d_r}{l}\right) - nD_2.$$

$$d_r = 2R\sin r\frac{\pi}{n}\ (r = 1, 2, \ldots, n-1),$$

and D_2 is found from Table 3-7.

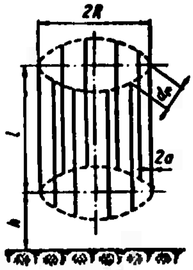


Fig. 3-57. *n* identical wires perpendicular to the boundary and arranged over the surface of a circular cylinder.

When $\varepsilon = \varepsilon_0$

$$C\,(\mathrm{pF}) \simeq \frac{24{,}16nl}{F_4}. \qquad (3\text{-}78a)$$

3. *A wire in the form of a circular ring parallel to a boundary* (Fig. 3-58):

$$C = 2C', \qquad (3\text{-}79)$$

where C' determined from formula (3-54).

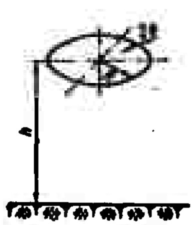


Fig. 3-58. A circular ring lying in a plane parallel to the flat surface of a conducting medium.

4. Γ*-shaped wires lying in planes perpendicular to a boundary.*

a) One wire (Fig. 3-59):

$$C \simeq \frac{2\pi\varepsilon\,(l_1 + l_2)}{\frac{l_2}{l_1 + l_2}\left[\ln\frac{2(h + l_1)}{a} - 2{,}303D_1\right] + \frac{l_1}{l_1 + l_2}\left(\ln\frac{l_1}{a} - 2{,}303D_2\right) + 2{,}303D_3}, \qquad (3\text{-}80)$$

where coefficient D_1 is determined from Table 3-6 at $d = 2(h + l_1)$, $l = l_2$; coefficient D_2 from Table 3-7 at $m = h$, $l = l_1$, and coefficient D_3 from Table 3-10.

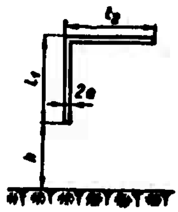


Fig. 3-59. Γ-shaped wire arranged in a plane perpendicular to a boundary.

Table 3-10. The values of the coefficient D_3 which enters formula (3-80), when $l_2/l_1 \leq 1$.

$\frac{l_2}{l_1}$	h/l_1						l_1/h				
	0	0,2	0,4	0,6	0,8	1,0	0,8	0,6	0,4	0,2	0
0,0	0	0	0	0	0	0	0	0	0	0	0
0,1	0,130	0,137	0,141	0,144	0,146	0,147	0,147	0,150	0,153	0,155	0,159
0,2	0,169	0,200	0,207	0,213	0,216	0,218	0,221	0,224	0,228	0,232	0,239
0,3	0,222	0,237	0,247	0,254	0,260	0,265	0,269	0,272	0,275	0,282	0,291
0,4	0,241	0,259	0,271	0,279	0,285	0,290	0,295	0,300	0,306	0,314	0,325
0,5	0,250	0,271	0,285	0,295	0,302	0,307	0,312	0,318	0,325	0,335	0,348
0,6	0,254	0,277	0,292	0,303	0,310	0,317	0,323	0,330	0,338	0,349	0,363
0,7	0,254	0,279	0,295	0,306	0,314	0,322	0,329	0,336	0,346	0,357	0,373
0,8	0,252	0,278	0,295	0,307	0,316	0,324	0,331	0,340	0,350	0,362	0,379
0,9	0,248	0,275	0,293	0,306	0,315	0,323	0,330	0,339	0,350	0,364	0,382
1,0	0,243	0,271	0,290	0,303	0,313	0,321	0,329	0,338	0,350	0,365	0,383

when $l_2/l_1 > 1$

$\frac{l_1}{l_2}$	h/l_1						l_1/h				
	0	0,2	0,4	0,6	0,8	1,0	0,8	0,6	0,4	0,2	0
0	0	0	0	0	0	0	0	0	0	0	0
0,1	0,055	0,064	0,072	0,078	0,083	0,088	0,093	0,097	0,106	0,125	0,158
0,2	0,099	0,116	0,129	1,137	0,146	0,155	0,165	0,174	0,187	0,207	0,239
0,3	0,135	0,157	0,173	0,184	0,195	0,206	0,214	0,226	0,241	0,262	0,291
0,4	0,164	0,189	0,207	0,222	0,233	0,243	0,252	0,263	0,276	0,296	0,325
0,5	0,186	0,214	0,233	0,248	0,260	0,269	0,278	0,290	0,305	0,323	0,348
0,6	0,204	0,233	0,253	0,267	0,278	0,286	0,297	0,309	0,323	0,340	0,363
0,7	0,218	0,247	0,267	0,282	0,293	0,302	0,311	0,322	0,335	0,352	0,373
0,8	0,229	0,258	0,278	0,292	0,302	0,311	0,320	0,330	0,342	0,358	0,379
0,9	0,237	0,265	0,285	0,298	0,308	0,317	0,326	0,336	0,347	0,362	0,382
1,0	0,213	0,271	0,290	0,303	0,313	0,321	0,329	0,338	0,350	0,365	0,383

When $\varepsilon = \varepsilon_0$

$$C\,(\mathrm{pF}) \simeq \frac{24{,}16\,(l_1 + l_2)}{\frac{l_2}{l_1 + l_2}\left[\lg\frac{2(h + l_1)}{a} - D_1\right] + \frac{l_1}{l_1 + l_2}\left(\lg\frac{l_1}{a} - D_2\right) + D_3}. \qquad (3\text{-}80a)$$

Example 3-18. To determine the capacity between grounds and an antenna consisting of a horizontal wire $l_2 = 30$ m long and $2a = 6$ mm in diameter arranged at an altitude of $l_1 + h = 15$ m, and of a vertical overhang of the same diameter $l_1 = 12$ m.

Using formula (3-80a), we compute the quantities preliminarily entering it. At the assigned parameters of a system

$$a = 0{,}003\ \text{m};\ h = 3\ \text{m};\ \lg\frac{2(h+l_1)}{a} = 40;$$

$$\lg\frac{l_1}{a} = \lg\frac{12}{0{,}003} = 3{,}602.$$

Concerning $\frac{d}{l} = \frac{2(h+l_1)}{l_2} = \frac{30}{30} = 1$ from Table 3-6 we find that $D_1 = 0.336$; concerning $\frac{m}{l} = \frac{h}{l_1} = \frac{3}{12} = 0{,}25$ from Table 3-7 we find that $D_2 = 0.291$; according to known relationships $\frac{l_1}{l_2} = \frac{12}{30} = 0{,}4$ and $\frac{h}{l_1} = \frac{3{,}0}{12} = 0{,}25$ from Table 3-10 we find that $D_3 = 0.194$.

Thus, we obtain that

$$C \simeq \frac{24{,}16\,(30+12)}{0{,}714\,(4{,}0-0{,}336)+0{,}286\,(3{,}602-0{,}291)+0{,}194} = \frac{1020}{3{,}769} = 271\ \text{pF}.$$

Let us note that during the determination of capacitance of the antenna being considered by the addition of the capacitances of horizontal and vertical wires (see examples 3-13 and 3-16) its value proves to be equal to 283 pF, i.e., 4.5% more than that calculated using formula (3-80a), considering the mutual effect of the wires.

b) n parallel wires (Fig. 3-60).

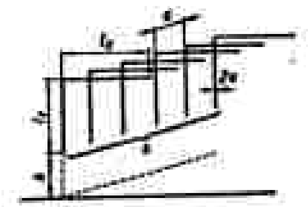


Fig. 3-60. Γ-shaped wires lying in parallel planes perpendicular to a boundary.

$$C \simeq \frac{2\pi\varepsilon\,(l_1+l_2)}{2{,}303L}, \tag{3-81}$$

where

$$L = \frac{1}{n}\left\{\frac{l_2}{l_1+l_2}\left(\lg\frac{2(h+l_1)}{a} - D_1\right) + \frac{l_1}{l_1+l_2}\left(\lg\frac{l_1}{a} - D_2\right) + (n-1)\left[\frac{l_2}{l_1+l_2}\left(\lg\frac{2(h+l_1)}{d} - D_1\right) + \frac{l_1}{l_1+l_2}\left(\lg\frac{l_1}{d} - D_2\right)\right]\right\} + D_3 - B_n.$$

and coefficient D_1, D_2 and D_3 are determined just as in the case of a single Γ-shaped wire, and coefficient B_n is found from Table 3-8.

At $\varepsilon = \varepsilon_0$

$$C \simeq \frac{24{,}16\,(l_1 + l_2)}{L} \text{ pF}. \tag{3-81a}$$

Example 3-19. To determine capacitance between grounds and by an antenna formed by pairwise connection of each of the horizontal and vertical wires considered in examples 3-14 and 3-17.

In this case $\frac{l_1}{l_2} = \frac{12}{30} = 0{,}4$ and $\frac{h}{l_1} = 0{,}25$ and from Table 3-10 we find, that $D_3 = 0.194$. Then, using data obtained in examples 3-14 and 3-17, we find that

$$L = \frac{1}{6}\left\{\frac{30}{30+12}(4{,}0 - 0{,}336) + \frac{12}{30+12}(3{,}602 - 0{,}291) + 5\left[\frac{30}{30+12}(1{,}695 - 0{,}336) + \frac{12}{30+12}(1{,}301 - 0{,}291)\right]\right\} - 0{,}252 + 0{,}194 = 1{,}588.$$

Substituting the obtained values into formula (3-81a), we have

$$C \simeq \frac{24{,}16\,(30+12)}{1{,}588} = 643 \text{ pF}.$$

If it is simple to summarize the capacitances obtained in examples 3-14 and 3-17, then $C = 744$ pF, which is 15.7% more than the quantity calculated using formula (3-82a), taking into account the mutual effect of horizontal and vertical wires.

5. *T-shaped wires lying in the planes perpendicular to the boundary.*

a) One wire (Fig. 3-61)

$$C \simeq \frac{2\pi\varepsilon\,(l_1 + 2l_2)}{\frac{2l_2}{l_1+2l_2}\left[\ln\frac{2(h+l_1)}{a} - 2{,}303D_1\right] + \frac{l_1}{l_1+2l_2}\left(\ln\frac{l_1}{a} - 2{,}303D_2\right) + N}, \tag{3-82}$$

$$N = 2{,}303\,\frac{2(l_1+l_2)}{l_1+2l_2}\,D_3.$$

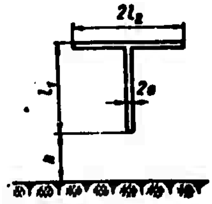


Fig. 3-61. The T-shaped wire arranged in the plane perpendicular to a boundary.

where the coefficient D_1 is determined from Table 3-6 when $d = 2(h + l_1)$; $l = l_2$; coefficient D_2 from Table 3-7 when $m = h$; $l = l_1$, and coefficients D_3 from Table 3-10.

When $\varepsilon = \varepsilon_0$

$$C \simeq \frac{24{,}16\,(l_1 + 2l_2)}{\frac{2l_2}{l_1 + 2l_2}\left[\lg\frac{2(h + l_1)}{a} - D_1\right] + \frac{l_1}{l_1 + 2l_2}\left(\lg\frac{l_1}{a} - D_2\right) + N}\ \text{pF}, \quad (3\text{-}82a)$$

$$N = \frac{2(l_1 + l_2)}{l_1 + 2l_2} D_3.$$

Example 3-20. To determine the capacitance between grounds and by a T-shaped antenna if its horizontal and vertical wires have the same sizes as in examples 3-13 and 3-16.

In this case $2l_2 = 30$ m; $l_1 = 12$ m; $h = 3$ m, $a = 0{,}003$

At such values $\frac{2l_2}{2l_2 + l_1} = \frac{30}{42} = 0{,}714$; $\frac{l_1}{2l_2 + l_1} = \frac{12}{42} = 0{,}286$. $\frac{2l_2 + 2l_1}{2l_2 + l_1} = \frac{54}{42} = 1{,}286$; $D_1 = 0{,}336$; $D_2 = 0{,}291$.

From Table 3-10 we further find that when $\frac{h}{l_1} = 0{,}25$ and $\frac{l_1}{l_2} = 0{,}8$ $D_3 = 0{,}263$.

Substituting the obtained quantities in formula (3-82a), we find

$$C \approx \frac{24{,}16\,(30 + 12)}{3{,}901} = 262\ \text{pF}.$$

Using this value, it can be established that simple addition of the capacitance of wires making up an antenna gives error of the order of 8%, and the relative difference in the values of the capacitance of T- and Γ-shaped antennas at the same length of horizontal and vertical wires is about 3.5% (compare example 3-18).

b) n parallel wires (Fig. 3-62)

$$C \simeq \frac{2\pi\varepsilon (l_1 + 2l_2)}{2{,}303T}, \tag{3-83}$$

where

$$T = \frac{1}{n}\left\{\frac{2l_2}{l_1 + 2l_2}\left[\lg\frac{2(l_1 + h)}{a} - D_1\right] + \frac{l_1}{l_1 + 2l_2}\left(\lg\frac{l_1}{a} - D_2\right) + \right.$$
$$\left. + (n-1)\left[\frac{2l_2}{l_1 + 2l_2}\left(\lg\frac{2(l_1 + h)}{d} - D_1\right) + \frac{l_1}{l_1 + 2l_2}\left(\lg\frac{l_1}{d} - D_2\right)\right]\right\} -$$
$$- B_n + \frac{2(l_1 + l_2)}{l_1 + 2l_2} D_3,$$

and coefficients D_1, D_2 and D_3 are determined just as in the case of a single T-shaped wire, and coefficient B_n is found from Table 3-8.

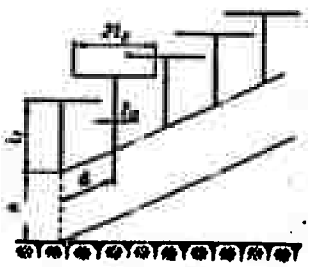


Fig. 3-62. Several T-shaped wires lying in parallel planes perpendicular to a boundary.

When $\varepsilon = \varepsilon_0$

$$C \simeq \frac{24{,}16(l_1 + 2l_2)}{T} \text{ pF}. \tag{3-83a}$$

6. *A V-shaped wire parallel to a boundary* (Fig. 3-63).

$$C \simeq \frac{2\pi\varepsilon (l_1 + l_2)}{\frac{l_1}{l_1 + l_2}\left(\ln\frac{2h}{a_1} - 2{,}303D_1\right) + \frac{l_2}{l_1 + l_2}\left(\ln\frac{2h}{a_2} - 2{,}303D_1'\right) + N}, \tag{3-84}$$

$$N = 2{,}303(Y_1 - Y_2)$$

where coefficient D_1 is determined from Table 3-6 at $l = l_1$, $d = 2h$; coefficient D_1' also from Table 3-6, but at $l = l_2$, $d = 2h$; and coefficients Y_1 and Y_2 from Table 3-11 and 3-12 respectively from the value of the angle θ and the relationships $2h/l_1$ and l_2/l_1.

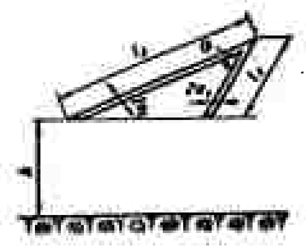


Fig. 3-63. A V-shaped wire lying in a plane parallel to a boundary.

When $\varepsilon = \varepsilon_0$

$$C \simeq \frac{24{,}16\,(l_1 + l_2)}{\frac{l_1}{l_1 + l_2}\left(\lg \frac{2h}{a_1} - D_1\right) + \frac{l_2}{l_1 + l_2}\left(\lg \frac{2h}{a_2} - D_1'\right) + Y_1 - Y_2}. \qquad (3\text{-}84a)$$

Example 3-21. To determine the capacitance between grounds and horizontal V-shaped antenna, at an altitude of $h = 15$ m and formed by wires $2a = 6$ mm in diameter and $l_1 = 30$ m and $l_2 = 15$ m long intersecting at an angle of $\theta = 45°$.

In this case $\frac{2h}{a} = 10\,000$, $\frac{2h}{l_1} = 1$; $\frac{2h}{l_2} = 2$, and from Table 3-6 we find that $D_1 = 0{,}336$, and $D_1' = 0{,}541$.

From Table 3-11 we obtain that at $\theta = 45°$ and $\frac{l_2}{l_1} = \frac{1}{2}$ coefficients $Y_1 = 0.497$, and from Table 3-12 we find that $Y_2 = 0.131$. Then $Y = Y_1 - Y_2 = 0.366$.

From formula (3-84a) the capacitance sought is

$$C \simeq \frac{24{,}16\,(30 + 15)}{3{,}962} = 276 \text{ pF}.$$

3-6. Capacitance in a System of Many Wires

In the present paragraph formulas are given for the calculation

l/t / θ, deg.	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
180	0,3010	0,3004	0,2983	0,2942	0,2873	0,2764	0,2598	0,2346	0,1957	0,1323
165	3029	3023	3002	2960	2891	2781	2613	2359	1967	1329
150	3086	3080	3056	3016	2944	2832	2660	2400	1999	1318
135	1185	3179	3156	3112	3037	2920	2741	2469	2053	1380
120	3334	3326	3303	3255	3176	3051	2860	2573	2134	1427
105	3542	3534	3508	3457	3370	3234	2028	2714	2244	1492
90	3828	3820	3780	3732	3635	3483	3254	2911	[illegible]	1578
85	3945	3936	3905	3844	3743	3584	3346	2989	2453	1612
80	4075	4066	4033	3970	3863	3697	3448	3076	2518	1650
75	4220	4211	4176	4109	2997	3823	3560	3172	2591	1691
70	4383	4372	4336	4265	4146	3962	3686	3277	2670	1736
65	4565	4554	4515	4440	4313	4118	3825	3395	2755	1786
60	4771	4759	4718	4636	4501	4292	3981	3526	3857	1842
55	5004	4992	4946	4859	4713	4489	4156	3678	2966	1903
50	5271	5257	5208	5112	4954	4712	4354	3838	3089	1971
45	5579	5563	5509	5404	5230	4966	4580	4025	3227	2048
40	5937	5920	5859	5742	5550	5260	4839	4239	3354	2136
35	6360	6340	6272	6140	5925	5603	5139	4486	3566	2236
30	6870	6846	6767	6616	6371	6009	5494	4778	3780	2354
25	7498	7470	7376	7198	6915	6502	5923	5128	4035	2494
20	8299	8264	8148	7933	7598	7118	6457	5563	4351	2668
15	9376	9330	9180	8909	8499	7926	7155	6129	4762	2892
10	10960	10892	10681	10318	9793	9082	8149	6934	5345	3210
5	13789	13663	13314	12771	12034	11079	9863	8320	6346	3787

of the partial capacitances of typical systems of the many infinitely long rectilinear wires arranged either in an infinite space or near an infinite flat conducting boundary.

Whole systems considered below are considered electroneutral (see § V-1), in connection with which only mutual partial capacitances are determined for them.

In this paragraph formulas are given for the capacitance between two wires in the presence of other uncharged conductors.

1. *A three-wire line in infinite space* (Fig. 3-64).

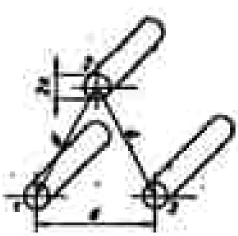


Fig. 3-64. A symmetric three-wire line in an infinite medium.

Table 3-12. The values of the coefficient γ_2, which enters formula (3-84).

$2h/l_1$ / θ, deg.	0,2	0,5	1,0	2,0	5,0	Note
0	0,648	0,359	0,203	0,106	0,043	$\frac{l_2}{l_1} = 1$
15	584	349	202	106	043	
30	497	328	197	106	043	
45	432	304	191	1045	043	
60	384	282	185	103	043	
75	348	264	178	102	043	
90	321	249	172	101	043	
105	300	237	167	099	043	
120	285	228	163	098	0425	
135	274	221	160	097	0425	
150	267	216	158	097	0425	
165	262	213	156	096	0425	
180	261	212	156	096	0425	
0	0,571	0,312	0,175	0,091	0,037	$\frac{l_2}{l_1} = 0{,}75$
15	528	306	174	091	037	
30	461	292	171	091	037	
45	406	274	167	091	037	
60	364	257	163	090	037	
75	331	242	158	089	037	
90	307	230	154	088	037	
105	288	220	150	087	037	
120	274	212	147	086	037	
135	264	206	144	086	037	
150	257	202	142	085	037	
165	253	199	141	085	037	
180	251	198	141	085	037	
0	0,432	0,239	0,135	0,071	0,029	$\frac{l_2}{l_1} = 0{,}5$
15	414	236	135	071	029	
30	379	229	133	071	029	
45	343	221	131	0705	029	
60	313	210	129	070	029	
75	289	200	126	0695	029	
90	270	192	124	069	029	
105	255	186	121	069	029	
120	244	180	1195	068	029	
135	235	175	118	068	0285	
150	230	172	117	0675	0285	
165	225	171	116	067	0285	
180	223	170	116	067	0285	
0	0,238	0,136	0,079	0,042	0,017	$\frac{l_2}{l_1} = 0{,}25$
15	235	136	079	042	017	
30	226	134	079	042	017	
45	0,215	0,131	0,78	0,042	0,017	$\frac{l_2}{l_1} = 0{,}25$
60	204	128	775	042	017	
75	194	126	077	042	017	
90	185	122	076	042	017	
105	178	120	075	042	017	
120	172	117	074	042	017	
135	167	116	074	041	017	
150	164	114	073	041	017	
165	162	113	073	041	017	
180	161	113	073	041	017	
0	0,099	0,059	0,035	0,019	0,008	$\frac{l_2}{l_1} = 0{,}1$
15	099	059	035	019	008	
30	097	059	035	019	008	
45	096	058	035	019	008	
60	092	058	035	019	008	
75	092	057	035	019	008	
90	020	057	035	019	008	
105	088	056	035	019	008	
120	086	056	034	019	008	
135	085	055	034	019	008	
150	084	055	034	019	008	
165	084	055	034	019	008	
180	0835	065	034	019	008	

Partial capacitances are determined by the formula

$$C'_{12l} = C'_{23l} = C'_{31l} = \frac{2\pi\varepsilon}{3\ln\frac{d\sqrt{3}}{a}}. \qquad (3\text{-}85)$$

The capacitance between any two wires in the presence of a third is determined by the formula

$$C_l \simeq \frac{\pi\varepsilon}{\ln\frac{d\sqrt{3}}{a}}. \qquad (3\text{-}86)$$

2. *A two-wire line over a flat conducting boundary* (Fig. 3-65).

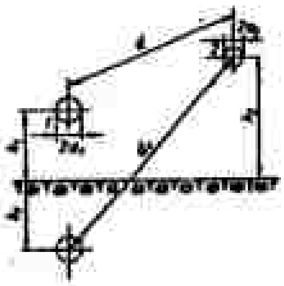


Fig. 3-65. A two-wire line over a flat conducting boundary (grounds).

a) General case:

$$\left.\begin{aligned}
C'_{10l} &\simeq 2\pi\varepsilon\frac{\ln\frac{2h_2}{a_2}-\ln\frac{d_1}{d}}{\ln\frac{2h_1}{a_1}\ln\frac{2h_2}{a_2}-\ln^2\frac{d_1}{d}};\\
C'_{20l} &\simeq 2\pi\varepsilon\frac{\ln\frac{2h_1}{a_1}-\ln\frac{d_1}{d}}{\ln\frac{2h_1}{a_1}\ln\frac{2h_2}{a_2}-\ln^2\frac{d_1}{d}};\\
C'_{12l} &\simeq 2\pi\varepsilon\frac{\ln\frac{d_1}{d}}{\ln\frac{2h_1}{a_1}\ln\frac{2h_2}{a_2}-\ln^2\frac{d_1}{d}}.
\end{aligned}\right\} \qquad (3\text{-}87)$$

b) Both wires are the same distance from the boundary: $a_1 = a_2 = a$; $h_1 = h_2 = h$.

$$\left.\begin{aligned} C'_{10t} = C'_{20t} &\simeq \frac{2\pi\varepsilon_0}{\ln\left[\frac{2h}{a}\sqrt{1+\left(\frac{2h}{d}\right)^2}\right]}; \\ C'_{12t} &= \frac{2\pi\varepsilon_0 \ln\sqrt{1+\left(\frac{2h}{d}\right)^2}}{\ln\left[\frac{2h}{a}\sqrt{1+\left(\frac{2h}{d}\right)^2}\right]\ln\left[\frac{2h}{a\sqrt{1+(2h/d)^2}}\right]}. \end{aligned}\right\} \tag{3-88}$$

c) Both wires are in a plane perpendicular to the boundary.

In this instance in formulas (3-87) it is necessary to place

$$d = h_2 - h_1; \quad d_1 = h_1 + h_2.$$

The capacitance between wires in the presence of a boundary in any of the cases *a*, *b*, or *c* is determined by the formula

$$C'_t = C'_{12t} + \frac{C'_{10t}C'_{20t}}{C'_{10t} + C'_{20t}}. \tag{3-89}$$

3. *A three-wire line over a flat conducting boundary (ground)* (Fig. 3-66).

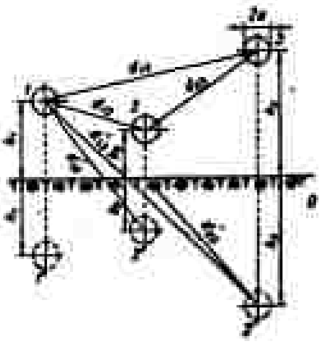


Fig. 3-66. A three-wire line over flat conducting boundary (ground) .

a) General case.

Partial capacitances are determined by the formulas:

$$
\left.
\begin{aligned}
C'_{10l} &= \frac{\alpha_{23}(\alpha_{13}-\alpha_{23})+\alpha_{22}(\alpha_{33}-\alpha_{13})+\alpha_{12}(\alpha_{23}-\alpha_{33})}{D};\\
C'_{20l} &= \frac{\alpha_{13}(\alpha_{23}-\alpha_{13})+\alpha_{11}(\alpha_{33}-\alpha_{23})+\alpha_{12}(\alpha_{13}-\alpha_{33})}{D};\\
C'_{30l} &= \frac{\alpha_{12}(\alpha_{23}-\alpha_{12})+\alpha_{11}(\alpha_{22}-\alpha_{23})+\alpha_{13}(\alpha_{12}-\alpha_{22})}{D};\\
C'_{12l} &= \frac{\alpha_{12}\alpha_{33}-\alpha_{13}\alpha_{23}}{D};\\
C'_{13l} &= \frac{\alpha_{13}\alpha_{22}-\alpha_{12}\alpha_{23}}{D};\\
C'_{23l} &= \frac{\alpha_{11}\alpha_{23}-\alpha_{12}\alpha_{13}}{D},
\end{aligned}
\right\}
\qquad (3\text{-}90)
$$

where
$$
D=\begin{vmatrix}\alpha_{11}, & \alpha_{12}, & \alpha_{13}\\ \alpha_{21}, & \alpha_{22}, & \alpha_{23}\\ \alpha_{31}, & \alpha_{32}, & \alpha_{33}\end{vmatrix}=\alpha_{11}(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+
$$
$$
+\alpha_{12}(\alpha_{31}\alpha_{23}-\alpha_{21}\alpha_{33})+\alpha_{13}(\alpha_{21}\alpha_{32}-\alpha_{22}\alpha_{31});
$$

$$
\alpha_{11}\simeq\frac{1}{2\pi\varepsilon}\ln\frac{2h_1}{a};\quad \alpha_{12}=\alpha_{21}\simeq\frac{1}{2\pi\varepsilon}\ln\frac{d'_{12}}{d_{12}};
$$
$$
\alpha_{22}\simeq\frac{1}{2\pi\varepsilon}\ln\frac{2h_2}{a};\quad \alpha_{13}=\alpha_{31}\simeq\frac{1}{2\pi\varepsilon}\ln\frac{d'_{13}}{d_{13}};
$$
$$
\alpha_{33}\simeq\frac{1}{2\pi\varepsilon}\ln\frac{2h_3}{a};\quad \alpha_{23}=\alpha_{32}\simeq\frac{1}{2\pi\varepsilon}\ln\frac{d'_{23}}{d_{23}}.
$$

d_{ik} is the distance between the *i*-th and *k*-th by wires; d'_{ik} is the distance between the *i*-th wire and the mirror image of the *k*-th wire.

b) The wires lie in one plane (parallel to the boundary) at equal distances from one another: ($h_1 = h_2 = h_3 = h$ and $d_{12} = d_{23} = d$).

Partial capacitances are determined by formulas (3-90) when

$$
\alpha_{11}=\alpha_{22}=\alpha_{33}\simeq\frac{1}{2\pi\varepsilon}\ln\frac{2h}{a};\quad \alpha_{12}=\alpha_{23}\simeq\frac{1}{2\pi\varepsilon}\ln\sqrt{1+\left(\frac{2h}{d}\right)^2};
$$
$$
\alpha_{13}\simeq\frac{1}{2\pi\varepsilon}\ln\sqrt{1+\left(\frac{h}{d}\right)^2}.
$$

4. *A four-wire line two wires of which are united* (Fig. 3-67).

When $d/a > 2$ the capacitance between wires 1 and 2 is determined by the formula

$$C_l \simeq \frac{2\pi\varepsilon}{\ln\frac{2d}{a} - 0{,}6\left(\frac{a}{d}\right)^2}. \qquad (3\text{-}91)$$

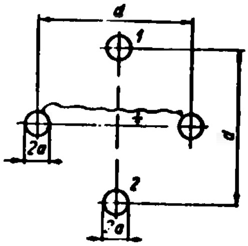


Fig. 3-67. A four-wire line consisting of two pairs of identical wires lying in mutually perpendicular planes.

5. *Two wires inside a cylindrical shell* (Fig. 3-68).

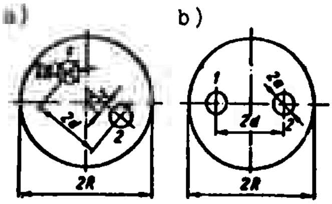


Fig. 3-68. Two wires of infinite inside a grounded cylindrical shell: a) the wires are located eccentrically relative to the axis of the shell; b) the wires are symmetrical relative to the axis of the shell.

The capacitance between wires 1 and 2 is determined by the formulas:

a) in the case of an asymmetric system (Fig. 3-68a)

$$C_l \simeq \frac{\pi\varepsilon}{\ln\frac{2d}{a} - \frac{2d^2R^2}{(R^2-c^2)^2} - \left(\frac{a}{2d}\right)^2\left[1 - \frac{4d^2R^2}{(R^2-c^2)^2}\right]}; \qquad (3\text{-}92)$$

b) in the case of a symmetric system (Fig. 3-68b)

$$C_l \simeq \frac{\pi\varepsilon}{\ln\left(\frac{R^2-d^2}{R^2+d^2}\cdot\frac{2d}{a}\right)}. \qquad (3\text{-}93)$$

6. *Two wires arranged between two grounded planes* (Fig. 3-69).

The capacitance between wires is determined by the formula:

$$C_1 \simeq \frac{2\pi\varepsilon}{\ln\left(\frac{2h}{\pi a}\,\mathrm{tg}\,\frac{\pi d}{2h}\right)}. \qquad (3\text{-}94)$$

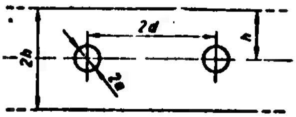


Fig. 3-69. Two wires arranged between two grounded planes.

7. *Three wires inside a cylindrical shell* (Fig. 3-70).

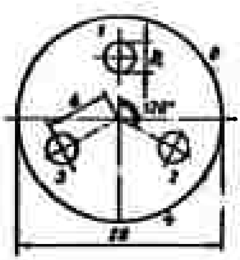


Fig. 3-70. Three infinitely long wires inside a cylindrical shell.

Partial capacitances are determined by the formula

$$C_{12l} = C_{13l} = C_{23l} \simeq \frac{2\pi\varepsilon}{3\ln\left[\frac{\sqrt{3}\,d}{a}\cdot\frac{R^2-d^2}{\sqrt{d^4+R^4+R^2d^2}}\right]} - \frac{1}{3}C'_{il}, \qquad (3\text{-}95)$$

where

$$C'_{il} \simeq \frac{2\pi\varepsilon}{\ln\left(\frac{R^3}{3d^2a}\left[1-\left(\frac{d}{R}\right)^6\right]\right)}, \quad (i = 1, 2, 3). \qquad (3\text{-}96)$$

8. *A wire and two cylindrical shells coaxial with it, one of which (the interior) is not closed* (Fig. 3-71).

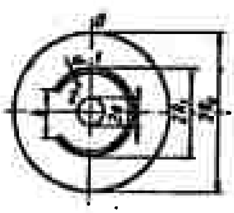


Fig. 3-71. An infinitesimally long wire surrounded by an open cylindrical shell and inside a cylindrical tube.

Partial capacitances are determined by the formulas:

$$C'_{10l} \simeq \frac{2\pi\varepsilon}{\ln\frac{R_2}{R_1}}; \qquad (3\text{-}97)$$

$$C'_{21l} \simeq \frac{2\pi\varepsilon}{\ln\frac{R_1}{a}}; \qquad (3\text{-}98)$$

$$C_{20l} \simeq \left(\frac{b}{\pi R_1}\right)^2 e^{-\frac{\pi d}{b}-1}\,\frac{C'_{10l}\cdot C'_{21l}}{\pi\varepsilon}. \qquad (3\text{-}99)$$

9. *A central wire and wire on the circumference inside a cylindrical shell* (Fig. 3-72).

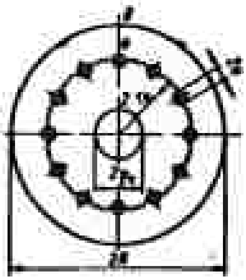


Fig. 3-72. A central wire and wires on a circumference inside a cylindrical shell.

$$C'_{10l} \simeq \pi\varepsilon\,\frac{\ln\frac{R}{\rho_0}}{\ln\frac{r_0}{\rho_0}\ln\frac{R}{r_0}+\frac{1}{2n}\ln\frac{R}{\rho_0}\ln\left\{\frac{r_0}{na}\left[1-\left(\frac{r_0}{R}\right)^{2n}\right]\right\}}; \qquad (3\text{-}100)$$

$$C'_{21l} \simeq \pi\varepsilon\,\frac{\ln\frac{R}{r_0}-\ln\frac{r_0}{\rho_0}}{\ln\frac{r_0}{\rho_0}\ln\frac{R}{r_0}+\frac{1}{2n}\ln\frac{R}{\rho_0}\ln\left\{\frac{r_0}{na}\left[1-\left(\frac{r_0}{R}\right)^{2n}\right]\right\}}, \qquad (3\text{-}101)$$

where n is the number of wires.

10. *Two wires on different sides of a flat plate of finite thickness, having a cut* (Fig. 3-73).

Partial capacitances are determined by the formulas

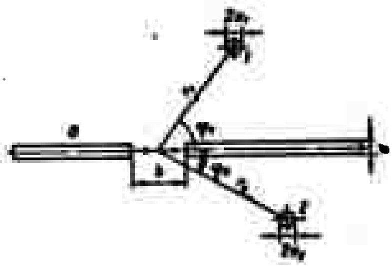


Fig. 3-73. Two wires on different sides of a plane with a slit.

$$C'_{12l} \simeq \left(\frac{2b}{\pi}\right)^2 e^{-\frac{\pi d}{b}-2} \cdot \frac{\sin\varphi_1 \cdot \sin\varphi_2}{r_1 r_2} \cdot \frac{C'_{10l} \cdot C'_{20l}}{\pi\varepsilon}; \qquad (3\text{-}102)$$

$$C'_{10l} \simeq \frac{2\pi\varepsilon}{\ln\left(\frac{r_1 \sin\varphi_1}{2a_1}\right) + \sqrt{\left(\frac{r_1 \sin\varphi_1}{2a_1}\right)^2 - 1}}; \qquad (3\text{-}103)$$

$$C'_{20l} \simeq \frac{2\pi\varepsilon}{\ln\left(\frac{r_2 \sin\varphi_2}{2a_2}\right) + \sqrt{\left(\frac{r_2 \sin\varphi_2}{2a_2}\right)^2 - 1}}. \qquad (3\text{-}104)$$

11. *Two wires on different sides of an infinite grating of plates of finite thickness.*

a) The wires are located at random (Fig. 3-74a):

$$C'_{12l} \simeq \left(\frac{2b}{\pi}\right)^2 e^{-\frac{\pi d}{b}-2} \frac{C'_{10l} \cdot C'_{20l}}{\pi\varepsilon} \sum_{\nu=1}^{n} \frac{\sin\varphi_{1\nu} \cdot \sin\varphi_{2\nu}}{r_{1\nu} \cdot r_{2\nu}}; \qquad (3\text{-}105)$$

$$C'_{10l} \simeq \frac{2\pi\varepsilon}{\ln\left[\frac{r_{1\nu} \sin\varphi_{1\nu}}{2a_1} + \sqrt{\frac{r_{1\nu} \sin\varphi_{1\nu}}{2a_1} - 1}\right]}; \qquad (3\text{-}106)$$

$$C'_{20l} \simeq \frac{2\pi\varepsilon}{\ln\left[\frac{r_{2\nu} \sin\varphi_{2\nu}}{2a_2} + \sqrt{\frac{r_{2\nu} \sin\varphi_{2\nu}}{2a_2} - 1}\right]}, \qquad (3\text{-}107)$$

where ν is the number of the plate nearest the wire.

b) The wires are located symmetrically relative to the grating (Fig. 3-74b):

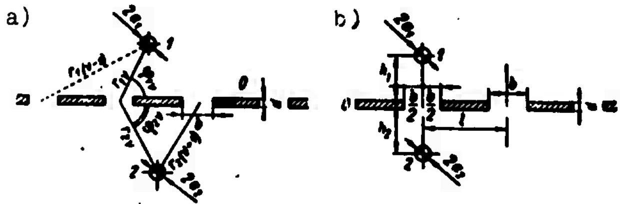


Fig. 3-74. Two wires on different sides of an infinite system of plates: a) wires located at random; b) wires located in a plane perpendicular to plates.

$$C'_{21l} \simeq \left(\frac{2b}{\pi}\right)^2 e^{-\frac{\pi d}{b}-2}\,\frac{C'_{10l}C'_{20l}}{\pi\varepsilon}\begin{cases}\dfrac{1}{h_1\cdot h_2} & \text{when } h_1,\ h_2 \ll l;\\[2mm] \dfrac{1}{l\,(h_1+h_2)} & \text{when } h_1 h_2 \gg l;\end{cases} \tag{3-108}$$

when $h_1 = h_2 = h$

$$C'_{21l} \simeq \left(\frac{2b}{\pi}\right)^2 e^{-\frac{\pi d}{b}-2}\,\frac{C'_{10l}C'_{20l}}{\pi\varepsilon}\,\frac{1}{2lh}\left(\operatorname{cth}\frac{\pi h}{l} + \frac{\frac{\pi h}{l}}{\operatorname{sh}^2\frac{\pi h}{l}}\right), \tag{3-109}$$

where C'_{10l} and C'_{20l} are determined by formulas (3-106) and (3-107).

C H A P T E R 4

CAPACITANCE OF FLAT PLATES

4-1. General Remarks

1. The present chapter contains formulas, tables and graphs for determining the capacitance of conductors having the form of flat plates. In all cases when nothing is said to the contrary, it is assumed that the thickness of the plates is infinitesimal.

2. Data are given on the capacitance of solitary plates, capacitors, formed by plates of finite or infinite dimensions and also about partial capacitances in a system of three infinitely long plates. In this case one ought to have in view that the concept of the capacitance of solitary infinitely long plates does not have meaning.

4-2. Capacitance of Solitary Plates

The present paragraph contains formulas, tables, and graphs for the determination of the capacitance of solitary plates of the following form: a circular disc; a semi-circular plate; an elliptical disc; a rectangular plate; a circular ring; a conductor formed by the union of either two coaxial circular plates, or two coplanar circular discs, or two rectangular plates lying in parallel planes, or two coplanar rectangular plates.

In using the materials of the present paragraph one should have in mind that the capacitance of plates of complex form can be evaluated on the basis of the general features of capacitance (see § V-2), using the given data on the capacitance of circular and elliptical discs.

1. *Circular disc* (Fig. 4-1).

$$C_0 \doteq 8\varepsilon a. \tag{4-1}$$

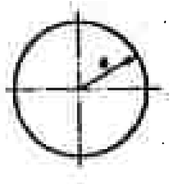


Fig. 4-1. Circular disc.

2. *Semi-circular disc* (Fig. 4-2).

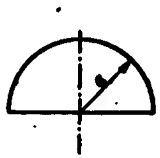


Fig. 4-2. Semi-circular disc.

The value of the capacitance of a semi-circular disc satisfies the following inequalities (compare example 2-4):

$$8\varepsilon a > C_0 > 8\varepsilon a \cdot 0{,}729. \tag{4-2}$$

3. *Elliptical disc* (Fig. 4-3).

$$C_0 = 8\varepsilon a \cdot \frac{\pi}{2K(k)}. \tag{4-3}$$

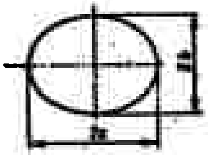
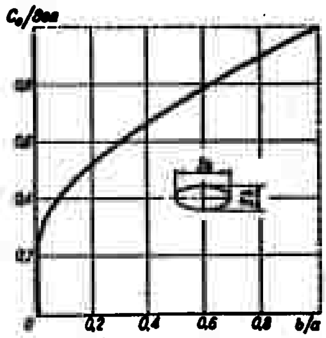


Fig. 4-3. Elliptical disc.

where $K(k)$ — a complete elliptical integral of the first kind (see Appendix I) with modulus $k=\sqrt{1-\left(\frac{b}{a}\right)^2}$.

If the ratio of the axes of an elliptical disc a/b monotonically rises, then at constant area its capacitance also monotonically rises.

The numerical values of the functions $C_0/8\varepsilon a = f(b/a)$ are given in Fig. 4-4.



b/a	0	0.003	0.006	0.008
$\frac{C_0}{8\varepsilon a}$	0	0,2229	0,2366	0,2476
b/a	0.01	0,02	0,03	0,05
$\frac{C_0}{8\varepsilon a}$	0,2630	0,2897	0,3207	0,3562
b/a	0,1	0,2	0,3	0,5
$\frac{C_0}{8\varepsilon a}$	0,4720	0,5208	0,5978	0,7284
b/a	0,7	0,8	1,0	
$\frac{C_0}{8\varepsilon a}$	0,8433	0,8972	1,0	

Fig. 4-4. The relationship of the capacitance of an elliptical disc with semi-axes a and b ($a > b$) to the capacitance of a circular disc of radius a.

4. *A rectangular disc* (Fig. 4-5).

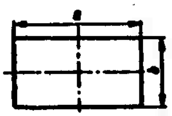


Fig. 4-5. A flat disc of rectangular form.

The accurate value of the capacitance of a conductor in the form of a rectangular (including a square) disc is unknown.[1]

The values of capacitance of a rectangular disc calculated by the method of grounds (see § 1-3), are given in Fig. 4-6.

Furthermore, the following approximation formulas can be used:

a)

$$C_0 \simeq \frac{2\pi\varepsilon a}{m \ln \frac{1+\sqrt{1+m^2}}{m} + \ln\left(m+\sqrt{1+m^2}\right) + \frac{1}{3m} + N}; \qquad (4\text{-}4)$$

$$N = \frac{m^2}{3} - \frac{(1+m^2)\sqrt{1+m^2}}{3m},$$

where $m = a/b$ (see example 1-3):

b)

$$C_0 \simeq \frac{2\pi\varepsilon a}{\ln\left(4\frac{a}{b}\right)}. \qquad (4\text{-}5)$$

---

[1]Determination of the capacitance of a disc of square form was the subject of a number of works. The fundamental results of these works are characterized by the following data for the quantity C_1 (C_1 is the ratio of the capacitance of a square disc to the capacitance of a circular disc with radius equal to the side of the disc):

1. G. Kavendish and J. Maksvell, 1879 [4-1] $C_1 \approx 1.1332$
2. J. Maxwell, 1893 [4-2] $C_1 \approx 0.5666$
3. Rayleigh 1894 [4-3] $C_1 > 0.56418$
4. G. Howe, 1919 [4-4] $C_1 \approx 0.5287$
5. G. Polya and G. Sege, 1951 [1-3] $0.56418 < C_1 < 0.59018$
6. D. Allen and S. Dennis, 1953 [4-5] $C_1 < 0.5682$
7. E. Gross and R. Wise, 1955 [4-6] $C_1 \approx 0.559$
8. D. Reitan and T. Higgins, 1957 [4-7] $C_1 \approx 0.559$

The most complete analysis of the capacitance of rectangular plates is contained in the last two of the works, from the results of which the basic data given in para. 4 were obtained.

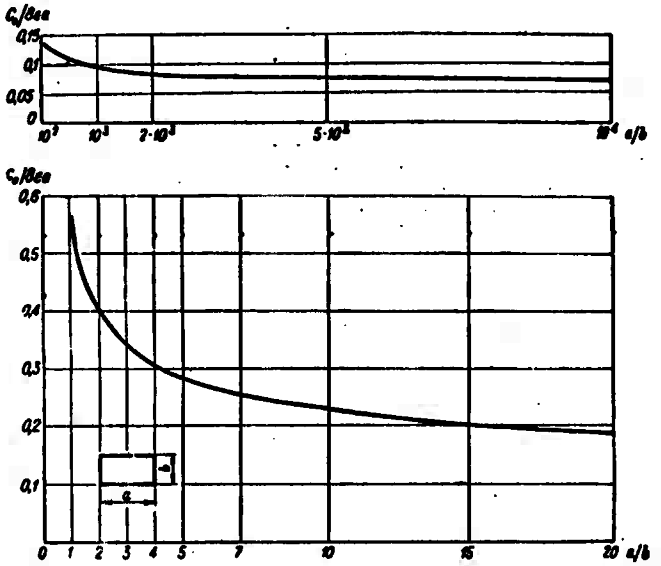


Fig. 4-6. A graph for determination of the capacitance of a flat rectangular disc.

a/b	1,0	1,5	2,0	3,0	4,0	20	10^2	10^3	10^4
$C_0/8\cdot a$	0,566	0,454	0,401	0,339	0,308	0,188	0,130	0,094	0,074

5. *Circular ring* (Fig. 4-7)[1]

$$C_0 = 8a \cdot \int_0^{\operatorname{arcsec}\frac{b}{a}} H(\theta)\,d\theta, \qquad (4\text{-}6)$$

[1]Accurate expressions for the capacitance of a flat circular ring have been obtained for a comparatively long time [4-8 to 4-10]; however; they are so complex that they are of only theoretical interest. The results of Nicholson [4-10] were obtained insufficiently correctly and referred to some particular relationships between radii of a ring. Higgins and Reitan [4-11] and Smayt [4-12] obtained rather accurate numerical results and Smayt also gave approximation formulas. The most complete results for the capacitance of a flat circular ring were obtained by Cook [4-13], who gave an accurate expression for the calculation of capacitance and conducted numerical calculations for the typical relationships of the radii of a ring.

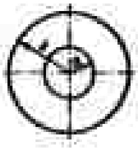


Fig. 4-7. A flat circular ring.

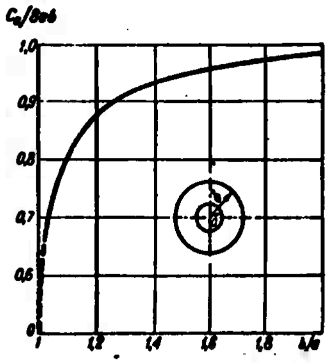
where function $H(\theta)$ is found from solution of the integral equation

$$H(\theta)\cdot\sin\theta\cdot\cos^2\theta+\left(\frac{2}{\pi}\right)^2\int\limits_0^{\operatorname{arcsec}\frac{b}{a}}H(\varphi)\cdot K(\theta,\ \varphi)\,d\varphi=1 \qquad (4\text{-}7)$$

with the nucleus

$$K(\theta,\ \varphi)=\frac{\sin^2\varphi\cdot\sec\varphi\cdot\ln\operatorname{tg}\frac{\varphi}{2}-\sin^2\theta\cdot\sec\theta\cdot\ln\operatorname{tg}\frac{\theta}{2}}{\sec^2\theta-\sec^2\varphi}.$$

The numerical values of the relationship of the capacitance of a ring to the capacitance of a circular disc of radius b are given in Fig. 4-8.



b/a	[illegible]	[illegible]	1.125
$C_0/8\varepsilon b$	[illegible]	0.8011	[illegible]
b/a	1.20	1.25	1.50
$C_0/8\varepsilon b$	[illegible]	[illegible]	0.9494
b/a	2.0	3.0	4.0
$C_0/8\varepsilon b$	0.9610	[illegible]	[illegible]

Fig. 4-8. A graph for calculation of capacitance of a flat circular ring (dotted line – extrapolation).

The capacitance of the ring can also be approximately determined with the aid of the following formulas:

$$C_0 \sim 8\varepsilon b \cdot \frac{2}{\pi}\left[\arccos\frac{a}{b} + \sqrt{1-\left(\frac{a}{b}\right)^2}\cdot \operatorname{Arth}\frac{a}{b}\right]\times$$

$$\times\left[1+\left(0{,}0143\frac{b}{a}\right)\cdot \operatorname{tg}^2\left(1{,}28\frac{a}{b}\right)\right] \qquad (4\text{-}8)$$

$[\delta < 0{,}1\%$ when $b/a > 1{,}1]$;

$$C_0 \simeq \frac{2\pi^2\varepsilon(a+b)}{\ln\left(16\frac{a+b}{b-a}\right)}, \qquad (4\text{-}9)$$

$[\delta < 0{,}1\%$ when $b/a < 1{,}1]$.[1]

6. *Two interconnected coaxial circular discs* (Fig. 4-9).

$$C_0 = 16\varepsilon a\int_0^1 f(t)\,dt, \qquad (4\text{-}10)$$

where $f(t)$ is found from solution of the integral equation

$$f(x) + \frac{1}{\pi}\int_0^1 f(t)\cdot\frac{l/a}{(x-t)^2+\left(\frac{l}{a}\right)^2}\,dt = 1. \qquad (4\text{-}11)$$

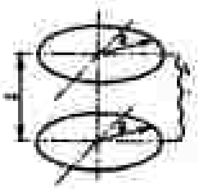


Fig. 4-9. Two interconnected coaxial discs.

The numerical values of the function $\frac{C_0}{8\varepsilon a} = f\left(\frac{l}{a}\right)$ are given in Fig. 4-10. The following approximation formulas can also be used;

[1]If a greater error is allowable, formula (4-9) can be used also for $b/a > 1.1$. Thus when $b/a = 1.25$ $\delta = 0.57\%$, and when $b/a = 2$ $\delta = 2.6\%$.

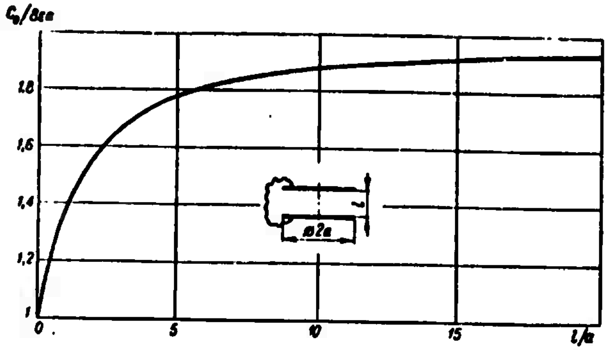


Fig. 4-10. A graph for the calculation of the capacitance of a solitary conductor, formed by the union of two identical coaxial discs.

l/a	0,0	0,4	0,6	0,8	1,0	1,2	1,5
$\frac{C}{8\varepsilon a}$	1,000	1,2064	1,2728	1,3312	1,3824	1,4276	1,4874
l/a	2,0	2,5	3,0	5,0	10,0	20,0	
$\frac{C}{8\varepsilon a}$	1,5631	1,6226	1,6684	1,7792	1,8810	1,937	

a) when $l/a > 1{,}5$

$$C_0 \simeq \frac{16\varepsilon a}{1+\frac{2}{\pi}\frac{a}{l}\left[1-\frac{7}{12}\left(\frac{a}{l}\right)^2+\frac{33}{40}\left(\frac{a}{l}\right)^4\right]} \qquad (4\text{-}12)$$

$[\delta < 3{,}8\%$ when $l/a \geqslant 1{,}5$; $\delta < 0{,}5\%$ when $l/a > 2]$;

b) when $l/a \geqslant 1$

$$C_0 \simeq \frac{16\varepsilon a}{1+\frac{2}{\pi}\operatorname{arcctg}\frac{l}{a}}, \qquad (4\text{-}13)$$

$[\delta < 3{,}6\%$ when $l/a \geqslant 1$; $\delta < 0{,}9\%$ when $l/a > 2{,}5]$

or

$$C_0 \simeq \frac{16\varepsilon a}{1+\frac{2}{\pi}\frac{a}{l}} \qquad (4\text{-}14)$$

$[\delta < 3\%$ when $l/a \geqslant 2$; $\delta < 0{,}3\%$ when $l/a > 5]$.

7. *Two interconnected coplanar circular discs* (Fig. 4-11).

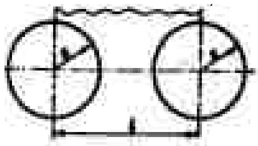


Fig. 4-11. Two interconnected coplanar discs.

The accurate value of capacitance is unknown. At rather high l/a the following approximation formulas can be used, the first of which is more accurate:

$$C_0 \simeq \frac{16\varepsilon a}{1+\frac{2}{\pi}\frac{a}{l}\left[1+\frac{7}{24}\left(\frac{a}{l}\right)^2+\frac{99}{320}\left(\frac{a}{l}\right)^4\right]}, \quad (4\text{-}15)$$

$$C_0 \simeq \frac{16\varepsilon a}{1+\frac{2}{\pi}\ \frac{a}{l}}. \quad (4\text{-}15a)$$

When $l/a \geq 3$ the values of capacitance, calculated from formula (4-15a) differ from the values determined from formula (4-15) by not more than 0.7%.

8. *Two parallel rectangular discs interconnected* (Fig. 4-12). Numerical values of $C_0/8\pi\varepsilon a = f(d/b)$ at short distances between discs are given in Table 4-1.

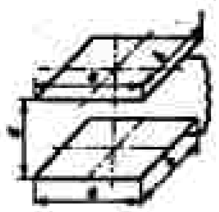


Fig. 4-12. Two interconnected rectangular discs lying in parallel planes.

The following approximation formulas can also be used:

Table 4-1. Relative values of the capacitance of the conductor formed by the union of two rectangular discs lying in parallel planes

$\left(\text{when } \frac{d}{b} < 1\right)$

$\frac{C_0}{4\pi\varepsilon a}$

a/b \ d/b	0.001	0.005	0.01	0.02
1	0,357	0,359	0,361	—
2	0,255	0,256	0,257	0,259
3	0,217	0,218	0,219	0,220
4	0,196	0,1965	0,197	0,199

a) when $d/a < 2,\ a/b > 1$

$$C_0 \simeq \frac{4\pi\varepsilon a}{\ln\frac{4a^2}{b\cdot d} - \frac{1}{2} + \frac{d}{a} - \frac{1}{4}\left(\frac{d}{a}\right)^2 + \frac{1}{32}\left(\frac{d}{a}\right)^4}; \qquad (4\text{-}16)$$

b) when $d/a \gg 1,\ \ a/b > 1$

$$C_0 \simeq 2C_{01}\frac{1}{1+\frac{C_{01}}{4\pi\varepsilon d}}, \qquad (4\text{-}17)$$

where C_{01} is the capacitance of a single disc determined from the data of p. 4 of the present paragraph.

9. *Two coplanar rectangular discs interconnected* (Fig. 4-13).

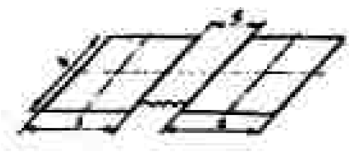


Fig. 4-13. Two interconnected coplanar rectangular discs.

The accurate value of capacitance is unknown.

When $d/a \gg 1,\ a/b > 1$

$$C_0 \simeq 2C_{01}\frac{1}{1+\frac{C_{01}}{4\pi\epsilon d}}, \tag{4-18}$$

where C_{01} is the capacitance of a single disc determined from data given in clause 4 of the present paragraph, specifically

$$C_0 \simeq \frac{4\pi\epsilon a}{\ln\left(4\frac{a}{b}\right)+\frac{a}{2d}}. \tag{4-19}$$

4-3. Capacitor Capacitance of Discs of Finite Dimensions

In the present paragraph formulas, tables, and graphs are given for the determination of the capacitance between two conductors that are the flat plates of finite dimensions or are formed by the union of several plates. Such conductors are coaxial circular discs; rectangular (specifically, square) plates, both arranged in parallel planes, and coplanar; concentric coplanar rings; a coaxial circular disc and ring arranged inside a cylinder with an impenetrable surface; and a circular disc arranged between two infinite planes.

1. *Two coaxial circular discs* (capacitor with circular plates) (Fig. 4-14).[1]

$$C = 4\epsilon a\cdot\int_0^1 f(t)\,dt, \tag{4-20}$$

where the function $f(t)$ is found from solving the integral equation

$$f(x)-\frac{1}{\pi}\int_0^1 f(t)\frac{l/a}{(x-t)^2+\left(\frac{l}{a}\right)^2}\,dt = 1.$$

[1]Determination of the capacitance of a capacitor with circular plates is the subject of a very big number of works [4-14]-[4-17]. An accurate solution to the problem is obtained in [4-18].

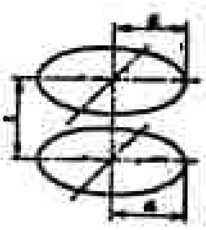


Fig. 4-14. Two coaxial discs.

The numerical values of the function $C/8\varepsilon a = f(l/a)$ are given in Fig. 4-15. The following approximation formulas can also be used:

a) when $l/a < 1$

$$C \simeq \varepsilon a\left[\pi \cdot \frac{a}{l} + \ln\left(16\pi \cdot \frac{a}{l}\right) - 1\right] \qquad (4\text{-}21)$$

$[\delta < 5{,}8\% \text{ when } l/a < 0{,}4]$

or

$$C \simeq \varepsilon \cdot \frac{\pi a^2}{l} \qquad (4\text{-}21a)$$

$[\delta < 15\% \text{ when } l/a < 0{,}1]$;

b) when $l/a > 1$

$$C \simeq \frac{4\varepsilon a}{1 - \frac{2}{\pi}\,\mathrm{arcctg}\,\frac{l}{a}} \qquad (4\text{-}22)$$

$[|\delta| < 2{,}4\% \text{ when } l/a > 2;\ |\delta| < 0{,}7\% \text{ when } l/a > 3]$

or

$$C \simeq \frac{4\varepsilon a}{1 - \frac{2}{\pi} \cdot \frac{a}{l}} \qquad (4\text{-}23)$$

$[|\delta| < 2{,}9\% \text{ when } l/a > 2{,}5;\ |\delta| < 0{,}4\% \text{ when } l/a > 5]$[1].

[1]Formulas (4-22) and (4-23) give an overstated value of capacitance.

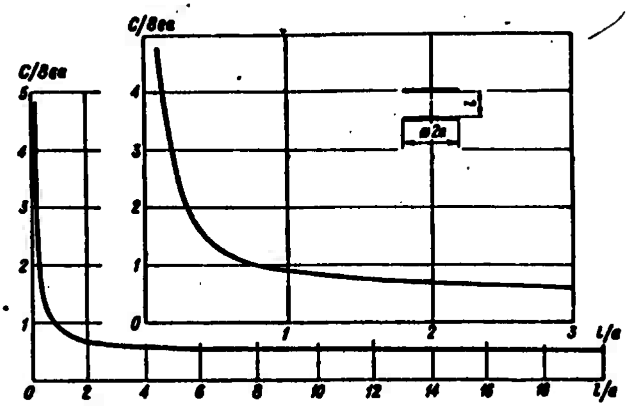


Fig. 4-15. Graph of function which characterizes relative capacitance between circular coaxial discs.

l/a	0,0	0,1	0,4	0,6	0,8	1,0	1,2
$\frac{C}{8\varepsilon a}$	∞	4,618	1,5514	1,1978	1,0186	0,9104	0,8380
l/a	1,5	2,0	2,5	3,0	5,0	10,0	20,0
$\frac{C}{8\varepsilon a}$	0,7044	0,6834	0,6517	0,6240	0,5708	0,5338	0,518

2. *Two identical rectangular plates* (a capacitor with rectangular plates).

The accurate value of capacitance is unknown.

a) Parallel plates (Fig. 4-16).

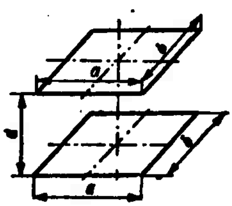


Fig. 4-16. Two identical parallel plates.

The approximation numerical values of capacitance at some values of a/d and b/d are given in Table 4-2, and for square plates ($a = b$) in Fig. 4-17.

Table 4-2. Relative values of capacitance between two rectangular plates lying in parallel planes.

$\frac{a}{d}$	0.50	0,666	0.83	0.85	1.0	1.5	2.0	2.66	3.0	3.33	4.0	5.0	5.1	6.0
$\frac{b}{d}$	0.283	0.50	0.50	0.283	0.50	0.50	1.0	2.0	1.0	2.0	2.0	3.0	1.7	3.0
$\frac{C}{4\pi\varepsilon a}$	0,136	0,164	0,170	0,123	0,159	0,130	0,203	0,312	0,188	0,301	0,300	0,309	0,251	0,379

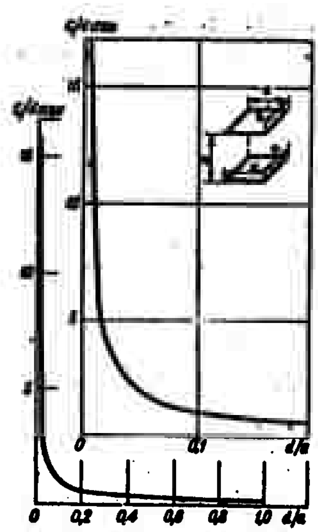


Fig. 4-17. Graph of function which characterizes relative capacitance between two identical square plates lying in parallel planes.

d/a	0	0,005	0,025
$C/4\pi\varepsilon a$	∞	15,97	3,4103
d/a	0,05	0,10	0,20
$C/4\pi\varepsilon a$	1,8266	1,0192	0,5986
d/a	0.50	1,00	∞
$C/4\pi\varepsilon a$	0,3375	0,2521	0,1764

The following approximation formulas can also be used:

when $a/d \ll 1$, $b/d \ll 1$

$$C_0 \simeq \frac{C_{01}}{2} \frac{1}{1 - \frac{C_{01}}{4\pi\varepsilon d}},$$

where C_{01} is the capacitance of a single plate (p. 4, § 4-2).

when $a/d>3,\ b/d>3$

$$C \simeq \varepsilon \cdot ab/d[1 + 1/\pi \cdot d/a(1 + \ln 2\pi a/d)] \times \times [1 + 1/\pi \cdot d/b(1 + \ln 2\pi b/d)]; \quad (4\text{-}25)$$

when $a/d > 3,\ b/d \gg 1$

$$C \simeq \varepsilon a \cdot b/d\left[1 + \frac{1}{\pi} \cdot d/a(1 + \ln 2\pi a/d)\right]; \quad (4\text{-}26)$$

when $a/d > 10,\ b/d > 10$

$$C \simeq \varepsilon \cdot ab/d \quad (4\text{-}26a)$$

$[\delta < 10\%]$.

b) Coplanar plates (Fig. 4-18).

When $b/a \gg 1$

$$C \simeq \varepsilon b \frac{K(k')}{K(k)}, \quad (4\text{-}27)$$

where

$$k = \frac{1}{1 + 2\frac{a}{d}}; \quad k' = \sqrt{1 - k^2}.$$

The values of function $\frac{C}{\varepsilon b} = \frac{K(k')}{K(k)} = f(a/d)$ are given in Fig. 4-19).

Fig. 4-18. Two identical oppositely charged coplanar plates.

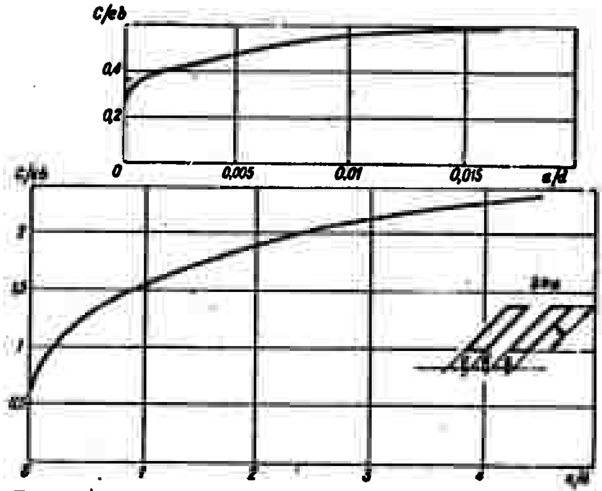


Fig. 4-19. Graph of function which characterizes relative capacitance between two identical coplanar plates.

a/d	0.00025	0.0025	0.0270	0.0590	0.2071	0.4129	1.081	1.736	4.500	49.50	[illegible]
$C/\varepsilon b$	0.3246	0.4361	0.6254	0.7353	1.000	1.211	1.599	1.828	2.347	3.814	5.280

When $b/a \gg 1,\ a/d \gg 1$

$$C \simeq \frac{2}{\pi}\varepsilon b \ln\left[4\left(1+2\frac{a}{d}\right)\right]. \tag{4-28}$$

When $b/a \gg 1,\ a/d \ll 1$

$$C \simeq \frac{\pi\varepsilon b}{\ln\left[4\left(1+\frac{d}{a}\right)\right]}. \tag{4-29}$$

When $d/a \gg 1$ and random b/a

$$C \simeq \frac{C_{01}}{2}\cdot\frac{1}{1-\frac{C_{01}}{4\pi\varepsilon d}}, \tag{4-30}$$

where C_{01} is the capacitance of a single plate determined from data given in clause 4, § 4-2.

3. *Concentric coplanar rings.*

a) The general case (Fig. 4-20).

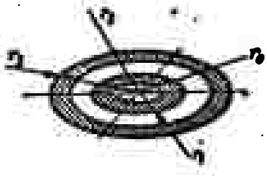


Fig. 4-20. Two concentric coplanar rings.

The numerical values of the function $\frac{C}{4\pi\varepsilon r_3 \cdot 0.9} = f\left(\frac{r_1}{r_3}\right)$ at $\frac{r_4}{r_3} = 6$ and various r_2/r_3 are given in Fig. 4-21.

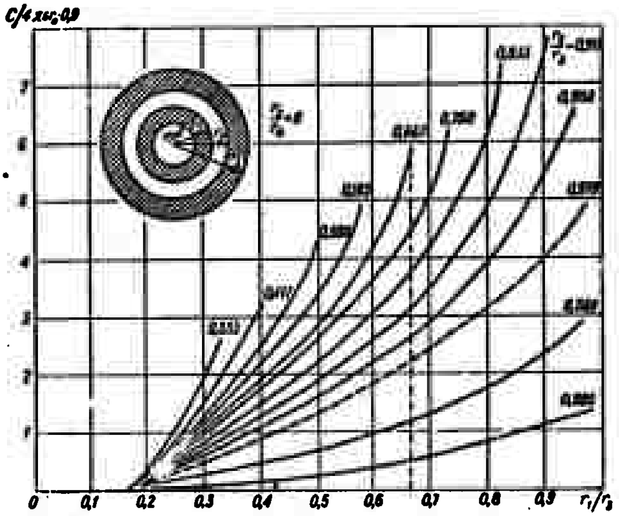


Fig. 4-21. Graph of function which characterizes relative capacitance between two concentric coplanar rings.

$\frac{r_1}{r_3}$	0.417	0.417	0.500	0.583	0.667
$\frac{r_2}{r_3}$	0.500	0.667	0.667	0.667	0.833
$\frac{C}{4\pi\varepsilon r_3 \cdot 0.9}$	2.6[illegible]	2.15	2.86	[illegible].86	[illegible].73

If $r_1 \leqslant 1{,}5\,(r_3 - r_2)$, then capacitance practically little depends upon r_3, i.e., the external radius of the external ring can be considered infinite.

Example 4-1. To determine the capacitance of the circular capacitor being used during measurement of the dampness of wood, at the following dimensions: $r_0 = 0.5$ cm; $r_1 = 1.5$ cm; $r_2 = 2$ cm; $r_3 = 3$ cm.

In this case

$$\frac{r_3}{r_0} = \frac{3}{0{,}5} = 6; \quad \frac{r_1}{r_3} = \frac{1{,}5}{3} = 0{,}5; \quad \frac{r_2}{r_3} = \frac{2}{3} = 0{,}667.$$

Using Fig. 4-21, we find that for the relationships of radii shown $\frac{C}{4\pi\varepsilon r_0 \cdot 0{,}9} = 2{,}86$, whence $C = 2{,}86 \cdot 4\pi \frac{\varepsilon'}{4\pi \cdot 9 \cdot 10^9} \cdot 0{,}5 \cdot 10^{-2} \cdot 0{,}9 = \varepsilon' \cdot 1{,}43 \cdot 10^{-12}$ F $= \varepsilon' \cdot 1{,}43$ pF, ε' is the relative specific inductive capacitance of the medium.

b) Disc in the circular cut of an infinite plane (Fig. 4-22)

$$C \simeq 8\varepsilon r_1 \left(1 + \frac{r_1}{r_2}\right) \int_0^K \frac{\operatorname{sn} \xi}{1 + k \operatorname{sn}^2 \xi}\, d\xi, \tag{4-31}$$

where K is a complete elliptical integral of the first kind with modulus $k = \frac{r_1}{r_2}$; $\operatorname{sn} \xi$ — an elliptical sine (see Appendix 1).

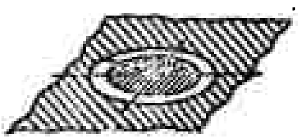


Fig. 4-22. A disc located in a circular cut of an infinite plane.

The numerical values of the function $\frac{C}{8\varepsilon r_1} = f\left(\frac{r_1}{r_2}\right)$ are given in Fig. 4-23.

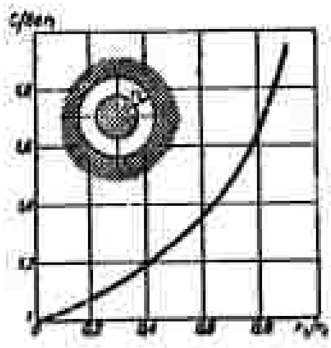


Fig. 4-23. Graph of a function which characterizes relative capacitance between a plane with circular cut and a disc in this cut (dotted line - extrapolation).

From (4-31) can be obtained the formula

$$C \simeq 2\varepsilon r_2\left[1+\frac{r_1}{r_2}+\sqrt{1-\left(\frac{r_1}{r_2}\right)^2}\right]\ln\frac{1+\frac{r_1}{r_2}}{1-\frac{r_1}{r_2}}, \qquad (4\text{-}32)$$

which gives the results differing from the data of calculation from formula (4-31) not more than 3%.

When $r_1/r_2 \ll 1$

$$C \simeq 8\varepsilon r_1, \qquad (4\text{-}33)$$

i.e., the value of the capacitance between a disc and plane when the radius of a disc is much less than the radius of the cut is approximately equal to the value of the capacitance of a solitary disc (compare clause 1, § 4-2).

<u>Example</u> 4-2. A 1 × 1 × 1 m tank made from thin insulating material with specific inductive capacitance which insignificantly differs from ($\varepsilon = 83\ \varepsilon_0$). On the bottom of the tank is a thin metal sheet which possesses in the center circular cut r_2 = 5 cm in radius with a symmetrically metal sheet in it r_1 = 1 cm in radius that possesses the same thickness as the plate.

To find the capacitance between disc and a plate.

In view of the considerable significant dimensions of the sheet in comparison with the radius of cut, it is possible to consider the sheet an infinite conducting plane in an infinite medium. Taking account furthermore of the fact that the specific inductive capacitance of air, it is possible to assume with sufficient accuracy that the plane considered is separated from the lower half-space by an impenetrable boundary (see § V-2).

In this instance an electrostatic field exists practically only in the half space. Using the principle of mirror image, (§ V-2), desired capacitance c can be determined according to one of the formulas (4-31)-(4-33) or from Fig. 4-23 with calculation of the relationship $c = 1/2\ C$.

At $r_1/r_2 = 0.01/0.05 = 0.2$ from the data of Fig. 4-23 we find

$$C/8\varepsilon r_1 = 1{,}07.$$

Therefore,

$$\check{C} = 1/2\cdot 8\varepsilon r_1\cdot 1{,}07 = \frac{1}{2}\cdot 8\cdot 83\cdot\frac{1}{4\pi\cdot 9\cdot 10^9}\cdot 0{,}01\cdot 1{,}07 = 31{,}2\cdot 10^{-12}\ \mathrm{F} = 31{,}2\ \mathrm{pF}.$$

If we use for calculation formula (4-33), then

$$\check{C}' = 1/2\cdot 8\varepsilon r_1 = 29{,}3\ \mathrm{pF}.$$

The relative error in determining the capacitance between the conductors being considered from formulas (4-31) and (4-33) is

$$\delta = \frac{\check{C}-\check{C}'}{\check{C}}\cdot 100\% = \frac{31{,}2-29{,}3}{31{,}2}\cdot 100\% = 6{,}1\%.$$

4. *Coaxial disc and ring inside a circular cylinder with an impenetrable surface* (Fig. 4-24).

$$C = \frac{\varepsilon b}{\frac{1}{\pi}\cdot\frac{l}{b}+a}. \qquad (4\text{-}34)$$

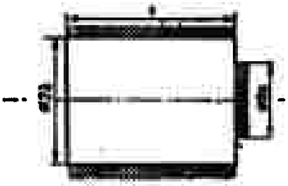


Fig. 4-24. Coaxial disc and ring inside a circular cylinder with impenetrable surface.

The numerical values of the parameter α are given in Fig. 4-25.

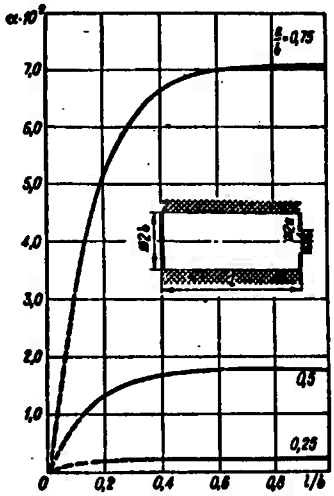


Fig. 4-25. Graph for determination of parameter α which enters formula (4-34) (dotted line – extrapolation).

$\alpha \cdot 10^{9}$ (a)

a/b \ l/b	0.10	0.20	0.30	0.40	0.50	0.60	0.80	1.00	∞	Пред. абс. погрешность (a)
0,25	0.144	0.193	0.207	0,212	0,213	0,213	0.214	0.214	0.214	0,003
0,50	0,83	1,33	1,57	1.70	1,75	1,78	1,79	1,79	1,79	0,02
0,75	3,21	5,07	6,08	6,59	6,88	6,97	7,04	7,07	7,07	0,13

KEY: (a) Absolute error limit.

5. *Circular disc and two infinite planes parallel to it* (Fig. 4-26).

$$C = 8\varepsilon b \int_0^1 g(t)\,dt. \tag{4-35}$$

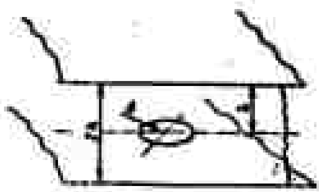


Fig. 4-26. Circular disc, between the infinite planes, interconnected.

Function g(t) is found from solution of the integral equation

$$g(\bar{t})-\frac{1}{\pi}\int_{-b}^{b}K(\bar{t}-\bar{s})\,g(\bar{s})\,d\bar{s}=1$$

with nucleus

$$K(\bar{t}-\bar{s})=\frac{1}{h}\sum_{n=0}^{\infty}(-1)^{n}\frac{A_{2n}}{2n!}\cdot\left(\frac{\bar{t}-\bar{s}}{h}\right)^{2n},$$

where

$$A_{2n}=\int_{0}^{\infty}\frac{2u^{2n}}{e^{2u}+1}\,du=\begin{cases}\ln 2 \text{ when } n=0;\\ 2^{-4n}(2^{2n}-1)(2n!)\zeta(2n+1); \text{ (when } n>0)\end{cases}$$

$\zeta(2n+1)=\sum_{\kappa=1}^{\infty}\frac{1}{\kappa^{2n+1}}$ the zeta-function of Riemann (see Appendix 1).

For the relationships $b/h \ll 1$

$$\frac{C}{8\varepsilon b}\simeq 1+\alpha\cdot\frac{b}{h}+\alpha^{2}\cdot\left(\frac{b}{h}\right)^{2}+\left[\alpha^{3}-\frac{\zeta(3)}{4\pi}\right]\cdot\left(\frac{b}{h}\right)^{3}+$$
$$+\left[\alpha^{4}-\frac{\ln 2}{\pi^{3}}\cdot\zeta(3)\right]\cdot\left(\frac{b}{h}\right)^{4}, \qquad (4\text{-}36)$$

where $\alpha=\frac{2\ln 2}{\pi}$; $\zeta(3)=1{,}202$.

4-4. Capacitor Capacitance of Plates of Infinite Length

The present paragraph contains formulas, tables and graphs for the determination of the capacitance between two conductors that are flat plates of infinite length or formed by the union of several plates. Among the conductors being considered there are two coplanar plates; three coplanar plates two of which are connected; two mutually perpendicular plates; two parallel plates; two plates at an angle with eath other; plates perpendicular to two infinite planes; and plates parallel to two infinite planes.

1. *Two coplanar plates.*

Formulas for determining capacitance per unit of length between two infinitely long plates lying in one plane are given in Table 4-3.

Example 4-3. To find capacitance per unit of length between two plates a = 10 cm wide in a medium with specific inductive capacitance ε, if the distance between plates is d = 1 cm.

In accordance with clause 2 of Table 4-3 the calculation is made from the formula

$$C_l = \varepsilon \frac{K(k')}{K(k)},$$

where the moduli of elliptical integrals are

$$k^2 = \frac{1}{\left(1 + 2\frac{a}{d}\right)^2} = 0{,}0022676,$$

$$k'^2 = 1 - k^2 = 0{,}9977324.$$

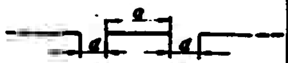
From the values of moduli found with the aid of the table of Appendix 2 we establish that

$$K(k) = 1{,}57169; \quad K(k') = 4{,}43287.$$

Table 4-3. Formulas for determining capacitance between two coplanar plates.

No. in order	Form of conductors	Calculation diagram	Calculation formulas: Accurate	Calculation formulas: Approximation	Relative error of approximation formulas, %
1	Two plates of different width		$C_l = 2\varepsilon \frac{K'}{K}$, where $k^2 = \frac{1+\frac{a}{d}+\frac{b}{d}}{\left(1+\frac{a}{d}\right)\left(1+\frac{b}{d}\right)}$	$C_l \simeq \frac{2\varepsilon}{\pi} \ln\left[16\frac{\left(1+\frac{a}{d}\right)\left(1+\frac{b}{d}\right)}{1+\frac{a}{d}+\frac{b}{d}}\right]$ at $\frac{a}{d} > 1, \frac{b}{d} > 1$; $C_l \simeq \frac{2\pi\varepsilon}{\ln\left[16\left(1+\frac{d}{a}\right)\left(1+\frac{d}{b}\right)\right]}$ at $\frac{a}{d} < 1; \frac{b}{d} < 1$	-1.5 at $\frac{a}{d} = 10$, $\frac{b}{d} = 20$; 0,07 at $\frac{a}{d} = 0.1$, $\frac{b}{d} = 0.2$
2	Two plates of the same width		$C_l = 2\varepsilon \frac{K_1'}{K_1} = \varepsilon \frac{K'}{K}$, where $k_1^2 = \frac{1+2\frac{a}{d}}{\left(1+\frac{a}{d}\right)^2}$, $k^2 = \frac{1}{\left(1+2\frac{a}{d}\right)^2}$	$C_l \simeq \frac{2\varepsilon}{\pi} \ln\left[4\left(1+2\frac{a}{d}\right)\right]$ at $\frac{a}{d} > 1$; $C_l \simeq \frac{\pi\varepsilon}{\ln\left[4\left(1+\frac{d}{a}\right)\right]}$ at $\frac{a}{d} < 1$	-0.01 at $\frac{a}{d} = 10$; 0,04 at $\frac{a}{d} = 0.1$
3	Plate and a half-plane		$C_l = 2\varepsilon \frac{K'}{K}$, where $k^2 = \frac{1}{1+\frac{a}{d}}$	$C_l \simeq \frac{2\varepsilon}{\pi} \cdot \ln\left[16\left(1+\frac{a}{d}\right)\right]$ at $\frac{a}{d} > 1$; $C_l \simeq \frac{2\pi\varepsilon}{\ln\left[16\left(1+\frac{d}{a}\right)\right]}$ at $\frac{a}{d} < 1$	-0.9 at $\frac{a}{d} = 10$; 0,9 at $\frac{a}{d} = 0.1$

Table 4-3 (Cont'd).

No. in order	Form of conductors	Calculation diagram	Calculation formulas: Accurate	Calculation formulas: Approximation	Relative error of approximation formulas, %
4	A plate and a plane with an asymmetric groove		$C_l = 2\varepsilon \frac{K}{K'}$, where $k^2 = \frac{1+\frac{d_1}{a}+\frac{d_2}{a}}{\left(1+\frac{d_1}{a}\right)\left(1+\frac{d_2}{a}\right)}$	$C_l \simeq \frac{2\pi\varepsilon}{\ln\left[16\frac{\left(1+\frac{d_1}{a}\right)\left(1+\frac{d_2}{a}\right)}{1+\frac{d_1}{a}+\frac{d_2}{a}}\right]}$ at $\frac{d_1}{a} > 1,\ \frac{d_2}{a} > 1$; $C_l \simeq \frac{2\varepsilon}{\pi}\cdot\ln\left[16\left(1+\frac{a}{d_1}\right)\left(1+\frac{a}{d_2}\right)\right]$ at $\frac{d_1}{a} < 1,\ \frac{d_2}{a} < 1$	1,5 at $\frac{d_1}{a} = 10$, $\frac{d_2}{a} = 20$; −0,1 at $\frac{d_1}{a} = 0{,}1$, $\frac{d_2}{a} = 0{,}2$
5	A plate and a plane with a symmetric cut		$C_l = 2\varepsilon \frac{K}{K'} = 4\varepsilon \frac{K_1}{K_1'}$, where $k^2 = \frac{1+2\frac{d}{a}}{\left(1+\frac{d}{a}\right)^2}$; $k_1^2 = \frac{1}{\left(1+2\frac{d}{a}\right)^2}$	$C_l \simeq \frac{2\pi\varepsilon}{\ln\left[4\left(1+2\frac{d}{a}\right)\right]}$ at $\frac{d}{a} > 1$; $C_l \simeq \frac{4\varepsilon}{\pi}\ln\left[4\left(1+\frac{a}{d}\right)\right]$ at $\frac{d}{a} < 1$	0,01 at $\frac{d}{a} = 10$; −0,04 at $\frac{d}{a} = 0{,}1$

Substituting numerical values into the given accurate formula, we obtain

$$C_l = \varepsilon \cdot \frac{4,43287}{1,57169} = 2,82045 \cdot \varepsilon.$$

If we make use of the approximation formula given in clause 2 of Table 4-3, then

$$C_{l\,np} = \frac{2\varepsilon}{\pi} \ln\left[4\left(1 + 2\frac{a}{d}\right)\right] = \frac{2\varepsilon}{\pi} \cdot \ln\left[4\left(1 + 2 \cdot \frac{10}{1}\right)\right] = 2,82075 \cdot \varepsilon.$$

Thus, the relative error of the approximation formula for the case considered is

$$\delta = \frac{C_l - C_{l\,np}}{C_l} \, 100\% = \frac{2,82045\varepsilon - 2,82075\varepsilon}{2,82045\varepsilon} \cdot 100\% = -0,011\%.$$

With increase in the ratio of a/d this error in absolute value becomes still less.

2. *Three coplanar plates, two of which are interconnected.*

The formula for determining capacitance per unit of length between two joint plates and the third plate (two plates have identical width and are equidistant from the third) are given in Table 4-4.

3. *Two mutually perpendicular plates.*

a) Plates of identical width (Fig. 4-27).

$$C_l = \varepsilon \cdot \frac{\pi}{\ln \frac{1}{q}}. \qquad (4\text{-}37)$$

Table 4-4. Formulas for determining capacitor capacitance in a system of three coplanar plates two of which have identical width and are equidistant from the third.

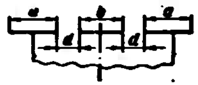
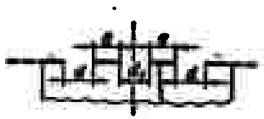
No. in order	Form of conductors	Calculation diagram	Calculation formulas: Accurate	Calculation formulas: Approximation	Relative error of approximation formulas, %
1	Symmetrical plates united		$C_l = \varepsilon\frac{K'}{K}$, where $k^2 = \left(1+\frac{2\frac{a}{d}}{2+\frac{b}{d}}\right)^2 \times \frac{1+\frac{b}{d}}{\left(1+\frac{a}{d}\right)\left(1+\frac{a}{d}+\frac{b}{d}\right)}$	$C_l \simeq \frac{4\varepsilon}{\pi}\ln\frac{16\frac{a}{d}\cdot\frac{b}{d}\left(\frac{a}{d}+\frac{b}{d}\right)}{\left(\frac{b}{d}+2\frac{a}{d}\right)^2}$ at $\frac{a}{d}\gg 1,\ \frac{b}{d}\gg 1$; $C_l \simeq \frac{4\pi\varepsilon}{\ln\left\{32\frac{d}{b}\left[\frac{d}{a}\cdot\frac{d}{b}+2\left(\frac{d}{a}+\frac{d}{b}\right)\right]\right\}}$ at $\frac{a}{d}\ll 1,\ \frac{b}{d}\ll 1$	2,3 at $\frac{a}{d}=20$; $\frac{b}{d}=10$ 0,65 at $\frac{a}{d}=0{,}2$; $\frac{b}{d}=0{,}1$
2	Asymmetrical plates united		$C_l = \varepsilon\cdot\left(\frac{K_1'}{K_1}+\frac{K_2'}{K_2}\right)$, where $k_1^2 = \frac{1+\frac{b}{d}}{\left(1+\frac{a}{d}\right)\left(1+\frac{a}{d}+\frac{b}{d}\right)}$ $k_2 = \left(1+\frac{2\frac{a}{d}}{2+\frac{b}{d}}\right)^{-1}$	$C_l \simeq \frac{2\varepsilon}{\pi}\ln\frac{16\frac{a}{d}\left(\frac{a}{d}+\frac{b}{d}\right)}{\frac{b}{d}+2\frac{a}{d}}$ at $\frac{a}{d}\gg 1,\ \frac{b}{d}\gg 1$; $C_l \simeq \pi\varepsilon\left\{\frac{1}{\ln\frac{16\left(1+2\frac{a}{d}+\frac{b}{d}\right)}{\frac{a}{d}\left(2+\frac{a}{d}+\frac{b}{d}\right)}}+\frac{1}{\ln\left\{32\frac{d}{b}\left[\frac{d}{a}\cdot\frac{d}{b}+2\left(\frac{d}{a}+\frac{d}{b}\right)\right]\right\}}\right\}$ at $\frac{a}{d}\ll 1,\ \frac{b}{d}\ll 1$	0,6 at $\frac{a}{d}=20$, $\frac{b}{d}=10$ 2,6 at $\frac{a}{d}=0{,}2$; $\frac{b}{d}=0{,}1$

Table 4-4 (Cont'd).

No. in order	Form of conductors	Calculation diagram	Calculation formulas: Accurate	Calculation formulas: Approximation	Relative error of approximation formulas, %
3	Two symmetrical; the third plate is an infinite plane with cut		$C_l = 4\varepsilon\frac{K}{K'}$, where $k^2 = \frac{1+\frac{d_1}{a}}{\left(1+\frac{d}{a}\right)\left(1+\frac{d}{a}+\frac{d_1}{a}\right)}$	$C_l \simeq \frac{4\pi\varepsilon}{\ln\frac{16\left(1+\frac{d}{a}\right)\left(1+\frac{d}{a}+\frac{d_1}{a}\right)}{1+\frac{d_1}{a}}}$ at $\frac{d}{a}>1$, $\frac{d_1}{a}>1$; $C_l \simeq \frac{4\varepsilon}{\pi}\ln\frac{16\left(1+\frac{d}{a}\right)\left(1+\frac{d}{a}+\frac{d_1}{a}\right)}{\frac{d}{a}\left(2+\frac{d}{a}+\frac{d_1}{a}\right)}$ at $\frac{d}{a}<1$, $\frac{d_1}{a}<1$	0,6 at $\frac{d_1}{a}=20$, $\frac{d}{a}=10$; −1,9 at $\frac{d_1}{a}=0{,}2$; $\frac{d}{a}=0{,}1$
4	Asymmetrical plates are united; one of the plates is an infinite plane with cut		$C_l = \varepsilon\left(\frac{K_1}{K_1'}+\frac{K_2}{K_2'}\right)$, where $k_1^2 = \frac{1+\frac{d_1}{a}}{\left(1+\frac{d}{a}\right)\left(1+\frac{d}{a}+\frac{d_1}{a}\right)}$; $k_2 = \left(1+\frac{2\frac{d}{a}}{2+\frac{d_1}{a}}\right)\cdot k_1$	$C_l \simeq \pi\varepsilon\left[\frac{1}{\ln\frac{16\left(1+\frac{d}{a}\right)\left(1+\frac{d}{a}+\frac{d_1}{a}\right)}{1+\frac{d_1}{a}}} + \frac{1}{\ln\frac{16\left(1+\frac{d}{a}\right)\left(1+\frac{d}{a}+\frac{d_1}{a}\right)\left(1+\frac{d_1}{2a}\right)^2}{\left(1+\frac{d_1}{a}\right)\left(1+\frac{d}{a}+\frac{d_1}{2a}\right)^2}}\right]$ at $\frac{d}{a}>1$, $\frac{d_1}{a}>1$; $C_l \simeq \frac{2\varepsilon}{\pi}\ln\frac{16\left(1+\frac{d}{a}\right)\left(2+\frac{d_1}{a}\right)\left(1+\frac{d}{a}+\frac{d_1}{a}\right)}{\frac{d}{a}\cdot\frac{d_1}{a}\left(2+\frac{d}{a}+\frac{d_1}{a}\right)}$ at $\frac{d}{a}<1$, $\frac{d_1}{a}<1$	1,9 at $\frac{d_1}{a}=20$, $\frac{d}{a}=10$; 0,3 at $\frac{d_1}{a}=0{,}2$, $\frac{d}{a}=0{,}1$

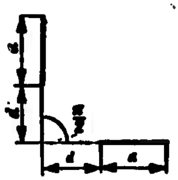


Fig. 4-27. Two mutually perpendicular infinitely long plates of the same width.

The parameter $q\,(0 < q < 1)$ is determined from the system of equations (for example, by means of exclusion of the unknown parameter ρ):

$$\lambda = \frac{\vartheta_0\left(\rho - \frac{1}{4}\right)}{\vartheta_0(\rho)} = \frac{1 - 2q\cdot\sin 2\pi\rho - 2q^4\cdot\cos 4\pi\rho + 2q^9\cdot\sin 6\pi\rho + \ldots}{1 - 2q\cos 2\pi\rho + 2q^4\cos 4\pi\rho - 2q^9\cdot\cos 6\pi\rho + \ldots},$$

$$\lambda = \frac{\vartheta_0'\left(\rho - \frac{1}{4}\right)}{\vartheta_0'(\rho)} = \frac{-q\cdot\cos 2\pi\rho + 2q^4\sin 4\pi\rho + 3q^9\cos 6\pi\rho - \ldots}{q\sin 2\pi\rho - 2q^4\sin 4\pi\rho + 3q^9\sin 6\pi\rho - \ldots}, \quad (4\text{-}38)$$

where

$$\lambda = \sqrt{\frac{1}{1 + \frac{a}{d}}}, \quad 0 < \rho < \frac{1}{2};$$

$\vartheta_0(x)$, $\vartheta_0'(x)$ is the theta-function and its derivative (see Appendix 1).

The approximation value of q can be determined from the formula

$$q_1 \simeq \frac{1}{2}\cdot\frac{1-\lambda}{\sqrt{1+\lambda^2}}. \quad (4\text{-}39)$$

For values of $\lambda > 0.4$ this formula gives the value of the parameter q with error exceeding 1%.

A more accurate value of q can be found from the formula

$$q_2 \approx \frac{1}{2}(1-\lambda)\frac{\sqrt{\lambda^2 + (1 - q_1^4\cdot\mu)^2}}{\lambda^2 + 1 - q_1^4\mu} - q_1^4 \times$$

$$\times \frac{(1+\lambda)\left[\lambda^2 - (1 - q_1^4\mu)^2\right]}{(\lambda^2 + 1 - q_1^4\mu)\cdot\sqrt{\lambda^2 + (1 - q_1^4\mu)^2}}, \quad (4\text{-}40)$$

where

$$p = 8 \cdot \frac{1+\lambda}{1-\lambda}.$$

The numerical values of the function $C_l/\varepsilon = f(a/d)$ see in Fig. 4-34 at $\varphi = 90°$.

b) A plate and a half-plane (Fig. 4-28).

$$C_l = \varepsilon \frac{2\pi}{\ln \frac{1}{q}}, \qquad (4\text{-}41)$$

where $q\,(0 < q < 1)$ is determined from formulas (4-38)-(4-40) with replacement in them of dimensionless parameter λ with $\lambda_1 = \sqrt{\lambda}$.

Fig. 4-28. Mutually perpendicular plates and half-plane.

c) A plane in which one plate passing through the middle of another plate is located.

The formulas for determining the capacitance of systems of this type are given in Table 4-5.

4. *Two parallel plates.*

a) Two plates of different width (Fig. 4-29).

Dependence $C_l/4\pi\varepsilon = f(b_2/d)$ at some fixed values of b_1/b_2 is depicted in Fig. 4-30.

Table 4-5. Formulas for the determination of capacitance between two mutually perpendicular plates, the plane of one of which passes through the middle of the other.

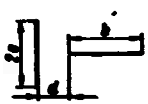
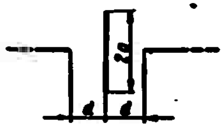
No. order	Form of conductors	Calculation diagram	Calculation formulas: Accurate	Calculation formulas: Approximation	Relative error of approximation formulas. %
1	Two plates of different width		$C_l = 2\varepsilon \cdot \frac{K'}{K}$, where $k = \frac{\frac{a}{d} + \sqrt{\left(1+\frac{b}{d}\right)^2 + \left(\frac{a}{d}\right)^2}}{\left(1+\frac{b}{d}\right)\left[\frac{a}{d} + \sqrt{1+\left(\frac{a}{d}\right)^2}\right]}$	$C_l \simeq \frac{4\varepsilon}{\pi} \ln \frac{8\frac{b}{d}}{1+\sqrt{1+\left(\frac{b}{a}\right)^2}}$ at $\frac{a}{d} > 1$, $\frac{b}{d} > 1$; $C_l \simeq \frac{2\pi\varepsilon}{\ln \frac{8\left(1+\frac{b}{d}\right)^2\left(1+\frac{d}{a}\right)^2}{\left(1+\frac{b}{d}\right)^2\left(1+\frac{d}{a}\right) - \left(1+\frac{b}{a}+\frac{d}{a}\right)}}$ at $\frac{a}{d} < 1$, $\frac{b}{d} < 1$	0,6 at $\frac{a}{d} = 10$, $\frac{b}{d} = 20$; 0,3 at $\frac{a}{d} = 0{,}1$, $\frac{b}{d} = 0{,}2$
2	Plate and half-plane		$C_l = 2\varepsilon \frac{K'}{K}$, where $k = \frac{1}{\frac{a}{d} + \sqrt{1+\left(\frac{a}{d}\right)^2}}$	$C_l \simeq \frac{4\varepsilon}{\pi} \ln \left\{4\left[\frac{a}{d} + \sqrt{1+\left(\frac{a}{d}\right)^2}\right]\right\}$ at $\frac{a}{d} > 1$; $C_l \simeq \frac{2\pi\varepsilon}{\ln\left\{8\left[1+\sqrt{1+\left(\frac{d}{a}\right)^2}\right]\right\}}$ at $\frac{a}{d} < 1$	−0,07 at $\frac{a}{d} = 10$; 2,2 at $\frac{a}{d} = 0{,}1$
3	Plate in the cut of a plane (an asymmetrical system)		$C_l = 2\varepsilon \frac{K}{K'}$, $k^2 = \frac{2(N+M)}{(1+N)(1+M)}$, $N = \left[1+\left(\frac{d_1}{a}\right)^2\right]^{1/2}$, $M = \left[1+\left(\frac{d_2}{a}\right)^2\right]^{1/2}$	$C_l \simeq \frac{4\pi\varepsilon}{\ln\left[16\left(1+\frac{d_1}{a}\cdot\frac{d_2}{a}\right)\right]}$ at $\frac{d_1}{a} > 1$, $\frac{d_2}{a} > 1$; $C_l \simeq \frac{4\varepsilon}{\pi} \cdot \ln\left[16\left(1+\frac{a}{d_2}\ \frac{a}{d_1}\right)\right]$ at $\frac{d_1}{a} < 1$, $\frac{d_2}{a} < 1$	1,4 at $\frac{d_1}{a} = 20$, $\frac{d_2}{a} = 10$; −0,1 at $\frac{d_1}{a} = 0{,}2$, $\frac{d_2}{a} = 0{,}1$

Table 4-5 (Cont'd).

No. in order	Form of conductors	Calculation diagram	Calculation formulas: Accurate	Calculation formulas: Approximation	Relative error of approximation formules, %
4	Plate in the cut of a plane (a symmetrical system)		$C_l = 4\varepsilon \frac{K}{K'}$, where $k^2 = \frac{1}{1+\left(\frac{d}{a}\right)^2}$	$C_l \approx \frac{4\pi\varepsilon}{\ln\left\{16\left[1+\left(\frac{d}{a}\right)^2\right]\right\}}$ at $\frac{d}{a} > 1$ $C_l \approx \frac{4\varepsilon}{\pi} \cdot \ln\left\{16\left[1+\left(\frac{a}{d}\right)^2\right]\right\}$ at $\frac{d}{a} < 1$	$-0{,}1$ at $\frac{d}{a} = 10$ $-0{,}06$ at $\frac{d}{a} = 0{,}1$

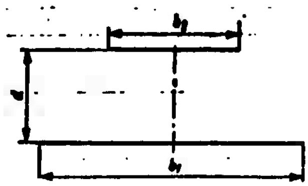


Fig. 4-29. Two parallel plates of different width.

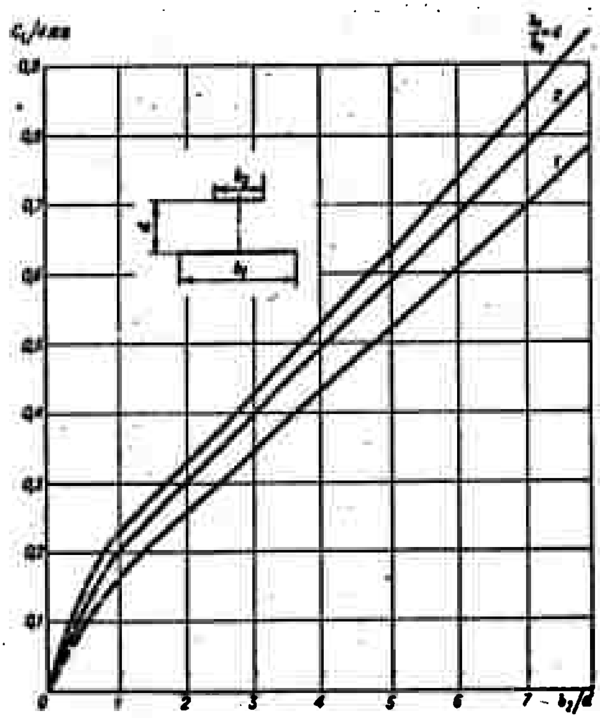


Fig. 4-30. Graph for determination of capacitance between two parallel plates of different width (dotted line – extrapolation).

b) Two plates of identical width (Fig. 4-31).

$$C_l = \varepsilon \frac{K}{K'}, \qquad (4\text{-}42)$$

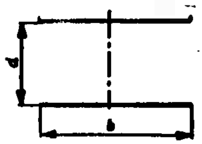


Fig. 4-31. Two parallel infinitely long plates of identical width.

and the modulus of k elliptical integrals is found from the equation

$$\frac{b}{d}=\frac{2}{\pi}[\mathrm{K}\cdot E(\beta,k)-\mathrm{E}\cdot F(\beta,k)]=\frac{2}{\pi}\cdot\mathrm{K}\cdot Z(\beta,k),$$

$$\beta=\arcsin\sqrt{\frac{1}{k^2}\left(1-\frac{\mathrm{E}}{\mathrm{K}}\right)},$$

where $F(\beta,k)$, K; $E(\beta,k)$, E are elliptical integrals of the first and second kind; $Z(\beta,k)$ is the zeta-function of Jacoby (see Appendix 1).

Numerical values of C_l/ε depending on b/d are given in Table 4-6 and in Fig. 4-32.

Table 4-6.

b/d	0,0076	0,0170	0,0308	0,0491	0,0723	0,0998	0,1336	0,1731	0,2213	0,2787
C_l/ε	0,5019	0,5773	0,6408	0,7142	0,7817	0,8508	0,9231	1,0000	1,0833	1,1783
b/d	0,3481	0,4338	0,5425	0,8075	1,3606	2,1775	2,5036	3,0018	4,8829	6,0888
C_l/ε	1,2792	1,4001	1,6400	1,9823	2,4347	3,4587	3,8198	5,0829	6,3862	7,6304
b/d	7,3113	8,5470	9,7816	11,024	14,136	17,259	20,392	23,533	26,676	30,838
C_l/ε	8,9127	10,186	11,460	12,722	15,915	19,098	22,282	25,465	28,649	31,831

For approximation calculation of the capacitance plates being considered between the following formulas can be used.

At $b/d \gg 1$

$$C_l \simeq \varepsilon\cdot\frac{b}{d}\cdot\left[1+\frac{1}{\pi}\cdot\frac{d}{b}\left(1+\ln 2\pi\frac{b}{d}\right)\right] \tag{4-43}$$

$[\delta < 1,5\%$ at $4 < b/d < 28]$.

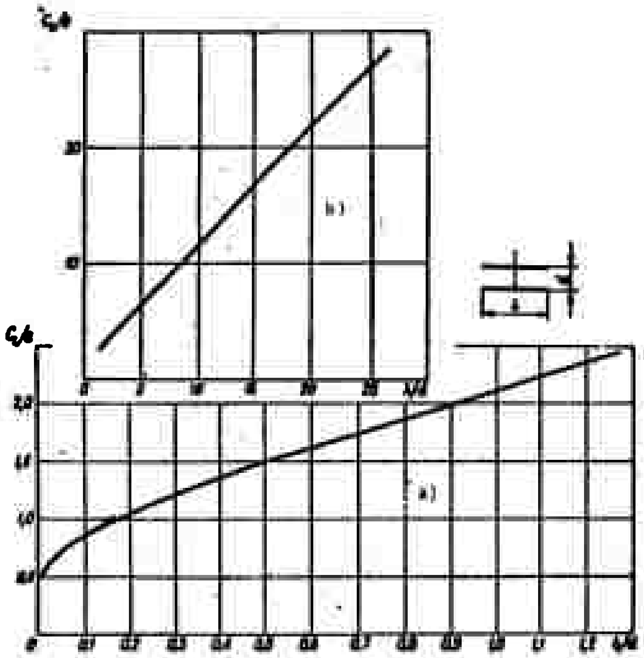


Fig. 4-32. A graph for the determination of capacitance between two parallel plates of identical width.

At $14 < b/d < 30$

$$C_l \simeq \varepsilon \cdot \frac{b}{d} \cdot \left\{1 + \frac{1}{\pi} \cdot \frac{d}{b} \cdot \left[1 + \ln\left(1 + \pi \cdot \frac{b}{d}\right)\right]\right\} \qquad (4\text{-}44)$$

$[\delta < 1{,}5\%]$.

At $b/d > 32$

$$C_l \simeq \varepsilon \cdot \frac{b}{d} \qquad (4\text{-}45)$$

$[\delta < 3\%]$.

At $b/d \ll 1$

$$C_l \simeq \frac{\pi\varepsilon}{\ln\left(4\frac{d}{b}\right)} \qquad (4\text{-}46)$$

$[|\delta| < 0{,}3\%$ at $b/d < 0{,}25]$.

5. *Two plates at an angle to one another.*

a) Plates of identical width (Fig. 4-33)

$$C_l = \varepsilon \cdot \frac{\pi}{\ln \frac{1}{q}}, \tag{4-47}$$

where $q\,(0 < q < 1)$ is determined from the system of equations (for example, by means of exclusion of the unknown parameter ρ):

$$\lambda = \frac{\vartheta_0\left(\rho - \frac{\varphi}{2\pi}\right)}{\vartheta_0(\rho)} =$$

$$= \frac{1 - 2q\cos(2\pi\rho - \varphi) + 2q^4\cos 2(2\pi\rho - \varphi) - 2q^9\cos 3(2\pi\rho - \varphi) + \ldots}{1 - 2q\cos 2\pi\rho + 2q^4\cos 4\pi\rho - 2q^9\cos 6\pi\rho + \ldots};$$

$$\lambda = \frac{\vartheta_0'\left(\rho - \frac{\varphi}{2\pi}\right)}{\vartheta_0'(\rho)} =$$

$$= \frac{q\cdot\sin(2\pi\rho - \varphi) - 2q^4\sin 2(2\pi\rho - \varphi) + 3q^9\sin 3(2\pi\rho - \varphi) - \ldots}{q\cdot\sin 2\pi\rho - 2q^4\sin 4\pi\rho + 3q^9\sin 6\pi\rho - \ldots}; \tag{4-48}$$

$$\lambda = \sqrt{\frac{1}{1 + \frac{e}{d}}}, \quad 0 < \rho < \frac{1}{2},$$

$\vartheta_0(x)$, $\vartheta_0'(x)$ is the theta-function and its derivative (see Appendix 1).

Fig. 4-33. Two plates of equal width, located at an angle to each other.

The approximation value of q can be determined from the formula

$$q_1 = \frac{1}{2}\,\frac{1 - \lambda}{\sqrt{1 + \lambda^2 - 2\lambda\cos\varphi}}. \tag{4-49}$$

A more accurate value of q is found from the formula

$$q_2 = \frac{(1-\lambda)\alpha^{1/2}}{2\beta} + \frac{\alpha(\cos 2\varphi - \lambda) + \gamma}{\alpha^{1/2}\beta},$$

where

$$\begin{aligned} \alpha &= (\cos\varphi - \lambda)^2 + (1 + q_1^4\mu)^2 \sin^2\varphi; \\ \beta &= (\cos\varphi - \lambda)^2 + (1 + q_2^4\mu) \sin^2\varphi; \\ \gamma &= (4\cos\varphi - \lambda)(1 + q_1^4\mu) \sin^2\varphi \cos\varphi; \\ \mu &= \frac{8\lambda(1-\cos\varphi)\cdot(1+\lambda)}{(\cos\varphi - \lambda)(1-\lambda)}. \end{aligned} \tag{4-50}$$

Numerical values of C_l/ε depending on a/d are given in Fig. 4-34.

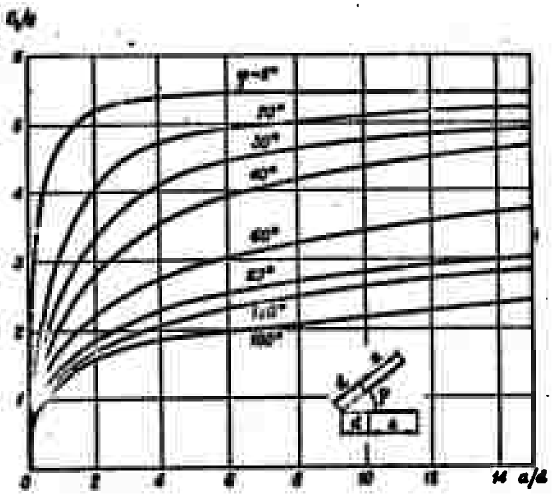


Fig. 4-34. A graph for determination of the capacitance between two plates at an angle to one another (dotted line – extrapolation).

b) Plate and half-plane (Fig. 4-35).

$$C_l = \varepsilon\cdot\frac{2\pi}{\ln\frac{1}{q}}, \tag{4-51}$$

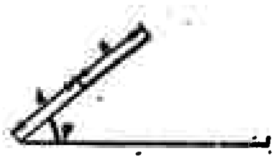


Fig. 4-35. An infinitely long plate and half-plane at an angle to one another.

where $q\ (0 < q < 1)$ is determined from the formulas (4-48)-(4-50), in which the quantity λ is replaced with $\lambda_1 = \gamma\lambda$.

6. *Plates perpendicular to two infinite planes.*

The formulas for the determination of capacitor capacitance per unit of length in systems consisting of one or two plates between two planes are given in Table 4-7.

7. *Plates parallel to two infinite planes.*

Formulas for determination of capacitor capacitance per unit of length for one or two plates arranged halfway between two planes are given in Table 4-8.

4-5. Partial Capacitances in a System of Many Infinitely Long Plates

Formulas for determination of partial capacitances per unit of length between strips in a system of three infinitely long plates are given in Table 4-9.

Example 4-4. To determine complete and partial capacitances per unit of length between strips in a shielded connected strip line with odd wave mode (Fig. 4-36), if $2h = 1$ cm, $b = 0.5$ cm, $2d = 0.5$ cm and the dielectric is air.

We find first the partial capacitance between strips C_{23l}, using the formula of clause 5 of Table 4-9.

Table 4-7. Formulas for the determination of capacitor capacitance in systems formed by two infinite planes and plates perpendicular to them.

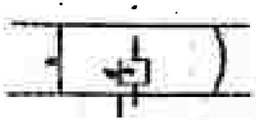
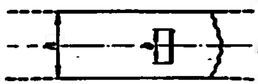
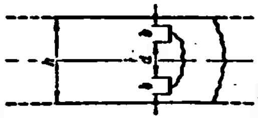
No. in order	Form of conductors	Calculation diagram	Calculation formulas: Accurate	Calculation formulas: Approximation	Relative error of approximation formulas, %
1	Two infinite planes and plate arranged asymmetrically relative to them		$C_l = 2\varepsilon\frac{K}{K'}$, where $k^2 = \frac{\sin\left(\frac{\pi}{2}\cdot\frac{b}{h}\right)\sin\left(\frac{\pi}{2}\cdot\frac{2d+b}{h}\right)}{\sin^2\left(\frac{\pi}{2}\cdot\frac{d+b}{h}\right)\cos^2\alpha}$ where $\alpha = \frac{\pi d}{2h}$	$C_l \simeq \frac{2\varepsilon}{\pi}\ln\frac{16}{1-k^2}$ at $\frac{b}{h}\simeq\frac{1}{2}$, $\frac{d}{h}\ll 1$; $C_l \simeq \frac{2\pi\varepsilon}{\ln\frac{16}{k^2}}$ at $\frac{b}{h}\ll 1$; $\frac{d}{h}\simeq\frac{1}{2}$	$-0{,}19$ at $\frac{b}{h}=0{,}4$, $\frac{d}{h}=0{,}1$ $0{,}55$ at $\frac{b}{h}=0{,}01$, $\frac{d}{h}=0{,}5$
2	Two infinite planes and the plate, arranged symmetrically relative to them		$C_l = 4\varepsilon\frac{K}{K'}$, where $k^2 = \sin^2\left(\frac{\pi}{2}\cdot\frac{b}{h}\right)$	$C_l \simeq \frac{8\varepsilon}{\pi}\ln\frac{4}{\cos\left(\frac{\pi}{2}\cdot\frac{b}{h}\right)}$ at $\frac{b}{h}\simeq\frac{1}{2}$; $C_l \simeq \frac{2\pi\varepsilon}{\ln\frac{4}{\sin\left(\frac{\pi}{2}\cdot\frac{b}{h}\right)}}$ at $\frac{b}{h}\ll 1$	$-0{,}19$ at $\frac{b}{h}=0{,}9$ $0{,}19$ at $\frac{b}{h}=0{,}1$
3	Two infinite planes and two plates, arranged symmetrically with respect to planes		$C_l = 4\varepsilon\frac{K}{K'}$, where $k^2 = \frac{\sin\frac{\pi b}{h}\sin\frac{\pi(b+d)}{h}}{\cos^2\frac{\pi d}{2h}}$	$C_l \simeq \frac{4\varepsilon}{\pi}\ln\frac{16}{1-k^2}$ at $\frac{b}{h}\simeq\frac{1}{2}$, $\frac{d}{h}\ll 1$, $C_l \simeq \frac{4\pi\varepsilon}{\ln\frac{16}{k^2}}$ at $\frac{b}{h}\ll 1$, $\frac{d}{h}\ll 1$	$-0{,}19$ at $\frac{b}{h}=0{,}4$, $\frac{d}{h}=0{,}1$ $0{,}79$ at $\frac{b}{h}=0{,}05$, $\frac{d}{h}=0{,}1$

Table --5. The formulas for determination of capacitance in systems formed by two infinite planes and plates parallel to them.

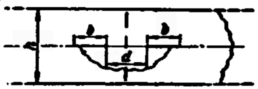
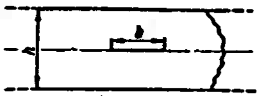
No. in order	Form of conductors	Calculation diagram	Calculation formulas: Accurate	Calculation formulas: Approximation	Relative error of approximation formulas, δ, %
1	Two infinite planes and two plates arranged halfway between them		$C_l = 4\varepsilon \cdot \frac{K}{K'}$, where $k^2 = \frac{\mathrm{sh}\,\frac{\pi b}{h}\,\mathrm{sh}\,\frac{\pi(b+d)}{h}}{\mathrm{ch}^2\,\frac{\pi(2b+d)}{2h}}$	$C_l \simeq \frac{4\varepsilon}{\pi}\cdot\ln\frac{16}{1-k^2}$ at $\frac{b}{h} \simeq 1,\ \frac{d}{h} < 1$ $C_l \simeq \frac{4\pi\varepsilon}{\ln\frac{16}{k^2}}$ at $\frac{b}{h} < 1,\ \frac{d}{h} < 1$	$-0{,}025$ at $\frac{b}{h} = 1,\ \frac{d}{h} = 0{,}1$ $0{,}65$ at $\frac{b}{h} = 0{,}05,\ \frac{d}{h} = 0{,}1$
2	Two infinite planes and plates arranged halfway between them		$C_l = 4\varepsilon \cdot \frac{K}{K'}$, where $k^2 = \mathrm{th}^2\left(\frac{\pi}{2}\cdot\frac{b}{h}\right)$	$C_l \simeq \frac{8\varepsilon}{\pi}\ln\left[4\,\mathrm{ch}\left(\frac{\pi}{2}\cdot\frac{b}{h}\right)\right]$ at $\frac{b}{h} > 1$; $C_l \simeq \frac{2\pi\varepsilon}{\ln\frac{4}{\mathrm{th}\left(\frac{\pi}{2}\cdot\frac{b}{h}\right)}}$ at $\frac{b}{h} < 1$	$-0{,}036$ at $\frac{b}{h} = 2$ $0{,}19$ at $\frac{b}{h} = 0{,}1$

Table 4-9. Formulas for determination of partial capacitances in a system of three infinitely long plates.

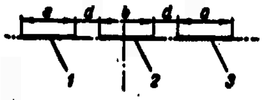
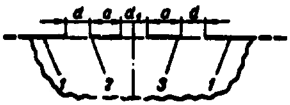
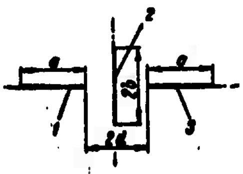
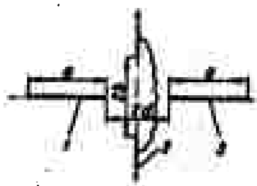
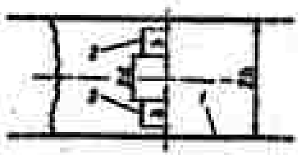
No. in order	System of conductors	Calculation diagram	Calculation formulas
1	Three coplanar plates, two of which have identical width and are located at equal distances from the third		$C_{13l} = \varepsilon\left(\frac{K'}{K} - \frac{K_1}{K_1'}\right)$, $C_{12l} = C_{23l} = 2\varepsilon \cdot \frac{K_1}{K_1'}$, where $k^2 = \frac{1+\frac{b}{d}}{\left(1+\frac{a}{d}\right)\left(1+\frac{a}{d}+\frac{b}{d}\right)}$, $k_1^2 = \frac{a}{d}\left(\frac{\frac{b}{d}}{2+\frac{b}{d}}\right)^2 \cdot \frac{2+\frac{a}{d}+\frac{b}{d}}{\left(1+\frac{a}{d}\right)\left(1+\frac{a}{d}+\frac{b}{d}\right)}$
2	A plane in the cut of which two plates of identical width are symmetrically located		$C_{23l} = \varepsilon\left(\frac{K'}{K} - \frac{K_1}{K_1'}\right)$, $C_{12l} = C_{13l} = 2\varepsilon \cdot \frac{K_1}{K_1'}$, where $k^2 = \frac{d}{a}\left(\frac{\frac{d_1}{a}}{2+\frac{d_1}{a}}\right)^2 \cdot \frac{2+\frac{d}{a}+\frac{d_1}{a}}{\left(1+\frac{d}{a}\right)\left(1+\frac{d}{a}+\frac{d_1}{a}\right)}$, $k_1^2 = \frac{1+\frac{d_1}{a}}{\left(1+\frac{d}{a}\right)\left(1+\frac{d}{a}+\frac{d_1}{a}\right)}$
3	Two plates arranged symmetrically relative to a third perpendicular to them		$C_{13l} = \varepsilon \cdot \left(\frac{K'}{K} - \frac{K_1}{K_1'}\right)$, $C_{12l} = C_{23l} = 2\varepsilon \cdot \frac{K_1}{K_1'}$, where $k^2 = \frac{1}{\left(1+\frac{a}{d}\right)^2}$, $k_1^2 = \frac{1}{1+\left(\frac{d}{b}\right)^2} \cdot \frac{\frac{a}{d}\left(2+\frac{a}{d}\right)}{\left(1+\frac{a}{d}\right)^2}$

Table 4-9 (Cont'd).

No. in order	System of conductors	Calculation diagram	Calculation formulas
4	Two plates arranged symmetrically relative to a cut in an infinite plane		$C_{12l} = \varepsilon\cdot\left(\frac{K'}{K} - \frac{K_1}{K_1'}\right)$, $C_{13l} = C_{23l} = 2\varepsilon\cdot\frac{K_1}{K_1'}$, where $k^2 = \frac{1}{\left(1+\frac{a}{d}\right)^2}$, $k_1^2 = \frac{1}{1+\frac{1+\left(\frac{b}{d}\right)^2}{\frac{a}{d}\cdot\left(2+\frac{a}{d}\right)}}$
5	Two united planes halfway between which there are two plates parallel to them		$C_{12l} = \varepsilon\cdot\left(\frac{K'}{K} - \frac{K_1}{K_1'}\right)$, $C_{13l} = C_{23l} = 2\varepsilon\cdot\frac{K_1}{K_1'}$, where $k^2 = \frac{\mathrm{sh}^2\left(\frac{\pi}{2}\cdot\frac{d}{h}\right)}{\mathrm{sh}^2\left(\frac{\pi}{2}\cdot\frac{d+b}{h}\right)}$, $k_1^2 = \frac{\mathrm{sh}\left(\frac{\pi}{2}\cdot\frac{2d+b}{h}\right)\cdot\mathrm{sh}\left(\frac{\pi}{2}\cdot\frac{b}{h}\right)}{\mathrm{ch}^2\left(\frac{\pi}{2}\cdot\frac{d+b}{h}\right)}$
6	Two united planes between which two plates perpendicular to them are symmetrically located		$C_{12l} = \varepsilon\cdot\left(\frac{K'}{K} - \frac{K_1}{K_1'}\right)$, $C_{13l} = C_{23l} = 2\varepsilon\cdot\frac{K_1}{K_1'}$, where $k^2 = \mathrm{tg}^2\left(\frac{\pi}{2}\cdot\frac{d}{h}\right)\cdot\mathrm{ctg}^2\left(\frac{\pi}{2}\cdot\frac{d+b}{h}\right)$, $k_1^2 = \frac{\sin\left(\frac{\pi}{2}\cdot\frac{2d+b}{h}\right)\cdot\sin\left(\frac{\pi}{2}\cdot\frac{b}{h}\right)}{\cos^2\left(\frac{\pi}{2}\cdot\frac{d}{h}\right)}$

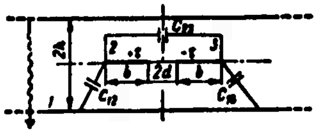


Fig. 4-36. A shielded connected strip line with odd wave mode.

Calculating the moduli of elliptical integrals, we have

$$k^2 = \frac{\text{sh}^2\left(\frac{\pi}{2}\cdot\frac{0,25}{0,5}\right)}{\text{sh}^2\left(\frac{\pi}{2}\cdot\frac{0,25+0,5}{0,5}\right)} = 0,0276.$$

$$k_1^2 = \frac{\text{sh}\left(\frac{\pi}{2}\cdot\frac{0,5+0,5}{0,5}\right)\cdot\text{sh}\left(\frac{\pi}{2}\cdot\frac{0,5}{0,5}\right)}{\text{ch}^2\left(\frac{\pi}{2}\cdot\frac{0,25+0,5}{0,5}\right)} = 0,9381;$$

$$(k')^2 = 1 - k^2 = 1 - 0,0276 = 0,9724;$$

$$(k_1')^2 = 1 - k_1^2 = 1 - 0,9381 = 0,0619.$$

Further with the aid of Appendix 2 we find

$$K = 1,582;\ K' = 3,196;\ K_1 = 2,806;\ K_1' = 1,596.$$

Substituting the numerical values into the formula for determination of capacitance we obtain

$$C_{23} = \varepsilon\left(\frac{K'}{K} - \frac{K_1}{K_1'}\right) = 8,854\cdot10^{-12}\left(\frac{3,196}{1,582} - \frac{2,806}{1,596}\right) = 2,33 \text{ pF/m}.$$

The partial capacitances of each of the strips relative to the grounded planes are determined analogously

$$C_{21} = C_{13} = 2\varepsilon\cdot\frac{K_1}{K_1'} = 2\cdot8,854\cdot10^{-12}\cdot\frac{2,806}{1,596} = 31,1 \text{ pF/m}.$$

The full capacitance between plates is

$$C_1 = C_{23} + \frac{C_{21}}{2} = \varepsilon\cdot\frac{K'}{K} = 8,854\cdot10^{-12}\cdot\frac{3,196}{1,582} = 17,9 \text{ pF/m}.$$

CHAPTER 5

CAPACITANCE OF SHELLS

5-1. General Remarks

1. In the present chapter formulas, tables, and graphs are given for the determination of the capacitance of conductors in the form of open and closed shells.

Especially considered are open shells of random form, and also any (including infinitely long) shells enveloping other conductors. The thickness of the open shells in all cases (if not contrary) is assumed infinitesimal.

2. Closed shells not enveloping other conductors, in an electrostatic sense are equivalent to the continuous conductors of the same form.

5-2. The Capacitance of Solitary Open Shells

In the present section data are given on the capacitance of solitary conductors in the form of open shells which possess the form of a hollow spherical segment, a hollow paraboloidal segment, or a cylindrical tube of finite length.[1]

[1]Also known are the results on an electrostatic field, and respectively the capacitances of hollow spherical shells with one [5-1] or two [5-2] circular cuts. These results, however, are so complex, that their utilization for computation of capacitance is quite difficult; therefore, data on the capacitance of the shells in the present paragraph are not given.

1. *A hollow spherical segment* (Fig. 5-1).

Fig. 5-1. A hollow spherical segment.

a) General case:

$$C_0 = 4\pi\varepsilon a\left(1 - \frac{\theta - \sin\theta}{\pi}\right). \quad (5\text{-}1)$$

The values of capacitance can be determined also from Fig. 5-2.

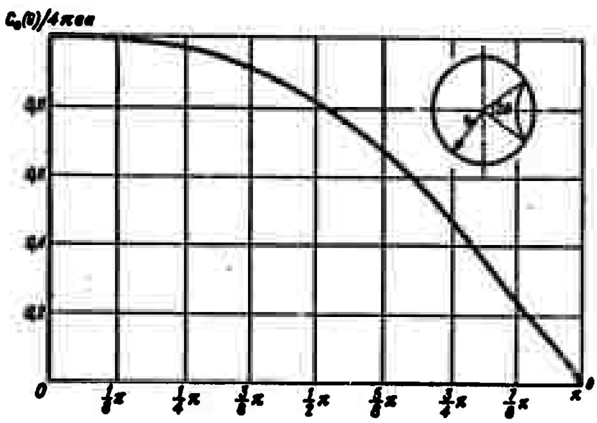


Fig. 5-2. A graph for the determination of the capacitance of a hollow spherical segment (dotted line – extrapolation).

θ	$\frac{1}{8}\pi$	$\frac{3}{16}\pi$	$\frac{1}{4}\pi$	$\frac{5}{16}\pi$	$\frac{3}{8}\pi$	$\frac{7}{16}\pi$	$\frac{1}{2}\pi$
$\frac{C_0(\theta)}{4\pi\varepsilon a}$	0.99690	0.98933	0.97499	0.95221	0.91907	0.87467	0.81831

(Continued)

θ	$\frac{9}{16}\pi$	$\frac{5}{8}\pi$	$\frac{11}{16}\pi$	$\frac{3}{4}\pi$	$\frac{13}{16}\pi$	$\frac{7}{8}\pi$	$\frac{15}{16}\pi$
$\frac{C_0(\theta)}{4\pi\varepsilon a}$	0,74970	0,66814	0,57714	0,47512	0,36422	0,24678	0,12468

b) A hemispheric shell ($\theta = \pi/2$):

$$C_0 = 4\pi\varepsilon a\left(\frac{1}{2} + \frac{1}{\pi}\right) = 4\pi\varepsilon a \cdot 0{,}8183. \qquad (5\text{-}2)$$

2. *A hollow paraboloidal segment* (Fig. 5-3).

$$C_0 = 8\varepsilon \cdot a\int_0^1 \psi(\tau)\,d\tau, \qquad (5\text{-}3)$$

where the function $\psi(\tau)$ is found from solution of the integral equation of Fredholm with a continuous nucleus:

$$\psi(\xi) - \frac{\rho}{\pi}\int_0^1 \psi(\tau)\cdot K(\mu, \nu)\,d\tau = 1, \quad 0 < \xi < 1,$$

and

$$\rho = h/a,$$

$$K(\mu, \nu) = \frac{1-\mu^2}{\mu}\cdot[\mathbf{K}(\mu) - \mathbf{E}(\mu)] + \frac{1-\nu^2}{\nu}[\mathbf{K}(\nu) - \mathbf{E}(\nu)],$$

K, **E** are complete elliptical integrals of the first and the second kind (see Appendix 1) with moduli

$$\mu = \frac{\rho(\tau-\xi)}{\sqrt{1+\rho^2(\tau-\xi)^2}}, \quad \nu = \frac{\rho(\tau+\xi)}{\sqrt{1+\rho^2(\tau+\xi)^2}}.$$

The dependence of capacitance on the quantity h/a is represented in Fig. 5-4. Furthermore, at rather low h/a the following approximation formula can be used:

$$C_0 \simeq 8\epsilon a\left[\left(1+\frac{1}{12}\left(2\frac{h}{a}\right)^2-\frac{1}{48}\left(2\frac{h}{a}\right)^4+\frac{31}{3360}\left(2\frac{h}{a}\right)^6-\right.\right.$$

$$\left.-\frac{1829}{128\cdot 2835}\cdot\left(2\frac{h}{a}\right)^8+\frac{99\,149}{1024\cdot 31\,185}\cdot\left(2\frac{h}{a}\right)^{10}-\ldots\right]=$$

$$=8\epsilon a\left[1+0{,}08333\left(2\frac{h}{a}\right)^2-0{,}02083\left(2\frac{h}{a}\right)^4+0{,}00923\left(2\frac{h}{a}\right)^6-\right.$$

$$\left.-0{,}00504\left(2\frac{h}{a}\right)^8+0{,}00310\left(2\frac{h}{a}\right)^{10}-\ldots\right] \qquad (5\text{-}4)$$

$[\delta < 0{,}1\%$ at $0 \leqslant h/a \leqslant 1/2]$.

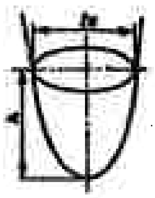
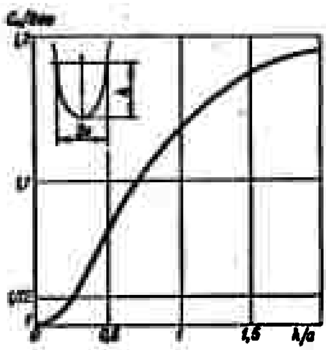


Fig. 5-3. Full paraboloidal segment.



$\frac{h}{a}$	0,1	0,2	0,3	0,4
$\frac{C_0}{8\epsilon a}$	1,0033	1,0128	1,0276	1,0467

(Continued)

$\frac{h}{a}$	0,5	0,6	0,8	1,0
$\frac{C_0}{8\epsilon a}$	1,0697	1,085	1,111	1,134

(Continued)

$\frac{h}{a}$	1,2	1,5	2,0
$\frac{C_0}{8\epsilon a}$	1,153	1,173	1,189

Fig. 5-4. Graph for the determination of the capacitance of a hollow paraboloidal segment.

If h/a < 0.3, then the capacitance of the paraboloidal segment is approximately equal to the capacitance of a circular disc a in radius (the error of such replacement does not exceed 2.7%).

3. *Cylindrical tube of finite length* (Fig. 5-5).

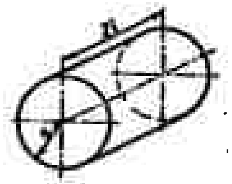


Fig. 5-5. Cylindrical tube of finite length.

The numerical values of the capacitance of a cylindrical tube of finite length are given in Table 5-1 and in Fig. 5-6.[1]

The following approximation formulas are valid also for the computation of capacitance:

$$C_0 \simeq 4\pi\varepsilon a \frac{\pi}{\ln\left(16\frac{a}{l}\right)} \quad \text{when} \quad \frac{l}{a}<4; \tag{5-5}$$

$$C_0 \simeq 4\pi\varepsilon a \frac{\pi^2 \cdot \frac{l}{a}}{\left(\ln\frac{16l}{a}\right)^2+\frac{\pi^2}{12}} \quad \text{when} \quad 9>\frac{l}{a}>4. \tag{5-6}$$

Note. The values $C_0/8\varepsilon a$ when $h/a \leq 0.5$ are determined from formula (5-4) with an error of <0.1%; at $h/a > 0.5$ by means of numerical integration with error <1%.

$$C_0 \simeq \frac{4\pi\varepsilon a \frac{l}{a}}{\ln\left(4\frac{l}{a}\right)-1}\left[1+\frac{4-\frac{\pi^2}{3}}{4\left(\ln\frac{4l}{a}-1\right)^2}\right]. \tag{5-7}$$

[1]The values shown were obtained on the basis of the results of works [5-3 thru 5-5], and also from the data of numerical calculations, politely given to the authors by Professor L. A. Vaynshteyn.

Table 5-1. Relative values of capacitance of a finite cylindrical tube.

$\frac{l}{a}$	0,1	0,3	0,5	0,7	0,9	1,1	1,3	1,5	
$\frac{C_0}{4\pi\varepsilon a}$	0,6192	0,7922	0,9122	1,0141	1,1066	1,1929	1,2748	1,3534	
$\frac{l}{a}$	1,7	1,9	2,1	2,3	2,5	2,7	2,9	3,1	
$\frac{C_0}{4\pi\varepsilon a}$	1,4291	1,5025	1,5739	1,6436	1,7118	1,7786	1,8441	1,9086	
$\frac{l}{a}$	3,5	3,9	4,5	4,9	5,9	6,9	7,9	8,9	9,9
$\frac{C_0}{4\pi\varepsilon a}$	2,0346	2,1571	2,3354	2,4514	2,7314	3,0015	3,2620	3,5158	3,7636
$\frac{l}{a}$	10,5	11,5	12,7	20	25	60	100	1000	
$\frac{C_0}{4\pi\varepsilon a}$	3,9097	4,1494	4,4314	6,0519	7,0928	13,632	20,332	68,900	

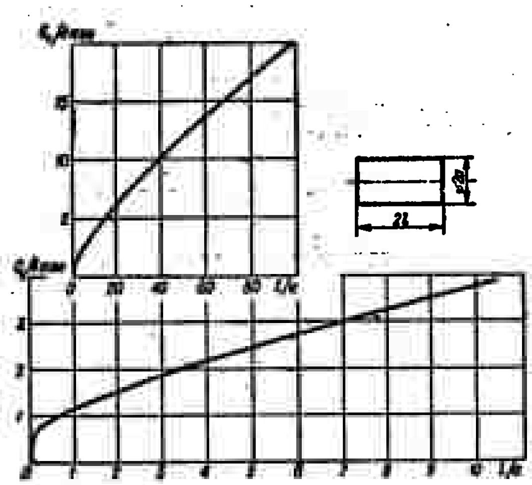


Fig. 5-6. Graph for the determination of the capacitance of a cylindrical tube of finite length.

When $l/a >> 1$ the conductor considered becomes a rectilinear wire, and the formulas given in § 3-2 can be used to calculate its capacitance.

The errors of formulas (5-5)-(5-7) are characterized by the curves of Fig. 5-7.

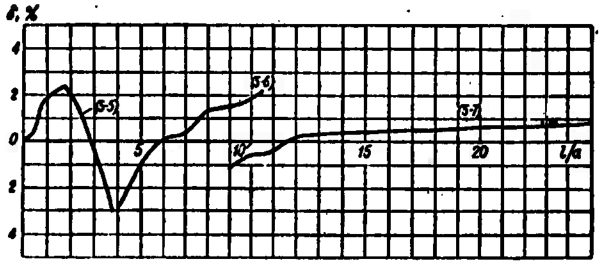


Fig. 5-7. Graph for determination of the inaccuracy of formulas (5-5)-(5-7).

5-3. The Capacitance of Solitary Closed Shells

The conductors considered in the present section can be divided into the following groups: conductors bounded by spherical surfaces, conductors of ellipsoidal form, conductors of toroidal form, a cylindrical conductor of finite length, and conductors in the form of regular polyhedrons.

The capacitance of conductors of more complex configuration can be evaluated on the basis of results for the capacitance of a sphere, a cylinder, a tetrahedron, a cube and an octahedron (see § V-2, and also [1-3]).

1. *Sphere* (Fig. 5-8).

$$C_0 = 4\pi\varepsilon \cdot a. \tag{5-8}$$

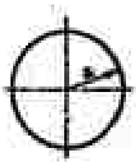
2. *Two nonintersecting spheres.*

a) General case (Fig. 5-9):

$$C_0 = 4\pi\varepsilon\cdot ab\cdot \operatorname{sh}\alpha\cdot\sum_{n=1}^{\infty}\left[\frac{1}{b\cdot\operatorname{sh}n\alpha + a\operatorname{sh}(n-1)\alpha} + \frac{1}{a\operatorname{sh}n\alpha + b\operatorname{sh}(n-1)\alpha} - \frac{1}{l\cdot\operatorname{sh}n\alpha}\right], \qquad (5\text{-}9)$$

where

$$\alpha = \operatorname{Arch}\frac{(2l)^2 - a^2 - b^2}{2ab}.$$



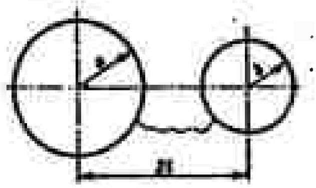


Fig. 5-8. Fig. 5-9.

Fig. 5-8. Sphere.

Fig. 5-9. Conductor formed by the union of two nonintersecting spheres of different radii.

At low values of the parameter $a/2l$ the approximation formula can be used

$$C_0 \simeq 4\pi\varepsilon(a+b)\cdot\frac{1-\frac{1}{l}\cdot\frac{ab}{a+b}}{1-\frac{ab}{(2l)^2}} \qquad (5\text{-}10)$$

$[\delta < 1{,}0\%$ at $a/2l < 0{,}5;\ b/a < 0{,}5]$.

Accurate and approximation numerical values of the function $C_0/4\pi\varepsilon a = f(b/a)$ at various values of the parameter $a/2l$ are given in Fig. 5-10.

b) Two intersecting spheres of identical radii (Fig. 5-11):

$$C_0 = 8\pi\varepsilon a\cdot\operatorname{sh}\beta\cdot\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\operatorname{sh}n\beta}, \qquad (5\text{-}11)$$

where $\beta = \operatorname{Arch} l/2a$.

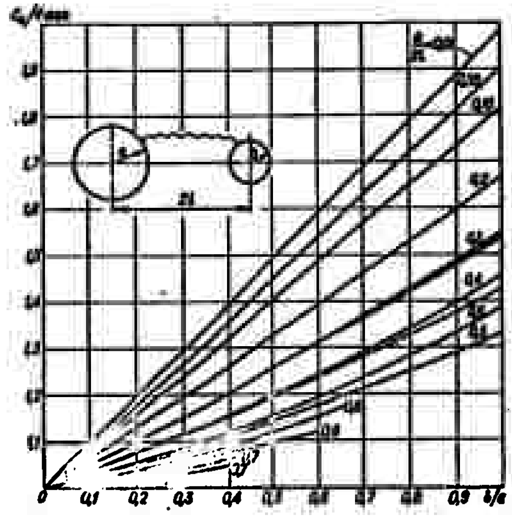


Fig. 5-10. Graph for determining the capacitance of a solitary conductor formed by the union of two nonintersecting spheres of different radii. ———— accurate values, – – – – – approximation values.

a/2l \ b/a	0,05	0,10	0,20	0,40	0,60	0,80	1,00
0,01	1,049	1,098	1,196	1,392	1,588	1,784	1,980
0,05	1,045	1,090	1,181	1,361	1,542	1,723	1,906
0,10	1,040	1,081	1,162	1,325	1,489	1,653	1,818
0,20	1,032	1,064	1,129	1,260	1,393	1,529	1,669
0,30	1,025	1,049	1,100	1,203	1,313	1,429	1,548
0,40	1,018	1,036	1,074	1,157	1,247	1,345	1,455
0,50	1,013	1,026	1,054	1,116	1,195	1,284	1,386
0,60	1,008	1,017	1,037	1,088	1,153	—	—
0,70	1,005	1,010	1,024	1,062	—	—	—
0,80	1,002	1,006	1,014	—	—	—	—

At $2l = a + b$ (adjoining spheres)

b/a	0,111	0,176	0,250	0,333	0,429	0,538	0,667	0,818	1,00
$\frac{C_0}{4\pi\varepsilon a}$	1,003	1,008	1,020	1,040	1,070	1,115	1,178	1,267	1,386

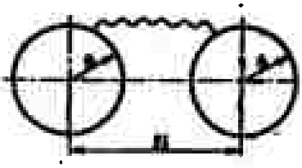


Fig. 5-11. Conductor formed by the union of two nonintersecting spheres of identical radii.

For a rather low value of the parameter $a/2l$ an approximation formula can also be used

$$C_0 \simeq 8\pi\varepsilon a \cdot \frac{1}{1+\frac{a}{2l}} \quad (5\text{-}12)$$

$[\delta < 0{,}3\%$ at $a/2l < 0{,}5]$.

The numerical values of the function $C_0/4\pi\varepsilon a = f(a/l)$ are given in Fig. 5-12.

c) Two touching sphere of different radii (Fig. 5-13).

$$C_0 = 4\pi\varepsilon \cdot \frac{ab}{a+b}\left[2+\frac{a}{b}+\frac{b}{a}-2\gamma-\psi\left(1+\frac{a}{a+b}\right)-\right.$$
$$\left.-\psi\left(1+\frac{b}{a+b}\right)\right], \quad (5\text{-}13)$$

where $\psi(1 + x)$ is a psi-function (see Appendix 1), γ is the Euler constant (γ = 0.5772...).

The table of values $\psi(1 + x)$ is contained in Appendix 6.

The data of the table to Fig. 5-10 can be used to determine the capacitance of two tangent spheres provided $2l = a + b$.

The approximation formula for rather low b/a has the form

$$C_0 \simeq 4\pi\varepsilon (a+b)\left[1-\frac{\frac{b}{a}}{1+\frac{b}{a}+\left(\frac{b}{a}\right)^2}\right] \quad (5\text{-}14)$$

$[\delta < 0{,}8\%$ at $b/a < 0{,}25]$.

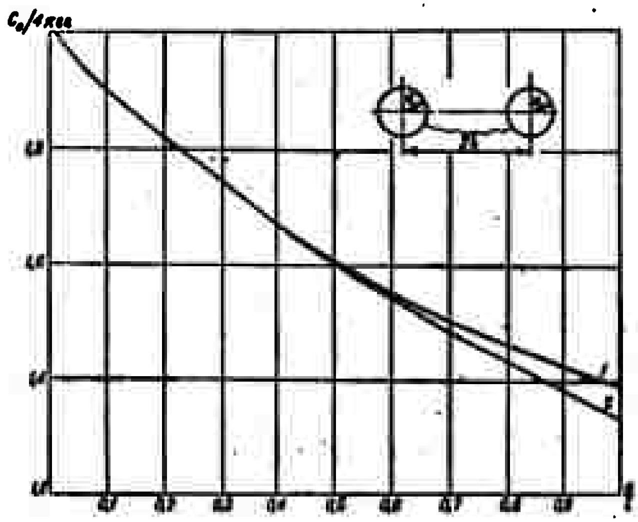


Fig. 5-12. A graph for determining the capacitance of a solitary conductor formed by the union of two nonintersecting spheres of identical radius. 1 ———— accurate values, 2 – – – – – approximation values.

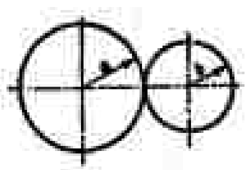


Fig. 5-13. Conductor formed by two touching spheres of different radii.

d) Two intersecting spheres of identical radii (Fig. 5-14).

$$C_0 = 8\pi\varepsilon a \cdot \ln 2 = 4\pi\varepsilon a \cdot 1{,}3862. \qquad (5\text{-}15)$$

3. *Conductors bounded by two intersecting spheres.*

a) General case (Fig. 5-15).

$$0 < \theta < \pi,$$
$$0 < \omega < 2\pi.$$

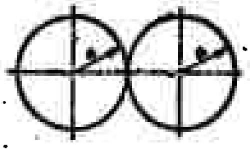


Fig. 5-14. Conductor formed by two touching spheres of identical radii.

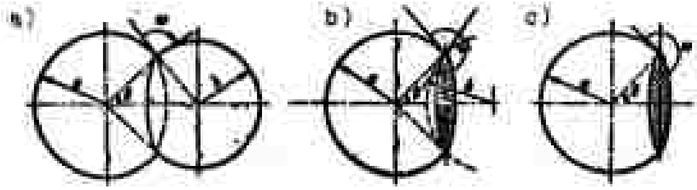


Fig. 5-15. Conductor bounded by two intersecting spheres.

At $\omega < \pi$ the conductor has the form shown in Fig. 5-15a; at $\omega > \pi$ (Fig. 5-15b) the conductor has the form of a spherical hole; at $\omega = \pi$ (limiting case) the conductor degenerates into a single sphere.

Radius a is always finite, radius b is found from the expression

$$b = \frac{a \cdot \sin\theta}{|\sin(\omega - \theta)|}$$

and can assume infinite values.

In the latter case ($\theta = \omega - \pi$; $\pi < \omega < 2\pi$) the conductor has the form of a spherical segment (Fig. 5-15c).

At $\theta = \omega/2$ ($\omega < \pi$) and at $\theta = \pi - \omega/2$ ($\omega > \pi$) the radii of the spheres are identical: $a = b$.

For any of the conductors of Fig. 5-15 capacitance is determined from the formula

$$C_0 = \frac{a \cdot \sin\theta}{\omega} \cdot \left\{ \frac{\pi}{\sin\frac{\pi\theta}{\omega}} - 8 \sum_{n=1}^{\infty} \sin^2\left(n \frac{\pi\theta}{\omega}\right) \times \right.$$

$$\left. \times \left[\psi\left(n + \frac{1}{2}\right) - \psi\left(n \frac{\pi}{\omega} + \frac{1}{2}\right) + \ln \frac{\pi}{\omega} \right] \right\}^{*}, \tag{5-16}$$

where $\psi(x)$ is a psi-function (see Appendix 1).

If ω is a rational fraction, multiplied by π or 2π, then the capacitance of any of the conductors in Fig. 5-15 can be expressed in finite form:

when $\omega = \frac{(2n-1)\pi}{2m} < 2\pi \quad (1 < n \leqslant 2m)$

$$C_0 = \frac{4\pi\varepsilon a \cdot \sin\theta}{2n-1} \times$$

$$\times \left\{ \sum_{s=1}^{n-1} \sum_{t=0}^{2m-1} \frac{(-1)^{s+1} \cdot \sin\frac{2s\pi}{2n-1}}{\sin\left(\frac{t\pi}{2m} + \frac{\theta}{2n-1}\right) \cdot \left[\sin\left(\frac{t\pi}{2m} + \frac{\theta}{2n-1}\right) + \sin\frac{s\pi}{2n-1}\right]} + \right.$$

$$+ 2 \sum_{s=1}^{n-1} \sum_{t=0}^{2m-1} \frac{(-1)^{s+1} \cdot \sin\frac{2s\pi}{2n-1}}{\cos\frac{t\pi}{m} - \cos\frac{2s\pi}{2n-1}} - (2n-1) \times$$

$$\left. \times \sum_{t=1}^{2m-1} \frac{1}{\sin\frac{(2n-1)t\pi}{2m}} \right\}; \tag{5-17}$$

at

$$\omega = \frac{2n\pi}{2m-1} < 2\pi \; (1 < n < 2m-1)$$

$$C_0 = \frac{4\pi\varepsilon a \cdot \sin\theta}{n} \times$$

$$\times \left\{ \sum_{s=1}^{n-1} \sum_{t=0}^{2m-2} \frac{(-1)^{s+1} \cdot \sin^2\frac{s\pi}{n}}{\left[\cos\left(\frac{2t\pi}{2m-1} + \frac{\theta}{n}\right) - \cos\frac{s\pi}{n}\right]} \times \right.$$

---

*At high n the series contained in (5-16) decreases as $1/n^2$.

$$\times\left[\frac{1-\frac{1}{\pi}\left(\frac{2l\pi}{2m-1}+\frac{\theta}{n}\right)}{\sin\left(\frac{2l\pi}{2m-1}+\frac{\theta}{n}\right)}-\frac{1-\frac{s}{n}}{\sin\frac{s\pi}{n}}\right]+$$

$$+\sum_{l=1}\sum_{s=1}\frac{(-1)^{s+1}\cdot\left(1-\frac{s}{n}\right)\cdot\sin\frac{s\pi}{n}}{\cos\frac{2l\pi}{2m-1}-\cos\frac{s\pi}{n}}+$$

$$+\frac{1}{\pi}-n\sum_{l=1}^{2m-2}\frac{1-\frac{2l}{2m-1}}{\sin\frac{2nl\pi}{2m-1}}\Bigg\}. \qquad (5\text{-}18)$$

For the case $\omega < \pi$ (Fig. 5-15a) the formulas are still more simplified and have the form:

at $\omega = 2\pi/m$ $(m = 3, 4, \ldots)$

$$C_0 = 4\pi\varepsilon a\cdot\left\{1-\frac{\theta-\sin\theta}{\pi}+\right.$$

$$\left.+\sin\theta\cdot\sum_{l=1}^{m-1}\left[\frac{1-\left(\frac{2l}{m}+\frac{\theta}{\pi}\right)}{\sin\left(\frac{2\pi l}{m}+\theta\right)}-\frac{1-\frac{2l}{m}}{\sin\frac{2\pi l}{m}}\right]\right\}; \qquad (5\text{-}19)$$

at $\omega = \pi/m$ $(m = 2, 3, \ldots)$

$$C_0 = 4\pi\varepsilon a\left\{1+\sin\theta\sum_{l=1}^{m-1}\left[\frac{1}{\sin\left(\frac{l\pi}{m}+\theta\right)}-\frac{1}{\sin\frac{l\pi}{m}}\right]\right\}. \qquad (5\text{-}20)$$

The numerical values of the quantity $C_0/4\pi\varepsilon a$ for some ω and θ are given in Table 5-2.

The numerical values of the capacitance of spherical segments $C_0/4\pi\varepsilon a = f(\omega)$ are given in Fig. 5-16.

Table 5-2. Relative values of capacitance of a solitary conductor formed by two intersecting spheres.

$\frac{C_0}{4\pi\varepsilon a}$

ω \ θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
$\frac{\pi}{3}$	1,35	1,36	—	—
$\frac{\pi}{2}$	1,08	1,29	—	—
$\frac{3\pi}{2}$	0,997	0,987	0,846	0,768
2π	0,991	0,[illegible]75	0,818	0,475

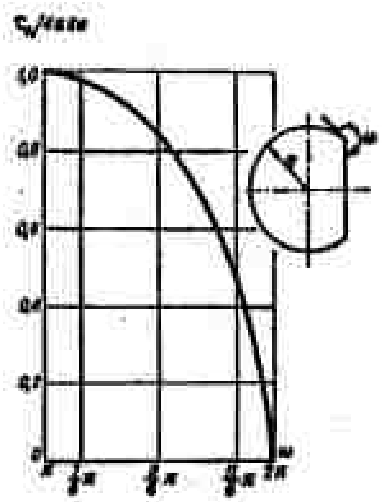


Fig. 5-16. Graph for the determination of the capacitance of a spherical segment.

b) Particular cases.

The formulas for the determination of the capacitance of some typical conductors formed by the intersection of two spheres are given in Table 5-3.

Table 5-3. Formulas for determination of the capacitance of some conductors formed by the intersection of two spheres.

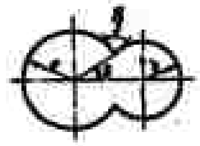
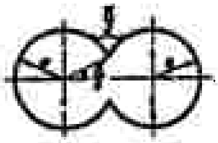
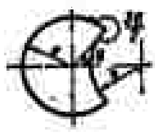
№ in order	Conductor: name	Conductor: diagram	Calculation formulas
1	Two intersecting spheres at $\omega = \pi/3$		$C_0 = 4\pi\varepsilon a\left\{1 + \sin\theta\left[\frac{1}{\sin\left(\frac{\pi}{3}+\theta\right)} + \frac{1}{\sin\left(2\frac{\pi}{3}+\theta\right)} - \frac{4}{\sqrt{3}}\right]\right\}$
2	The same, at equal radii of spheres at ($\omega = \pi/3$, $\theta = \pi/6$)		$C_0 = 4\pi\varepsilon a\left(2{,}5 - \frac{2}{\sqrt{3}}\right) = 4\pi\varepsilon a \cdot 1{,}3459$
3	Two orthogonally intersecting spheres at $\omega = \pi/2$		$C_0 = 4\pi\varepsilon a(1 + \operatorname{tg}\theta - \sin\theta) = 4\pi\varepsilon\left(a + b - \frac{ab}{\sqrt{a^2+b^2}}\right)$
4	The same, at equal radii of spheres at ($\omega = \pi/2$, $\theta = \pi/4$)		$C_0 = 4\pi\varepsilon a\left(2 - \frac{\sqrt{2}}{2}\right) = 4\pi\varepsilon a \cdot 1{,}2929$
5	A spherical hole at the orthogonal intersection of spheres ($\omega = 3\pi/2$)		$C_0 = 4\pi\varepsilon a\frac{\sin\theta}{\sqrt{3}}\left[\sqrt{3} - \frac{4}{3} + \frac{1}{\left(\sin\frac{\theta}{3}+\frac{\sqrt{3}}{2}\right)2\sin\frac{\theta}{3}} + \frac{1}{\left(\cos\frac{\theta}{3}+\frac{\sqrt{3}}{2}\right)2\cos\frac{\theta}{3}}\right]$
6	The same, at equal radii of spheres ($\omega = 3\pi/2$, $\theta = \pi/4$)		$C_0 = 4\pi\varepsilon a \cdot \frac{\sqrt{6}}{6}\left(\sqrt{3} + \frac{4}{2+\sqrt{6}} - \frac{3}{4}\right) = 4\pi\varepsilon a \cdot 0{,}7679$

Table 5-3 (Continued).

№ in order	Conductor		Calculation formulas
	name	diagram	
7	Spherical segment at $\omega = 3\pi/2$ ($\theta = \pi/2$) (hemi-sphere)		$C_0 = 4\pi\varepsilon a\cdot 2\left(1-\frac{1}{\sqrt{3}}\right) = 4\pi\varepsilon a\cdot 0{,}8453$

4. *Three intersecting spheres* (Fig. 5-17).

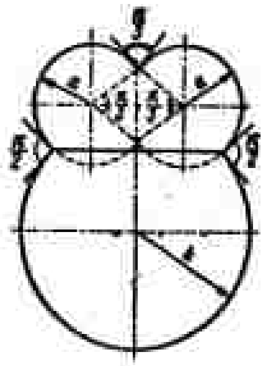


Fig. 5-17. Conductor formed by the intersection of three spheres.

If a conductor is formed by two identical spheres of radius a, intersecting at an angle of $\pi/3$, and by a third sphere of radius b, which intersects each of the identical spheres at right angles, then

$$C_0 = 4\pi\varepsilon\cdot\left[b + a\left(\frac{5}{2} - \frac{2}{3}\sqrt{3}\right) - 2ab\left(\frac{1}{\sqrt{a^2+b^2}} - \frac{1}{\sqrt{a^2+3b^2}} + \frac{1}{2\sqrt{a^2+4b^2}}\right)\right]. \quad (5\text{-}21)$$

At identical radii of all spheres $(a = b)$

$$C_0 = 4\pi\varepsilon \cdot a\left(\frac{9}{2} - \frac{2\sqrt{3}}{3} - \right.$$
$$\left. - \sqrt{2} - \frac{1}{\sqrt{5}}\right) = 4\pi\varepsilon a \cdot 1{,}4839. \qquad (5\text{-}22)$$

5. *Ellipsoids.*

a) Triaxial ellipsoid ($a > b > c$) (Fig. 5-18):

$$C_0 = 4\pi\varepsilon a \cdot \frac{\sqrt{1-\left(\frac{c}{a}\right)^2}}{F(\varphi, k)}, \qquad (5\text{-}23)$$

where

$$k^2 = \frac{1-\left(\frac{b}{a}\right)^2}{1-\left(\frac{c}{a}\right)^2}; \quad \varphi = \arcsin\sqrt{1-\left(\frac{c}{a}\right)^2};$$

$F(\phi, k)$ is an incomplete elliptical integral of the first kind (Appendix 1).

If the semiaxes of an ellipsoid are equal respectively to a, $a(1 + \alpha)$, $a(1 + \alpha\cdot\beta)$, and $|\alpha\cdot\beta| < 1$, then

$$C_0 \simeq 4\pi\varepsilon a \cdot \left[1 + \frac{1}{3}\alpha(1+\beta) - \frac{1}{45}\alpha^2(1-\beta+\beta^2)\right]. \qquad (5\text{-}24)$$

Example 5-1. To determine the capacitance of a conductor in the form of a triaxial ellipsoid which is in distilled water ($\varepsilon \simeq 83\ \varepsilon_0$), if its semiaxes are respectively $a = 10$ cm; $b = 8$ cm, $c = \sqrt{28}$ cm.

Using formula (5-23), let us predetermine the modulus and argument of an elliptical integral of the first kind $F(\phi, k)$.

At assigned dimensions

$$k^2 = \frac{1-\left(\frac{8}{10}\right)^2}{1-\left(\frac{\sqrt{28}}{10}\right)^2} = 0{,}5;$$

$$\varphi = \arcsin \sqrt{1-\left(\frac{\sqrt{28}}{10}\right)^2} = 58°.$$

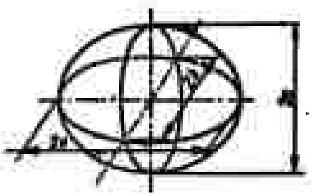


Fig. 5-18. Conductor in the form of a triaxial ellipsoid.

From the table of elliptical integrals we find that

$$F(\varphi, k) = F\left(58°, \sqrt{0,5}\right) = 1,099.$$

Substituting the found value of $F(\phi, k)$ in formula (5-23), we obtain

$$C_0 = 4\pi \frac{83}{4\pi \cdot 9 \cdot 10^9} \cdot 0,1 \cdot \frac{\sqrt{1-\left(\frac{\sqrt{28}}{10}\right)^2}}{1,099} = 8,35 \cdot 10^{-10} \text{ F} = 835 \text{ pF}$$

b) Condensed spheroid $(a = b > c)$ (Fig. 5-19):

$$C_0 = 4\pi\varepsilon a \frac{\sqrt{1-\left(\frac{c}{a}\right)^2}}{\arccos \frac{c}{a}}. \qquad (5\text{-}25)$$

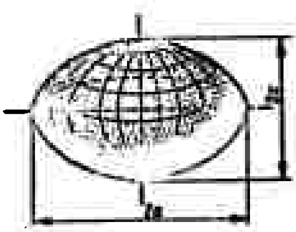


Fig. 5-19. Conductor in the form of a condensed spheroid.

c) A drawn out spheroid *(a > b = c)* (Fig. 5.20):

$$C_0 = 4\pi\varepsilon a \frac{\sqrt{1-\left(\frac{b}{a}\right)^2}}{\operatorname{Arch}\frac{a}{b}}. \qquad (5\text{-}26)$$

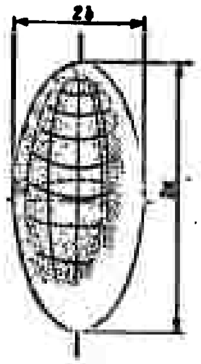


Fig. 5-20. A conductor in the form of a drawn out spheroid.

6. *Torus.*

a) Torus of circular section[1] (Fig. 5-21):

$$C_0 = 8\pi\varepsilon l \cdot \sqrt{1-\left(\frac{a}{l}\right)^2} \cdot \left[\frac{Q_{-\frac{1}{2}}(\operatorname{ch}\alpha)}{P_{-\frac{1}{2}}(\operatorname{ch}\alpha)} + 2\sum_{n=1}^{\infty}\frac{Q_{n-\frac{1}{2}}(\operatorname{ch}\alpha)}{P_{n-\frac{1}{2}}(\operatorname{ch}\alpha)}\right]. \qquad (5\text{-}27)$$

[1] A more general case is that of a torus of oval section; however, the calculation of the capacitance of such a conductor [5-6] requires preliminary tabulation of a number of special functions and that is why it is not considered here.

where

$$P_{n+\frac{1}{2}}(\mathrm{ch}\,\alpha)=\frac{1}{\pi}\int_0^{\pi}\frac{d\beta}{(\mathrm{ch}\,\alpha+\mathrm{sh}\,\alpha\cos\beta)^{n+\frac{1}{2}}},$$

$$Q_{n+\frac{1}{2}}(\mathrm{ch}\,\alpha)=\int_0^{\infty}\frac{d\beta}{(\mathrm{ch}\,\alpha+\mathrm{sh}\,\alpha\,\mathrm{ch}\,\beta)^{n+\frac{1}{2}}}$$

are Legendre functions of the first and second kind (see Appendix 1), $\mathrm{ch}\,\alpha = l/a$.

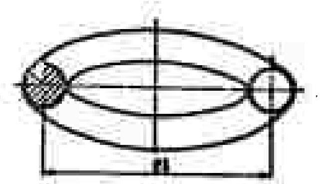


Fig. 5-21. A conductor of toroidal form.

The numerical values of function $C_0/4\pi\varepsilon l = f(a/l)$ are given in Fig. 5-22.

The following approximation formulas can also be used:

$$C_0\simeq 8\pi\varepsilon l\sqrt{1-\left(\frac{l}{a}\right)^2}\cdot\left(\frac{K'}{K}+2\frac{K'-E'}{E}\right), \qquad (5\text{-}28)$$

where K, K', E, E' are complete elliptical integrals of the first and second kind (see Appendix 1) with modulus $k=\frac{2\sqrt{l^2-a^2}}{l+\sqrt{l^2-a^2}}$,

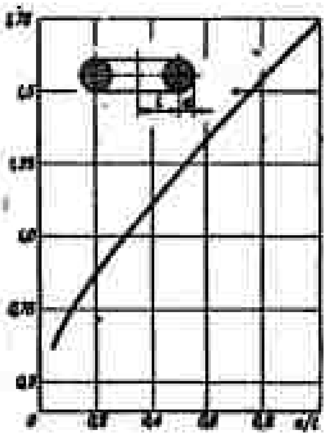
$|\delta<1\%$ at $a/l<0{,}45|$,

$$C_0\simeq 4\pi\varepsilon l\cdot\frac{\pi}{\ln\left(8\frac{l}{a}\right)} \qquad (5\text{-}29)$$

$[\delta < 1\%$ at $a/l < 0,12$; $\delta < 4\%$ at $a/l < 0,30]$.

$$C_0 \simeq 4\pi\epsilon l\,(0,68 + 1,07a/l) \tag{5-30}$$

$[\delta < 1\%$ at $a/l > 0,30]$.



$\frac{a}{l}$	0,05	0,10	0,15	0,20
$\frac{C_0}{4\pi\epsilon l}$	0,616	0,727	0,800	0,868

(Continued)

$\frac{a}{l}$	0,25	0,30	0,35	0,40
$\frac{C_0}{4\pi\epsilon l}$	0,932	0,992	1,050	1,106
$\frac{a}{l}$	0,45	0,50	0,60	0,70
$\frac{C_0}{4\pi\epsilon l}$	1,161	1,216	1,323	1,429

(Continued)

$\frac{a}{l}$	0,80	0,90	1,00
$\frac{C_0}{4\pi\epsilon l}$	1,534	1,638	1,741

Fig. 5-22. Graph for determining capacitance of a conductor of toroidal form.

b) A torus without an opening (formed by the rotation of a circle around a tangent) (Fig. 5-23):

$$C_0 = 16\epsilon a \int_0^{\infty} \frac{K_0(x)}{I_0(x)}\,dx, \tag{5-31}$$

where $I_0(x)$, $K_0(x)$ are Bessel functions of an imaginary argument (see Appendix 1).

$$C_0 \simeq 4\pi\epsilon a \cdot 1,7413528. \tag{5-32}$$

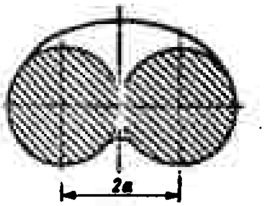


Fig. 5-23. Conductor formed by rotation of a circle around a tangent.

7. *Cylinder of finite length* (Fig. 5-24).

At $0 \leq l/a \leq 8$

$$C_0 \simeq 4\pi\varepsilon a\left[0{,}6372 + 0{,}5535\cdot\left(\frac{l}{a}\right)^{0{,}76}\right] \quad [\delta < 0{,}2\%]. \tag{5-33}$$

The numerical values of the function $C_0/4\pi\varepsilon a = f(l/a)$ are given in Fig. 5-25.

At $l/a > 10$

$$C_0 \simeq \frac{4\pi\varepsilon l}{\operatorname{Arsh}\frac{2l}{a} + \frac{a}{2l} - \sqrt{\left(\frac{a}{2l}\right)^2 + 1}} \quad [\delta < 5\%]. \tag{5-34}$$

At $l/a > 50$

$$C_0 \simeq \frac{4\pi\varepsilon l}{\ln\frac{4l}{a} - 1}. \tag{5-35}$$

See also clause 1 of § 3-2.

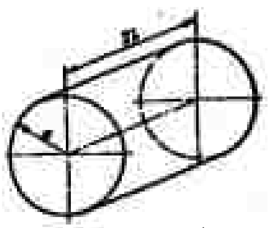


Fig. 5-24. A conductor in the form of a cylinder of finite length.

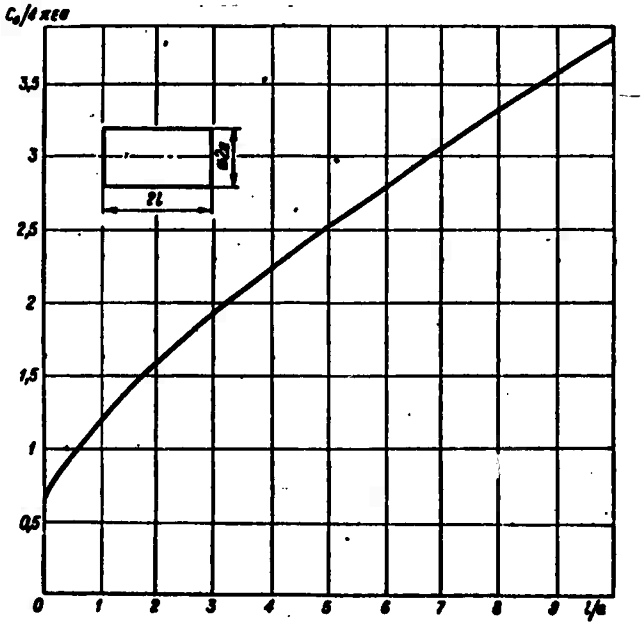


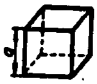
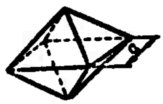
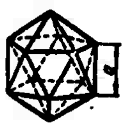
Fig. 5-25. A graph for determination of the capacitance of a conductor of cylindrical form.

$\frac{l}{a}$	0,000	0,125	0,250	0,333	0,500	0,600	0,800
$\frac{C_0}{4\pi\varepsilon a}$	0,63733	0,7512	0,8302	0,8777	0,9643	1,0126	1,1050

(Continued)

$\frac{l}{a}$	1,0	2	4	8	100
$\frac{C_0}{4\pi\varepsilon a}$	1,1913	1,5255	2,2547	3,3250	20,8

Table 5-4. Upper and lower boundaries of capacitance of conductors in the form of regular polyhedrons.

No. in order	Conductor: Name	Conductor: General form	Boundaries of values of the quantity $C_1 = C_0/4\pi\varepsilon a$ (C_0 is the capacitance of the conductor; $4\pi\varepsilon a$ is the capacitance of a sphere of radius a, equal to the length of the edge of a polyhedron).
1	Tetrahedron		$0{,}745 < C_1 < 0{,}9032$
2	Cube (hexahedron)		$0{,}6393 < C_1 < 0{,}6675$, $C_1 \approx 0{,}65565$
3	Octahedron		$0{,}591 < C_1 < 0{,}6327$
4	Dodecahedron		$0{,}5049 < C_1 < 0{,}5627$
5	Icosahedron		$0{,}5036 < C_1 < 0{,}5419$

8. *Regular polyhedrons.*

Upper and lower limits of values of capacitances of conductors in the form of regular polyhedrons are given in Table 5-4.[1] The data of Table 5-4 are given in relation to the capacitance of a sphere with radius equal to the length of an edge. The length of the edge of polyhedrons can be calculated from their assigned volume or area of surfaces with the aid of the data given in Table 5-5.

Table 5-5. Geometric parameters of regular polyhedrons (a – length of edge).

Name of conductor	Number of boundaries and their form	Number		Complete surface	Volume
		edge	vertexes		
Tetrahedron	4 triangles	6	4	$1.7321\ a^2$	$0.1179\ a^3$
Cube (hexahedron)	6 squares	12	8	$6.0\ a^2$	$1.0\ a^3$
Octahedron	8 triangles	12	6	$3.4641\ a^2$	$0.4714\ a^3$
Dodecahedron	12 pentagons	30	20	$20.6457\ a^2$	$7.6631\ a^3$
Icosahedron	20 triangles	30	12	$8.6603\ a^2$	$2.1817\ a^3$

[1]It is not without interest to observe the development of works on determination of the capacitance of a cube. There is an assumption [1-3], that the approximation value of the capacitance of a cube was known already to Dirichlet; however, the main results on determination of the capacitance of a cube were obtained only in the last two decades and are characterized by the following data:

1. G. Polya, 1947-48 [5-7, 5-8], $0{,}62211 < C_1 < 0{,}7105$.
2. G. Polya and G. Sege, 1951 [1-3], $0{,}632 < C_1 < 71055$.
3. T. I. Higgins and D. K. Reitan, 1951 [5-9], $C_1 \approx 0{,}6555$.
4. W. Gross, 1952 [5-10], $C_1 \approx 0{,}6464;\ |C_1 - 0{,}6464| < 0{,}032$.
5. R. I. Mc-Maxon, 1953 [5-11], $C_1 > 0{,}639273$.
6. L. Daboni, 1953 [5-12], $C_1 < 0{,}676$.
7. W. E. Parr, 1961 [5-15], $C_1 < 0{,}6675$.
8. I. Van Bladel and K. Mei, 1962 [5-16], $C_1 \approx 0{,}65565$.

Comparison of these results leads to the data shown in graph 2 of Table 5-4.

Furthermore, the capacitance of a cube was evaluated in the works of L. E. Payne and H. F. Weinberger [5-13, 5-14].

5-4. Capacitance Between Two Infinitely Long Shells

In the present section formulas, tables, and graphs are given for the determination of capacitance per unit of length of conductors that are infinitely long shells. These conductors are:

cylindrical shells of circular and an elliptical section; shells having in a section an equilateral triangle; shells of rectangular and square sections; shells of regular n-angular section; and circular and arched shells.

1. *Shells of circular and elliptical sections.*

a) Shells of circular section.

Formulas for determination of capacitance per unit of length between infinitely long shells of circular section are shown in Table 5-6.

b) Confocal shells of elliptical section (Fig. 5-26)

$$C_l = \frac{2\pi\varepsilon}{\operatorname{Arch}\frac{a_1a_2 - b_1b_2}{c^2}} = \frac{2\pi\varepsilon}{\ln\frac{a_1+\sqrt{a_1^2-c^2}}{a_2+\sqrt{a_2^2-c^2}}}, \qquad (5\text{-}36)$$

where $c^2 = a_1^2 - b_1^2 = a_2^2 - b_2^2$.

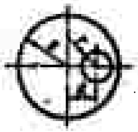
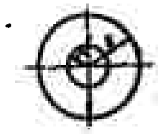
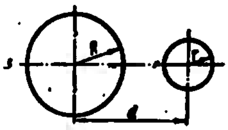
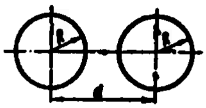
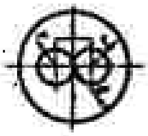
c) Coaxial circular and elliptical shells (Fig. 5-27).

$$C_l = \frac{4\pi\varepsilon\cdot K'(k)}{K(k) - F(\varphi, k)}, \qquad (5\text{-}37)$$

where K and K' are complete elliptical integrals of the 1st kind (see Appendix 1) with moduli

$$k = \frac{R^2-a^2}{R^2+a^2}\,\frac{R^2+b^2}{R^2-b^2} \text{ and } k' = \sqrt{1-k^2}.$$

Table 5-6. Formulas for determination of capacitance per unit of length between infinitely long cylindrical shells of a circular section.

No. in order	Location of shells	Diagram	Calculation formulas
1	One of the shells is inside the other		$C_l = \frac{2\pi\varepsilon}{\mathrm{Arch}\,\frac{r^2+R^2-d^2}{2rR}}$
2	Shells are coaxial (cylindrical capacitor)		$C_l = \frac{2\pi\varepsilon}{\ln\frac{R}{r}}$
3	One of the shells is outside the other		$C_l = \frac{2\pi\varepsilon}{\mathrm{Arch}\,\frac{d^2-(r^2+R^2)}{2rR}}$
4	The same, as clause 3, with equal radii of shells		$C_l = \frac{\pi\varepsilon}{\mathrm{Arch}\,\frac{d}{2R}}$
5	Two touching shells inside a third, the axis of which coincides with the line of contact of the first two		$C_l \approx \frac{2\pi\varepsilon}{\ln\left(\frac{2}{\pi}\cdot\frac{R}{r}\right)}$
6	Two identical connected shells inside the third symmetrically relative to its axis		at $r^2 \ll R^2$, $d^2 \ll R^2$, $\frac{4\pi\varepsilon}{\ln\frac{R^2}{rd}} < C_l < \frac{4\pi\varepsilon}{\ln\frac{R^2}{r\cdot\sqrt{d^2+4r^2}}}$

$F(\phi, k)$ is an incomplete elliptical integral of kind I (see Appendix 1) with modulus k and argument

$$\varphi = \arcsin\left(\frac{b}{a}\,\frac{R^2-a^2}{R^2+b^2}\right).$$

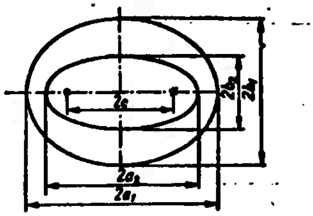


Fig. 5-26.

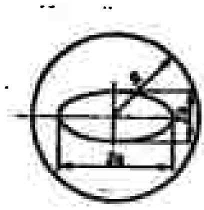


Fig. 5-27.

Fig. 5-26. Confocal shells of elliptical section.

Fig. 5-27. Coaxial circular and elliptical shells.

d) Off-axial circular and elliptical shells (Fig. 5-28).

At $\rho \ll c$

$$C_l \simeq \frac{2\pi\varepsilon}{\ln\frac{c}{4\rho} - \mu_1 - 2\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\mu_1}\left[\frac{\operatorname{ch}^2 n\mu_0 \cos^2 n\nu_0}{\operatorname{ch} n\mu_1} + \frac{\operatorname{sh}^2 n\mu_0 \sin^2 n\nu_0}{\operatorname{sh} n\mu_1}\right]}, \qquad (5\text{-}38)$$

where μ_1 = Arch a/c = Arsh b/c, $c = \sqrt{a^2 - b^2}$, and the quantities μ_0 and ν_0 are defined as the solution of the system of equations

$$2x_0 = c \operatorname{ch}\mu_0 \cos\nu_0; \quad 2y_0 = c \operatorname{sh}\mu_0 \sin\nu_0.$$

2. *Shells having in a section an equilateral triangle* (Fig. 5-29).

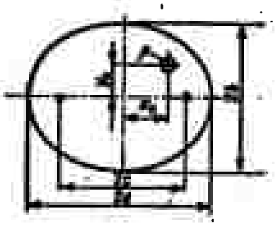


Fig. 5-28.

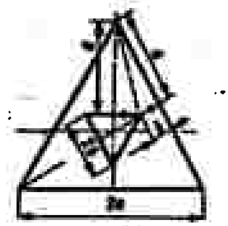


Fig. 5-29.

Fig. 5-28. Coaxial circular and elliptical shells.

Fig. 5-29. Infinitely long shells of triangular section $b = a$ tg 15° = = 0.268 a.

Regardless of the dimensions of the sides

$$C_l = 6\varepsilon. \tag{5-39}$$

3. *Shells of rectangular and square sections.*

a) Rectangular shells with parallel sides enveloping each other (Fig. 5-30).

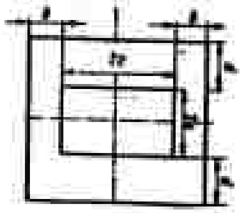
$$C_l \simeq 4\varepsilon\left[\frac{a}{b} + \frac{c}{d} + \frac{2}{\pi}\left(\ln\frac{b^2+d^2}{4bd} + \frac{b}{d}\,\mathrm{arctg}\,\frac{d}{b} + \frac{d}{b}\,\mathrm{arctg}\,\frac{b}{d}\right)\right]. \tag{5-40}$$

b) Rectangular shells with parallel sides not enveloping each other (Fig. 5-31).

The values of capacitance per unit of length of the conductors considered are given in Fig. 5-32.

Numerical values are determined with error of the order of 1%.

<u>Example</u> 5-2. To determine the capacitance C between the sections of two parallel bars far from ends and in ethanol ($\varepsilon \approx 26\varepsilon_0$) (Fig. 5-31), if $a = 2$ cm; $b = 4$ cm; $d = 2$ cm, and the length of section is $l = 5$ cm.



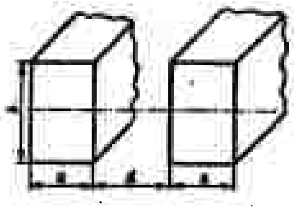


Fig. 5-30.   Fig. 5-31.

Fig. 5-30. Coaxial rectangular shells with parallel sides.

Fig. 5-31. Rectangular shells with parallel sides.

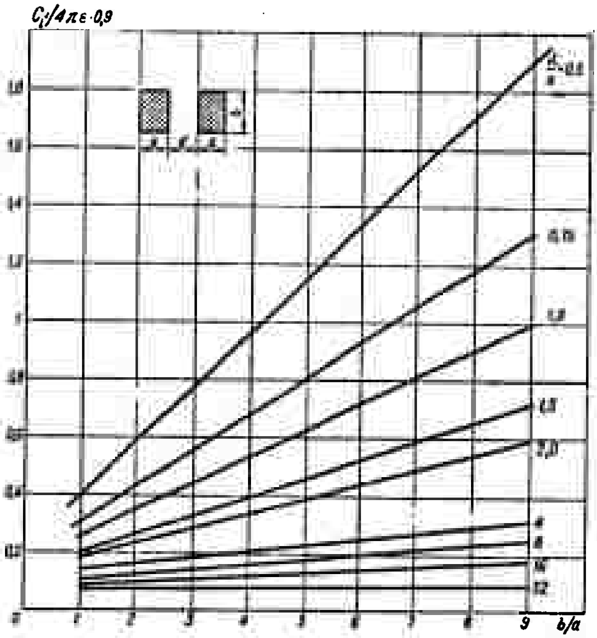


Fig. 5-32. A graph for determining capacitance per unit of length between two rectangular shells.

For assigned relations $b/a = 2$ and $d/a = 1$ with the aid of Fig. 5-32 we find the capacitance of the system per unit of length:

$$C_l \simeq 4\pi\varepsilon\cdot 0{,}9\cdot 0{,}35 = 3{,}96\varepsilon.$$

The capacitance between the sections considered is obtained by means of multiplication of the obtained value of C_l by the length of a section

$$C = C_l l \simeq 3{,}96\cdot 26\cdot \frac{1}{4\pi\cdot 9\cdot 10^9}\cdot 5\cdot 10^{-2} = 4{,}56\cdot 10^{-11}\ \mathrm{F} \simeq 46\ \mathrm{pF}.$$

c) Square shells with parallel sides enveloping each other (Fig. 5-33).

$$C_l = 8\varepsilon \frac{K_0}{K_0'}, \qquad (5\text{-}41)$$

where K_0, K_0' are complete elliptical integrals of the first kind with moduli k_0 and $k_0' = \sqrt{1 - k_0^2}$, respectively (see Appendix 1). The modulus $k_0 = (k_1 - k_1'/k_1 + k_1')^2$, and the parameters k_1 and $k_1' = \sqrt{1 - k_1^2}$ are determined from the equation

$$\frac{K(k_1)}{K(k_1')} = 2\frac{a}{d} - 1,$$

$K(k_1)$, $K(k_1')$ are complete elliptical integrals of the 1st kind with moduli k_1 and $k_1' = \sqrt{1 - k_1^2}$, respectively (see Appendix 1).

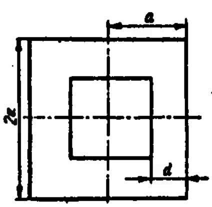


Fig. 5-33. Coaxial square shells with parallel sides.

The dependence $k_1 = f(a/d)$ is given in Fig. 5-34.

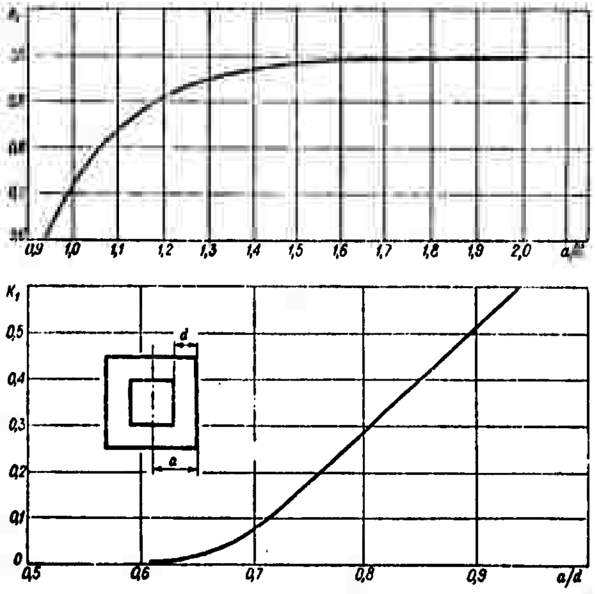


Fig. 5-34. Graph for determination of parameter $k_1 = f(a/d)$.

a/d	0.59461	0,60915	0,63108	0,67925	0,71305	0,80597	0,89085	0,93872
k_1	0,001	0,00[illegible]	0,01	0,05	0,1	0,3	0,5	0,6
a/d	1,06993	1,18911	1,29043	1,41399	1,76341	2,02059	2,4071	2,77361
k_1	0,8	0,9	0,94868	0,9747	0,99499	0,99749	0,99995	0,999995

Example 5-3. To determine capacitance per unit of length for a coaxial transmission system with square transverse section of central and external conductors (Fig. 5-33), if $2(a - d) = 1$ cm, $2a = 4$ cm, and the dielectric is air.

To determine the capacitance of the system we find the ratio

$$\frac{a}{d} = \frac{2}{1,5} = 1,33$$

and with the aid of the curve of Fig. 5-34 we establish that

$$k_1 = 0,96; \quad k_1' = \sqrt{1 - 0,96^2} = 0,28.$$

Using the obtained values of k_1 and k_1', we obtain

$$k_0 = \left(\frac{0,96 - 0,28}{0,96 + 0,28}\right)^2 = 0,3.$$

From modulus k_0 with the aid of Appendix 2 we find that

$$\frac{K_0}{K_0'} = 0,61,$$

and from formula (5-41) we obtain

$$C_l = 0,61 \cdot 8\varepsilon_0 = \frac{4,88 \cdot 10^{-9}}{36\pi} = 43,2 \text{ pF/m}.$$

d) Coaxial shells of the square section turned 45° relatively to each other (Fig. 5-35).

At $a = b$

$$C_l = 8\varepsilon. \qquad (5\text{-}42)$$

4. *Shells of regular n-angular section* (Fig. 5-36).

If the mutual location of the sections is such that their centers coincide, the middles of the sides of the external polygon are placed against the vertexes of the interior polygon, and furthermore, the distance between these points is equal to b, then

$$C_l = 2n\varepsilon, \qquad (5\text{-}43)$$

where n is the number of sides of each polygon.

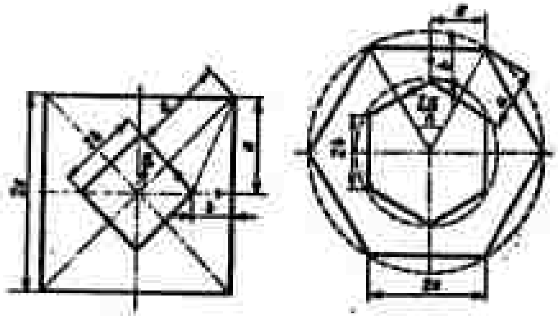


Fig. 5-35. Fig. 5-36.

Fig. 5-35. Shells of square section turned 45° relative to each other ($b = 0.414\ a$).

Fig. 5-36. Shells of regular n-angular section $\left(b = a\,\mathrm{tg}\frac{\pi}{n}\right)$.

5. *Infinitely long circular and arched shells.*

a) Two coaxial arched shells of identical radius (Fig. 5-37).

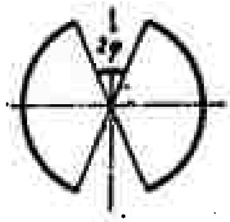
$$C_l \simeq \frac{\varepsilon}{\pi^2} \ln\left(\mathrm{ctg}\frac{\varphi}{2} + \sqrt{\mathrm{ctg}^2\frac{\varphi}{2} - 1}\right). \qquad (5\text{-}44)$$

At low ϕ

$$C_l \simeq \frac{\varepsilon}{\pi^2} \cdot \ln\frac{4}{\varphi}. \qquad (5\text{-}44a)$$

b) A circular shell and two interconnected identical arched shells coaxial with it (Fig. 5-38).

$$C_l \simeq \frac{2\pi\varepsilon}{\ln\frac{R}{r\cdot\sqrt{\cos\varphi}}}. \qquad (5\text{-}45)$$



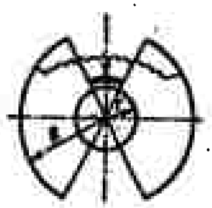


Fig. 5-37. Fig. 5-38.

Fig. 5-37. Two coaxial arched shells of identical radius and length.

Fig. 5-38. A circular shell and two intersected identical arched shells coaxial with it.

5-5. Capacitance Between Infinitely Long Shells and Plates

In the present section formulas, tables and graphs are given for determining the capacitance between conductors that are infinitely long shells and plates.

They include a plate inside a shell of circular section; a plate outside a shell of circular section; a plate inside and outside a shell of elliptical section; a plate inside a shell of rectangular section; a circular disc and cylindrical shell of circular section.

1. *A plate inside a shell of circular section.*

a) General case (Fig. 5-39).

$$C_l = \varepsilon \cdot \frac{2\pi}{\ln \frac{1}{q}}, \qquad (5\text{-}46)$$

where the parameter $q (0 < q < 1)$ is determined from the formulas (4-49) and (4-50), in which the quantity λ is replaced with

$$\lambda_2 = \sqrt{\frac{\frac{a}{d} - \frac{b}{d} - 1}{\left(\frac{b}{d}+1\right)\left(\frac{a}{d}-1\right)}},$$

and $a = 2R \sin \phi/2$.

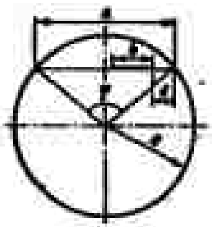


Fig. 5-39. A plate of finite width inside an infinite shell of circular section.

The numerical values of the function $C_l/\varepsilon = f(\lambda_2)$ are given in Fig. 5-40. Values of λ_2 depending on b/d at various a/d are given in Fig. 5-41.

b) The plate is inside a shell in a plane passing through its axis.

The formulas for the determination of capacitance per unit of length between the conductors being considered at different relationship of their sizes are given in Table 5-7.

c) The plate is between two interconnected concentric circular shells (Fig. 5-42).

At $R_a = r \cdot R/R_i$

$$C_l = \frac{4\varepsilon K(\varkappa)}{K'(\varkappa)}, \qquad (5\text{-}47)$$

where K and K' are the complete and supplementary elliptical integrals of the first kind (see Appendix 1) with modulus $\varkappa = k \cdot \mathrm{sn}\,[pK'(k), k']$ sn x is an elliptical sine (see Appendix 1), and

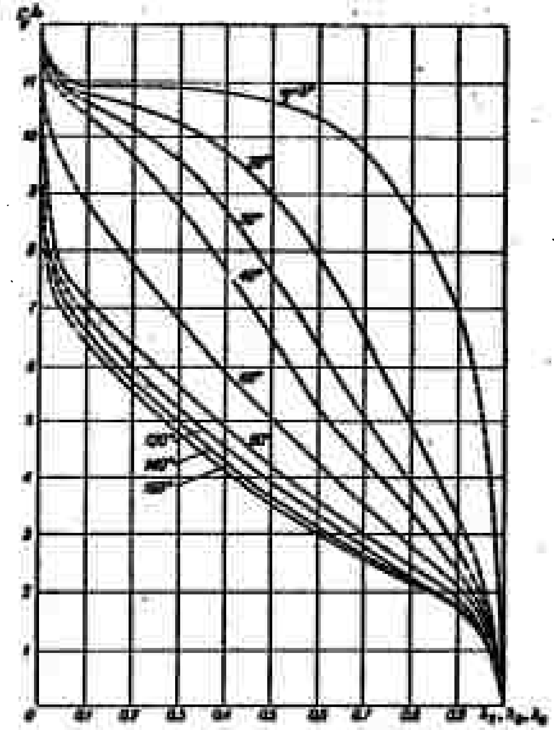


Fig. 5-40. Graph for determination of capacitance between a circular shell and a plate inside or outside it (dotted line – extrapolation).

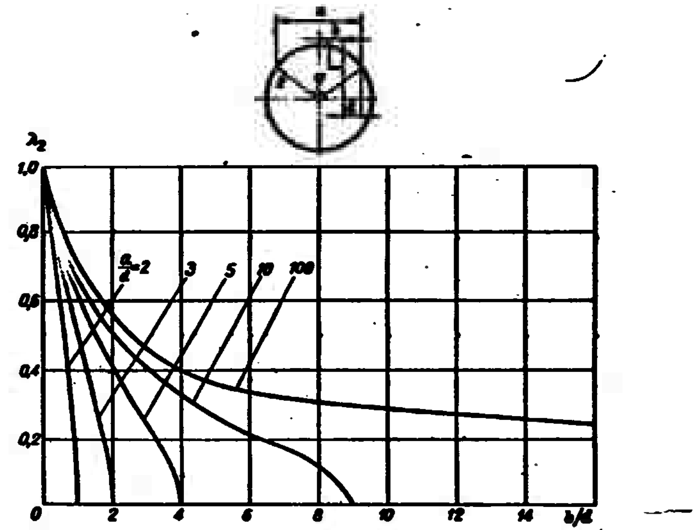


Fig. 5-41. Graph for the determination of parameter λ_2 necessary during calculation of capacitance between a shell and a plate inside it.

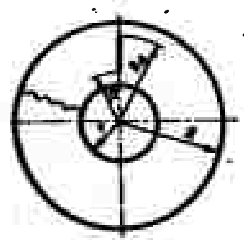


Fig. 5-42. Plate between two joint concentric circular shells.

Table 5-7. Formula for determination of capacitance between a circular shell and a plate passing through its axis.

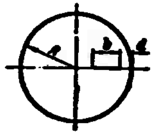
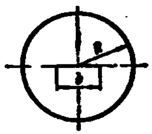
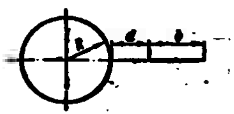
№ in order	Location of plate relative to shell	Calculation diagram	Calculation formulas: accurate	Calculation formulas: approximation	Relative error of the approximation formulas (%)
1	Plate inside a shell, arranged asymmetrically		$C_l=2\varepsilon\frac{K'}{K}$, where $k=\frac{d}{b+d}\cdot\frac{1-\frac{b+d}{2R}}{1-\frac{d}{2R}}$	$C_l\approx\frac{4\varepsilon}{\pi}\ln\frac{4\cdot\left(1-\frac{d}{2R}\right)}{\frac{d}{b+d}-\frac{d}{2R}}$ at $\frac{b}{R}\approx1$, $\frac{d}{R}<1$, $C_l\approx\frac{2\pi\varepsilon}{\ln\frac{16\left(1+\frac{b}{d}\right)^2\left(1-\frac{d}{2R}\right)^2}{\frac{b}{d}\left(2+\frac{b}{d}\right)-\frac{b}{R}\left(1+\frac{b}{d}\right)}}$ at $\frac{b}{R}<1$, $\frac{d}{R}\approx1$	-0.01 at $\frac{b}{R}=0.8$, $\frac{d}{R}=0.1$, 0.2 at $\frac{b}{R}=0.05$, $\frac{d}{R}=0.9$
2	Plate inside a shell, arranged symmetrically		$C_l=2\varepsilon\frac{K'}{K}$, where $k=\left(\frac{1-\frac{b}{2R}}{1+\frac{b}{2R}}\right)^2$	$C_l\approx\frac{8\varepsilon}{\pi}\ln\left[2\cdot\frac{1+\frac{b}{2R}}{1-\frac{b}{2R}}\right]$ at $\frac{b}{2R}\approx1$, $C_l\approx\frac{2\pi\varepsilon}{\ln\left(2\cdot\frac{2R}{b}\right)}$ at $\frac{b}{2R}<1$	-0.03 at $\frac{b}{2R}=0.9$, -0.02 at $\frac{b}{2R}=0.1$
3	Plate of finite width outside a shell		$C_l=2\varepsilon\frac{K'}{K}$, where	$C_l\approx\frac{2\pi\varepsilon}{\ln\frac{4\left(1+\frac{2R}{d}\right)^2}{\frac{R}{d}\left(1+\frac{R}{d}\right)-\frac{R}{b+d}\left(1+\frac{R}{b+d}\right)}}$ at $\frac{d}{2R}>1$, $\frac{b}{R}\approx1$	0.04 at $\frac{d}{2R}=10$, $\frac{b}{R}=1$,

Table 5-7 (Continued).

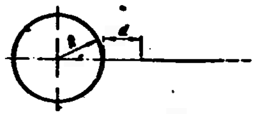
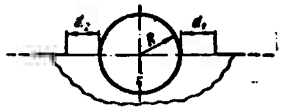
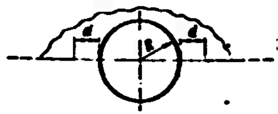
№ in order	Location of plate relative to shell	Calculation diagram	Calculation formulas		Relative error of the approximation formulas (%)
			accurate	approximation	
			$k = \frac{d}{b+d} \frac{1+\frac{b+d}{2R}}{1+\frac{d}{2R}}$	$C_l \simeq \frac{4\pi}{\pi} \ln \frac{4\left(1+\frac{2R}{d}\right)}{1+\frac{2R}{b+d}}$ at $\frac{d}{2R} < 1$, $\frac{b}{R} \simeq 1$	$-0{,}5$ at $\frac{d}{2R} = 0{,}1$, $\frac{b}{R} = 1$
4	Shell and half-plane		$C_l = 2\pi \frac{K'}{K}$, where $k = \frac{1}{1+\frac{2R}{d}}$	$C_l \simeq \frac{2\pi\varepsilon}{\ln \frac{4\left(1+\frac{2R}{d}\right)^2}{\frac{R}{d}\left(1+\frac{R}{d}\right)}}$ at $\frac{d}{2R} > 1$, $C_l \simeq \frac{4\pi}{\pi} \ln \left[4\left(1+\frac{2R}{d}\right)\right]$ at $\frac{d}{2R} < 1$	$2{,}1$ at $\frac{d}{2R} = 10$, $-0{,}04$ at $\frac{d}{2R} = 0{,}1$
5	Shell in the cut of an infinite plane, arranged asymmetrically with respect to cut		$C_l = 2\pi \frac{K'}{K}$, where $k = \frac{1}{\left(1+\frac{2R}{d_1}\right)\left(1+\frac{2R}{d_2}\right)}$	$C_l \simeq \frac{4\pi}{\pi} \ln \left[4\left(1+\frac{2R}{d_1}\right)\left(1+\frac{2R}{d_2}\right)\right]$ at $\frac{2R}{d_1} > 1$, $\frac{2R}{d_2} > 1$ $C_l \simeq \frac{2\pi\varepsilon}{\ln \frac{16}{1-\frac{1}{\left(1+\frac{2R}{d_1}\right)^2\left(1+\frac{2R}{d_2}\right)^2}}}$ at $\frac{2R}{d_1} < 1$, $\frac{2R}{d_2} < 1$	$-0{,}001$ at $\frac{2R}{d_1} = 10$, $\frac{2R}{d_2} = 20$, $8{,}1$ at $\frac{2R}{d_1} = 0{,}1$, $\frac{2R}{d_2} = 0{,}2$

Table 5-7 (Continued).

№ in order	Location of plate relative to shell	Calculation diagram	Calculation formulas		Relative error of the approximation formulas (%)
			accurate	approximation	
6	The same, as in clause 5 with symmetrical location of shell		$C_l = 2\pi \frac{K'}{K}$, where $k = \frac{1}{\left(1+\frac{2R}{d}\right)^2}$	$C_l \approx \frac{8\varepsilon}{\pi} \ln\left[2\left(1+\frac{2R}{d}\right)\right]$ at $\frac{2R}{d} > 1$, $C_l \approx \frac{2\pi\varepsilon}{\ln \frac{16}{1-\frac{1}{\left(1+\frac{2R}{d}\right)^4}}}$ at $\frac{2R}{d} < 1$	0,2 at $\frac{2R}{d} = 10$, 4,7 at $\frac{2R}{d} = 0,1$

$$p = \frac{\ln \frac{rR}{R_i^2}}{\ln \frac{R}{r}},$$

and k is determined from a transcendental equation

$$\frac{K'(k)}{K(k)} = \frac{1}{2\pi} \ln \frac{R}{r};$$

2. *Plate outside shell of circular section.*

a) The general case (Fig. 5-43).

$$C_l = \varepsilon \cdot \frac{2\pi}{\ln \frac{1}{q}}, \qquad (5\text{-}48)$$

where the parameter $q (0 < q < 1)$ is determined from the formulas (4-49) and (4-50), in which the quantity λ is replaced with

$$\lambda_3 = \sqrt{\frac{1 + \frac{a}{d} + \frac{b}{d}}{\left(1 + \frac{a}{d}\right)\left(1 + \frac{b}{d}\right)}},$$

and $a = 2R \sin \phi/2$.

The values of λ_3 depending on b/d at various a/d are given in Fig. 5-44.

At $b = \infty$ (shell and half-plane)

$$\lambda_3 = \sqrt{\frac{1}{1 + \frac{a}{d}}}.$$

Numerical values of C_l are found with the aid of the graph given in Fig. 5-40.

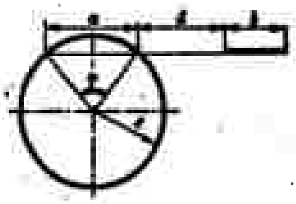


Fig. 5-43. Circular shell and plate of the finite width outside it.

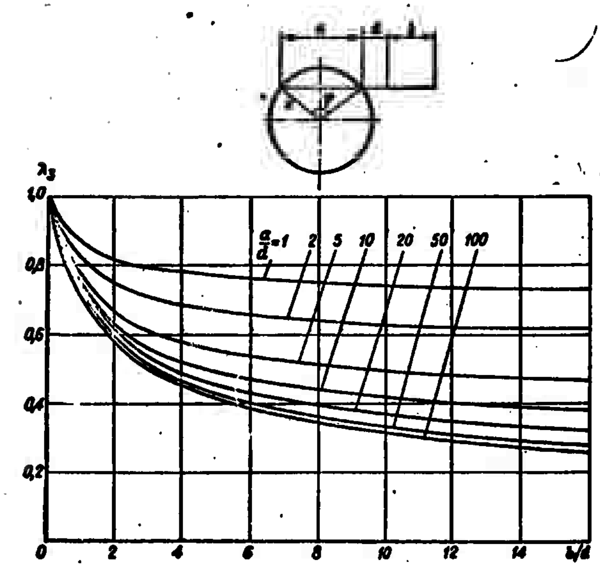


Fig. 5-44. Graph for the determination of the parameter λ_3 necessary in calculation of capacitance between a circular shell and a plate of finite width outside it.

b) Shell in the cut of an infinite plane (Fig. 5-45).

$$C_l = \varepsilon \cdot \frac{2\pi}{\ln\frac{1}{q}}, \qquad (5\text{-}49)$$

where the parameter $q\,(0 < q < 1)$ is determined from the formulas (4-49) and (4-50), in which λ is replaced by

$$\lambda_4 = \sqrt{\frac{1}{\left(1+\frac{a}{d_1}\right)\left(1+\frac{a}{d_2}\right)}}.$$

and $a = 2R \cdot \sin \phi/2$.

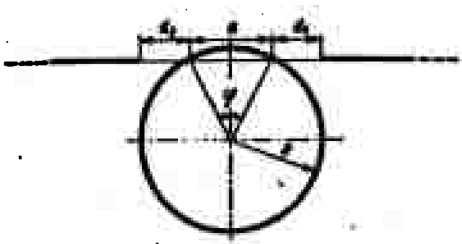


Fig. 5-45. Circular shell in a cut of an infinite plane.

The values of λ_4 depending on a/d_1 at various a/d_2 are given in Fig. 5-46, and the numerical values of capacitance per unit of length are found with the aid of the graph given in Fig. 5-40.

c) Plate in plane passing through axis of shell.

The formulas for the determination of capacitance per unit of length between the conductors being considered at different relationship of their sizes are given in Table 5-7 (see clause 1 of the present section).

3. *Plate and shell of elliptical section.*

a) Plate inside shell (Fig. 5-47).

If the edges of the plate coincide with the foci of an ellipse $(c = \sqrt{a^2 - b^2})$, then

$$C_l = \frac{2\pi\varepsilon}{\operatorname{Arch}\frac{a}{c}}. \qquad (5\text{-}50)$$

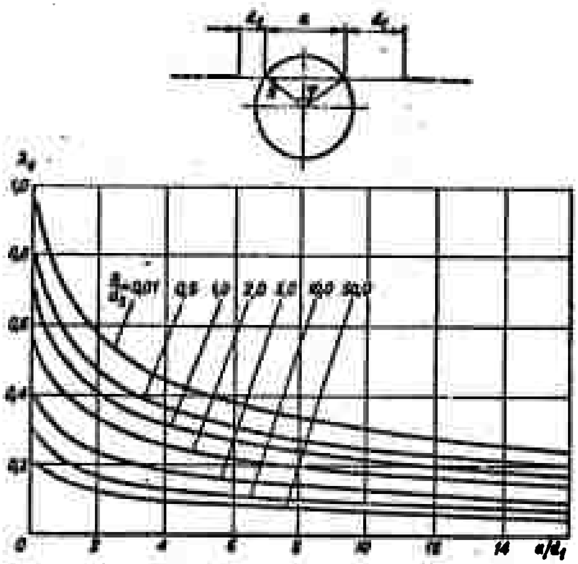


Fig. 5-46. Graph for determination of the parameter λ_4 necessary during calculation of capacitance between a circular shell and infinite plane, in the cut of which is a shell.

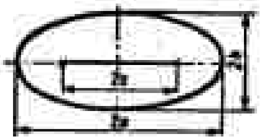


Fig. 5-47. Infinitely long elliptical shell and plate inside it.

b) Plate outside shell.

Formulas for determination of capacitance per unit of length between conductors considered are given in Table 5-8.

Table 5-8. Formulas for determination of capacitance between an elliptical shell and a plate outside a shell in the plane of symmetry.

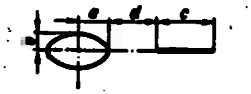
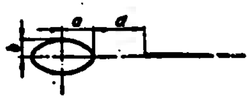
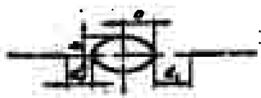
No. in order	Type of plate	Section of system	Calculation formulas
1	Plate of finite width		$C_l = 2\pi \frac{K'}{K}$, where $k^2 = \frac{(1-m_1)(1+m_2)}{(1+m_1)(1-m_2)}$, $m_1 = \frac{a^2-b^2}{a(a+d) - b\sqrt{(a+d)^2-(a^2-b^2)}}$, $m_2 = \frac{a^2-b^2}{a(a+d+c) - b\sqrt{(a+d+c)^2-(a^2-b^2)}}$
2	Half-plane		$C_l = 2\pi \cdot \frac{K'}{K}$, where $k^2 = \frac{1-m}{1+m}$, $m = \frac{a^2-b^2}{a(a+d) - b\cdot\sqrt{(a+d)^2-(a^2-b^2)}}$
3	Plane with asymetrical cut		$C_l = 2\pi \frac{K}{K'}$, where $k^2 = \frac{2(m_1+m_2)}{(1+m_1)(1+m_2)}$, $m_{1,2} = \frac{a^2-b^2}{a(a+d_{1,2}) - b\sqrt{(a+d_{1,2})^2-(a^2-b^2)}}$

Table 5-8 (Continued).

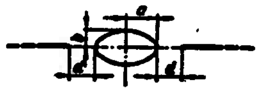
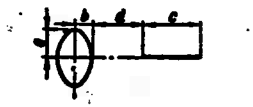
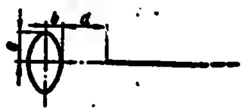
No. in order	Type of plate	Section of system	Calculation formulas
4	Plane with symmetrical cut		where $C_l = 4\varepsilon \cdot \frac{K}{K'}$, $k = \frac{a^2-b^2}{a(a+d)-b\sqrt{(a+d)^2-(a^2-b^2)}}$
5	Plate of finite width		where $C_l = 2\varepsilon \frac{K'}{K}$, $k = \frac{m_2+\sqrt{1+m_2^2}}{m_1+\sqrt{1+m_1^2}}$, $m_1 = \frac{a^2-b^2}{a(b+d)-b\sqrt{(b+d)^2+(a^2-b^2)}}$, $m_2 = \frac{a^2-b^2}{a(b+c+d)-b\sqrt{(b+d+c)^2+(a^2-b^2)}}$
6	Half-plane		where $C_l = 2\varepsilon \cdot \frac{K}{K'}$, $k^2 = \frac{2}{1+\sqrt{1+m^2}}$, $m = \frac{a(b+d)-b\sqrt{(b+d)^2+(a^2-b^2)}}{a^2-b^2}$

Table 5-8 (Continued).

No. in order	Type of plate	Section of system	Calculation formulas
7	Plane with asymmetrical cut		where $C_l = 2\varepsilon \frac{K}{K'}$, $k^2 = \frac{2\left(m_1 \cdot \sqrt{1+m_2^2} + m_2\sqrt{1+m_1^2}\right)}{\left(m_1+\sqrt{1+m_1^2}\right)\left(m_2+\sqrt{1+m_2^2}\right)}$, $m_{1,2} = \frac{a^2-b^2}{a(b+d_{1,2}) - b\sqrt{(b+d_{1,2})^2+(a^2-b^2)}}$.
8	Plane symmetrical cut		where $C_l = 4\varepsilon \frac{K}{K'}$, $k^2 = \frac{1}{1+m^2}$, $m = \frac{a(b+d) - b\sqrt{(b+d)^2+(a^2-b^2)}}{a^2-b^2}$

4. *Plate inside a shell of rectangular section* (Fig. 5-48).

$$C_l = 2\pi \cdot \frac{K_0}{K_0'}, \qquad (5\text{-}51)$$

where the modulus of the complete elliptical integral of the first kind K_0 (see Appendix 1) is

$$k_0^2 = \frac{2[\operatorname{sn}(u_1, k) - \operatorname{sn}(u_2, k)]}{[1 + \operatorname{sn}(u_1, k)] \cdot [1 - \operatorname{sn}(u_2, k)]};$$

$$u_1 = \left(1 - 2\frac{d}{l}\right)K, \quad u_2 = \left(1 - 2\frac{b+d}{l}\right)K.$$

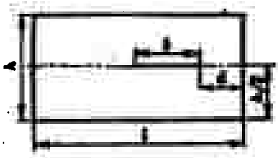


Fig. 5-48. Infinitely long elliptical shell and plate inside it.

The modulus *k* of an elliptical integral K and functions sn (*u*, *k*) (see Appendix 1) is found from the equation

$$\frac{h}{l} = \frac{K'}{K} = \frac{1}{\pi} \cdot \ln\frac{1}{q}$$

or directly from the formula

$$k = \sqrt{q} \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}}\right)^4, \quad q = e^{-\pi \cdot \frac{h}{l}}.$$

(The quantity *k* can be found from the assigned ratio *h/l* with the aid of Appendix 2).

At the symmetric location of a plate (Fig. 5-49)

$$C_l = 4\varepsilon \cdot \frac{K_0}{K'_0}, \qquad (5\text{-}52)$$

where $k_0 = \mathrm{sn}\left(\frac{b}{l} K;\ k\right)$, and modulus k is defined just as above.

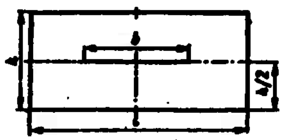


Fig. 5-49. Shell and plate symmetrically arranged inside it.

When the section of the shell has square form, the formulas for calculation of capacitance between the conductors being considered take the form shown in Table 5-9.

Example 5-4. To determine the capacitance per unit of length of the system shown in Fig. 5-48, if $h/l = 0.78$; $d/l = 0.19$; $d + b/l = 0.40$, and the dielectric is air.

Solution. From the assigned ratio $h/l = K'/K = 0.78$ with the aid of Appendix 2 we find that $k^2 = 0.75$; $K = 2.16$.

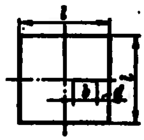
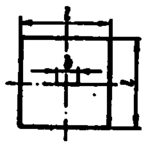
The values of elliptical sines which enter formula (5-51) can be calculated directly from their tables, short extracts from which are given in Appendix 5, or from tables of elliptical integrals. Let us make use in this case of the latter method.

Calculating the arguments of elliptical sines, we have:

$$u_1 = (1 - 2\cdot 0{,}19)\cdot 2{,}16 = 1{,}34;$$
$$u_2 = (1 - 2\cdot 0{,}40)\cdot 2{,}16 = 0{,}43.$$

Turning then to the table contained in [Appendix 5], we find that at $u_1 = 1.34$ and $k^2 = 0.75$ amplitude $\phi_1 \simeq 65°$. Analogously $\phi_2 \simeq 24°$.

Table 5-9. Formula for determination of capacitance between a shell of square section and a plate inside a shell in the plane of its symmetry.

No. in order	Location of plate	Calculation diagrams	Calculation formulas
1	Plate is in the plane of symmetry passing through the middles of the opposite sides of a shell		$C_l = 2\varepsilon \cdot \frac{K}{K'}$, where $k^2 = \frac{2\left[\mathrm{sn}\left(u_1; \sqrt{0,5}\right) - \mathrm{sn}\left(u_2; \sqrt{0,5}\right)\right]}{\left[1 + \mathrm{sn}\left(u_1; \sqrt{0,5}\right)\right] \times \left[1 - \mathrm{sn}\left(u_2; \sqrt{0,5}\right)\right]}$, $u_1 = \left(1 - 2\frac{d}{l}\right) \cdot K\left(\sqrt{0,5}\right)$, $K\left(\sqrt{0,5}\right) = 1,85407$, $u_2 = \left(1 - 2\frac{b+d}{l}\right) \cdot K\left(\sqrt{0,5}\right)$. $\mathrm{sn}(u, k_1)$ is an elliptical sine (see Appendix 1)
2	The same, as in clause 1, with symmetric location of plate		$C_l = 4\varepsilon \cdot \frac{K}{K'}$, where $k = \mathrm{sn}\left[K\left(\sqrt{0,5}\right) \cdot \frac{b}{l}; \sqrt{0,5}\right]$
3	Plate in diagonal plane		$C_l = 2\varepsilon \cdot \frac{K'}{K}$, where $k = \frac{\mathrm{cn}\left(u_1; \sqrt{0,5}\right) \cdot \left[2 - \mathrm{cn}\left(u_2; \sqrt{0,5}\right)\right]}{\mathrm{cn}\left(u_2; \sqrt{0,5}\right) \cdot \left[2 - \mathrm{cn}\left(u_1; \sqrt{0,5}\right)\right]}$, $u_1 = \sqrt{2} \cdot K\left(\sqrt{0,5}\right) \cdot \frac{d}{l}$, $\sqrt{2} \cdot K\left(\sqrt{0,5}\right) = 2,62204$, $u_2 = \sqrt{2} \cdot K\left(\sqrt{0,5}\right) \cdot \frac{b+d}{l}$. $\mathrm{cn}(u, k_1)$ is an elliptical cosine (see Appendix 1)

On the basis of formula sn u = sin ϕ we have:

$$\mathrm{sn}\, u_1 = \sin 65° = 0{,}906;$$
$$\mathrm{sn}\, u_2 = \sin 24° = 0{,}407.$$

We find then the modulus k_0 of elliptical integrals

$$k_0^2 = \frac{2\,(0{,}906 + 0{,}407)}{(1 + 0{,}906)\,(1 + 0{,}407)} = 0{,}979.$$

Then K_0 = 3.336, K_0' = 1.579 and using formula (5-51) we obtain

$$C_l = 2 \cdot 8{,}85 \cdot 10^{-12} \cdot \frac{3{,}336}{1{,}579} = 37{,}4 \text{ pF/m}.$$

5. *Circular disc and cylindrical shell of circular section.*

a) The shell is infinitely long and coaxial with the disc (Fig. 5-50).

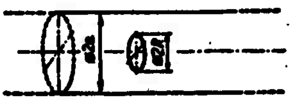
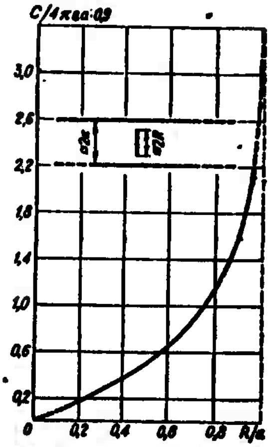


Fig. 5-50. Circular disc inside infinitely long shell of circular section.

The numerical values of the function $C/4\pi\varepsilon a \cdot 0.9 = f(R/a)$ are given in Fig. 5-51.

b) The shell is closed and is coaxial with the disc (Fig. 5-52).

The numerical values of the function $C/4\pi\varepsilon a \cdot 0.9 = f(R/a)$ at various l/a are given in Table 5-10 and in Fig. 5-53.



$\frac{R}{a}$	0,1	0,2	0,3	0,4
$\frac{C}{4\pi\varepsilon a\cdot 0,9}$	0.07500	0.15916	0.25564	0.30889

$\frac{R}{a}$	0,5	0,6	0,7
$\frac{C}{4\pi\varepsilon a\cdot 0,9}$	0.42843	0,65817	0.86509

$\frac{R}{a}$	0,8	0,9	0,95
$\frac{C}{4\pi\varepsilon a\cdot 0,9}$	1,16531	1,6486	2,130

Fig. 5-51. Graph for determination of capacitance between an infinitely long shell of circular section and a circular disc inside it (dotted line – extrapolation).

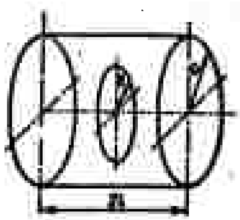


Fig. 5-52. Circular disc inside a closed shell of circular section.

Table 5-10. Values of capacitance between a closed shell of circular section and a circular disc inside it.

$\frac{R}{a}$	$\frac{C}{4\pi\varepsilon a\cdot 0,9}$	Maximum absolute error	$\frac{C}{4\pi\varepsilon a\cdot 0,9}$	Maximum absolute error	$\frac{C}{4\pi\varepsilon a\cdot 0,9}$	Maximum absolute error
	$l/a = 0.25$		$l/a = 0.5$		$l/a = 1.0$	
0,25	0,2848	0,00002	0,2251	0,0003	0,2072	0,0020
0,50	0,8447	0,0006	0,5790	0,0062	0,5042	0,0489
0,75	1,7615	0,0037	1,1563	0,0367	1,0060	0,1948

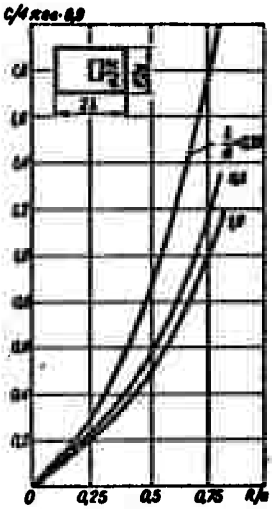


Fig. 5-53. Graph for determination of capacitance between a closed shell of circular section and a circular disc inside it (dotted line – extrapolation).

5-6. Capacitor Capacitance of Closed Shells

In the present section formulas, tables and graphs are given for the determination of capacitance between conductors, at least one of which is a closed shell.

The surfaces of conductors considered below are spheres (one of which is inside or outside the other); confocal ellipsoids; coaxial tori of circular section.

A separate group is made up of the systems formed by a sphere or by a spheroid inside shells or near infinite planes.

1. *Two spheres.*

a) Two concentric spheres (spherical capacitor), (Fig. 5-54)

$$C = 4\pi\varepsilon \cdot \frac{rR}{R-r}. \tag{5-53}$$

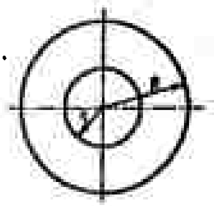


Fig. 5-54. Two concentric spheres.

b) Two conconcentric spheres one of which is inside the other (Fig. 5-55).

$$C = 4\pi\varepsilon \cdot 2r \cdot \operatorname{sh} \alpha_1 \sum_{n=0}^{\infty} \frac{e^{-(2n+1)\alpha_1}}{1-e^{-(2n+1)\alpha_2}}, \tag{5-54}$$

where

$$\alpha_1 = \operatorname{Arch} \frac{R^2 - r^2 - d^2}{2rd},$$

$$\alpha_2 = \operatorname{Arch} \frac{R^2 + r^2 - d^2}{2rR}.$$

At $\frac{d}{r} \ll 1, \frac{r}{R} \ll 1$

$$C \simeq 4\pi\varepsilon \frac{rR}{R-r}\left[1-\frac{R}{r}\cdot\frac{d^2}{(R-r)(R^2-r^2)}\right] \quad (5\text{-}55)$$

$[\delta < 0{,}2\%$ when $d/r \leqslant 0{,}04,\ r/R < 0{,}2]$.

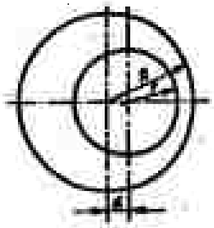


Fig. 5-55. Two non-concentric spheres one of which is inside the other.

c) Two spheres, one of which is inside the other (Fig. 5-56).

$$C = 4\pi\varepsilon \cdot rR \cdot \operatorname{sh}\alpha \cdot f(r, R, d), \quad (5\text{-}56)$$

where

$$f(r, R, d) = \frac{N\sum_{n=1}^{\infty}\frac{1}{r\operatorname{sh} n\alpha + R\operatorname{sh}(n-1)\alpha} - \left(\frac{1}{2d}\sum_{n=1}^{\infty}\frac{1}{\operatorname{sh} n\alpha}\right)^2}{\sum_{n=1}^{\infty}\frac{1}{R\operatorname{sh} n\alpha + r\operatorname{sh}(n-1)\alpha} + M},$$

$$N = \sum_{n=1}^{\infty}\frac{1}{R\operatorname{sh} n\alpha + r\operatorname{sh}(n-1)\alpha}$$

$$M = \frac{1}{r\operatorname{sh} n\alpha + R\operatorname{sh}(n-1)\alpha} - \frac{1}{d\cdot\operatorname{sh} n\alpha};$$

$$\alpha = \operatorname{Arch}\frac{(2d)^2 - (r^2+R^2)}{2rR}.$$

Specifically, at $r = R$

$$C = 2\pi\varepsilon R\operatorname{sh}\beta\sum_{n=1}^{\infty}\frac{1}{\operatorname{sh} n\beta}, \quad (5\text{-}56a)$$

where $\beta = \operatorname{Arch}\frac{d}{R}$.

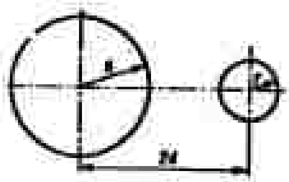


Fig. 5-56. Two spheres one of which is outside the other.

The numerical values of the function $\frac{C}{4\pi\varepsilon R}=f\left(\frac{r}{R}\right)$ at various values of $R/2d$ are given in Fig. 5-57.

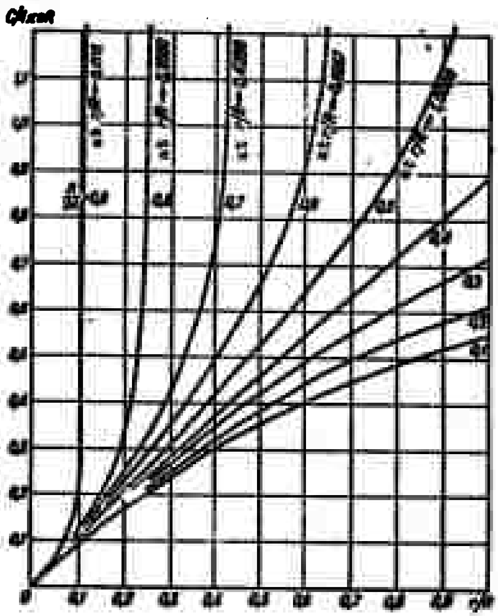


Fig. 5-57. Graph for determination of capacitance between two spheres, one of which is located outside the other (dotted line - extrapolation).

At $R/2d \ll 1$

$$C \simeq 4\pi\varepsilon\cdot\frac{rR}{r+R}\cdot\frac{1}{1-\frac{1}{d}\cdot\frac{rR}{r+R}} \qquad (5\text{-}57)$$

$[|\delta| = 0{,}73\%$ when $R/2d = r/R = 0{,}2]$.

When $R/2d \ll 1$ and $r = R$

$$C \simeq \frac{2\pi\varepsilon R}{1-\frac{R}{2d}} \tag{5-57a}$$

$[\delta < 0.24\%$ when $R/2d < 0.2]$.

2. *Two confocal ellipsoids.*

a) Triaxial ellipsoids (Fig. 5-58):

$$C = \frac{4\pi\varepsilon d}{F_1(\varphi_1, k) - F_2(\varphi_2, k)}, \tag{5-58}$$

where $F(\phi, k)$ are elliptical integrals of the first kind (see Appendix 1) with modulus

$$k^2 = \frac{a_1^2 - b_1^2}{a_1^2 - c_1^2} = \frac{a_2^2 - b_2^2}{a_2^2 - c_2^2}$$

and arguments

$$\varphi_1 = \arcsin\sqrt{1-\left(\frac{c_1}{a_1}\right)^2}; \quad \varphi_2 = \arcsin\sqrt{1-\left(\frac{c_2}{a_2}\right)^2},$$

and

$$d = \sqrt{a_1^2 - c_1^2} = \sqrt{a_2^2 - c_2^2}, \quad [a > b > c].$$

Example 5-5. To find the capacitance of the air capacitor formed by confocal triaxial ellipsoids the semiaxes of which are $a_1 = 5$ cm; $b_1 = 3$ cm; $c_1 = 2$ cm; $a_2 = 7$ cm; $b_2 = \sqrt{33}$ cm; $c_2 = \sqrt{28}$ cm.

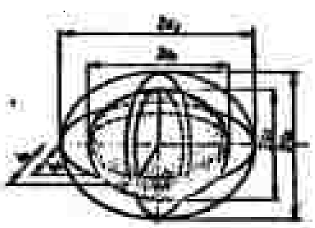


Fig. 5-58. Triaxial confocal ellipsoids.

To determine capacitance it is necessary to find in advance the values of the elliptical integrals $F(\phi, k)$. We first compute their modulus k and arguments ϕ_1 and ϕ_2:

$$k^2 = \frac{1-\left(\frac{b_1}{a_1}\right)^2}{1-\left(\frac{c_1}{a_1}\right)^2} = \frac{1-\left(\frac{3}{5}\right)^2}{1-\left(\frac{2}{5}\right)^2} = 0{,}7619,$$

$$\varphi_1 = \arcsin\sqrt{1-\left(\frac{2}{5}\right)^2} = 66°25',$$

$$\varphi_2 = \arcsin\sqrt{1-\left(\frac{\sqrt{28}}{7}\right)^2} = 40°53'.$$

Using then table [Appendix 4], we obtain

$$F_1(\varphi_1, k) = 1{,}3954; \quad F_2(\varphi_2, k) = 0{,}7632.$$

Substituting in formula (5-58) the numerical values of the parameters entering it, we find the sought capacitance

$$C = 4\pi\varepsilon \frac{d}{F_1(\varphi_1, k) - F_2(\varphi_2, k)} = 4\pi \cdot \frac{1}{4\pi \cdot 9 \cdot 10^9} \times$$
$$\times \frac{\sqrt{5^2 - 2^2} \cdot 10^{-2}}{1{,}3954 - 0{,}7632} = 0{,}8054 \cdot 10^{-11}\ \text{ф} \approx 8\ \text{pF}.$$

b) Drawn out spheroids (Fig. 5-59):

$$C = \frac{8\pi\varepsilon \cdot d}{\ln\left|\frac{a_1+d}{a_1-d} \cdot \frac{a_2-d}{a_2+d}\right|}, \tag{5-59}$$

where $d = \sqrt{a_1^2 - c_1^2} = \sqrt{a_2^2 - c_2^2}$ $[a = b < c]$.

c) Condensed spheroids (Fig. 5-60):

$$C = \frac{4\pi\varepsilon \cdot d}{\left|\arccos\frac{c_1}{a_1} - \arccos\frac{c_2}{a_2}\right|}, \tag{5-60}$$

where

$$d = \sqrt{a_1^2 - c_1^2} = \sqrt{a_2^2 - c_2^2} \quad [a = b > c].$$

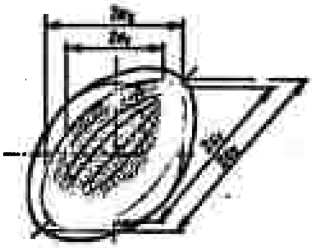


Fig. 5-59. Drawn out confocal spheroids.

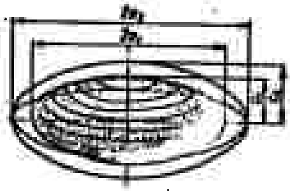


Fig. 5-60. Condensed confocal spheroids.

3. *Coaxial tori of circular section* (Fig. 5-61).

$$C = \frac{4\pi^2\varepsilon d}{\ln\frac{R}{r}} \cdot \left[1-\left(\frac{R}{d}\right)^2 \cdot \left(1-\frac{r^2}{R^2}+2\frac{r}{R}\cdot\ln\frac{r}{R}\right)\right]. \quad (5\text{-}61)$$

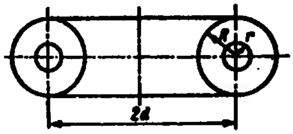


Fig. 5-61. Coaxial tori.

Having found capacitance per unit length (by dividing by $2\pi d$) and having approached infinity, for two concentric circular cylinders we obtain $C = \frac{2\pi\varepsilon}{\ln\frac{R}{r}}$, which coincides with formula 2 of Table 5-6.

4. *Sphere inside a cube* (Fig. 5-62).

$$C \simeq \frac{4\pi\varepsilon R}{1-\left[1{,}7476+\frac{16{,}468}{\left(\frac{a}{R}\right)^6-234{,}63}\right]\frac{R}{a}}. \quad (5\text{-}62)$$

At $a/R > 2.5$ the capacitance of the system considered can be calculated as the capacitance of a spherical capacitor (see 5-53), the radius of the external plate of which is equal to 0.5722 a.

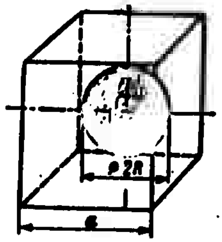


Fig. 5-62. A sphere inside a cube.

5. *Sphere inside an infinitely long cylinder* (Fig. 5-63).

$$C = 4\pi\varepsilon R\cdot A_0, \tag{5-63}$$

where A_0 is determined from the infinite system of equations:

$$\frac{(2p)!}{(4p+1)R^{2p}}\cdot A_{2p} - \frac{2R}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+p}R^{2n}J(2n+2p,\,a)}{(4n+1)(2n+2p+1)(2n)!}\cdot A_{2n} = \frac{\delta_{2p}}{R},$$

where

$$J(2n+2p,\,a) = \int_0^{\infty}\frac{t^{2n+2p}}{I_0^2(ta)}\,dt,$$

$I_0(ta)$ - the Bessel function of an imaginary argument (see Appendix 1).

$$\delta_{2p} = \begin{cases} 0, & \text{if } 2p \neq 0; \\ 1, & \text{if } 2p = 0. \end{cases}$$

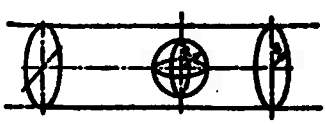


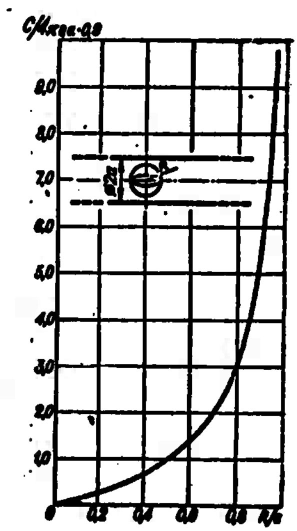
Fig. 5-63. Sphere inside infinitely long cylinder.

The numerical values of the function $\frac{C}{4\pi\varepsilon\cdot 0{,}9} = f(R/a)$ is given in Fig. 5-64. The following approximation formula can also be used:

$$C \simeq 4\pi\varepsilon R\left[1 + 0{,}8707\frac{R}{a} + 0{,}7581\left(\frac{R}{a}\right)^2 + \right.$$

$$\left. + 0{,}6601\left(\frac{R}{a}\right)^3 + 0{,}5747\left(\frac{R}{a}\right)^4 + 0{,}5004\left(\frac{R}{a}\right)^5\right] \quad (5\text{-}64)$$

$[\delta < 1\%$ when $R/a < 0{,}5]$.



$\frac{R}{a}$	0,1	0,2	0,3	0,4
$\frac{C}{4\pi\varepsilon a\cdot 0{,}9}$	0,12188	0,28045	0,45183	0,66308

$\frac{R}{a}$	0,5	0,6	0,7
$\frac{C}{4\pi\varepsilon a\cdot 0{,}9}$	0,90642	1,40887	2,08236

$\frac{R}{a}$	0,8	0,9	0,95
$\frac{C}{4\pi\varepsilon a\cdot 0{,}9}$	3,08518	5,41943	8,70674

Fig. 5-64. Graph for determination of the capacitance of a sphere inside an infinitely long cylinder.

6. *Spheroid inside infinitely long cylinder* (Fig. 5-65).

The numerical values of $C/4\pi\varepsilon a\cdot 0.9 = f(b/a)$ for two values of b/a are given in Fig. 5-66.

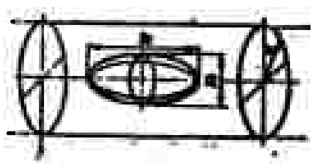


Fig. 5-65. A spheroid inside an infinitely long cylinder.

b/a	0.1	0.2	0.3	0.4
$\frac{C}{4\pi\varepsilon a\cdot 0.9}$ drawn out (1/2)	0.16316	0.37760	0.65364	1.02710
$\frac{C}{4\pi\varepsilon a\cdot 0.9}$ condens. (2/1)	0.09916	0.21517	0.36318	0.52120
b/a	0.5	0.6	0.7	
$\frac{C}{4\pi\varepsilon a\cdot 0.9}$ drawn out (1/2)	1.53536	2.26163	3.3710	
$\frac{C}{4\pi\varepsilon a\cdot 0.9}$ condens. (2/1)	0.73236	1.01033	1.40331	
b/a	0.8	0.9	0.95	
$\frac{C}{4\pi\varepsilon a\cdot 0.9}$ drawn out (1/2)	—	—	—	
$\frac{C}{4\pi\varepsilon a\cdot 0.9}$ condens. (2/1)	2.03230	3.36782	5.1466	

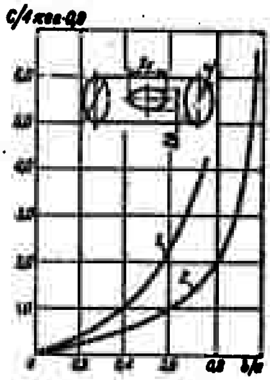


Fig. 5-66. A graph for determination of the capacitance of a spheroid inside an infinitely long cylinder: 1 - drawn out (the ratio of axes 1/2); 2 - condensed (ratio of axes 2/1); dotted line - extrapolation.

7. *Sphere inside flat ring* (Fig. 5-67).

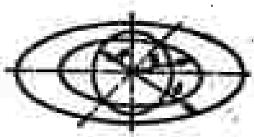


Fig. 5-67. Sphere inside flat ring ("ring of Saturn").

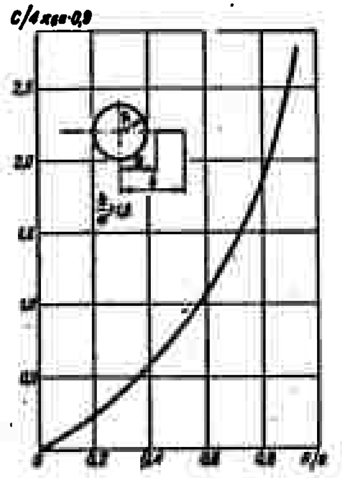
At $1.5 < b/a \leq \infty$

$$C \simeq 4\pi\varepsilon\cdot k(k')^2 \int_0^K \frac{\operatorname{sn}\xi\, d\xi}{\operatorname{dn}^2\left(\arccos\frac{1}{\sqrt{1+k}}\right)\xi - k\operatorname{cn}^2\left(\operatorname{arctg}\sqrt{k}\right)\xi}, \qquad (5\text{-}65)$$

where sn u, cn u, dn u are elliptical functions (see Appendix 1);

$$k = \left(\frac{R}{a}\right)^2.$$

The numerical values of function $C/4\pi\varepsilon a = f(R/a)$ are given in Fig. 5-68.



$\frac{R}{a}$	0,42	0,64	0,78	0,90
$\frac{C}{4\pi\varepsilon a \cdot 0,9}$	0,62	1,22	1,76	2,54

Fig. 5-68. Graph for determination of the capacitance of a sphere inside a flat ring (at $b/a > 1.5$) (the dotted line - extrapolation).

8. *Sphere between infinite planes* (Fig. 5-69).

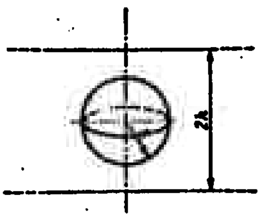


Fig. 5-69. Sphere between infinite planes.

At R/h not too close to 1,

$$C \approx 4\pi\varepsilon R\left[1 + \sum_{n=1}^{4} \rho^n\right], \qquad (5\text{-}66)$$

where $\rho = \frac{R}{h} \ln 2$.

At values of R/h comparable with 1,

$$C \approx 4\pi\varepsilon R \frac{R}{h-R} \frac{3}{4}. \qquad (5\text{-}67)$$

The approximation numerical values of the function $C/4\pi\varepsilon R = f(R/h)$ are given in Fig. 5-70.

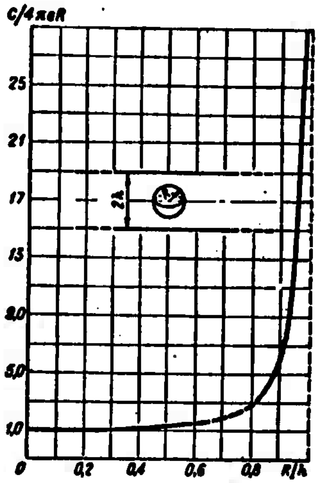


Fig. 5-70. Graph for determination of the capacitance of a sphere between infinite planes (dotted line - extrapolation).

APPENDIX 1

SPECIAL FUNCTIONS USED TO CALCULATE ELECTRICAL CAPACITANCE

1. *Elliptical integrals.*

The integrals

$$F(\varphi, k)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}},\quad E(\varphi, k)=\int_0^{\varphi}\sqrt{1-k^2\sin^2\psi}\,d\psi,$$
$$\Pi(\varphi, n, k)=\int_0^{\varphi}\frac{d\psi}{(1+n\sin^2\psi)\sqrt{1-k^2\sin^2\psi}} \tag{1}$$

are called incomplete elliptical integrals of the first, second, and third kind, respectively. The quantity ϕ is called an argument or amplitude, n a parameter, and k a modulus.

The number $k'=\sqrt{1-k^2}$ is called a supplementary modulus, and integrals (1) with modulus k' are called supplementary integrals. Frequently the quantity α = arc sin k is introduced, which is called a modular angle.

At $\phi = \frac{\pi}{2}$ integrals (1) are called complete elliptical integrals and are labeled

$$\mathbf{K}=\mathbf{K}(k)=\int_0^{\pi/2}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}};\quad \mathbf{E}=\mathbf{E}(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\psi}\,d\psi;$$
$$\Pi(n, k)=\int_0^{\pi/2}\frac{d\psi}{(1+n\sin^2\psi)\sqrt{1-k^2\sin^2\psi}}. \tag{2}$$

Complete supplementary elliptical integrals are frequently marked with a prime

$$K' = K(k'); \quad E' = E(k'); \quad \Pi'(n, k) = \Pi(n, k').$$

For the most frequently utilized complete elliptical integrals of the first kind the following expansions are valid:

$$K = \frac{\pi}{2} \cdot \left[1 + \left(\frac{1}{2}\right)^2 \cdot k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \cdot k^4 + \ldots\right] \quad (3)$$

(is used when $k \ll 1$);

$$K = \ln\frac{4}{k'} + \left(\frac{1}{2}\right)^2 \cdot \left(\ln\frac{4}{k'} - \frac{2}{1 \cdot 2}\right) \cdot k'^2 + \ldots \quad (4)$$

(is used when $k \approx 1$).

More detailed information about elliptical integrals is given in [Appendices Literature 1-3].

The tables of values K, K' and also K'/K and K/K' are given in Appendix 2. More complete tables of elliptical integrals of the first, second, and third kind are contained in [Appendices Literature 4], and also in [Appendices Literature 3, 5].

2. *Elliptical functions of Jacoby.*

The function opposite to the elliptical integral of the first kind is called an elliptical sine and is designated

$$\operatorname{sn} u \equiv \operatorname{sn}(u, k) = \sin\varphi = \sin \operatorname{am} u.$$

The overhead limit ϕ of an integral is called amplitude, and the quantity u is called argument. The dependence of an argument upon amplitude is written:

$$u = \arg\varphi.$$

The functions

$$\operatorname{cn} u = \cos\varphi = \cos \operatorname{am} u,$$

$$\operatorname{dn} u = \sqrt{1 - k^2 \sin^2\varphi} = \frac{d\varphi}{du}$$

By definition

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1, \quad \mathrm{dn}^2 u + k^2 \mathrm{sn}^2 u = 1; \quad \mathrm{dn}^2 u - k^2 \mathrm{cn}^2 u = k'^2.$$

For elliptical functions the following ideas in the form of exponential series are valid:

$$\left.\begin{aligned} \mathrm{sn}\, u &= u - \frac{1+k^2}{3!}u^3 + \frac{1+14k^2+k^4}{5!}u^5 - \\ &\quad - \frac{1+135(k^2+k^4)+k^6}{7!}u^7 + \ldots \\ \mathrm{cn}\, u &= 1 - \frac{1}{2!}u^2 + \frac{1+4k^2}{4!}u^4 - \frac{1+44k^2+16k^4}{6!}u^6 + \ldots \\ \mathrm{dn}\, u &= 1 - \frac{k^2}{2!}u^2 + \frac{k^2(4+k^2)}{4!}u^4 - \frac{k^2(16+44k^2+k^4)}{6!}u^6 + \ldots \end{aligned}\right\} \qquad (5)$$

The zeta-function of Jacoby is determined as an expression of the form

$$Z(\beta, k) = E(\beta, k) - \frac{E}{K}F(\beta, k), \qquad (6)$$

where

$$\beta = \arcsin\sqrt{\frac{K-E}{k^2 K}}.$$

More detailed information about elliptical functions is contained in [Appendices Literature 2, 3, 6]. Short extracts from the tables of elliptical functions are given in Appendices 3 and 4. More detailed tables of functions sn u, cn u, dn u are given in [Appendices Literature 8], Part II, and of function $K \cdot Z(\beta, k)$ in [Appendices Lit. 3]. The graphs of the values of elliptical functions at three different values of modulus are given in Appendices Figs. 1, 2, and 3.

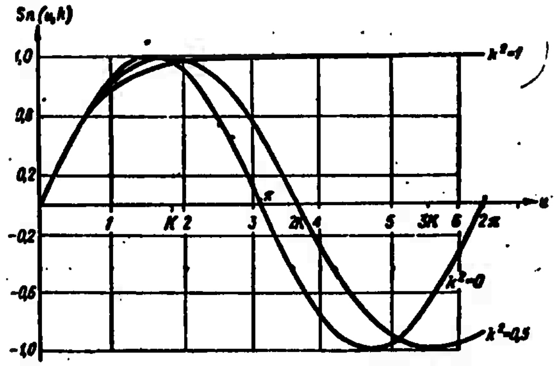


Fig. 1

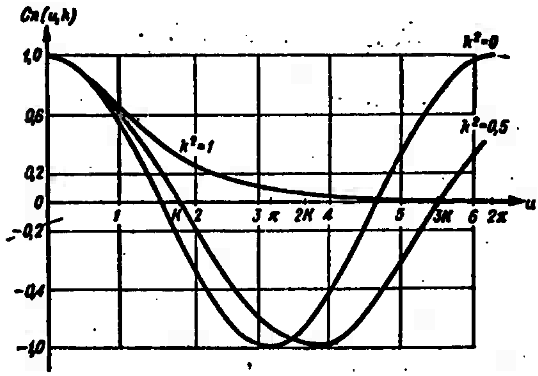


Fig. 2.

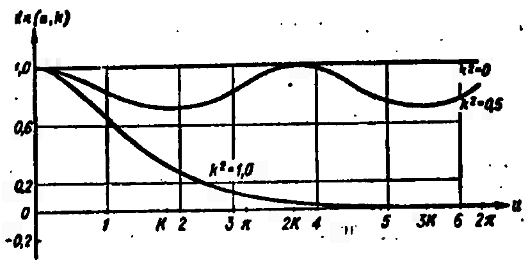


Fig. 3.

3. *Theta-function.*

Theta-function is defined as the sum of the series[1]

$$\vartheta_0(x) = 1 - 2q\cos 2\pi x + 2q^4\cos 4\pi x - 2q^9\cos 6\pi x + \ldots$$
$$\ldots + (-1)^n q^{n^2}\cos 2n\pi x + \ldots, \qquad (7)$$

where

$$q = e^{-\pi\rho};\quad \rho = \frac{K'}{K}.$$

The theta-function depends on two parameters - the argument x and the modulus of elliptical integrals k since the latter determines the values of q.

When q is close to one the expansion takes place

$$\vartheta_0(x) = 2\sqrt{\rho'}\, e^{-\frac{x'^2}{\pi\rho'}}\left[q'^{\frac{1}{4}}\operatorname{ch} x' + q'^{\frac{9}{4}}\operatorname{ch} 3x' + q'^{\frac{25}{4}}\operatorname{ch} 5x' + \ldots + \right.$$
$$\left. + q'^{\frac{(2n+1)^2}{4}}\operatorname{ch}(2n+1)x' + \ldots\right], \qquad (8)$$

where

$$\rho' = \frac{1}{\rho};\quad q' = e^{-\pi\rho'};\quad x' = \frac{\pi x}{\rho}.$$

The short table of values of function $\vartheta_0(x)$ is given in Appendix 5. A graph of dependence of the parameter q upon k^2 is given in Appendices Fig. 4, and the values of function $\ln\frac{1}{q} = f(k^2)$ are given in Appendices Fig. 5.

4. *Bessel functions.*

Linearly independent solutions of the Bessel equation of zero order

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + u = 0$$

[1]The given expression defines only one of the four introduced Jacoby theta-functions; for more detail see [Appendices Literature 2, 6].

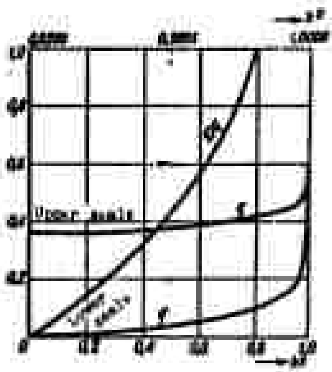


Fig. 4.

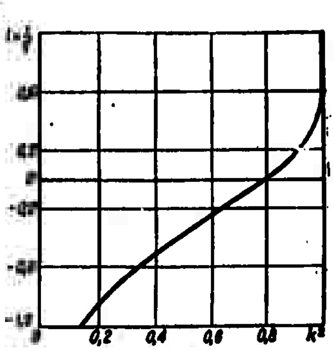


Fig. 5.

are the functions

$$J_0(z) = 1 - \left(\frac{1}{2}z\right)^2 + \frac{\left(\frac{1}{2}z\right)^4}{1^2\cdot 2^2} - \frac{\left(\frac{1}{2}z\right)^6}{1^2\cdot 2^2\cdot 3^2} + \ldots, \tag{9}$$

(Bessel functions of the first kind of zero order) and

$$N_0(z) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{z}{2}\right)J_0(z) + \left(\frac{1}{2}z\right)^2 - \frac{\left(\frac{1}{2}z\right)^4}{1^2\cdot 2^2}\left(1 + \frac{1}{2}\right) + \right.$$
$$\left. + \frac{\left(\frac{1}{2}z\right)^6}{1^2\cdot 2^2\cdot 3^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \ldots\right]. \tag{10}$$

where $\gamma = 0.5772157$ is the Euler constant (the Bessel function of the second kind of zero order).

The function

$$H_0^{(1)}(z) = J_0(z) + jN_0(z) \tag{11}$$

is called the Bessel function of the third kind or the Hankel function.

During calculations frequent use is made of the functions $I_0(z)$ and $K_0(z)$, connected with $J_0(z)$ and $H_0^1(z)$ by the dependences

$$\left.\begin{aligned} I_0(z) &= J_0(jz); \\ K_0(z) &= j\frac{\pi}{2}H_0^{(1)}(jz). \end{aligned}\right\} \tag{12}$$

Functions $I_0(z)$ and $K_0(z)$ are called the Bessel functions of an imaginary argument or the modified Bessel functions of zero order. The function $K_0(z)$ is known also as the MacDonald function.

Bessel functions are the topic of vast literature (see, for example, [Appendices Literature (AL) 1, 7,9]), and they are completely comprehensively tabulated [AL 5, 9, 10] (in [AL 8] the information about tables is given).

5. *Legendre functions of the first and second kind.*

In the book Legendre functions with coefficient, equal to half of an odd integer are used. These functions are linearly independent solutions of the equation

$$\frac{d^2u}{d\alpha^2} + \operatorname{cth}\alpha\frac{du}{d\alpha} - \left(n^2 - \frac{1}{4}\right)u = 0 \tag{13}$$

and have the form

$$\begin{aligned} P_{n+\frac{1}{2}}(\operatorname{ch}\alpha) &= \frac{1}{\pi}\int_0^{\pi}\frac{d\beta}{(\operatorname{ch}\alpha + \operatorname{sh}\alpha\cos\beta)^{n+\frac{1}{2}}}; \\ Q_{n+\frac{1}{2}}(\operatorname{ch}\alpha) &= \int_0^{\infty}\frac{d\beta}{(\operatorname{ch}\alpha + \operatorname{sh}\alpha\operatorname{ch}\beta)^{n+\frac{1}{2}}}. \end{aligned} \tag{14}$$

At $n = 0$ and $n = 1$ the Legendre functions are expressed through complete elliptical integrals of the first and the second kind (see clause 1 of this Appendix).

$$\left.\begin{aligned} &P_{1/2}(\operatorname{ch}\alpha) = 2\sqrt{k'}\cdot\mathrm{K}; \quad Q_{1/2}(\operatorname{ch}\alpha) = 2\sqrt{k'}\cdot\mathrm{K}'; \\ &P_{1+1/2}(\operatorname{ch}\alpha) = \frac{2}{\sqrt{k'}}\mathrm{E}; \quad Q_{1+1/2}(\operatorname{ch}\alpha) = \frac{2}{\sqrt{k'}}(\mathrm{K}' - \mathrm{E}'), \end{aligned}\right\} \tag{15}$$

where the modulus is

$$k = \frac{2}{1 + \operatorname{cth} a}, \quad k' = \sqrt{1 - k^2}.$$

More detailed information about the Legendre functions is given in [AL 6, 8, 11], and tables of the functions with coefficient equal to half of an odd integer are contained in [AL 12].

6. *Psi-function.*

The function $\psi(z)$ is the logarithmic derivative of a gamma function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}; \tag{16}$$

where

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

The function $\psi(z)$ satisfies the following functional relationships:

$$\left.\begin{aligned} \psi(z+1) &= \frac{1}{z} + \psi(z); \\ \psi(1-z) - \psi(z) &= \pi \operatorname{ctg} \pi z; \\ \psi(z) + \psi\left(z + \frac{1}{2}\right) + 2 \ln 2 &= 2\psi(2z). \end{aligned}\right\} \tag{17}$$

Computation $\psi(z)$ at special values of z can be carried out using the formulas:

$$\left.\begin{aligned} \psi(n+1) &= -\gamma + \sum_{k=1}^{n} \frac{1}{k}; \quad n = 1, 2, \ldots; \\ \psi\left(n + \frac{1}{2}\right) &= -\gamma - 2 \ln 2 + 2 \sum_{k=1}^{n} \frac{1}{2k-1}; \quad n = 1, 2 \ldots \end{aligned}\right\} \tag{18}$$

More detailed information about psi-function is given in [AL 8, 11]. The table of values $\psi(1 + x)$ is given in Appendix 6.

7. *The zeta-function of Riemann.*

The zeta-function of Riemann for Re $z > 1$ is determined by the formula

$$\zeta(z) = \frac{2^z}{(2^z - 1)\Gamma(z)} \int_0^\infty \frac{t^{z-1}e^t}{e^{2t} - 1} dt, \tag{19}$$

where $\Gamma(z)$ is a gamma function (see clause 6 of this Appendix).

A zeta-function satisfies the following functional relationships:

$$\left.\begin{aligned} 2^z\Gamma(1-z)\zeta(1-z)\sin\frac{z\pi}{2} &= \pi^{1-z}\zeta(z), \\ 2^{1-z}\Gamma(z)\zeta(z)\cos\frac{z\pi}{2} &= \pi^z\zeta(1-z); \\ \Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\zeta(z) &= \Gamma\left(\frac{1-z}{2}\right)\pi^{\frac{z-1}{2}}\zeta(1-z). \end{aligned}\right\} \tag{20}$$

Computation of $\zeta(z)$ at Re $z > 1$ can be carried out using the formula

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{21}$$

More detailed information about the zeta-function of Riemann is given in [AL 6].

The table of values of $\zeta(x)$ is given in Appendix 7.

APPENDIX 2

THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND

$$K(k) = K = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2 \sin^2\psi}}\,; K(k') = K' = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k'^2 \sin^2\psi}}\,;$$

$$k' = \sqrt{1-k^2}$$

k^2	K	K'	K'/K	K/K'	$(k')^2$
0,00	1,57080	∞	∞	0,00000	1,00
0,01	1,57475	3,69564	2,34682	0,42611	0,99
0,02	1,57874	3,35414	2,12457	0,47068	0,98
0,03	1,58278	3,15587	1,99388	0,50153	0,97
0,04	1,58687	3,01611	1,90067	0,52613	0,96
0,05	1,59100	2,90834	1,82799	0,54705	0,95
0,06	1,59519	2,82075	1,76828	0,56552	0,94
0,07	1,59942	2,74707	1,71754	0,58223	0,93
0,08	1,60371	2,68355	1,67334	0,59761	0,92
0,09	1,60805	2,62777	1,63414	0,61194	0,91
0,10	1,61244	2,57809	1,59887	0,62544	0,90
0,11	1,61689	2,53333	1,56680	0,63825	0,89
0,12	1,62139	2,49264	1,53734	0,65047	0,88
0,13	1,62595	2,45534	1,51009	0,66221	0,87
0,14	1,63058	2,42093	1,48471	0,67353	0,86
0,15	1,63526	2,38902	1,46094	0,68449	0,85
0,16	1,64000	2,35926	1,38258	0,69513	0,84
0,17	1,64481	2,33141	1,41744	0,70550	0,83
0,18	1,64968	2,30523	1,39738	0,71562	0,82
0,19	1,65462	2,28055	1,37829	0,72553	0,81
0,20	1,65962	2,25721	1,36007	0,73526	0,80
0,21	1,66470	2,23507	1,34262	0,74481	0,79
0,22	1,66985	2,21402	1,32588	0,75422	0,78
0,23	1,67507	2,19397	1,30978	0,76349	0,77
0,24	1,68037	2,17483	1,29425	0,77265	0,76
0,25	1,68575	2,15652	1,27926	0,78171	0,75
0,26	1,69121	2,13897	1,26476	0,79066	0,74
0,27	1,69675	2,12213	1,25070	0,79955	0,73
0,28	1,70237	2,10595	1,23707	0,80836	0,72
0,29	1,70809	2,09037	1,22381	0,81712	0,71
0,30	1,71389	2,07536	1,21091	0,82583	0,70
0,31	1,71978	2,06088	1,19834	0,83449	0,69
0,32	1,72577	2,04689	1,18607	0,84312	0,68
0,33	1,73186	2,03336	1,17409	0,85172	0,67
0,34	1,73805	2,02028	1,16238	0,86030	0,66
0,35	1,74435	2,00760	1,15091	0,86887	0,65
0,36	1,75075	1,99530	1,13986	0,87744	0,64
0,37	1,75727	1,98337	1,12867	0,88600	0,63
0,38	1,76390	1,97178	1,11786	0,89457	0,62
0,39	1,77065	1,96052	1,10723	0,90315	0,61
0,40	1,77752	1,94957	1,09679	0,91175	0,60
0,41	1,78452	1,93891	1,08652	0,92037	0,59
0,42	1,79165	1,92853	1,07640	0,92903	0,58
0,43	1,79892	1,91841	1,06642	0,93771	0,57
0,44	1,80633	1,90855	1,05659	0,91644	0,56

Continued.

k^2	K	K'	K'/K	K/K'	$(k')^2$
0,45	1,81388	1,89892	1,04688	0,95522	0,55
0,46	1,82159	1,88953	1,03730	0,96404	0,54
0,47	1,82946	1,88036	1,02782	0,97293	0,53
0,48	1,83749	1,87140	1,01845	0,98188	0,52
0,49	1,84569	1,86264	1,00918	0,99090	0,51
0,50	1,85407	1,85407	1,00000	1,00000	0,50
Values of modulus close to 0 and 1					
0,000001	1,57080	8,29405	5,28016	0,18939	0,999999
0,000002	1,57080	7,94748	5,05952	0,19765	0,999998
0,000003	1,57080	7,74475	4,93046	0,20282	0,999997
0,000004	1,57080	7,60091	4,83888	0,20666	0,999996
0,000005	1,57080	7,48934	4,76786	0,20974	0,999995
0,000006	1,57080	7,39818	4,70982	0,21232	0,999994
0,000007	1,57080	7,32111	4,66075	0,21456	0,999993
0,000008	1,57080	7,25434	4,61825	0,21653	0,999992
0,000009	1,57080	7,19545	4,58076	0,21830	0,999991
0,000010	1,57080	7,14277	4,54722	0,21991	0,999990
0,000100	1,57083	5,99159	3,81427	0,26217	0,999900
0,000200	1,57087	5,64512	3,59362	0,27827	0,999800
0,000300	1,57091	5,44249	3,46454	0,28864	0,999700
0,000400	1,57095	5,29875	3,37295	0,29648	0,999600
0,000500	1,57099	5,18727	3,30191	0,30286	0,999500
0,000600	1,57103	5,09620	3,24385	0,30828	0,999400
0,000700	1,57107	5,01921	3,19477	0,31301	0,999300
0,000800	1,57111	4,95253	3,15225	0,31723	0,999200
0,000900	1,57115	4,89373	3,11474	0,32105	0,999100
0,001000	1,57119	4,84113	3,08118	0,32455	0,999000
0,001100	1,57123	4,79356	3,05084	0,32778	0,998900
0,001200	1,57127	4,75014	3,02312	0,33078	0,998800
0,001300	1,57131	4,71020	2,99763	0,33360	0,998700
0,001400	1,57135	4,67322	2,97402	0,33624	0,998600
0,001500	1,57139	4,63880	2,95205	0,33875	0,998500
0,001600	1,57142	4,60661	2,93149	0,34112	0,998400
0,001700	1,57146	4,57638	2,91217	0,34339	0,998300
0,001800	1,57150	4,54788	2,89396	0,34555	0,998200
0,001900	1,57154	4,52092	2,87674	0,34762	0,998100
0,002000	1,57158	4,49535	2,86040	0,34960	0,998000
0,002100	1,57162	4,47103	2,84485	0,35151	0,997900
0,002200	1,57166	4,44784	2,83002	0,35335	0,997800
0,002300	1,57171	4,42569	2,81586	0,35513	0,997700
0,002400	1,57174	4,40448	2,80231	0,35685	0,997600
0,002500	1,57178	4,38414	2,78929	0,35851	0,997500
0,002600	1,57182	4,36461	2,77679	0,36013	0,997400
0,002700	1,57186	4,34581	2,76476	0,36170	0,997300
0,002800	1,57190	4,32769	2,75317	0,36322	0,997200
0,002900	1,57194	4,31022	2,74198	0,36470	0,997100
0,003000	1,57198	4,29334	2,73117	0,36614	0,997000

APPENDIX 3

FUNCTIONS sn(u, k), cn(u, k), dn(u, k)

u	$k^2 = 0,1$; K = 1,61244			$k^2 = 0,3$; K = 2,71389			$k^2 = 0,5$; K = 1,85507		
	sn (u, k)	cn (u, k)	dn (u, k)	sn (u, k)	cn (u, k)	dn (u, k)	sn (u, k)	cn (u, k)	dn (u, k)
0.00	0,00000	1,00000	1,00000	0,00000	1,00000	1,00000	0,00000	1,00000	1,00000
0.01	0,01000	0,99995	1,00000	0,01000	0,99995	0,99999	0,01000	0,99995	0,99998
0.03	0,03000	0,99955	0,99998	0,02999	0,99955	0,99987	0,02999	0,99955	0,99978
0.05	0,04998	0,99875	0,99988	0,04997	0,99875	0,99963	0,04997	0,99875	0,99938
0.10	0,09982	0,99501	0,99950	0,09978	0,99501	0,99851	0,09975	0,99501	0,99751
0.15	0,14938	0,98878	0,99888	0,14927	0,98880	0,99555	0,14916	0,98881	0,99442
0.20	0,19851	0,98009	0,99803	0,19828	0,98015	0,99409	0,19802	0,98020	0,99015
0.25	0,24715	0,96898	0,99694	0,24666	0,96910	0,99083	0,24616	0,96923	0,98473
0.30	0,29510	0,95547	0,99564	0,29425	0,95573	0,98633	0,29341	0,95599	0,97821
0.35	0,34224	0,93961	0,99413	0,34094	0,94009	0,98241	0,33963	0,94056	0,97073
0.40	0,38847	0,92146	0,99243	0,37657	0,92226	0,97733	0,38467	0,92305	0,96230
0.45	0,43365	0,90108	0,99055	0,43103	0,90234	0,97173	0,42842	0,90358	0,95301
0.50	0,47760	0,87853	0,98852	0,47422	0,88041	0,96568	0,47076	0,88227	0,94297
0.55	0,52046	0,85388	0,98636	0,51602	0,85658	0,95923	0,51168	0,85923	0,93228
0.60	0,56999	0,82722	0,98409	0,55635	0,83005	0,95244	0,55083	0,83462	0,92103
0.65	0,60184	0,79662	0,98172	0,59514	0,80262	0,94533	0,58843	0,80855	0,90933
0.70	0,64025	0,76817	0,97929	0,63230	0,77472	0,93811	0,62434	0,78115	0,89727
0.75	0,67704	0,73595	0,97681	0,66780	0,74434	0,93071	0,65851	0,75257	0,88497
0.80	0,71212	0,70206	0,97431	0,70157	0,71260	0,92322	0,69093	0,72292	0,87253
0.85	0,74542	0,66660	0,97182	0,73358	0,67961	0,91573	0,72159	0,69232	0,86003
0.90	0,77688	0,62965	0,96935	0,76379	0,64546	0,90829	0,75048	0,66090	0,84758
0.95	0,80643	0,59132	0,96694	0,79220	0,61027	0,90096	0,77761	0,62875	0,83526
1.00	0,83404	0,55171	0,96459	0,81877	0,57412	0,89380	0,8030	0,59598	0,82316
1.10	0,88319	0,46502	0,96021	0,86641	0,49934	0,88023	0,84867	0,52893	0,79993
1.20	0,92401	0,38236	0,95636	0,90668	0,42181	0,86797	0,88772	0,46039	0,77845
1.30	0,95625	0,28254	0,95318	0,93965	0,34212	0,85739	0,92045	0,39087	0,75920
1.40	0,97973	0,20033	0,95080	0,96538	0,26084	0,84877	0,94717	0,32074	0,74259
1.50	0,99431	0,10649	0,94928	0,98396	0,17840	0,84235	0,96818	0,25027	0,72892
1.60	0,99993	0,01130	0,94869	0,99546	0,09520	0,83828	0,98373	0,17964	0,71843
1.70	0,99655	−0,08238	0,94905	0,99993	0,01162	0,83668	0,99405	0,10886	0,71129
1.80	0,98419	−0,17710	0,95033	0,99740	−0,0721	0,83759	0,99927	0,03824	0,70762
1.90	0,96292	−0,26979	0,95251	0,98786	−0,15535	0,84098	0,99947	−0,03247	0,70748
1.99	0,93824	−0,35135	0,95517	0,97324	−0,22981	0,84608	0,98637	−0,09811	0,71037

Continued.

u	$k^2 = 0,7$; $K = 2,07536$			$k^2 = 0,9$; $K = 2,57809$			$k^2 = 1$; $K = \infty$		
	sn (u, k)	cn (u, k)	dn (u, k)	sn (u, k)	cn (u, k)	dn (u, k)	sn (u, k)	cn (u, k)	dn (u, k)
0,00	0,00000	1,00000	1,00000	0,00000	1,00000	1,00000	0,00000	0,00000	1,00000
0,01	0,01000	0,99995	0,99997	0,01000	0,99995	0,99996	0,01000	0,99995	0,99995
0,03	0,02999	0,99955	0,99969	0,02999	0,99955	0,89960	0,02999	0,99955	0,99955
0,05	0,04996	0,99875	0,99913	0,04996	0,99875	0,99888	0,04996	0,99875	0,99875
0,10	0,09972	0,99502	0,99651	0,09968	0,99502	0,99552	0,09967	0,99502	0,99502
0,20	0,19776	0,98025	0,98622	0,19750	0,98030	0,98229	0,19738	0,98033	0,98033
0,25	0,24566	0,96936	0,97865	0,24517	0,96948	0,97258	0,24492	0,96954	0,96954
0,30	0,29257	0,95624	0,96958	0,29173	0,95650	0,96094	0,29131	0,95663	0,95662
0,35	0,33833	0,94103	0,95910	0,33703	0,94150	0,94751	0,33638	0,94173	0,94173
0,40	0,38278	0,92384	0,94733	0,38089	0,92462	0,93243	0,37995	0,92501	0,92501
0,45	0,42581	0,90481	0,93439	0,42320	0,90604	0,91587	0,42190	0,90664	0,90664
0,50	0,46729	0,88410	0,92041	0,46384	0,88592	0,89795	0,46212	0,80682	0,88682
0,55	0,50715	0,86186	0,90552	0,50273	0,86444	0,87894	0,50052	0,86572	0,86572
0,60	0,54531	0,83823	0,88986	0,53980	0,85179	0,85892	0,53705	0,84355	0,84355
0,65	0,58173	0,81338	0,87356	0,57502	0,81814	0,83810	0,57167	0,82048	0,82048
0,70	0,61636	0,78746	0,85678	0,60837	0,79366	0,81664	0,60437	0,70671	0,79671
0,75	0,64919	0,76062	0,83963	0,63984	0,76851	0,79470	0,63515	0,77239	0,74239
0,80	0,68023	0,73300	0,82226	0,66945	0,74286	0,77243	0,66404	0,74770	0,74770
0,85	0,70947	0,70473	0,80477	0,69723	0,71685	0,74999	0,69107	0,72279	0,72270
0,90	0,73695	0,67595	0,78729	0,72323	0,69061	0,72749	0,71630	0,69779	0,69770
0,95	0,66270	0,64675	0,76994	0,74749	0,66427	0,70607	0,73978	0,67285	0,67285
1,00	0,78676	0,61726	0,75280	0,77009	0,63794	0,68284	0,76159	0,64805	0,64805
1,10	0,83002	0,55773	0,71954	0,81054	0,58569	0,63932	0,80050	0,59933	0,59933
1,20	0,86718	0,49799	0,68818	0,84518	0,53448	0,59758	0,83365	0,55229	0,55229

Continued.

u	$k^2 = 0.7$; $K = 2.07536$ sn (u, k)	cn (u, k)	dn (u, k)	$k^2 = 0.9$; $K = 2.57800$ sn (u, k)	cn (u, k)	dn (u, k)	$k^2 = 1$; $K = \infty$ sn (u, k)	cn (u, k)	dn (u, k)
1,30	0,88873	0,43850	0,65924	0,87463	0,48480	0,55815	0,86172	0,50738	0,50738
1,40	[illegible]	0,37957	0,63313	0,89950	0,43693	0,52136	0,88535	0,46492	0,46492
1,50	0,94695	0,32139	0,61017	0,92037	0,39104	0,48747	0,90515	0,42510	0,42510
1,60	0,96152	0,26402	0,59059	0,93779	0,34720	0,45661	0,92167	0,38798	0,38798
1,70	0,97825	0,20744	0,57456	0,95223	0,30538	0,42887	0,93541	0,35357	0,35357
1,80	0,98845	0,15157	0,56221	0,96412	0,26548	0,40427	0,94681	0,32180	0,32180
1,90	0,99536	0,09625	0,55361	0,97381	0,22737	0,38278	0,95624	0,29259	0,29259
2,00	0,99915	0,04129	0,54881	0,98162	0,19087	0,36440	0,96403	0,26580	0,26580
2,10	0,99991	—0,01349	0,54784	0,98779	0,15576	0,34905	0,97045	0,24129	0,24129
2,20	0,99766	—0,06834	0,55070	0,99255	0,12183	0,33669	0,97574	0,21892	0,21892
2,30	0,99235	—0,12345	0,55738	0,99605	0,08885	0,32727	0,98010	0,19852	0,19852
2,40	0,98385	—0,17902	0,56783	0,99840	0,05656	0,32075	0,98367	0,17995	0,17995
2,50	—	—	—	0,99969	0,02472	0,31710	0,98661	0,16307	0,16307
2,60	—	—	—	0,99998	—0,00693	0,31630	0,98903	0,14773	0,14773
2,70	—	—	—	0,99925	—0,03863	0,31834	0,99101	0,13381	0,13381
2,80	—	—	—	0,99750	—0,07063	0,32325	0,99263	0,12117	0,12117
3,00	—	—	—	0,99063	—0,13657	0,34174	0,99505	0,09933	0,09933
3,50	—	—	—	—	—	—	0,99818	0,06034	0,06034
4,00	—	—	—	—	—	—	0,99933	0,03662	0,03662
4,5	—	—	—	—	—	—	0,99975	0,02222	0,02222
5,0	—	—	—	—	—	—	0,99991	0,01348	0,01348
6,0	—	—	—	—	—	—	0,99999	0,00496	0,00496
6,5	—	—	—	—	—	—	1,00000	0,00301	0,00301

APPENDIX 4

FUNCTION $KZ(\beta, k)$

$\beta°$	$\alpha = 1°$	$\alpha = 15°$	$\alpha = 30°$	$\alpha = 45°$	$\alpha = 60°$	$\alpha = 75°$	$\alpha = 89°$
	$k^2 = 0,00030$	$k^2 = 0,06699$	$k^2 = 0,25000$	$k^2 = 0,50000$	$k^2 = 0,75000$	$k^2 = 0,93301$	$k^2 = 0,99970$
0°	0,000000	0,000000	0,000000	0,000000	0,000000	0,000000	0,000000
5°	0,000021	0,004688	0,018862	0,043755	0,082227	0,147228	0,386241
10°	0,000041	0,009238	0,037403	0,086448	0,162776	0,292070	0,768208
15°	0,000060	0,013513	0,054811	0,127026	0,239971	0,432134	1,141628
20°	0,000077	0,017387	0,070690	0,164459	0,312138	0,565367	1,502217
25°	0,000092	0,020743	0,084599	0,197748	0,377610	0,688264	1,845706
30°	0,000104	0,023479	0,096103	0,225942	0,434726	0,799407	2,167786
35°	0,000112	0,025510	0,104844	0,248154	0,481836	0,895883	2,464090
40°	0,000118	0,026774	0,110525	0,263583	0,517310	0,975016	2,730134
45°	0,000120	0,027228	0,112924	0,271538	0,539547	1,033555	2,961210
50°	0,000118	0,026855	0,111909	0,271473	0,547003	1,066585	3,152208
55°	0,000113	0,025602	0,107447	0,263028	0,535238	1,078397	3,297296
60°	0,000104	0,023683	0,099613	0,246077	0,512007	1,056317	3,389358
65°	0,000092	0,020976	0,088594	0,220781	0,467411	0,998480	3,418883
70°	0,000077	0,017619	0,074696	0,187640	0,404143	0,899033	3,371563
75°	0,000060	0,013718	0,058332	0,147536	0,322854	0,751288	3,222428
80°	0,000041	0,009390	0,040018	0,101748	0,225584	0,549278	2,917759
85°	0,000021	0,004769	0,020354	0,051923	0,116121	0,293208	2,291948
90°	0,000000	0,000000	0,000000	0,000000	0,000000	0,000000	0,000000

APPENDIX 5

FUNCTION $\vartheta_0(x)$

$2x$	$\alpha = 0°$	$\alpha = 9°$	$\alpha = 18°$	$\alpha = 27°$	$\alpha = 36°$
	$k^2 = 0,00000$	$k^2 = 0,02447$	$k^2 = 0,09549$	$k^2 = 0,20611$	$k^2 = 0,34549$
0,0	1,0000	0,9970	0,9874	0,9712	0,9471
0,1	1,0000	0,9970	0,9881	0,9725	0,9497
0,2	1,0000	0,9975	0,9899	0,9766	0,9572
0,3	1,0000	0,9982	0,9927	0,9831	0,9683
0,4	1,0000	0,9991	0,9961	0,9911	0,9836
0,5	1,0000	1,0000	1,0000	1,0000	1,0000
0,6	1,0000	1,001	1,004	1,009	1,016
0,7	1,0000	1,002	1,007	1,017	1,031
0,8	1,0000	1,003	1,010	1,023	1,043
0,9	1,0000	1,003	1,012	1,028	1,050
1,0	1,0000	1,003	1,013	1,029	1,053

$2x$	$\alpha = 45°$	$\alpha = 54°$	$\alpha = 63°$	$\alpha = 72°$	$\alpha = 81°$
	$k^2 = 0,50000$	$k^2 = 0,65451$	$k^2 = 0,79389$	$k^2 = 0,90451$	$k^2 = 0,97553$
0,0	0,9136	0,8680	0,8052	0,7152	0,5634
0,1	0,9198	0,8744	0,8147	0,7290	0,5698
0,2	0,9300	0,8931	0,8424	0,7691	0,6494
0,3	0,9493	0,9223	0,8853	0,7980	0,7429
0,4	0,9732	0,9592	0,9397	0,9110	0,8639
0,5	1,0000	1,0000	1,0000	1,0002	0,9956
0,6	1,027	1,041	1,060	1,088	1,131
0,7	1,051	1,078	1,115	1,168	1,254
0,8	1,070	1,107	1,159	1,231	1,353
0,9	1,080	1,126	1,186	1,272	1,417
1,0	1,086	1,132	1,195	1,286	1,439

APPENDIX 6

FUNCTION $\psi(1 + x)$

x	0	1	2	3	4	5	6	7	8	9
0,0	—0,5772	—0,5609	—0,5448	—0,5289	—0,5133	—0,4978	—0,4826	—0,4676	—0,4528	—0,4382
0,1	—0,4238	—0,4095	—0,3955	—0,3816	—0,3679	—0,3543	—0,3410	—0,3277	—0,3147	—0,3018
0,2	—0,2890	—0,2764	—0,2640	—0,2517	—0,2395	—0,2275	—0,2155	—0,2038	—0,1921	—0,1806
0,3	—0,1692	—0,1579	—0,1467	—0,1357	—0,1248	—0,1139	—0,1032	—0,0926	—0,0821	—0,0717
0,4	—0,0614	—0,0512	—0,0411	—0,0311	—0,0211	—0,0113	—0,0016	+0,0081	+0,0176	+0,0271
0,5	+0,0365	+0,0458	+0,0550	+0,0642	+0,0732	+0,0822	+0,0911	+0,1000	+0,1087	+0,1174
0,6	+0,1260	+0,1346	+0,1431	+0,1515	+0,1598	+0,1681	+0,1763	+0,1845	+0,1926	+0,2006
0,7	+0,2085	+0,2165	+0,2243	+0,2321	+0,2398	+0,2475	+0,2551	+0,2626	+0,2701	+0,2776
0,8	+0,2850	+0,2923	+0,2996	+0,3069	+0,3141	+0,3212	+0,3283	+0,3353	+0,3423	+0,3493
0,9	+0,3562	+0,3630	+0,3699	+0,3766	+0,3833	+0,3900	+0,3967	+0,4033	+0,4098	+0,4163

APPENDIX 7

FUNCTION $\zeta(x)$

x	0	1	2	3	4	5	6	7	8	9
0	—0,5000	—0,6030	—0,7339	—0,9046	—1,135	—1,460	—1,953	—2,778	—4,438	—9,430
1	∞	10,584	5,592	3,932	3,106	2,612	2,286	2,054	1,882	1,750
2	1,645	1,560	1,491	1,432	1,383	1,341	1,305	1,274	1,247	1,223
3	1,202	1,183	1,167	1,152	1,139	1,127	1,116	1,106	1,098	1,090
4	1,0823	1,0757	1,0698	1,0643	1,0593	1,0547	1,0505	1,0467	1,0431	1,0399
5	1,0369	1,0342	1,0317	1,0293	1,0272	1,0252	1,0234	1,0217	1,0201	1,0187
6	1,01734	1,01611	1,01496	1,01390	1,01292	1,01201	1,01116	1,01038	1,00965	1,00898
7	1,00835	1,00777	1,00723	1,00673	1,00626	1,00583	1,00542	1,00505	1,00470	1,00438
8	1,00408	1,00380	1,00354	1,00329	1,00307	1,00286	1,00266	1,00248	1,00231	1,00216
9	1,002008	1,001878	1,001744	1,001626	1,001515	1,001413	1,001317	1,001227	1,001144	1,001067
10	1,000995	1,000927	1,000865	1,000806	1,000752	1,000701	1,000654	1,000609	1,000568	1,000530

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