BOEING SCIENTIFIC RESEARCH LABORATORIES DOCUMENT D1-82-0972

BOEING
SCIENTIFIC RESEARCH LABORATORIES

MAY 1970

On Maximum Likelihood Estimators of Shape and Scale Parameters and Their Application in Constructing Confidence Contours

Sam C. Saunders

MATHEMATICAL & INFORMATION SCIENCES

D1-82-0972

ON MAXIMUM LIKELIHOOD ESTIMATORS OF SHAPE AND SCALE PARAMETERS AND THEIR APPLICATION IN CONSTRUCTING CONFIDENCE CONTOURS

by

Sam C. Saunders

Mathematical and Information Sciences Report No. 5

Mathematical and Information Sciences Laboratory

BOEING SCIENTIFIC RESEARCH LABORATORIES

May 1970

SUMMARY

If a family of distributions is of the form $F[(t/\beta)^{\alpha}]$ for $t > 0$, where F is known and α and β are the shape and scale parameters respectively, the maximum likelihood estimates $\hat{\alpha}$, $\hat{\beta}$, calculated from a complete sample or an incomplete sample censored under certain conditions or terminated at a given order observation, have the property that $U = \hat{\alpha}/\alpha$ and $V = (\hat{\beta}/\beta)^{\alpha}$, have distributions which are parameter-free. Thus $T = U \ln V$ has a parameter-free distribution. Consequently, a tabulation of these distributions for a specific sampling situation could be used to perform the usual statistical tests and to construct interval estimates for the parameters in the same way the sample variance, s, and "Student's" t-distribution do for the normal law.

Moreover, it is shown that this result can be used to obtain confidence contours along the entire distribution function analogous to the Kolmogorov-Smirnov bounds on the empiric cumulative distribution.

Taking the Weibull distribution and a sample of the first 3 order observations out of 5 as an example (in which case the parameter-free properties of U and V are not new), computational procedures are specified for determining the distribution of U and V by Monte Carlo methods using the latest random number generators. These computing times are given, as it is for the calculations necessary to compute a confidence contour along the entire distribution. Graphs of the distributions of U and V and the confidence contour for the distribution are presented for this case.

1. Introduction

In a paper by Thoman, Baine and Antle [12], it was shown that certain pivotal functions of the maximum likelihood estimators for parameters in the Weibull family have distributions which are parameter-free. (This fact was also mentioned in the mathematical appendix of the proprietary report [2].) By using Monte Carlo methods, this basic result made possible the production of tables of the percentile points of these distributions so that confidence intervals for the parameters can be determined. In turn this makes possible tests of hypotheses regarding the parameters, see [11].

It is the purpose of this note to generalize this result in several respects. Firstly, to point out that for any given continuous distribution with support on the positive line and unknown shape and scale parameters the same pivotal functions of the maximum likelihood estimators are parameter-free. Secondly, that the same conclusion holds for incomplete samples censored under certain conditions or terminated at a given ordered observation. Moreover in these cases it is possible to obtain confidence contours along the entire distribution function not just of tail probabilities as given in [6]. These contours are not defective near the tails as are the Kolmogorov-Smirnov bounds and by being more specific are narrower.

2. The Model and the Estimates

Let R denote a given survival distribution (unity minus the distribution) with support on the positive real line. We assume the density $f = -R'$ exists and is differentiable and we denote the hazard rate by $q = f/R$. We now make the assumption

1°: The observable random variable X has an unknown survival distribution H within the two parameter family defined for given R by

$$H(x) = P[X > x] = R[(x/\beta)^{\alpha}] \quad \text{for} \quad x > 0,$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter, often called the characteristic life.

Alternatively, we could formulate a model with unknown scale and location parameters and a fixed distribution by considering $\ell n X$ as an observable variate. Thus there is an obvious analogue for everything which follows within that formulation.

We shall now introduce a sampling situation which often arises in life testing, namely we observe either the time at which failure takes place or the time at which the life test is terminated with the component unfailed and we know which of these events occurred. For example, in fatigue life testing we are told the total time the specimen has been fatigued and whether or not it has broken. There are similar situations in clinical trials in medicine.

Let X denote the life length and Z the random time at which the test is terminated for any other reason than failure. Suppose we observe the random couple $\min(X,Z)$, $\{X < Z\}$ where $\{A\}$ is the indicator of the event A, taking the value one or zero, and $\min(X,Z)$ is the time the test was operating.

Let X_i for $i = 1,\ldots,n$ be mutually independent and identically distributed life length random variables with distribution F, density F' and survival distribution $\overline{F}$. Let Z_i for $i = 1,\ldots,n$ be a set of non-negative censoring random variables which may be dependent upon the X's.

Let $Y_i = \min(X_i,Z_i)$, $I_i = \{X_i < Z_i\}$ for $i = 1,\ldots,n$ and call the random set of indices of failed items

$$\Lambda = \{i = 1,\ldots,n : I_i = 0\} .$$

We now make assumption

> 2°: $(Z_1,\ldots,Z_n)$ are independent, for given $\Lambda = \lambda$ of $\{X_i : i \notin \lambda\}$ on the event $\bigcap_{i \notin \lambda} [X_i > Z_i]$.

This assumption means that if a given value of $(x_1,\ldots,x_n)$ should result in a set of $(z_1,\ldots,z_n)$ then for any i such that $x_j > z_j$ (i.e., $j \in \lambda$) any larger observed value of x_j would have resulted in the same set of $(z_1,\ldots,z_n)$ in distribution. Thus for each censored test the component being even longer lived would not alter the result any.

Lemma: If 2° holds the density of (Y_i,I_i) for $i = 1,\ldots,n$ is proportional to

$$(2.1) \qquad \prod_{i\in\lambda} F'(y_i) \prod_{j\notin\lambda} \bar{F}(y_j)$$

where the constant of proportionality depends upon the failure observations $\{y_i : i\in\lambda\}$ but not upon F.

Proof: Consider the event

$$[Y_1 = y_1,\ldots,Y_n = y_n,\ I_1 = \ldots = I_k = 1,\ I_{k+1} = \ldots = I_n = 0]$$

or equivalently

$$(2.2) \qquad \bigcap_{i=1}^{k} [X_i < Z_i][X_i = x_i] \bigcap_{i=k+1}^{n} [X_i > Z_i][Z_i = z_i] \ .$$

Let g denote the joint density of $Z_1,\ldots,Z_n$ given $X_1,\ldots,X_n$ assuming $Z_1,\ldots,Z_n$ independent of $X_{k+1},\ldots,X_n$ on $\bigcap_{i=k+1}^{n} [X_i > Z_i]$. The likelihood of the event specified in (2.2) is

$$\prod_{i=1}^{k} F'(x_i) \prod_{j=k+1}^{n} F(z_j) \int_{x_k}^{\infty} \cdots \int_{x_1}^{\infty} g(z_1,\ldots,z_n | x_1,\ldots,x_k) dz_1 \ldots dz_k.$$

Except for the altered notation in (2.1) this is the result. ||

Consider the special case when the Z_i's are independent of the X_i's. This would include censoring at a specific time determined by either chance or design. The case when the Z_i's are functionally dependent on the X_i's would include the case of censoring at a given failure. Clearly 2° is true in the case of Z_i's independent of X_i's. To check that 2° is true in the case $Z_i = X_{(k)}$ $i = 1,\ldots,n$, the k^{th} ordered observation, we examine the joint density of $X_{(k)}, X_{k+1},\ldots,X_n$ on the event $X_{(k)} < X_i$ for $i = k+1,\ldots,n$. It is

$$k\binom{n}{k}\bar{F}^{k-1}(x_{(k)})\prod_{i=k+1}^{n}F'(x_i) \quad x_{(k)} < x_i \quad i = k+1,\ldots,n$$

Note this is the condition specified.

From (1.1) we see the density of X is

$$f(x^\alpha/\beta^\alpha)\alpha x^{\alpha-1}\beta^{-\alpha} \quad \text{for} \quad x > 0 \ . \tag{2.3}$$

Following the notation introduced in the lemma we let $(y_1,\ldots,y_k)$ denote the set of observations of failed items, i.e., observations of X and $(y_{k+1},\ldots,y_n)$ denote the set of censored tests, i.e., observations of Z.

From (2.1), setting $F'(x)$ equal to (2.3), we may write the log-likelihood as

$$L = \sum_{i=1}^{k}[\ell n f(y_i^\alpha/\beta^\alpha) + \ell n(\alpha/\beta) + (\alpha-1)\ell n(y_i/\beta)] + \sum_{i=k+1}^{n}\ell n R(y_i^\alpha/\beta^\alpha) \ .$$

Hence

$$\frac{\partial L}{\partial\alpha} = \frac{k}{\alpha} + \sum_{i=1}^{k}\ell n(y_i/\beta) - \sum_{i=1}^{n}(y_i/\beta)^\alpha \ell n(y_i/\beta)\psi_{i,k}(y_i^\alpha/\beta^\alpha)$$

where in general

$$\psi_{i,k}(x) = q(x) - \{i \le k\}q'(x)/q(x) \quad i,k = 1,\ldots,n,$$

and similarly

$$\frac{\partial L}{\partial\beta} = -\frac{k\alpha}{\beta} + \frac{\alpha}{\beta}\sum_{i=1}^{n}(y_i/\beta)^\alpha\psi_{i,k}(y_i^\alpha/\beta^\alpha) \ .$$

Thus the joint maximum likelihood estimators $\hat{\alpha}$, $\hat{\beta}$ are the simultaneous solutions to the equations

$$\frac{1}{k}\sum_{i=1}^{n}\left(\frac{y_i}{\hat{\beta}}\right)^{\hat{\alpha}}\psi_{i,k}[(y_i/\hat{\beta})^{\hat{\alpha}}] = 1 \tag{2.4}$$

$$(2.5) \qquad \sum_{i=1}^{n} \left(\frac{y_i}{\hat{\beta}}\right)^{\hat{\alpha}} \ell n(y_i/\hat{\beta}) \psi_{i,k}[(y_i/\hat{\beta})^{\hat{\alpha}}] - \sum_{i=1}^{k} \ell n(y_i/\hat{\beta}) = \frac{k}{\hat{\alpha}} .$$

Note that if we assume a Weibull model, then $q(x) = 1$ for $x > 0$ and we have for all $i,k = 1,\ldots,n$

$$(2.6) \qquad \psi_{i,k}(x) = 1 \quad \text{for all} \quad x > 0 .$$

Naturally enough this results in considerable simplification. In particular (2.4) can be explicitly solved for $\hat{\beta}$.

For the log-normal model, namely

$$q(x) = \frac{\mathfrak{N}'(\ell nx)}{x\mathfrak{N}(-\ell nx)} \qquad \text{for} \quad x > 0 ,$$

where $\mathfrak{N}$ is the standard normal distribution, we find

$$(2.7) \qquad \psi_{i,k}(x) = \begin{cases} \dfrac{1 + \ell nx}{x} & i \le k \\ q(x) & i > k . \end{cases}$$

Substituting (2.6) and (2.7) into the equations (2.4) and (2.5) we find they reduce to those given by Cohen in [4] for the Weibull and log-normal cases, respectively.

Another case is an extreme value distribution with $q(x) = e^x$ for $x > 0$. Here we have

$$\psi_{i,k}(x) = \begin{cases} e^x & i > k \\ e^x - 1 & i \le k . \end{cases}$$

This also results in some simplification in (2.4) and (2.5).

Let W have survival distribution R and for $i = 1,\ldots,n$ set

$$(2.8) \qquad w_i = (y_i/\beta)^{\alpha} \qquad u = \hat{\alpha}/\alpha \qquad v = (\hat{\beta}/\beta)^{\alpha} .$$

Thus we see for $i = 1,\ldots,n$

$$(y_i/\hat{\beta})^{\hat{\alpha}} = (w_i/v)^u, \qquad (y_i/\hat{\beta})^{\alpha} = w_i/v$$

and simplifying (2.4) and (2.5) we obtain two equations in (u,v):

$$\frac{1}{k}\sum_{i=1}^{n} \xi_i \psi_{i,k}(\xi_i) = 1$$

(2.9)

$$\frac{1}{k}\sum_{i=1}^{n} \xi_i \ln\xi_i \psi_{i,k}(\xi_i) - \frac{1}{k}\sum_{i=1}^{k} \ln\xi_i = 1$$

where

$$\xi_i = (w_i/v)^u \quad i = 1,\ldots,n \; .$$

It follows from (2.8) that $W_1,\ldots,W_k$ are distributed independently of α and β. We shall assume

3° $(W_{k+1},\ldots,W_n)$ have a known distribution independent of α and β where k is the given number of failed (uncensored) items.

In the case of random sampling for a fixed time, it is clear that 2° would not be satisfied; on the other hand, it is clear that sampling until a fixed number of failures would satisfy it. There are other situations which arise in practice for which 2° holds. For example, in certain fatigue tests of structural details, failure takes place other than within the area of primary concern. Such fatigue may be unrepresentative of service if it is caused by the abnormalities of local stress induced from clamping the detail in the fatigue machine. The assumption is made that the shape parameters of the two distributions of fatigue at the different areas are the same while the scale parameters (characteristic lives) have a known ratio determined from the maximum deflection and the gross area stresses.

There follows immediately from equations (2.9) the

Theorem: If 1°, 2° and 3° hold, then the random variables $U = \hat{\alpha}/\alpha$, $V = (\hat{\beta}/\beta)^{\alpha}$ have a joint distribution independent of α and β.

Also we have the

Corollary: If 1°, 2° and 3° hold then the random variable $T = U\ell nV = \hat{\alpha}(\ell n\hat{\beta} - \ell n\beta)$ has a distribution independent of α and β.

The preceding theorem is a generalization of the results given in [12] for the Weibull distribution. It was not known when the survey article [7] was written although it may have been suspected from sampling results for this case, see the references in [7]. In [12] the percentiles of the marginal distributions of U and V are tabulated only for complete samples, for obvious reasons. It is mentioned there that the parameter-free property of the pivotal functions holds when the observations are censored at the k^{th} failure.

Obviously, if we have a log-normal distribution so that (2.7) holds we obtain the theorems concerning the log-normal model. These can be related to results about the t-distribution through an exponential transformation. Thus the corollary above says essentially that "Student's" result [10] about exact inference can be extended to any two-dimensional family known except for scale and location parameters when using the maximum likelihood estimates.

The importance of the preceding results is that it establishes the usefulness of the maximum likelihood estimators, whose optimum properties are well known, under a wide category of censoring and truncation of the data. Further it gives conditions under which the pivotal functions of the maximum likelihood estimators are parameter-free and thus it is possible, by varying R, to perform studies of the robustness of these estimators.

If in a given instance one wishes to determine a joint confidence region in both parameters, or in one parameter separately with the other unknown (or the corresponding tests) then we need to calculate the required percentile points of the distribution which can be done using some acceptable Monte Carlo procedure. This has already been accomplished for certain cases for the Weibull distribution in [11].

3. Confidence Contours on the Distribution

Although the need of calculating confidence contours along the entire distribution function arises often in reliability practice there are but a few methods of doing so.

Let $\tilde{H}$ be an estimate of the survival distribution $H \epsilon \mathcal{H}$. Previously in [9], we have called $\tilde{H}$ *ample* for H iff $\tilde{H}H^{-1}(p)$ for each $p \varepsilon (0,1)$ has a distribution independent of $H \epsilon \mathcal{H}$. (Here and in what follows juxtaposition of functions indicates composition.) For such estimators the analogue of the Kolmogorov-Smirnov statistic

$$\tilde{D}_n = \sqrt{n} \sup_x |\tilde{H}(x) - H(x)| = \sqrt{n} \sup_p |\tilde{H}H^{-1}(p) - p|$$

and the Cramér-von Mises statistic

$$\tilde{W}_n^2 = -n \int_{-\infty}^{\infty} |\tilde{H} - H|^2 dH = n \int_0^1 |\tilde{H}H^{-1}(p) - p|^2 dp$$

are distribution-free with respect to $\mathcal{H}$. For the model presented in §2 we obtain

$$(3.1) \qquad \hat{H}H^{-1}(p) = R\{[VR^{-1}(p)]^U\} \qquad \text{for} \quad 0 < p < 1 .$$

The importance of the probability integral transform to obtain estimates which are parameter-free is evident and is not new; for example, it was studied in [5]. Thus we see $\hat{H}(x) = R[(x/\hat{\beta})^{\hat{\alpha}}]$ is ample for any two parameter family with R specified, since clearly the distribution of the quantity $\hat{H}H^{-1}(p)$ for each $p \epsilon (0,1)$ does not depend upon the parameters α, β.

It is possible to obtain the distribution of $\tilde{D}$ by simulation and if tables of the percentage points were provided they could be used to determine confidence contours similar to the well-known Kolmogorov-Smirnov bounds, e.g., see [3]. However, we do not favor such bounds because of their very bad behavior in the tails of the distribution. This is usually the region where we are most interested in good behavior. Instead we turn to another method.

Suppose that there exists for each $\varepsilon\epsilon(0,1)$ a continuous monotone increasing function, say A_ε, mapping (0,1) onto (0,1) such that

$$(3.2) \qquad P[A_\varepsilon^{-1}\hat{H} \geq H] = P[\hat{H}H^{-1} \geq A_\varepsilon] = \varepsilon$$

where an inequality between functions indicates the inequality holds for their functional values at all points in their common domain. From (3.2) an upper confidence contour of level ε for H then would be $A_\varepsilon^{-1}\hat{H}$. At a later time we mention the alterations necessary to obtain either lower or two-sided confidence contours.

If we substitute (3.1) into (3.2) and apply R^{-1}, which is an order-reversing transformation, then take logarithms we obtain

$$(3.3) \qquad P[T + U \ \ell nR^{-1}(p) \leq \ell nR^{-1}A_\varepsilon(p) \quad \text{for all } p\epsilon(0,1)] = \varepsilon \ .$$

If we set

$$(3.4) \qquad \eta_\varepsilon(x) = \ell nR^{-1}A_\varepsilon R(e^x)$$

and make the change of variable $x = \ell nR^{-1}(p)$ in (3.3) we obtain

$$(3.5) \qquad \varepsilon = P[T \leq \inf_x(\eta_\varepsilon(x) - xU)] \ .$$

We now define a functional on the extended real line for any real valued function η and any $u > 0$ by

(3.6) $$\omega_\eta(u) = \inf_{-\infty<x<\infty} [\eta(x) - xu] .$$

The question is, can we find a function η_ε for which $\omega_{\eta_\varepsilon}(U)$ exists finitely with probability one and satisfies equation (3.5), to wit,

(3.7) $$\varepsilon = P[T \leq \omega_{\eta_\varepsilon}(U)] .$$

If so, it would be theoretically possible to obtain an upper confidence contour along the entire distribution H from knowledge of the joint distribution of U and T. We now seek conditions which help with the determination of η_ε.

Let $\mathcal{C}$ be the set of increasing convex functions mapping the real line onto itself each of which has a continuous derivative with the positive real line as its range. A proof is easily given for the

Remark: If $\eta\epsilon\mathcal{C}$ then $\omega_\eta(U)$ exists and is a random variable given by

(3.8) $$\omega_\eta(U) = \eta[(\eta')^{-1}(U)] - U(\eta')^{-1}(U) .$$

Note that if $\phi\epsilon\mathcal{C}$ then $\eta\epsilon\mathcal{C}$ where

(3.9) $$\eta(x) = \phi(x/b) + a \quad \text{for} \quad -\infty < x < \infty$$

for any real a and $b > 0$ and

(3.10) $$\omega_\eta(u) = a + \omega_\phi(bu) \quad \text{for} \quad u > 0.$$

It is clear that by proper choice of a,b the right hand side of (3.10) can be determined for given $\phi \epsilon \mathcal{C}$ so that (3.7) is satisfied, in theory, for each ε in the unit interval.

From the definition of η_ε in terms of A_ε in (3.4) we find, setting $\ell^{-1}(x) = e^x$

$$(3.11) \qquad A_\varepsilon = R\ell^{-1}\eta_\varepsilon \ell R^{-1} .$$

Thus the upper confidence contour of level ε is

$$(3.12) \qquad A_\varepsilon^{-1}\hat{H}(t) = R\ell^{-1}\eta_\varepsilon^{-1}[\hat{\alpha}\ell(t/\hat{\beta})] \quad \text{for} \quad t > 0.$$

In the preceding discussion an arbitrary choice of ϕ was made as an illustration of the feasibility of the computation which must be done. Of course a better choice of the parameterization could be made by considering the power for the test corresponding to the confidence contour for a specified alternative. It is possible that tests of an optimal nature could be constructed in certain instances against certain alternatives.

4. Lower and Two-Sided Confidence Contours

If we are concerned with obtaining lower confidence contours for H we would seek an appropriate monotone increasing function, say B_ε, for ε near unity, mapping (0,1) onto (0,1) such that

$$P[\hat{H} \leq B_\varepsilon H] = P[B_\varepsilon^{-1}\hat{H} \leq H] = \varepsilon.$$

Proceeding as before and making the same change of variable we obtain, in analogy with (3.5), an expression equivalent with the above, namely

$$(4.1) \qquad P[T \geq \sup_x (\nu_\varepsilon^{\#}(x) - xU)] = \varepsilon$$

where

$$(4.2) \qquad \nu_\varepsilon^{\#}(x) = \ell n R^{-1} B_\varepsilon R(e^x) \quad \text{for} \quad -\infty < x < \infty .$$

Define for any $u > 0$ and any function ν for which it exists

$$(4.3) \qquad \rho_\nu(u) = \sup_x [\nu(x) - xu].$$

If we set

$$(4.4) \qquad \nu^{\#}(x) = -\nu(-x) \quad \text{for} \quad -\infty < x < \infty$$

we see

$$(4.5) \qquad \rho_{\nu^{\#}}(u) = -\inf_x [\nu(x) - xu] = -\omega_\nu(u) .$$

Thus if $\nu \in \mathcal{C}$ then $\nu^{\#}$ is a monotone increasing concave function mapping the real line onto itself with a continuous derivative having the positive real line as its range. We label this set of functions $\mathcal{C}^{\#}$.

As a consequence of (4.5) we may relate some of our previous results to this case. For instance fix ϕ and $\eta \epsilon \mathcal{C}$ as before in (3.18) then

(4.6) $$(\eta^{\#})^{-1}(y) = -\eta^{-1}(-y) = -b\phi^{-1}(a-y)$$

and $\eta^{\#} \epsilon \mathcal{C}^{\#}$.

If we determine for a prescribed ϕ a pair (a,b) which by (3.9) defines an upper confidence contour of level ε, near zero, as given in (3.7) in terms of η then by the use of the transformation (4.4) and the identity (4.5) we obtain

(4.7) $$P[T \geq \omega_\eta(U)] = P[-T \geq \rho_\eta\#(U)] = 1 - \varepsilon.$$

Comparing (4.1), we see, in those situations where T has a distribution symmetric about zero (e.g., when we have complete samples and the underlying survival distribution H possesses the requisite symmetry) that $\eta^{\#}$ as given in (4.6) determines a lower confidence contour of level $1 - \varepsilon$ which is near unity.

In the more usual case where we do not have $-T$ with the same distribution as T a determination must be made of the appropriate (a,b) in (4.6) in order to satisfy (4.1). A discussion of the feasibility of this we defer until later.

To obtain both upper and lower confidence contours for H simultaneously we seek A and B both monotone increasing functions mapping the unit interval onto itself such that

$$\varepsilon = P[BH \geq \hat{H} \geq AH] = P[A^{-1}\hat{H} \geq H \geq B^{-1}\hat{H}] .$$

From the first equality above, we obtain as before

$$\varepsilon = P[\ell R^{-1}B(p) \leq T + U\ell R^{-1}Ap, \quad \text{for all} \quad p\in(0,1)]$$

$$(4.8) \qquad = P[\rho_{\nu}{}^{\#}(v) \leq T \leq \omega_{\eta}(v)]$$

where $\nu^{\#}\in \mathcal{C}^{\#}$ is defined in terms of B by (4.2), $\eta\in\mathcal{C}$ is defined in terms of A by (3.11) and ρ,ω were defined respectively by (4.3) and (3.6).

From (4.4) we seek both $\nu,\eta\in\mathcal{C}$ for which

$$(4.9) \qquad \varepsilon = P[-\omega_{\nu}(U) \leq T \leq \omega_{\eta}(U)] .$$

The determination of an appropriate ν and η in (4.9), or the corresponding η in (3.7) or $\nu^{\#}$ in (4.1) can be done rather straightforwardly if we fix $\phi\in\mathcal{C}$ and choose the appropriate values of the parameters (a,b) across a two dimensional subspace of $\mathcal{C}$ as defined in (3.9).

If we do so in the two sided case we have from (3.10)

$$\varepsilon = P[-\omega_{\phi}(b^{\#}U) - a^{\#} \leq T \leq a + \omega_{\phi}(bU)]$$

with the obvious use of notation. If we exercise two degrees of freedom and set $b^{\#} = b$, $a^{\#} = a$ we obtain

$$(4.10) \qquad \varepsilon = P[|T| \leq a + \omega_{\phi}(bU)]$$

from which the proper choice of the parameters may be more easily made. A method of choosing the appropriate parameters a,b we take up next.

5. Monte Carlo Techniques

This discussion is to make the point that with the latest random number generators available and the computing capability currently extant, the distribution of the relevant statistics from U,V can in practice be determined by simulation with as much precision and speed as could be done in many instances using numerical calculation were the distributions available in closed form. Moreover, the wide variety of sampling situations which can arise in practice makes it virtually impossible to provide tables of more than limited usefulness for even one specific choice of R.

We now outline a method for the determination of either a confidence interval on α with β unknown (or β with α unknown or both) or confidence contours along the entire distribution (or on one side only) for the Weibull failure model when a given number of ordered observations have been obtained. The general procedure may be inferred from this particular one.

Let $y_1,\ldots,y_k$ be the first ordered observations from $n \geq k$ independent machine generated exponential variates with unit mean. Recall, for the Weibull model, $R(x) = e^{-x}$. Solve for u in the equation $\chi(u) = 0$ where

$$\chi(u) = \frac{N'(u)}{N(u)} - \frac{1}{u} - \frac{1}{k}\sum_{i}^{k} \ell n y_i$$

and

$$N(u) = \sum_{i=1}^{k} y_i^u + (n-k)y_k^u$$

by using Newton-Raphson iteration procedure.

Now compute for u_1 determined as a solution of $X(u) = 0$ the value

$$v_1 = \left[\frac{N(u_1)}{k}\right]^{\frac{1}{u_1}}$$

The value (u_1, v_1) is one observation of (U,V). This process can be repeated m times from independent samples of y's to obtain an independent sample of size m from (U,V). The value of m can be made sufficiently large to determine the joint distribution of U,V (or the marginals) to a degree of accuracy limited by the machine procedure that generates random numbers. The desired percentiles of this empiric distribution are then tabulated, which can then be used to obtain confidence intervals or regions in the obvious way. We do not pursue this matter farther because of the similarity with work done previously in [6]. But as an example we present the empirical marginal distributions of U and V in Figures 1 and 2, respectively, for $k = 3$, $n = 5$ with $m = 4500$. Time for both computations was 5.67 minutes on the IBM 360.

The generated random numbers recommended here, and used in [2], are of the type called composite congruential generators. These second generation methods appear to be better, that is they satisfy more stringent statistical tests of randomness, than those of the simple congruential generators used previously. In the particular method utilized, three generators are mixed for the IBM 360 each of which will produce a full period of residues relatively prime to the modulus 2^{32}. Consequently, these mixed generators will produce 2^{30} distinct random numbers before repeating. This method is presented in detail and its practical advantage discussed in [8]. To obtain our exponential observations with unit mean we merely take the negative of the natural logarithm of the uniformly distributed observations generated by the

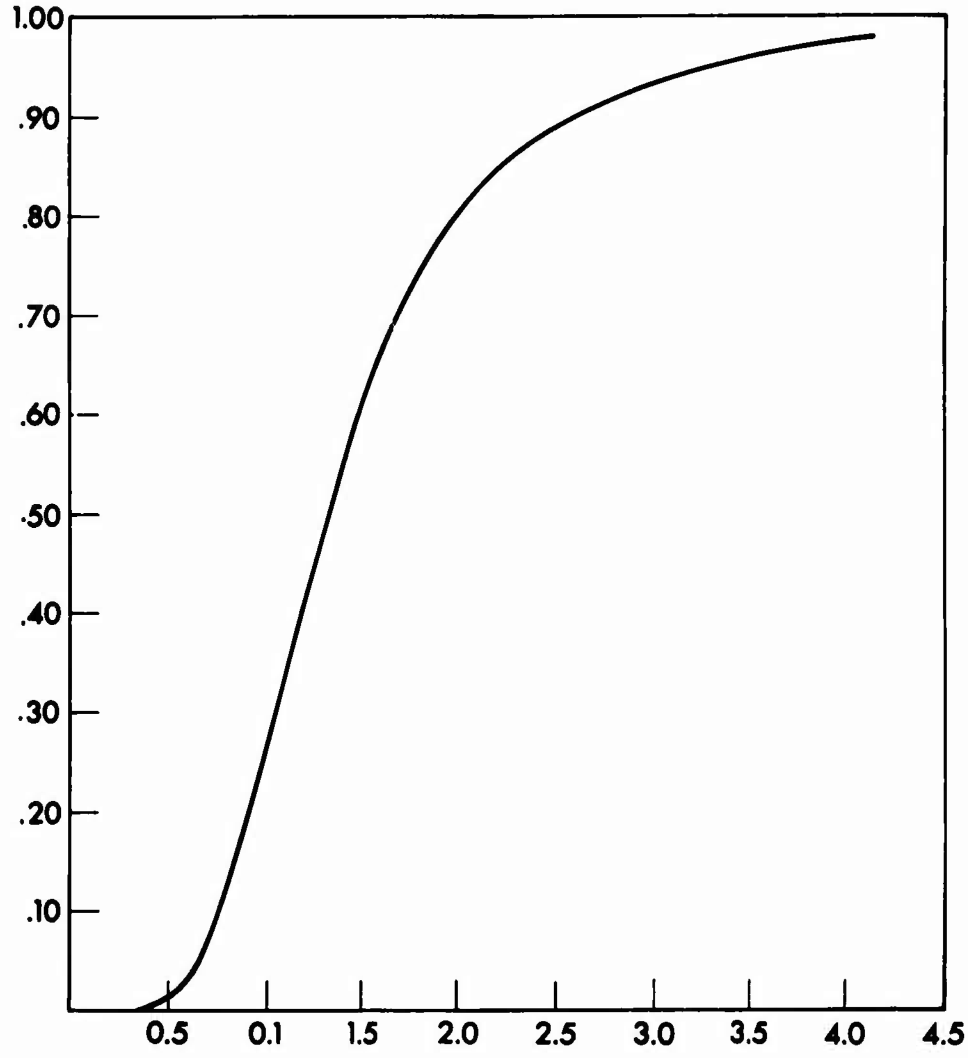

Figure 1. Distribution of $U = \hat{\alpha}/\alpha$, where α is the shape parameter of the Weibull distribution and $\hat{\alpha}$ is the maximum likelihood estimate based on the first three observations failing out of five.

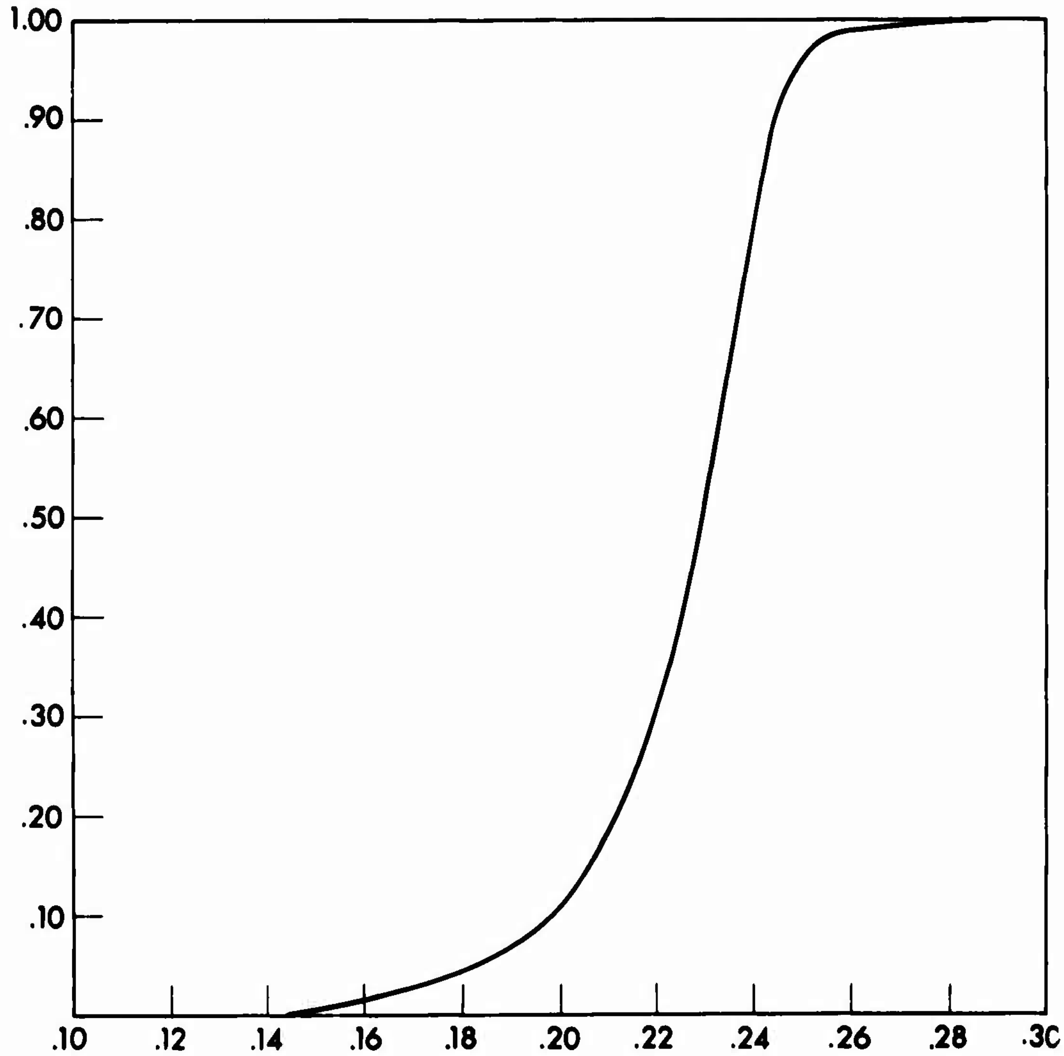

Figure 2. Distribution of $V = (\hat{\beta}/\beta)^{\alpha}$, where α, β are the shape and scale parameters of the Weibull distribution and the maximum likelihood estimate based on the first three observations failing out of five.

mixed congruential method.

We now outline the construction of confidence contours for the Weibull distribution by supposing ϕ was fixed and finding the appropriate a,b from the two-dimensional subset of C so that either (3.7), (4.7) or (4.9) is satisfied. Recall that $T = U\ell nV$ and let

$$r(a,b) = P[T \le a + \omega_\phi(bU)]$$

or we might replace T by $|T|$ as in (4.10).

A feasible procedure then is to take a mesh of values in the half plane $a,b > 0$ and on the basis of our sample $(U_1,V_1,\ldots,U_m,V_m)$ calculated from the appropriate random number generators, compute the relative frequencies of the occurrence of the appropriate events. We tabulate

$$r(a_i,b_j) = \frac{1}{m}\sum_{k=1}^{m} \{T_k \le a_i + \omega_\phi(b_j U_k)\}$$

where as before $\{\pi\}$ is the indicator of the relation π being one if true and zero otherwise. From the evidence of these results we may wish to interpolate to find more appropriate values and/or, as we proceed, to increase the sample size m and/or refine the mesh.

As a test case we chose

$$\phi(x) = \begin{cases} e^x - 1 & \text{for } x \ge 0 \\ -\ell n(1-x) & \text{for } x \le 0 \end{cases}$$

from which

$$\omega_\phi(u) = \begin{cases} u - 1 - u\ell nu & \text{if } u \ge 1 \\ 1 - u + \ell nu & \text{if } u \le 1 \end{cases}$$

A mesh, presented in Table I, for the range of values

$$a = -.6(.1)-.3 \quad \text{and} \quad .4(.1).8$$

$$b = .25(.25)2.75$$

with $k = 3$, $n = 5$ and $m = 4{,}500$ was computed on the IBM 360 in 5.09 minutes. (It was repeated to verify that all entries were accurate to two significant figures.) As an example the entry $a = .8$, $b = .25$ has $r(a,b) = .9484$ thus

$$\eta^{-1}(y) = \begin{cases} b\ell n(1 + y - a) & a \le y < \infty \\ b(1 - e^{a-y}) & -\infty < y \le a \end{cases}$$

can be used in conjunction with (3.12) to obtain an upper confidence contour of level .95. A graph is presented in Figure 3.

Table I

M=4500 N=5 K=3

b \ a	-0.60	-0.5	-0.40	-0.30	0.40	0.50	0.60	0.70	0.80
2.75	.009	.013	.018	.028	.518	.555	.578	.601	.619
2.50	.012	.016	.021	.035	.542	.575	.601	.624	.646
2.25	.017	.023	.031	.044	.568	.600	.626	.651	.671
2.00	.025	.032	.041	.056	.597	.634	.659	.680	.702
1.75	.035	.044	.054	.073	.636	.666	.694	.716	.733
1.50	.049	.058	.070	.092	.675	.709	.729	.749	.765
1.25	.071	.080	.094	.120	.722	.746	.772	.791	.807
1.00	.092	.104	.124	.152	.773	.799	.817	.831	.844
0.75	.127	.149	.160	.196	.826	.845	.858	.870	.879
0.50	.162	.180	.208	.247	.870	.885	.896	.908	.916
0.25	.204	.227	.260	.303	.914	.926	.935	.943	.948

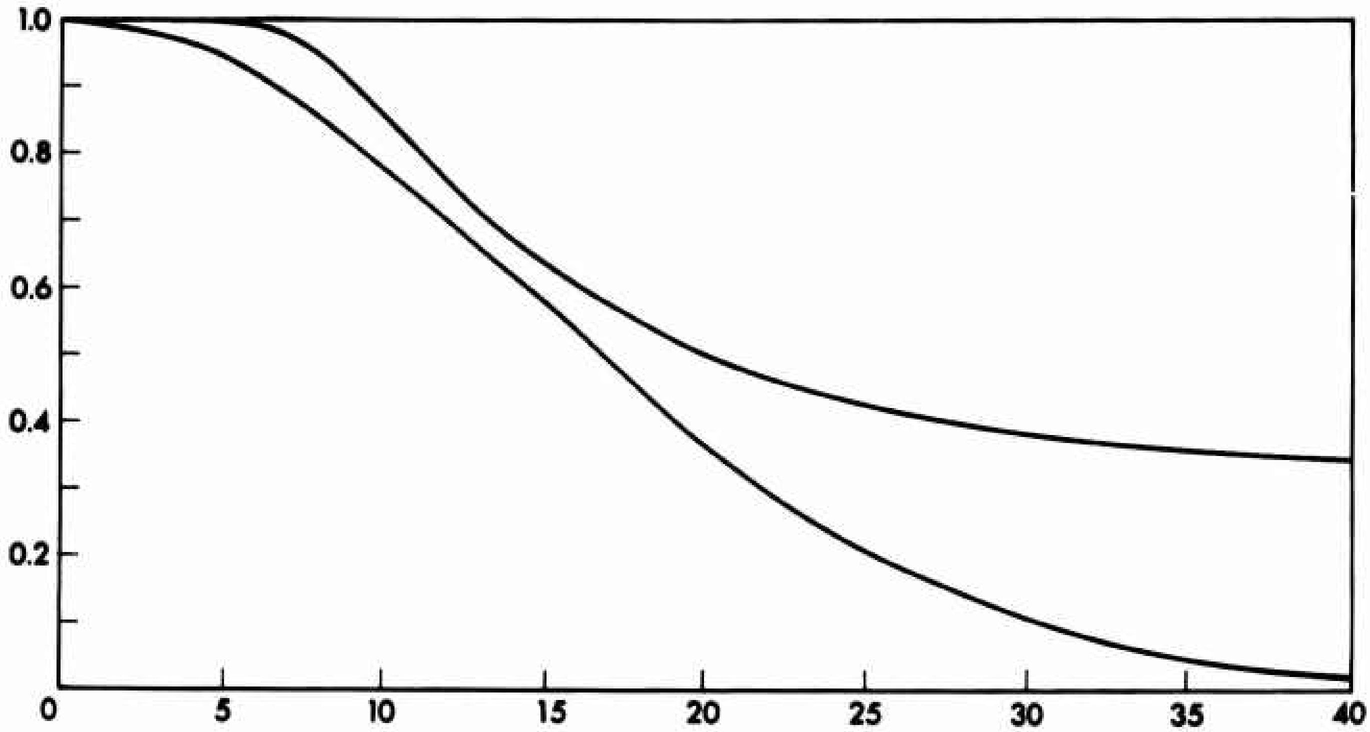

Figure 3. Upper confidence contour of level .95 for a Weibull distribution with $\alpha = 2$, $\beta = 20$ using $\hat{\alpha} = 1.776$, $\hat{\beta} = 21.36$ calculated from the first three ordered observations out of five, with contour function ϕ defined by $\phi(x) = e^x - 1$ for $x \geq 0$, $\phi(x) = -\ell n(1-x)$ for $x \leq 0$.

REFERENCES

[1] Bain, Lee J. and Antle, Charles E., 1967, Estimation of Parameters in the Weibull Distribution, *Technometrics*, 9, 621-627.

[2] Besuner, P. M. and Whittaker, I, C., 1969, A Reliability Analysis Approach to Fatigue Life Variability of Aircraft Structures, Air Force Materials Laboratory Technical Report, AFML-TR-69-65.

[3] Birnbaum, Z. W., 1962, Introduction to Probability and Mathematical Statistics, 278-283, Harper & Brothers, New York.

[4] Cohen, A. Clifford, Jr., 1965, Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and Censored Samples, *Technometrics*, 7, 579-588.

[5] David, F. N. and Johnson, N. L., 1948, The Probability Integral Transformation When Parameters Are Estimated from the Sample, *Biometrika*, 35, 182-190.

[6] Johns, M. W. and Liebermann, J. G., 1966, An Exact Asymptotically Efficient Confidence Bound for Reliability in the Case of the Weibull Distribution, *Technometrics*, 8, 135-175.

[7] Mann, Nancy R., 1968, Point and Interval Estimation Procedures for the Two-Parameter Weibull and the Extreme Value Distributions, *Technometrics*, 10, 231-256.

[8] Marsaglia, G. and Bray, T. A., 1968, One-Line Random Number Generators and Their Use in Combinations, *Communications of the*

ACM, 11, 757-759.

[9] Saunders, S. C., 1962, On the Use of a Distribution-Free Property in Determining a Transformation of One Variate Such That It Will Exceed Another with a Given Probability, *Sankhyā*, A-24, 239-250.

[10] "Student", 1908, The Probable Error of the Mean, *Biometrika*, 6, 1-25.

[11] Thoman, Darrel R. and Bain, Lee J., 1969, Two Sample Tests in the Weibull Distribution, *Technometrics*, 11, 805-815.

[12] Thoman, D. R., Bain, L. J. and Antle, C. E., 1969, Inferences on the Parameters of the Weibull Distribution, *Technometrics*, 11, 445-460.