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Abstract

Air Force requirements, such as the knowledge of the upper atmosphere environment of vehicles and the knowledge of the propagation characteristics of radio and radar signals, require the solutions of the equations of motion of fluid dynamics and of electromagnetic theory which are often very complicated. This report presents a new mathematical approach to the obtaining of such solutions. The vector field is represented in such a form that new techniques may be used to find the appropriate solutions. Some problems of fluid dynamics and electromagnetic theory are solved as an illustration of the new approach. In later reports, new techniques will be used in other problems related to Air Force needs.

In this report, eigenfunctions of the curl operator are introduced. The expansion of vector fields in terms of these eigenfunctions leads to a decomposition of such fields into three modes, one of which corresponds to an irrotational vector field, and two of which correspond to rotational circularly vector fields of opposite signs of polarization. Under a rotation of coordinates, the three modes which are introduced in this fashion remain invariant. Hence the Helmholtz decomposition of vector fields has been introduced in an irreducible, rotationally invariant form.

These expansions enable one to handle the curl and divergence operators simply. As illustrations of the use of the curl eigenfunctions, four problems are solved.

Contents

Eigenfunctions of the Curl Operator, Rotationally Invariant Helmholtz Theorem, and Applications to Electromagnetic Theory and Fluid Dynamics

I. INTRODUCTION

In problems which involve vector fields, the curl and divergence operators play a central role. For example, in electromagnetic theory, Maxwell's equations are expressed in terms of these operators, while in fluid dynamics it is convenient to decompose the velocity vector field into two parts: one for which the curl is zero, and the other for which the divergence is zero. In this report, eigenfunctions of the curl operator are introduced which considerably simplify problems involving vector fields, particularly when these fields are defined over the entire coordinate space. Vector fields in terms of these eigenfunctions will be expanded, and will show that each vector of a vector field can be expressed as a sum of three vectors: one is an irrotational vector, and the other two are rotational vectors which are circularly polarized and have opposite signs of polarization. Furthermore, this decomposition is rotationally invariant because in a frame of reference obtained from the original one by a rotation, the three vectors, into which the rotated vector is split, each maps separately into the corresponding vector into which the original vector was decomposed. That is, the irrotational vector goes into the irrotational vector, while each of the two rotational circularly-polarized vectors go into the corresponding circularly-polarized vector. This allows improvement upon the

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original Helmholtz decomposition of a vector into a rotational and irrotational part. Vector and scalar potentials will also be introduced in a rotationally invariant manner such that the irrotational mode is obtained from a scalar potential, and each of the circularly-polarized rotational modes is obtained as the curl of a corresponding vector potential. It will also be shown how to construct these vector and scalar potentials from the vector.

In addition to giving the expansion of the vectors in terms of cartesian variables for the arguments of the vector functions, expansions will be given in terms of spherical coordinates. In this case, the modes are conveniently expressed in terms of the vector spherical harmonics. This expansion will obviously be useful for cases of spherical symmetry.

The initial value problem for Maxwell's equations will be solved when the sources and currents are given functions of time and space, and will show how the use of the curl eigenfunctions yields in a very simple manner the uncoupling of the longitudinal and transverse modes. The velocity vector fields of fluid mechanics can also be interpreted in terms of these modes and show that the irrotational modes of the vector field correspond to irrotational motion of the fluid, while the circularly-polarized components correspond to the rotational motion of the fluid. The vorticity can be related simply to these circularly-polarized components. Such concepts will be used to obtain families of new exact solutions of the incompressible Navier-Stokes equations and solutions in the theory of linearized compressible viscous flow. Exact solutions will also be obtained for vertical shear motion in an ocean on the rotating earth.

The eigenfunctions of the curl and their properties were obtained by considering helicity representations of the rotation group. The general properties of such representations have been discussed in great detail by Moses (1970). The method by which the eigenfunctions were obtained is given in Appendix A. Having obtained the eigenfunctions, however, it is a straightforward matter to verify their properties by direct computation, and this verification will be indicated in the body of this report.

1. THE $\underset{\sim}{Q}_\lambda$ VECTORS

Let the variable λ take on the three values $\lambda = 1, 0, -1$, and $\underset{\sim}{\eta}$ be any unit vector. Then for every $\underset{\sim}{\eta} = (\eta_1, \eta_2, \eta_3)$ the three vectors $\underset{\sim}{Q}_\lambda(\underset{\sim}{\eta}) = [Q_{1\lambda}(\underset{\sim}{\eta}), Q_{2\lambda}(\underset{\sim}{\eta}), Q_{3\lambda}(\underset{\sim}{\eta})]$ are defined by

$$Q_0(\eta) = -\eta \ ,$$

$$Q_\lambda(\eta) = -\lambda(2)^{-1/2}\left[\frac{\eta_1(\eta_1+i\lambda\eta_2)}{1+\eta_3} - 1,\ \frac{\eta_2(\eta_1+i\lambda\eta_2)}{1+\eta_3} - i\lambda,\ \eta_1+i\lambda\eta_2\right], \tag{1}$$

for $\lambda = \pm 1$.

The following properties of Q_λ are easily verified:

$$\begin{aligned} &Q_\lambda^*(\eta)\cdot Q_\mu(\eta) = \delta_{\lambda\mu} \ , \\ &\sum_\lambda Q_{i\lambda}^*(\eta)Q_{j\lambda}(\eta) = \delta_{ij} \ , \\ &\eta\cdot Q_\lambda(\eta) = 0 \quad \text{for } \lambda = \pm 1 \ , \\ &\eta\times Q_\lambda(\eta) = -i\lambda Q_\lambda(\eta) \ . \end{aligned} \tag{2}$$

The above equations show that the unit vectors Q_λ span the three-dimensional vector space for each vector η .

The vectors Q_λ were introduced previously (in a somewhat different notation) as a means of reducing the electromagnetic fields and the electromagnetic vector potential to the irreducible representations of the Poincaré group (Moses, 1966a). This report extends the applicability of these vectors.

The following properties of Q_λ are useful when it is desired to impose reality conditions upon the vector fields:

$$\begin{aligned} &Q_\lambda^*(\eta) = -Q_{-\lambda}(\eta) \quad \text{for } \lambda = \pm 1, \\ &Q_0^*(\eta) = Q_0(\eta) \ . \end{aligned} \tag{3}$$

Also

$$Q_\lambda^*(-\eta) = -\left(\frac{\eta_1 - i\lambda\eta_2}{\eta_1 + i\lambda\eta_2}\right) Q_\lambda(\eta) \ . \tag{4}$$

Let p be any vector. Then $Q_\lambda(p)$ is defined by

$$Q_\lambda(p) \equiv Q_\lambda(\eta) \ . \tag{5}$$

where

$$\eta = \frac{\mathbf{p}}{p}, \quad p = |\mathbf{p}| . \tag{5a}$$

Then

$$Q_0(\mathbf{p}) = -\frac{\mathbf{p}}{p} ,$$

$$Q_\lambda(\mathbf{p}) = -\lambda(2)^{-1/2}\left[\frac{p_1(p_1+i\lambda p_2)}{p(p+p_3)} - 1, \ \frac{p_2(p_1+i\lambda p_2)}{p(p+p_3)} - i\lambda, \ \frac{p_1+i\lambda p_2}{p}\right], \tag{6}$$

for $\lambda = \pm 1$.

It will now be convenient to indicate how the vectors Q_λ behave under rotations of the frame of reference. As is well known, every rotation of a frame of reference can be described by means of a vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ where the direction of $\boldsymbol{\theta}$ gives the axis of rotation and $\theta = |\boldsymbol{\theta}|$ is the angle of rotation ($0 \leq \theta \leq \pi$). If a vector $\mathbf{x}$ has the three components x_i in one frame of reference, in a rotated frame it will have the components x_i' where

$$x_i' = \sum_j R_{ij}(\boldsymbol{\theta}) x_j , \tag{7}$$

and $R_{ij}(\boldsymbol{\theta})$ are the matrix elements of the rotation matrix $R(\boldsymbol{\theta})$.

$$R_{ij}(\boldsymbol{\theta}) = \delta_{ij}\cos\theta + \left[\frac{(1-\cos\theta)}{\theta^2}\right]\theta_i\theta_j + \sum_k \epsilon_{ijk}\theta_k \frac{\sin\theta}{\theta} , \tag{8}$$

where ϵ_{ijk} is the usual anti-symmetric tensor:

$$\epsilon_{jik} = \epsilon_{ikj} = -\epsilon_{ijk}, \quad \epsilon_{123} = 1 . \tag{8a}$$

It is convenient to define the vector $\mathbf{x}'$ as being the vector whose components in the unrotated frame are given by x_i' which are the components of $\mathbf{x}$ in the rotat. frame. Then Eq. (7) may be written as

$$\mathbf{x}' = R(\boldsymbol{\theta})\mathbf{x} . \tag{7a}$$

Let $\underset{\sim}{u}(\underset{\sim}{x})$ be a vector function of $\underset{\sim}{x}$. Then under a rotation parametrized by $\underset{\sim}{\theta}$, the components of this vector in the rotated frame will be components of the vector $\underset{\sim}{u}'(\underset{\sim}{x}')$ given by

$$\underset{\sim}{u}'(\underset{\sim}{x}) = R(\underset{\sim}{\theta})\underset{\sim}{u}[R(-\underset{\sim}{\theta})\underset{\sim}{x}] . \tag{7b}$$

In a similar way, one can define vector functions of the vector $\underset{\sim}{p}$ and give their transformation properties in terms of the vector $\underset{\sim}{\theta}$ which describes a rotation of coordinates. The vectors $\underset{\sim}{Q}_\lambda(\underset{\sim}{p})$ may be regarded as being vector functions of $\underset{\sim}{p}$. In a certain sense they are "eigenfunctions" of rotations. They satisfy the relation

$$R(\underset{\sim}{\theta})\underset{\sim}{Q}_\lambda[R(-\underset{\sim}{\theta})\underset{\sim}{p}] = \exp\{2i\lambda\Phi(\underset{\sim}{\theta},\underset{\sim}{\eta})\}\underset{\sim}{Q}_\lambda(\underset{\sim}{p}) , \tag{9}$$

where

$$\underset{\sim}{\eta} = \frac{\underset{\sim}{p}}{p} , \tag{9a}$$

and $\Phi(\underset{\sim}{\theta},\underset{\sim}{\eta})$ is the principal branch of

$$\tan\Phi(\underset{\sim}{\theta},\underset{\sim}{\eta}) = \frac{(\underset{\sim}{\theta}\cdot\underset{\sim}{\eta} + \theta_3)\tan(\theta/2)}{\theta(1+\eta_3) + (\underset{\sim}{\theta}\times\underset{\sim}{\eta})_3 \tan(\theta/2)} . \tag{9b}$$

The proof of Eqs. (9) and (9b) will be given in Section 5, where additional properties of the vectors $\underset{\sim}{Q}_\lambda$ will be given.

3. EIGENFUNCTIONS OF THE CURL OPERATOR, EXPANSION THEOREMS, AND INVARIANT HELMHOLTZ THEOREM

Let the vectors $\underset{\sim}{\chi}_\lambda(\underset{\sim}{x}|\underset{\sim}{p})$ be defined by

$$\chi_\lambda(\underset{\sim}{x}|\underset{\sim}{p}) = (2\pi)^{-3/2}e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\underset{\sim}{Q}_\lambda(\underset{\sim}{p}) . \tag{10}$$

Then from the properties of the Fourier transformation and the completeness and orthogonality properties of the vectors Q_λ in Eq. (2), the vectors $\underset{\sim}{\chi}_\lambda(\underset{\sim}{x}|\underset{\sim}{p})$ satisfy the following orthogonality and completeness relations:

$$\int d\underset{\sim}{x}\, \underset{\sim}{x}_\lambda^*(\underset{\sim}{x}|\underset{\sim}{p})\cdot \underset{\sim}{x}_\mu(\underset{\sim}{x}|\underset{\sim}{p}') = \delta_{\lambda\mu}\delta(\underset{\sim}{p}-\underset{\sim}{p}') ,$$
$$\sum_\lambda \int d\underset{\sim}{p}\, x_{i\lambda}^*(\underset{\sim}{x}|\underset{\sim}{p})\, x_{j\lambda}(\underset{\sim}{x}'|\underset{\sim}{p}) = \delta_{ij}\delta(\underset{\sim}{x}-\underset{\sim}{x}') , \tag{11}$$

where $x_{i\lambda}$ denotes the i'th component of the vector $\underset{\sim}{x}_\lambda$

From Eq. (2) it is seen that

$$\underset{\sim}{\nabla}\times \underset{\sim}{x}_\lambda(\underset{\sim}{x}|\underset{\sim}{p}) = p\lambda\, \underset{\sim}{x}_\lambda(\underset{\sim}{x}|\underset{\sim}{p}),$$
$$\underset{\sim}{\nabla}\cdot \underset{\sim}{x}_\lambda(\underset{\sim}{x}|\underset{\sim}{p}) = 0 \quad \text{for} \quad \lambda = \pm 1 , \tag{12}$$
$$\underset{\sim}{\nabla}\cdot \underset{\sim}{x}_0(\underset{\sim}{x}|\underset{\sim}{p}) = -ip(2\pi)^{-3/2}e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}$$

The first part of Eq. (12) shows that the vectors $\underset{\sim}{x}_\lambda$ are eigenfunctions of the curl operator with eigenvalue λp. The variable λ itself may be considered to be the eigenvalue of the operator $(-\underset{\sim}{\nabla}^2)^{-1/2}\underset{\sim}{\nabla}\times$ when this operator is properly interpreted.

The completeness and orthogonality relations of Eq. (11) enable us to expand any vector function $\underset{\sim}{u}(\underset{\sim}{x})$, subject to very general restrictions, as follows:

$$\underset{\sim}{u}(\underset{\sim}{x}) = \sum_\lambda \int d\underset{\sim}{p}\; \underset{\sim}{x}_\lambda(\underset{\sim}{x}|\underset{\sim}{p})g_\lambda(\underset{\sim}{p})$$
$$= \sum_\lambda \underset{\sim}{u}_\lambda(\underset{\sim}{x}) \tag{13}$$

where

$$\underset{\sim}{u}_\lambda(\underset{\sim}{x}) = \int d\underset{\sim}{p}\; \underset{\sim}{x}_\lambda(\underset{\sim}{x}|\underset{\sim}{p})\, g_\lambda(\underset{\sim}{p}) . \tag{13a}$$

Eqs. (13) and (13a) constitute our sharpening of the Helmholtz decomposition theorem, for from Eq. (2)

$$\underset{\sim}{\nabla}\cdot \underset{\sim}{u}_\lambda(\underset{\sim}{x}) = 0 \quad \text{for } \lambda = \pm 1,$$
$$\underset{\sim}{\nabla}\times \underset{\sim}{u}_0(\underset{\sim}{x}) = 0 . \tag{14}$$

One can also introduce vector and scalar potentials. For $\lambda = \pm 1$, the vector potentials $\underset{\sim}{A}_\lambda(\underset{\sim}{x})$ and the scalar potential $V(\underset{\sim}{x})$ are introduced. First note that the coefficients $g_\lambda(\underset{\sim}{p})$ of the expansion Eq. (13) can be obtained from $\underset{\sim}{u}(\underset{\sim}{x})$ as follows:

$$g_\lambda(\underset{\sim}{p}) = \int \underset{\sim}{\chi}_\lambda{}^*(\underset{\sim}{x}|\underset{\sim}{p}) \cdot \underset{\sim}{u}(\underset{\sim}{x})\, d\underset{\sim}{x} \,. \tag{13b}$$

Then

$$\underset{\sim}{A}_\lambda(\underset{\sim}{x}) = \lambda \int \left(\frac{d\underset{\sim}{p}}{p}\right) \underset{\sim}{\chi}_\lambda(\underset{\sim}{x}|\underset{\sim}{p})\, g_\lambda(\underset{\sim}{p}) \;,$$

$$V(\underset{\sim}{x}) = i(2\pi)^{-3/2} \int \left(\frac{d\underset{\sim}{p}}{p}\right) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, g_0(\underset{\sim}{p}) \,. \tag{15}$$

From Eq. (12) it follows that

$$\underset{\sim}{u}_\lambda(\underset{\sim}{x}) = \underset{\sim}{\nabla} \times \underset{\sim}{A}_\lambda(\underset{\sim}{x}), \quad \text{for } \lambda = \pm 1$$

$$\underset{\sim}{u}_0(\underset{\sim}{x}) = \underset{\sim}{\nabla} V(\underset{\sim}{x}) \,. \tag{16}$$

Thus the original Helmholtz theorem has been sharpened in two ways. First, two irrotational vectors have been introduced in the decomposition of a general vector, each of which is the curl of its own vector potential. Second, a procedure has been given for obtaining the vector and scalar potentials when the vector $\underset{\sim}{u}(\underset{\sim}{x})$ is given. Now it will be shown how our formulation is rotationally invariant.

Under the rotation of axes described by the vector $\underset{\sim}{\theta}$, the components of the vectors $\underset{\sim}{u}_\lambda(\underset{\sim}{x})$ are given in the new frame by the components of $\underset{\sim}{u}_\lambda{}'(\underset{\sim}{x})$, where

$$\underset{\sim}{u}_\lambda{}'(\underset{\sim}{x}) = R(\underset{\sim}{\theta})\,\underset{\sim}{u}_\lambda[R(-\underset{\sim}{\theta})\underset{\sim}{x}] \;, \tag{17}$$

See Eqs. (7a) and (7b).

On using Eq. (9), note that $g_\lambda{}'(\underset{\sim}{p})$, which replaces $g_\lambda(\underset{\sim}{p})$ in the expansion Eqs. (13) and (13a), is related to $g_\lambda(\underset{\sim}{p})$ by

$$g_\lambda{}'(\underset{\sim}{p}) = \exp\{2i\lambda\,\Phi(\underset{\sim}{\theta}, \underset{\sim}{\eta})\}\, g_\lambda[R(-\underset{\sim}{\theta})\underset{\sim}{p}] \;, \tag{18}$$

where the function $\Phi(\underset{\sim}{\theta}, \underset{\sim}{\eta})$ is given by Eq. (9b).

Thus, except for a phase, the decomposition given by Eqs. (13), (13a), and (13b) is invariant. Vectors belonging to values of λ do not mix, unlike the situation for the cartesian components.

The scalar potential and the vector potentials, which in the rotated frame are denoted by $V'(\underset{\sim}{x})$ and $A'(\underset{\sim}{x})$ are also uncoupled:

$$V'(\underset{\sim}{x}) = V[\, R(-\underset{\sim}{\theta})\underset{\sim}{x}\,]$$

$$\underset{\sim}{A}_\lambda{}'(\underset{\sim}{x}) = R(\underset{\sim}{\theta})\, \underset{\sim}{A}_\lambda [R(-\underset{\sim}{\theta})\underset{\sim}{x}] \ . \tag{19}$$

Vector and scalar potentials do not mix. Furthermore, there are no ambiguities in gauge in the present invariant set-up.

Now it will be shown how the requirement that $u(\underset{\sim}{x})$ be real affects the expansion Eq. (13) and puts restrictions on the functions $g_\lambda(\underset{\sim}{p})$. From Eq. (4) it is seen that a necessary and sufficient condition that the vector $\underset{\sim}{u}(\underset{\sim}{x})$ be real is

$$g_\lambda(\underset{\sim}{p}) = -\left(\frac{p_1 - i\lambda p_2}{p_1 + i\lambda p_2}\right) g_\lambda{}^*(-\underset{\sim}{p}) \ . \tag{20}$$

When the vector $\underset{\sim}{u}(\underset{\sim}{x})$ is real, it will be convenient to replace the expansions Eqs. (13), (13a) and (15) by

$$\underset{\sim}{u}(\underset{\sim}{x}) = \frac{1}{2}\left\{\sum_\lambda \int d\underset{\sim}{p}\ \underset{\sim}{\chi}_\lambda(\underset{\sim}{x}|\underset{\sim}{p}) g_\lambda(\underset{\sim}{p}) + \sum_\lambda \int d\underset{\sim}{p}\ \underset{\sim}{\chi}_\lambda{}^*(\underset{\sim}{x}|\underset{\sim}{p}) g_\lambda{}^*(\underset{\sim}{p})\right\}$$

$$= \sum_\lambda \underset{\sim}{u}_\lambda(\underset{\sim}{x}) \ , \tag{13c}$$

where

$$\underset{\sim}{u}_\lambda(\underset{\sim}{x}) = \frac{1}{2}\left\{\int d\underset{\sim}{p}\ \underset{\sim}{\chi}_\lambda(\underset{\sim}{x}|\underset{\sim}{p}) g_\lambda(\underset{\sim}{p}) + \int d\underset{\sim}{p}\ \underset{\sim}{\chi}_\lambda{}^*(\underset{\sim}{x}|\underset{\sim}{p}) g_\lambda{}^*(\underset{\sim}{p})\right\} \ . \tag{13d}$$

Also

$$\underset{\sim}{A}_\lambda(\underset{\sim}{x}) = \frac{\lambda}{2}\left\{\int \left(\frac{d\underset{\sim}{p}}{p}\right) \underset{\sim}{\chi}_\lambda(\underset{\sim}{x}|\underset{\sim}{p}) g_\lambda(\underset{\sim}{p}) + \int\left(\frac{d\underset{\sim}{p}}{p}\right) \underset{\sim}{\chi}_\lambda{}^*(\underset{\sim}{x}|\underset{\sim}{p}) g_\lambda{}^*(\underset{\sim}{p})\right\},$$

$$V(\underset{\sim}{x}) = \frac{i}{2}(2\pi)^{-3/2}\left\{\int\left(\frac{d\underset{\sim}{p}}{p}\right) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} g_0(\underset{\sim}{p}) - \int\left(\frac{d\underset{\sim}{p}}{p}\right) e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}} g_0{}^*(\underset{\sim}{p})\right\}. \tag{15a}$$

4. APPLICATIONS

This section gives some applications of the use of the curl eigenfunctions, and shows how the modes that have been introduced uncouple in many cases.

4.1 Application to Maxwell's Equations

4.1.1 THE INITIAL VALUE PROBLEM IN THE USUAL FORM.

In terms of Gaussian units, Maxwell's Equations are:

$$\nabla \times H(x, t) = \frac{4\pi}{c} j(x, t) + \frac{1}{c} \frac{\partial E(x, t)}{\partial t} , \tag{21}$$

$$\nabla \times E(x, t) = -\frac{1}{c} \frac{\partial H(x, t)}{\partial t} , \tag{22}$$

$$\nabla \cdot E(x, t) = 4\pi\rho(x, t) , \tag{23}$$

$$\nabla \cdot H(x, t) = 0 . \tag{24}$$

The current density $j(x, t)$ and the charge density $\rho(x, t)$ are taken as given functions of x and t, and it is assumed that the dielectric constant $\epsilon = 1$ and the magnetic permeability $\mu = 1$.

Then the general initial value problem posed by these equations will be solved. Actually, in terms of the present notation it will be shown how the modes uncouple. In paragraph 4.1.2, a more convenient, though less conventional, notation will be introduced, and then the solution of the initial value problem will be carried out to the end.

Expansion Eq. (13) for the vectors E and H may be written as

$$H(x, t) = \sum_\lambda \int dp \; \chi_\lambda(x \mid p) h_\lambda(p, t)$$

$$E(x, t) = \sum_\lambda \int dp \; \chi_\lambda(x \mid p) f_\lambda(p, t) , \tag{25}$$

where the functions $h_\lambda(p, t)$ and $f_\lambda(p, t)$ are regarded as unknown functions of their arguments.

Also, the known current density j is expanded as

$$j(x, t) = \sum_\lambda \int dp \; \chi_\lambda(x \mid p) k_\lambda(p, t) , \tag{26}$$

where the functions $k_\lambda(\underset{\sim}{p}, t)$ are known from Eq. (13b).

The scalar ρ is expanded in terms of an ordinary Fourier transformation as follows:

$$\rho(\underset{\sim}{x}, t) = (2\pi)^{-3/2} \int d\underset{\sim}{p}\; e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, r(\underset{\sim}{p}, t)\,, \tag{27}$$

where the function $r(\underset{\sim}{p}, t)$ is also known.

Substitute Eqs. (25) and (27) into Eqs. (23) and (24). On using Eq. (12), one obtains

$$-i(2\pi)^{-3/2} \int d\underset{\sim}{p}\; p\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, h_0(\underset{\sim}{p}, t) = 0\,,$$

$$-i(2\pi)^{-3/2} \int d\underset{\sim}{p}\; p\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, f_0(\underset{\sim}{p}, t) = (2\pi)^{-3/2} \int d\underset{\sim}{p}\; e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, r(\underset{\sim}{p}, t)\,. \tag{28}$$

Thus the longitudinal components of the electromagnetic field can be solved immediately:

$$h_0(\underset{\sim}{p}, t) \equiv 0\,,$$

$$f_0(\underset{\sim}{p}, t) = \frac{4\pi i}{p}\, r(\underset{\sim}{p}, t)\,. \tag{29}$$

Similarly, on using Eq. (25) and Eq. (26) in Eqs. (21) and (22), and using the fact that the vectors $\underset{\sim}{\chi}_\lambda(\underset{\sim}{x} \mid \underset{\sim}{p})$ are linearly independent,

$$\frac{\partial f_0(\underset{\sim}{p}, t)}{\partial t} = -4\pi\, k_0(\underset{\sim}{p}, t)\,, \tag{30}$$

and

$$\lambda p f_\lambda(\underset{\sim}{p}, t) = -\frac{1}{c}\, \frac{\partial h_\lambda(\underset{\sim}{p}, t)}{\partial t}\,, \quad \lambda = \pm 1\,, \tag{31}$$

$$\lambda p h_\lambda(\underset{\sim}{p}, t) = \frac{4\pi}{c}\, k_\lambda(\underset{\sim}{p}, t) + \frac{1}{c}\, \frac{\partial f_\lambda(\underset{\sim}{p}, t)}{\partial t}\,, \quad \lambda = \pm 1\,. \tag{32}$$

Note that the modes are completely uncoupled. This result is in sharp contrast to the situation in which one uses simply the Fourier transform of the cartesian components of the vectors, where considerable coupling occurs.

From the second part of Eqs. (29) and from Eq. (30), one obtains a necessary condition for the solubility of Maxwell's Equations:

$$\frac{\partial r(\underset{\sim}{p}, t)}{\partial t} = ipk_0(\underset{\sim}{p}, t) . \quad (33)$$

This equation is simply the equation of continuity

$$\frac{\partial \rho(\underset{\sim}{x}, t)}{\partial t} + \underset{\sim}{\nabla} \cdot \underset{\sim}{j}(\underset{\sim}{x}, t) = 0 \quad (33a)$$

when it is written in terms of modes through the substitution of ρ and $\underset{\sim}{j}$ by Eqs. (26) and (27). It is interesting to note that only the lamellar part of the current density $\underset{\sim}{j}$ is determined by the charge density ρ.

Since Eq. (29) gives $h_0(\underset{\sim}{p}, t)$ and $f_0(\underset{\sim}{p}, t)$, one need only find $h_\lambda(\underset{\sim}{p}, t)$ and $f_\lambda(\underset{\sim}{p}, t)$ for $\lambda = \pm 1$. Equations (31) and (32) are a pair of ordinary differential equations which are of first order and which are coupled. They are easily solved using standard, well-known techniques. However, instead of carrying out the solution in the present notation, Maxwell's Equations will be written in terms of another notation which will give the results more directly. The objective in this section was to show the effectiveness of the use of the curl eigenfunctions without the use of notational simplifications. Furthermore, it is believed that the techniques used in this section are capable of being extended in various directions. We may go further into the matter in later reports.

4.1.2 THE SOLUTION OF THE INITIAL VALUE PROBLEM IN TERMS OF AN ALTERNATIVE NOTATION

Define the complex vector $\underset{\sim}{\Psi}(\underset{\sim}{x}, t)$ by

$$\underset{\sim}{\Psi}(\underset{\sim}{x}, t) = \underset{\sim}{E}(\underset{\sim}{x}, t) - i\underset{\sim}{H}(\underset{\sim}{x}, t). \quad (34)$$

In terms of $\underset{\sim}{\Psi}$, Maxwell's Equations, Eqs. (21) through (24) become

$$\underset{\sim}{\nabla} \times \underset{\sim}{\Psi}(\underset{\sim}{x}, t) = -\frac{i}{c}\frac{\partial \underset{\sim}{\Psi}(\underset{\sim}{x}, t)}{\partial t} - \frac{4\pi i}{c}\underset{\sim}{j}(\underset{\sim}{x}, t) , \quad (35)$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\Psi}(\underset{\sim}{x}, t) = 4\pi \rho(\underset{\sim}{x}, t) . \quad (36)$$

The expression $\underset{\sim}{\Psi}(\underset{\sim}{x}, t)$ is expanded as follows:

$$\underset{\sim}{\Psi}(\underset{\sim}{x}, t) = \sum_\lambda \int d\underset{\sim}{p}\, \underset{\sim}{X}_\lambda(\underset{\sim}{x}|\underset{\sim}{p}) g_\lambda(\underset{\sim}{p}, t) . \quad (37)$$

Upon substitution into Eqs. (35) and (36) along with Eqs. (26) and (27), one obtains

$$g_0(\underset{\sim}{p}, t) = \frac{4\pi i}{p} r(\underset{\sim}{p}, t),$$

$$-\frac{i}{c}\frac{\partial g_0(\underset{\sim}{p}, t)}{\partial t} = \frac{4\pi i}{c}\, k_0(\underset{\sim}{p}, t), \tag{38}$$

$$\frac{\partial g_\lambda(\underset{\sim}{p}, t)}{\partial t} - icp\lambda g_\lambda(\underset{\sim}{p}, t) = -4\pi k_\lambda(\underset{\sim}{p}, t), \text{ for } \lambda = \pm 1 .$$

The first part of Eqs. (38) gives the irrotational or longitudinal part of the electromagnetic field immediately. For the second part of Eq. (38) to be valid, the consistency condition Eq. (30) is required as in the earlier treatment. The last part of Eq. (38) is an ordinary differential equation which gives the transverse field. The general solution is

$$g_\lambda(\underset{\sim}{p}, t) = g_\lambda(\underset{\sim}{p}) e^{ic\lambda pt} - 4\pi e^{ic\lambda pt} \int_0^t e^{-ic\lambda pt'} k_\lambda(\underset{\sim}{p}, t')dt' , \tag{39}$$

where $g_\lambda(\underset{\sim}{p})$ is a constant of integration and is given by

$$g_\lambda(\underset{\sim}{p}) = g_\lambda(\underset{\sim}{p}, 0) . \tag{40}$$

The first term on the right of Eq. (39) represents the radiation field, and the second term represents the field due to the currents. To find the significance of the variable λ in the electromagnetic case, let us consider the pure radiation field. Accordingly, set $k_\lambda(\underset{\sim}{p}, t) \equiv 0$ in Eq. (39), and take $g_\lambda(\underset{\sim}{p})$ to be given by

$$g_\lambda(\underset{\sim}{p}) = \delta_{\lambda, 1}\delta(p_1)\delta(p_2)\delta(p_3 - k) , \tag{41}$$

where $k > 0$.

The electric field is given by

$$E_1(\underset{\sim}{x}, t) = (\ .\pi)^{-3/2}(2)^{-1/2}\cos(kct+kz),$$

$$E_2(\underset{\sim}{x}, t) = -(2\pi)^{-3/2}(2)^{-1/2}\sin(kct+kz), \tag{42}$$

$$E_3(\underset{\sim}{x}, t) = 0 .$$

The electromagnetic wave is clearly a wave moving in the negative z-direction and is circularly polarized in the direction of propagation. Let us now take

$$g_\lambda(\underset{\sim}{p}) = \delta_{\lambda,-1}\delta(p_1)\delta(p_2)\delta(p_3-k) . \qquad (43)$$

The electric field is given by

$$\begin{aligned} E_1(\underset{\sim}{x},t) &= -(2\pi)^{-3/2}(2)^{-1/2}\cos(kct-kz), \\ E_2(\underset{\sim}{x},t) &= (2\pi)^{-3/2}(2)^{-1/2}\sin(kct-kz) \\ E_3(\underset{\sim}{x},t) &= 0 . \end{aligned} \qquad (44)$$

This electromagnetic wave propagates along the positive z-axis with a circular polarization opposite to the direction of propagation. Hence, in our representation, the radiation field consists of a superposition of circularly-polarized waves travelling in various directions. Of course, by picking the function $g_\lambda(\underset{\sim}{p})$ appropriately, plane polarized waves can also be represented.

4.1.3 GREEN'S FUNCTIONS, AND AN INVERSE PROBLEM

By using Eq. (13b), one can express $g_\lambda(\underset{\sim}{p})$ in terms of $\underset{\sim}{\Psi}(\underset{\sim}{x},0)$. Similarly, $k_\lambda(\underset{\sim}{p},t)$ can be expressed in terms of the current density $\underset{\sim}{j}(\underset{\sim}{x},t)$. Then, from the substitution of Eq. (39) into Eq. (37), it is seen that $\underset{\sim}{\Psi}(\underset{\sim}{x},t)$ is an expression involving Green's functions in $\underset{\sim}{x},t$ space. For the sake of brevity, this expression will not be given in this report.

It is also possible to solve an inverse problem. Assume that the current density $\underset{\sim}{j}(\underset{\sim}{x},t)$ is identically zero for $t < 0$, is "switched on" during the interval $0 < t < T$, and is identically zero again for $t > T$. During the times that the current density if "off", the transverse part of the electromagnetic field will be a radiation field. One can use Eq. (39) to solve the direct problem, in which one prescribes the radiation field before the current is switched on and calculates the radiation field after the current is switched off. The current density is assumed to be known. In the inverse problem, one prescribes the radiation field before the current density is switched on and the radiation field after the current is switched off. One asks for the current density, during the time that it is "on", which will take the initial radiation field into the final radiation field. This problem has been shown to have non-unique solutions (Moses, 1958) using a different form for the electromagnetic field. The present notation seems more useful for solving the inverse problem. In particular, it appears easier to impose "physical" conditions - such as the requirement that the current density be expressible in terms of multipoles contained within a given volume - upon the inverse problem to yield a meaningful unique solution. Perhaps we shall discuss this problem in later reports.

4.2 Application to Fluid Dynamics

4.2.1 IRROTATIONAL AND ROTATIONAL VELOCITY MODES, VORTICITY MODES, INTERPRETATION OF VORTICITY MODES AS SHEAR VELOCITY MODES, AND STRATIFIED VORTICITY MODES

Let us now regard the vector $\underset{\sim}{u}(\underset{\sim}{x})$ which is decomposed into the curl modes of Eqs. (13) or (13c) as being the velocity vector of a fluid, using the Eulerian description. The mode $u_0(\underset{\sim}{x})$ is clearly the irrotational mode and can be obtained from a scalar potential $V(\underset{\sim}{x})$ as in Eq. (16). For the present, only the rotational modes $\underset{\sim}{u}_\lambda(\underset{\sim}{x})$ for which $\lambda = \pm 1$ will be discussed. From Eq. (13c) it is seen that the most general real rotational velocity field $\underset{\sim}{u}(\underset{\sim}{x})$ can be written, on absorbing some factors in $g_\lambda(\underset{\sim}{p})$,

$$\underset{\sim}{u}(\underset{\sim}{x}) = \frac{1}{2}\left\{\sum_{\lambda=\pm 1}\int d\underset{\sim}{p}\,\underset{\sim}{Q}_\lambda(\underset{\sim}{\eta})e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}g_\lambda(\underset{\sim}{p}) + \sum_{\lambda=\pm 1}\int d\underset{\sim}{p}\,\underset{\sim}{Q}_\lambda{}^*(\underset{\sim}{\eta})e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}}g_\lambda{}^*(\underset{\sim}{p})\right\} \qquad (45)$$

where

$$(\underset{\sim}{\eta} = \frac{\underset{\sim}{p}}{p}) .$$

Thus the general rotational velocity vector $\underset{\sim}{u}(\underset{\sim}{x})$ is a superposition of the velocities $\underset{\sim}{u}_\lambda(\underset{\sim}{x}|\underset{\sim}{\eta}, p, \alpha)$ where

$$\underset{\sim}{u}_\lambda(\underset{\sim}{x}|\underset{\sim}{\eta}, p, \alpha) = \underset{\sim}{Q}_\lambda(\underset{\sim}{\eta})e^{ip(\underset{\sim}{\eta}\cdot\underset{\sim}{x}+\alpha)} + \underset{\sim}{Q}_\lambda{}^*(\underset{\sim}{\eta})e^{-ip(\underset{\sim}{\eta}\cdot\underset{\sim}{x}+\alpha)}, \qquad (46)$$

In Eq. (46), $\underset{\sim}{\eta}$ is any unit vector, p any real number, and α any other real number.

The velocity $\underset{\sim}{u}_\lambda(\underset{\sim}{x}|\underset{\sim}{\eta}, p, \alpha)$ will be called a "vorticity mode" of helicity λ, direction $\underset{\sim}{\eta}$, wave length $(2\pi/p)$, and phase α. Note that in a single vorticity mode, the phase can be made zero by changing the origin of coordinates appropriately. To find the velocity field determined by a single vorticity mode, let us take the phase $\alpha = 0$, and assume that the unit vector $\underset{\sim}{\eta}$ points in the positive z direction; that is, $\underset{\sim}{\eta} = (0, 0, 1)$. Then

$$u_{\lambda 1}(\underset{\sim}{x}|\underset{\sim}{\eta}, p, 0) = (2)^{1/2}\lambda \cos pz ,$$

$$u_{\lambda 2}(\underset{\sim}{x}|\underset{\sim}{\eta}, p, 0) = -(2)^{1/2} \sin pz,$$

$$u_{\lambda 3}(\underset{\sim}{x}|\underset{\sim}{\eta}, p, 0) = 0 . \qquad (47)$$

Thus the velocity vector $\mathbf{u}_\lambda(\mathbf{x}|\boldsymbol{\eta}, p, \alpha)$ is transverse to the direction $\boldsymbol{\eta}$, is constant in magnitude, and rotates clockwise as one moves in the positive η direction of $\lambda = 1$ and rotates counter clockwise if $\lambda = -1$. Thus, if an axis is assigned to the direction $\boldsymbol{\eta}$ and if in each plane orthogonal to this axis the velocity vector is drawn from the axis, the locus of the tip of the vector will be a negative helix if $\lambda = 1$ as one moves in the positive direction on the axis, and a positive helix if $\lambda = -1$. The pitch of the helix equals the wave length of the vorticity mode, namely $2\pi/p$.

It is clear that the vorticity mode corresponds to a kind of shear velocity field. The use of the curl eigenfunctions leads to the decomposition of rotational flows into shear flows. The notion of "little vortices", which one might expect to be useful in describing fluid motion with vorticity, plays no role.

It will be useful to introduce the superposition of vorticity modes which have the same direction $\boldsymbol{\eta}$. Such a superposition will be called "a stratified vorticity mode". Designating such a mode by $\mathbf{u}(\mathbf{x}|\boldsymbol{\eta})$,

$$\mathbf{u}(\mathbf{x}|\boldsymbol{\eta}) = \sum_{\lambda=\pm 1} \mathbf{Q}_\lambda(\boldsymbol{\eta}) \int_{-\infty}^{+\infty} dp\, e^{ip(\boldsymbol{\eta}\cdot\mathbf{x})} g_\lambda(p) , \qquad (48)$$

where p is a single real variable and $g_\lambda(p)$ is a complex function of p. The vector $\boldsymbol{\eta}$ is the direction of the stratified vorticity mode. One sees that $\mathbf{u}(\mathbf{x}|\boldsymbol{\eta})$ as a function of $\mathbf{x}$ depends only on $\mathbf{x}\cdot\boldsymbol{\eta}$. It is for this reason that the term "stratified" is used to describe the superposition. These modes satisfy the condition

$$[\mathbf{u}(\mathbf{x}|\boldsymbol{\eta})\cdot\nabla]\,\mathbf{u}(\mathbf{x}|\boldsymbol{\eta}) = 0 . \qquad (49)$$

As will be seen, a consequence of Eq. (49) is that one of the non-linear terms in the equations of fluid motion vanishes and that one is able to get some exact solutions of fluid flow.

From the first part of Eq. (3), a necessary and sufficient condition for $\mathbf{u}(\mathbf{x}|\boldsymbol{\eta})$ to be real is

$$g_\lambda^{*}(p) = -\, g_{-\lambda}(-p) . \qquad (50)$$

On using Eq. (50) in Eq. (48)

$$\underset{\sim}{u}(\underset{\sim}{x}\,|\,\underset{\sim}{\eta}) = \underset{\sim}{Q}_1(\underset{\sim}{\eta}) \int_{-\infty}^{+\infty} dp\, e^{ip(\underset{\sim}{\eta}\cdot\underset{\sim}{x})} g_1(p)$$

$$+ \underset{\sim}{Q}_1^*(\underset{\sim}{\eta}) \int_{-\infty}^{+\infty} dp\, e^{-ip(\underset{\sim}{\eta}\cdot\underset{\sim}{x})} g_1^*(p) \,. \tag{51}$$

Let us now define $f_\lambda(p)$ for $p > 0$ by

$$f_1(p) = g_1(p),\quad f_{-1}^*(p) = -g_1(-p) \,. \tag{52}$$

Then on using the first part of Eq. (3), Eq. (51) becomes

$$\underset{\sim}{u}(\underset{\sim}{x}\,|\,\underset{\sim}{\eta}) = \sum_{\lambda=\pm 1} \underset{\sim}{u}_\lambda(\underset{\sim}{x}\,|\,\underset{\sim}{\eta}) \,, \tag{53}$$

where

$$\underset{\sim}{u}_\lambda(\underset{\sim}{x}\,|\,\underset{\sim}{\eta}) = \underset{\sim}{Q}_\lambda(\underset{\sim}{\eta}) \int_0^{\infty} dp\, e^{ip(\underset{\sim}{\eta}\cdot\underset{\sim}{x})} f_\lambda(p)$$

$$+ \underset{\sim}{Q}_\lambda^*(\underset{\sim}{\eta}) \int_0^{\infty} dp\, e^{-ip(\underset{\sim}{\eta}\cdot\underset{\sim}{x})} f_\lambda^*(p) \,. \tag{53a}$$

Clearly, the vector $\underset{\sim}{u}_\lambda(\underset{\sim}{x}\,|\,\underset{\sim}{\eta})$ is a superposition of vorticity modes $\underset{\sim}{u}_\lambda(\underset{\sim}{x}\,|\,\underset{\sim}{\eta}, p, \alpha)$ belonging to the same value of λ and $\underset{\sim}{\eta}$.

4.2.2 EXACT SOLUTIONS OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATION

4.2.2.1 General Stratified Vorticity Motion

The Navier-Stokes equations for an incompressible fluid are

$$\frac{\partial \underset{\sim}{v}(\underset{\sim}{x}, t)}{\partial t} + [\underset{\sim}{v}(\underset{\sim}{x}, t)\cdot\underset{\sim}{\nabla}]\,\underset{\sim}{v}(\underset{\sim}{x}, t) + \frac{1}{\rho}\underset{\sim}{\nabla} P(\underset{\sim}{x}, t) - \nu\nabla^2 \underset{\sim}{v}(\underset{\sim}{x}, t) = 0 \,,$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{v}(\underset{\sim}{x}, t) = 0 \,. \tag{54}$$

In Eq. (54), $\underset{\sim}{v}(\underset{\sim}{x}, t)$ is the velocity vector, $P(\underset{\sim}{x}, t)$ is the pressure, ν is the kinematic viscosity μ/ρ, and ρ is the density taken to be constant, of course. The objective is to find solutions of this equation which correspond to a stratified vorticity mode. That this is possible is due to the fact that the non-linear term of the first part of Eq. (54) vanishes; see Eq. (49). Furthermore, it will be seen that the pressure P uncouples from the stratified vorticity mode which simplifies the problem even more. One hopes that one can get some insight into the nature of more general viscous flows and even, perhaps, a deeper understanding of turbulence. In a later report, we shall give other exact solutions of rotational, viscous flow which, in a sense, is complementary to the flow discussed in the present section.

Actually, a somewhat more general problem than the time-development of a stratified vorticity mode will be solved. It will be assumed that the velocity $\underset{\sim}{v}(\underset{\sim}{x}, t)$ is the sum of a velocity $\underset{\sim}{V}(t)$, which depends on time but not on position, and a stratified vorticity mode. That is, the stratified mode "rides" a velocity which varies with time but whose space variation is slow compared with the space variation (that is, pitch) of the helices in the stratified mode.

Thus

$$\underset{\sim}{v}(\underset{\sim}{x}, t) = \underset{\sim}{V}(t) + \underset{\sim}{u}(\underset{\sim}{x}, t) , \tag{55}$$

where

$$\underset{\sim}{u}(\underset{\sim}{x}, t) = \sum_{\lambda=\pm 1} \underset{\sim}{Q}_{\lambda}(\underset{\sim}{\eta}) \int_{-\infty}^{+\infty} dp \; e^{ip(\underset{\sim}{\eta} \cdot \underset{\sim}{x})} g_{\lambda}(p, t) . \tag{55a}$$

Clearly, $\underset{\sim}{v}(\underset{\sim}{x}, t)$ satisfies the second part of Eq. (54). Hence one need only be concerned with the first of these equations.

Let us write

$$P(\underset{\sim}{x}, t) = \int d\underset{\sim}{p} \; e^{i\underset{\sim}{p} \cdot \underset{\sim}{x}} r(\underset{\sim}{p}, t) - \rho \frac{\partial \underset{\sim}{V}(t)}{\partial t} \cdot \underset{\sim}{x} . \tag{56}$$

It should be noted that, while in Eq. (55a) the integration is over a single variable denoted by p, the integration in Eq. (56) is over the three-dimensional space whose variables are collectively denoted by $\underset{\sim}{p}$.

Using Eq. (49) with $\underset{\sim}{u}(\underset{\sim}{x}, t)$ for $\underset{\sim}{u}(\underset{\sim}{x} \,|\, \underset{\sim}{\eta})$, the nonlinear term in the first part of Eq. (54) disappears. On substituting Eqs. (55), (55a) and (56) into the first part of Eq. (54)

$$\sum_{\lambda=\pm 1} \underset{\sim}{Q}_\lambda(\underset{\sim}{\eta}) \int_{-\infty}^{+\infty} dp\, e^{ip(\underset{\sim}{\eta}\cdot\underset{\sim}{x})} \left\{ \frac{\partial g_\lambda(p,t)}{\partial t} + ip\,[\underset{\sim}{\eta}\cdot\underset{\sim}{V}(t)]\,g_\lambda(p,t) \right.$$

$$\left. + \nu p^2 g_\lambda(p,t) \right\} = -\frac{i}{\rho} \int d\underset{\sim}{p}\; e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, \underset{\sim}{p}\, r(\underset{\sim}{p},t) \,. \tag{57}$$

Multiply through by $e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}$ and integrate with respect to $\underset{\sim}{x}$:

$$\sum_{\lambda=\pm 1} \underset{\sim}{Q}_\lambda(\underset{\sim}{\eta})\,\delta(\underset{\sim}{k}-p\underset{\sim}{\eta}) \left\{ \frac{\partial g_\lambda(p,t)}{\partial t} + ip[\underset{\sim}{\eta}\cdot\underset{\sim}{V}(t)]\,g_\lambda(p,t) \right.$$

$$\left. + \nu p^2 g_\lambda(p,t) \right\} = -\frac{i}{\rho}\,\underset{\sim}{k}\, r(\underset{\sim}{k},t) \,. \tag{58}$$

Let us dot through by $\underset{\sim}{k}$ and use $\underset{\sim}{k}\cdot\underset{\sim}{Q}_\lambda(\underset{\sim}{\eta})\delta(\underset{\sim}{k}-p\underset{\sim}{\eta}) = p\,\underset{\sim}{\eta}\cdot\underset{\sim}{Q}_\lambda(\underset{\sim}{\eta})\delta(\underset{\sim}{k}-p\underset{\sim}{\eta}) = 0$. Then

$$\underset{\sim}{k}^2 r(\underset{\sim}{k},t) = 0 \tag{59}$$

for which the general solution is

$$r(\underset{\sim}{k},t) = A(t)\,\delta(\underset{\sim}{k}) \,, \tag{60}$$

where $A(t)$ is an arbitrary function of t, as can be shown from the theory of distributions. Thus from Eq. (56)

$$P(\underset{\sim}{x},t) = A(t) - \frac{\partial \underset{\sim}{V}(t)}{\partial t}\cdot\underset{\sim}{x} \,. \tag{61}$$

Except for the derivative of the space-independent wind, the pressure uncouples from the velocity field.

Now substitute Eq. (60) in Eq. (57). The right hand side of Eq. (57) vanishes. On letting $\xi = \underset{\sim}{\eta}\cdot\underset{\sim}{x}$, taking the Fourier transform with respect to ξ, and using the linear independence of the vectors $\underset{\sim}{Q}_\lambda(\underset{\sim}{\eta})$, one obtains the following <u>uncoupled</u> ordinary differential equations for $g_\lambda(p,t)$:

$$\frac{\partial g_\lambda(p,t)}{\partial t} + \left\{ ip[\underset{\sim}{\eta}\cdot\underset{\sim}{V}(t)] + p^2\nu \right\} g_\lambda(p,t) = 0 \,, \tag{62}$$

for which the general solution is

$$g_\lambda(p,t) = g_\lambda(p)\exp\left\{-ip\int_0^t dt'\,[\underset{\sim}{n}\cdot\underset{\sim}{V}(t')]\right\}\exp[-\nu p^2 t]\ , \tag{63}$$

where $g_\lambda(p)$ is the constant of integration: $g_\lambda(p) = g_\lambda(p,0)$.

For $\underset{\sim}{u}(\underset{\sim}{x},t)$ to be real, it is required that

$$g_{-1}(p) = -g_1^*(-p)\ . \tag{64}$$

Now define $f_\lambda(p)$ for $p > 0$ by

$$f_1(p) = g_1(p),\ f_{-1}(p) = -g_1^*(-p)\ . \tag{65}$$

Then substitute Eqs. (63), (64), and (65) into Eq. (55a). On using the first part of Eqs. (3), one obtains finally

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \sum_{\lambda=\pm1}\underset{\sim}{w}_\lambda\left(\underset{\sim}{x} - \int_0^t \underset{\sim}{V}(t')dt',\ t\right) \tag{66}$$

where $\underset{\sim}{w}_\lambda(\underset{\sim}{x},t)$ is given by

$$\underset{\sim}{w}_\lambda(\underset{\sim}{x},t) = \underset{\sim}{Q}_\lambda(\underset{\sim}{\eta})\int_0^\infty dp\ e^{ip(\underset{\sim}{\eta}\cdot\underset{\sim}{x})}e^{-\nu p^2 t}f_\lambda(p)$$

$$+\ \underset{\sim}{Q}_\lambda^*(\underset{\sim}{\eta})\int_0^\infty dp\ e^{-ip(\underset{\sim}{\eta}\cdot\underset{\sim}{x})}e^{-\nu p^2 t}f_\lambda^*(p)\ . \tag{67}$$

Clearly, $\underset{\sim}{u}(\underset{\sim}{x},t)$ of Eqs. (66) and (67) is a sum of stratified vorticity modes $\underset{\sim}{u}_\lambda(\underset{\sim}{x}|\underset{\sim}{\eta})$ given by Eq. (53a) where the amplitudes now include an explicit time factor. It is seen that even if there is a wind which drives the vorticity modes, the modes with smaller pitch or wavelength damp out much more readily than the longer wavelengths. It was our hope, when we first investigated this solution, that this vortex motion would have some of the characteristics of turbulence in which longer wavelength modes feed shorter wavelength modes. However, as can be seen, stratified vorticity motion does not have this property. We have investigated other exact solutions of the incompressible Navier-Stokes equation in which the non-linear term $\underset{\sim}{v}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{v}$ plays a more central role. Again, however, these special solutions, which will be discussed in later reports, do not have the properties of turbulent flow. We hope, in later work, to discuss more general solutions of the Navier-Stokes equations through the use of perturbation theory. In this report, one of the solutions of the linearized, compressible Navier-Stokes equations, which we shall discuss shortly, has some of the properties of turbulence. However, since this flow is longitudinal, it is probably not what is usually thought of as a turbulent flow.

4.2.2.2 Stratified Flows in Oceans on a Rotating Earth.

We will now consider the most general incompressible viscous flow which can occur in oceans on a rotating earth subject to the condition that the velocity field and pressure depend on the vertical coordinate only. It will be seen that such flows are stratified vorticity motions of the type just discussed, whose time dependence include an oscillatory factor which depends on the latitude and constitutes a sort of "tide".

Following Brunt (1952), the equations of motion of a fluid on the rotating earth expressed in terms of Cartesian coordinates in which the x-axis is horizontal and points east, the y axis is horizontal and points north, and the z-axis is vertical is

$$\frac{\partial \underset{\sim}{v}(\underset{\sim}{x}, t)}{\partial t} + [\underset{\sim}{v}(\underset{\sim}{x}, t) \cdot \underset{\sim}{\nabla}]\, \underset{\sim}{v}(\underset{\sim}{x}, t) + \frac{1}{\rho} \underset{\sim}{\nabla} P(\underset{\sim}{x}, t) + 2\underset{\sim}{\omega} \times \underset{\sim}{v}(\underset{\sim}{x}, t)$$

$$+ g\underset{\sim}{k} - \nu \nabla^2 \underset{\sim}{v}(\underset{\sim}{x}, t) = 0 , \qquad (68)$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v}(\underset{\sim}{x}, t) = 0 ,$$

where $\underset{\sim}{\omega}$ is the angular velocity of the earth about the North Pole, and $\underset{\sim}{k}$ is the unit vector in the direction of the vertical. Note that from the second part of Eq. (68), $\underset{\sim}{v}(\underset{\sim}{x}, t)$ can be expanded in terms of rotational modes as follows:

$$\underset{\sim}{v}(\underset{\sim}{x}, t) = \underset{\sim}{V}(t) + (2\pi)^{-3/2} \sum_{\lambda=\pm 1} \int d\underset{\sim}{p}\, \underset{\sim}{Q}_\lambda(\underset{\sim}{p})\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, g_\lambda(\underset{\sim}{p} \cdot t) . \qquad (69)$$

In Eq. (69), $\underset{\sim}{V}(\ )$ is a thus far undetermined function of time but not of space. While vectors $\underset{\sim}{v}(\underset{\sim}{x}, t)$, expanded in Eq. (69), satisfy the second part of Eq. (68), this expansion is the only possible expansion, if we assume that we are working in an infinite fluid, that is, the scale of variation in space of the fluid velocity is small compared to the distance of the boundaries of the fluid. Whether the fluid is of finite or infinite extent, a solution will be found for stratified flow which is unique in the case that the fluid is of infinite extent.

The requirement that $\underset{\sim}{v}(\underset{\sim}{x}, t)$ depends on the vertical coordinate $z = x_3$ only leads to the following condition on $g_\lambda(\underset{\sim}{p}, t)$:

$$g_\lambda(\underset{\sim}{p}, t) = \delta(p_x)\delta(p_y)g_\lambda(p_z, t) , \tag{70}$$

where $g_\lambda(p_z, t)$ is now a function of p_z and t only.

Then, on absorbing a factor of $(2\pi)^{-3/2}$ into the function g_λ and on renaming the variable p_z, our expression for $\underset{\sim}{v}(\underset{\sim}{x}, t)$ becomes

$$\underset{\sim}{v}(\underset{\sim}{x}, t) = \underset{\sim}{V}(t) + \underset{\sim}{u}(z, t) , \tag{71}$$

where $\underset{\sim}{u}(z, t)$ is given by

$$\underset{\sim}{u}(z, t) = \sum_{\lambda=\pm 1} \underset{\sim}{Q}_\lambda \int_{-\infty}^{+\infty} dp\, e^{ipz}\, g_\lambda(p, t), \tag{71a}$$

and

$$\underset{\sim}{Q}_\lambda = \underset{\sim}{Q}_\lambda(\underset{\sim}{k}) = (2)^{-1/2}\lambda(1, i\lambda, 0) . \tag{71b}$$

It is clear that the vector u(z, t) is the sum of two stratified vorticity modes with the direction $\underset{\sim}{\eta} = \underset{\sim}{k}$.

Note that $u_z = u_3 = 0$. Furthermore, the non-linear term simplifies in an essential way:

$$[\underset{\sim}{v}(\underset{\sim}{x}, t) \cdot \underset{\sim}{\nabla}]\underset{\sim}{v}(\underset{\sim}{x}, t) = V_z(t) \frac{\partial \underset{\sim}{u}(z, t)}{\partial z} . \tag{72}$$

Let θ be the co-latitude measure from the North Pole. Let $\omega = |\underset{\sim}{\omega}|$ be the a - solute value of the angular velocity of rotation of the earth, $\alpha = 2\omega \cos\theta$, $\beta = 2\omega \sin\theta$, $\hat{\underset{\sim}{V}}(t)$ be the vector whose components are $[V_x(t), V_y(t), 0]$, and W(t) be defined by $W(t) = V_z(t)$.

Finally, the dyadic $\underset{\sim}{A}$ is introduced whose components are given by the matrix A where A is defined below:

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{73}$$

The x- and y- components of Eq. (68) can be written as

$$\frac{\partial \hat{\underline{V}}(t)}{\partial t} + \alpha \underline{\underline{A}}\hat{\underline{V}}(t) + \beta \underline{i} W(t) + \frac{\partial \underline{u}(z,t)}{\partial t} + W(t)\frac{\partial \underline{u}(z,t)}{\partial z}$$

$$+ \alpha \underline{\underline{A}}\,\underline{u}(z,t) - \nu \nabla^2 \underline{u}(z,t) = 0\,, \tag{74}$$

where $\underline{i}$ is the unit vector in the x-direction: $\underline{i} = (1,0,0)$. The z-component of Eq. (68) is

$$\frac{\partial W(t)}{\partial t} - \beta V_x(t) - \beta u_x(z,t) + \frac{1}{\rho}\frac{\partial P(z,t)}{\partial z} + g = 0\,. \tag{75}$$

Equation (74) will first be solved by substituting Eq. (71a) for $\underline{u}(z,t)$. Note that the vectors $\underline{Q}_\lambda$ are eigenvectors of the dyadic $\underline{A}$ for $\lambda = \pm 1$:

$$\underline{A}\,\underline{Q}_\lambda = -i\lambda \underline{Q}_\lambda\,. \tag{76}$$

Then the orthogonality of the vectors $\underline{Q}_\lambda$ is used and a Fourier transformation is taken with respect to z. On defining

$$G_\lambda(t) = \underline{Q}_\lambda^{*}\cdot\hat{\underline{V}}(t) = \underline{Q}_\lambda^{*}\cdot\underline{V}(t),\quad (\lambda = \pm 1)\,, \tag{77}$$

The following Equation is obtained from Eq. (74):

$$\left[\frac{\partial G_\lambda(t)}{\partial t} - i\lambda\alpha G_\lambda(t) + (2)^{-1/2}\lambda\beta W(t)\right]\delta(p) +$$

$$\frac{\partial g_\lambda(p,t)}{\partial t} + \left[-i\lambda\alpha + ipW(t) + \nu p^2\right] g_\lambda(p,t) = 0\,. \tag{78}$$

From the theory of distributions, the coefficient of the delta-function must vanish as must the second line on the right of Eq. (78). Thus $G_\lambda(t)$ is given in terms of a differential equation as dependingon W(t). Note that

$$\hat{\underline{V}}(t) = \sum_{\lambda = \pm 1} G_\lambda(t)\underline{Q}_\lambda\,. \tag{79}$$

On integrating the differential equation for $G_\lambda(t)$ and then on substituting into Eq. (79), one obtains the following relationships between the components of $\underset{\sim}{V}(t)$:

$$\hat{\underset{\sim}{V}}(t) = \hat{\underset{\sim}{V}}(0)\cos\alpha t - \underset{\sim}{A}\,\hat{\underset{\sim}{V}}(0)\sin\alpha t - \beta\int_0^t dt'\,\underset{\sim}{G}(t-t')W(t') ,$$

$$V_z(t) = W(t) , \tag{80}$$

where $\underset{\sim}{G}(t)$ is a vector Green's function

$$\underset{\sim}{G}(t) = (\cos\alpha t,\ -\sin\alpha t, 0) . \tag{80a}$$

Thus the vector $\underset{\sim}{V}(t)$ can have only the form given by Eq. (80), where $\hat{\underset{\sim}{V}}(0)$ and $W(t)$ are arbitrary.

The expression $\underset{\sim}{u}(z,t)$ can also be found in much the same manner as in the previous section, where the most general solution for the stratified vorticity mode was found. The differential equation for $g_\lambda(p,t)$ is solved by the usual elementary methods, and the reality condition is used on $\underset{\sim}{u}(z,t)$ to obtain relations between the constants of integration. Our final result is

$$\underset{\sim}{u}(z,t) = \sum_{\lambda=\pm1} \hat{\underset{\sim}{w}}_\lambda\left(z-\int_0^t dt'\,W(t'), t\right) , \tag{81}$$

where

$$\hat{w}_\lambda(z,t) = \underset{\sim}{Q}_\lambda e^{i\lambda\alpha t}\int_0^\infty dp\, e^{ipz}\, e^{-\nu p^2 t} f_\lambda(p)$$

$$+ \underset{\sim}{Q}_\lambda{}^* e^{-i\lambda\alpha t}\int_0^\infty dp\, e^{-ipz}\, e^{-\nu p^2 t} f_\lambda{}^*(p) \tag{81a}$$

Note that if the rotation rate of the earth were zero or if we were at the Equator, the velocity $\underset{\sim}{u}(z,t)$ is identical to the stratified vorticity mode Eqs. (66) and (67) when $\underset{\sim}{\eta} = \underset{\sim}{k}$. Let us now take $f_\lambda(p)$, which in general is arbitrary, to be given by

$$f_\lambda(p) = A\delta(p-k) , \tag{82}$$

where A and k are positive. It is clear that the general $\underset{\sim}{w}_\lambda(z,t)$ is a superposition of such particular velocities. On absorbing a factor $(2)^{-1/2}$ into A, one obtains

$$w_{\lambda x}(z,t) = A\lambda \cos(\alpha t+\lambda kz)e^{-\nu k^2 t}, \quad w_{\lambda y}(z,t) = -A \sin(\alpha t+\lambda kz)e^{-\nu k^2 t},$$

$$w_{\lambda z}(z,t) = 0 \tag{83}$$

If $\underset{\sim}{V}(t)$ is picked to be identically zero and one of the $\underset{\sim}{w}_\lambda$ chosen to be equal to zero, then the velocity is given by Eq. (83). This velocity is seen to be a circularly-polarized wave travelling in the negative z- direction and polarized in the direction of motion if $\lambda = 1$, and travels in the positive- z direction and is polarized opposite to the direction of propagation when $\lambda = -1$. If the viscosity is not zero, this wave is damped in time.

It is now easy to integrate Eq. (75) for the pressure, since W(t), $V_x(t)$, $u_x(z,t)$ are known. Let us consider the special case where the velocity is given by Eq. (83). Then the pressure is given by

$$P(z,t) = P_o(t) - gz + \frac{2}{k} A\rho\beta\lambda e^{-\nu k^2 t} \sin\frac{kz}{2} \cos\left(\alpha t+\lambda \frac{kz}{2}\right), \tag{84}$$

where $P_o(t)$ is the pressure at $z=0$ and is an arbitrary function of the time.

It is perhaps interesting to note that Eqs. (83) and (84) provide a solution for possible motions under the surface of a "quiet" ocean in which the surface of the ocean is taken to be at $z=0$. Then $P_o(t)$ is the atmospheric pressure at the surface. Such solutions are the simplest vorticity solutions in the ocean which it is possible to have.

4.2.3 PROPAGATION OF SMALL DISTURBANCES IN A COMPRESSIBLE, VISCOUS GAS

This section treats the propagation of a small disturbance in a compressible, viscous gas. It will be assumed that the fluid flow is adiabatic and the flow is general in that it may have vorticity as well as irrotational flow. The restriction to adiabatic flow is not essential, for equations could be included having energy transport. The adiabatic flow, however, is simpler and illustrates more directly the use of the modes which have been introduced. It should be mentioned that Morse and Ingard (1968) have treated closely related problems. They take into account heat conduction. They also include the case where vorticity is present, but they separate the vorticity by using well-known vector calculus relations instead of using the curl eigenfunctions, as in this report. Furthermore they are interested primarily in time-harmonic motions and in solutions near boundaries. By contrast, our interest is mainly in initial value problems solved in the infinite domain.

Following Landau and Lifshitz (1959), the equations of motion for a viscous fluid are

$$\frac{\partial \underset{\sim}{v}(\underset{\sim}{x},t)}{\partial t} + [\underset{\sim}{v}(\underset{\sim}{x},t)\cdot\underset{\sim}{\nabla}]\,\underset{\sim}{v}(\underset{\sim}{x},t) + \frac{1}{\rho}\underset{\sim}{\nabla} P(\underset{\sim}{x},t) - \frac{\mu}{\rho}\nabla^2 \underset{\sim}{v}(\underset{\sim}{x},t)$$

$$- \frac{1}{\rho}\left(\frac{1}{3}\mu+\zeta\right) \underset{\sim}{\nabla}[\underset{\sim}{\nabla}\cdot\underset{\sim}{v}(\underset{\sim}{x},t)] = 0\,,$$

$$\frac{\partial \rho(\underset{\sim}{x},t)}{\partial t} + [\underset{\sim}{v}(\underset{\sim}{x},t)\cdot\underset{\sim}{\nabla}]\,\rho(\underset{\sim}{x},t) + \rho(x,t)\underset{\sim}{\nabla}\cdot\underset{\sim}{v}(\underset{\sim}{x},t) = 0\,. \tag{85}$$

In the first part of Eq. (85), μ is the usual "dynamic" viscosity coefficient and ζ is the "second" or "bulk" viscosity coefficient.

It will also be assumed that the pressure is a function of density only:

$$P = F(\rho)\,. \tag{86}$$

Equation (86) holds for adiabatic flow. It also holds for isothermal flow.

We will now investigate how a small disturbance propagates. This disturbance is regarded as being a disturbance about a steady state whose velocity, density, and pressure are constants denoted respectively by $\underset{\sim}{V}$, ρ_0, and P_0, and is written as:

$$\underset{\sim}{v}(\underset{\sim}{x},t) = \underset{\sim}{V} + \underset{\sim}{u}(\underset{\sim}{x},t)$$

$$P(\underset{\sim}{x},t) = P_0 + \hat{P}(\underset{\sim}{x},t)$$

$$\rho(\underset{\sim}{x},t) = \hat{\rho}_0 + \hat{\rho}(\underset{\sim}{x},t)\,. \tag{87}$$

In Eq. (87) $\underset{\sim}{u}$, $\hat{P}$, and $\hat{\rho}$ are regarded as small quantities as in the usual perturbation approaches. By substituting Eq. (87) in Eq. (85) and neglecting products of the small quantities, the linearized equations of fluid flow are obtained below:

$$\frac{\partial \underset{\sim}{u}(\underset{\sim}{x},t)}{\partial t} + \underset{\sim}{V}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{u}(\underset{\sim}{x},t) + \frac{c^2}{\rho_0}\nabla\hat{\rho}(x,t) - \nu\nabla^2\underset{\sim}{u}(\underset{\sim}{x},t)$$

$$- \frac{1}{3}(\nu+4\xi)\underset{\sim}{\nabla}[\underset{\sim}{\nabla}\cdot\underset{\sim}{u}(\underset{\sim}{x},t)] = 0\,,$$

$$\frac{\partial\hat{\rho}(\underset{\sim}{x},t)}{\partial t} + \underset{\sim}{V}\cdot\underset{\sim}{\nabla}\hat{\rho}(\underset{\sim}{x},t) + \rho_0[\underset{\sim}{\nabla}\cdot\underset{\sim}{u}(\underset{\sim}{x},t)] = 0\,,$$

$$\hat{P}(\underset{\sim}{x},t) = c^2\hat{\rho}(\underset{\sim}{x},t)\,. \tag{88}$$

In Eq. (88), we use

$\nu = (\mu/\rho_0)$ = kinematic viscosity,

$$\xi = (3\zeta/4\rho_0), \ c^2 = F'(\rho_0) . \tag{88a}$$

In Eq. (88a), the prime means the derivative with respect to the argument. The quantity c in the usual treatments of linearized compressible flow turns out to be the velocity of the sound waves. It will play the same role in the present treatment. The first two parts of Eq. (88) are regarded as being a pair of simultaneous linear partial differential equations for the unknown quantities $\underset{\sim}{u}(\underset{\sim}{x}, t)$ and $\hat{\rho}(\underset{\sim}{x}, t)$. The last part of Eq. (88) allows one to obtain $\hat{P}(\underset{\sim}{x}, t)$ from $\hat{\rho}(\underset{\sim}{x}, t)$.

From Eq. (13), it may be written that

$$\underset{\sim}{u}(\underset{\sim}{x}, t) = \sum_{\lambda} \int d\underset{\sim}{p}\, \underset{\sim}{X}_{\lambda}(\underset{\sim}{x}|\underset{\sim}{p}) g_{\lambda}(\underset{\sim}{p}, t) , \tag{89}$$

where the summation over λ includes all three values.

It is clear that since $\hat{\rho}$ is a scalar, it should be expanded as an ordinary Fourier integral:

$$\hat{\rho}(\underset{\sim}{x}, t) = (2\pi)^{-3/2} \int d\underset{\sim}{p}\ e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} r(\underset{\sim}{p}, t) . \tag{90}$$

Equations (89) and (90) will be used in the first two parts of Eq. (88), and ordinary differential equations will be obtained in the variable t for the functions $g_{\lambda}(\underset{\sim}{p}, t)$ and $r(\underset{\sim}{p}, t)$ which are uncoupled to a high degree.

On substituting Eqs. (89) and (90) into the second part of Eq. (88), on using Eq. (12), and finally on taking a Fourier transform with respect to $\underset{\sim}{x}$, there is found one of the equations we wish to solve:

$$\frac{\partial r(\underset{\sim}{p}, t)}{\partial t} + i(\underset{\sim}{V}\cdot\underset{\sim}{p}) r(\underset{\sim}{p}, t) - ip\rho_0 g_0(\underset{\sim}{p}, t) = 0 . \tag{91}$$

[In Eq. (91) $p = |\underset{\sim}{p}|$, as usual] .

Let us now substitute Eqs. (89) and (90) into the first part of Eq. (88), and use the second and third parts of Eq. (12). On taking the Fourier transformation with respect to $\underset{\sim}{x}$, one obtains:

$$\sum_{\lambda} \underset{\sim}{Q}_{\lambda}(\underset{\sim}{p}) \left[\frac{\partial g_{\lambda}(\underset{\sim}{p}, t)}{\partial t} + i(\underset{\sim}{V} \cdot \underset{\sim}{p}) g_{\lambda}(\underset{\sim}{p}, t) + \nu p^2 g_{\lambda}(\underset{\sim}{p}, t) \right]$$

$$+ \underset{\sim}{p} \left[\frac{c^2}{\rho_0} i r(\underset{\sim}{p}, t) - \frac{1}{3} (\nu + 4\xi) p g_0(\underset{\sim}{p}, t) \right] = 0 . \tag{92}$$

On substituting $-p\underset{\sim}{Q}_0(\underset{\sim}{p})$ for $\underset{\sim}{p}$ in Eq. (92) and using the fact that the vectors $\underset{\sim}{Q}_{\lambda}(\underset{\sim}{p})$ are linearly independent, one obtains

$$\frac{\partial g_0(\underset{\sim}{p}, t)}{\partial t} + i(\underset{\sim}{V} \cdot \underset{\sim}{p}) g_0(\underset{\sim}{p}, t) + \frac{4}{3} p^2 \tau g_0(\underset{\sim}{p}, t) - i \frac{c^2}{\rho_0} p r(\underset{\sim}{p}, t) = 0 \tag{93}$$

where

$\tau = \nu + \xi$,

and

$$\frac{\partial g_{\lambda}(\underset{\sim}{p}, t)}{\partial t} + i(\underset{\sim}{V} \cdot \underset{\sim}{p}) g_{\lambda}(\underset{\sim}{p}, t) + \nu p^2 g_{\lambda}(\underset{\sim}{p}, t) = 0 \tag{94}$$

for $\lambda = \pm 1$.

Note that the differential equations for $g_{\lambda}(\underset{\sim}{p}, t)$ uncouple completely for the vorticity modes $\lambda = \pm 1$. Equations (91) and (93) couple the density amplitude $r(\underset{\sim}{p}, t)$ and the irrotational velocity amplitude $g_0(\underset{\sim}{p}, t)$. Thus the vorticity disturbances are completely uncoupled from the pressure and density disturbances. In fact, Eq. (94) would also be valid for incompressible flows, if perturbation theory had been used on the incompressible Navier-Stokes equation. In a sense, the discussion of the problem of small disturbances in a compressible fluid bears strong resemblances to our earlier discussion on Maxwell's Equations.

Since the rotational and irrotational components of $\underset{\sim}{u}(\underset{\sim}{x}, t)$ uncouple, it is convenient to write

$$\underset{\sim}{u}(\underset{\sim}{x}, t) = \underset{\sim}{u}_r(x, t) + \underset{\sim}{u}_{ir}(\underset{\sim}{x}, t) , \tag{95}$$

where $\underset{\sim}{u}_r$ and $\underset{\sim}{u}_{ir}$ are, respectively, the rotational and irrotational parts of $\underset{\sim}{u}$ and are thus given by

$$\underset{\sim}{u}_r(\underset{\sim}{x}, t) = \sum_{\lambda=\pm 1} \int d\underset{\sim}{p} \ \underset{\sim}{\chi}_\lambda(\underset{\sim}{x}|\underset{\sim}{p}) g_\lambda(\underset{\sim}{p}, t) ,$$

$$\underset{\sim}{u}_{ir}(\underset{\sim}{x}, t) = \int d\underset{\sim}{p} \ \underset{\sim}{\chi}_0(\underset{\sim}{x}|\underset{\sim}{p}) g_0(\underset{\sim}{p}, t) . \tag{95a}$$

In Eq. (95), the real part of the expressions are taken because, as explained in connection with Eq. (13c), the reality conditions on the vectors lead to conditions on the amplitudes $g_\lambda(\underset{\sim}{p}, t)$ which allow us to assume that the imaginary part of the expansions are zero.

Equation (94) is easily solved. The solution is, in fact,

$$g_\lambda(\underset{\sim}{p}, t) = g_\lambda(\underset{\sim}{p}) \exp[-i(\underset{\sim}{V} \cdot \underset{\sim}{p})t - \nu p^2 t], \tag{96}$$

where $g_\lambda(\underset{\sim}{p}) = g_\lambda(\underset{\sim}{p}, 0)$ is the constant of integration.

We have immediately

$$\underset{\sim}{u}_r(\underset{\sim}{x}, t) = \sum_{\lambda=\pm 1} \underset{\sim}{w}_\lambda(\underset{\sim}{x} - \underset{\sim}{V}t, t) , \tag{97}$$

where

$$\underset{\sim}{w}_\lambda(\underset{\sim}{x}, t) = \int d\underset{\sim}{p} \ \underset{\sim}{\chi}_\lambda(\underset{\sim}{x}|\underset{\sim}{p}) g_\lambda(\underset{\sim}{p}) e^{-\nu p^2 t} . \tag{97a}$$

It is clear that $\underset{\sim}{u}_r$ is a superposition of vorticity modes which are being carried along with the steady wind whose velocity is $\underset{\sim}{V}$ and whose amplitude is being attenuated with the time factor $e^{-\nu p^2 t}$. Hence the vorticity modes with shorter wavelength damp out at a much faster rate than those with a longer wavelength. That is, the small scale vorticity modes are damped out faster than the larger scale modes. Thus, the rotational velocity $\underset{\sim}{u}_r$ does not at all behave like turbulence in this respect.

We shall now show that the irrotational velocity $\underset{\sim}{u}_{ir}$ and the corresponding density increment $\hat{\rho}$ can be written as the sum of two parts: one which can be interpreted as a damped sound wave, and the other which is a damped standing wave. The damping of the sound wave is such that the longer wavelengths persist longer, while for the standing wave the shorter wavelengths maintain themselves for a longer time. Surprisingly, then, this standing irrotational wave has some of the characteristics of turbulent flow in that small scale phenomena persist over large scale ones.

Differential Eqs. (91) and (93) will now be solved. Let $R(\underset{\sim}{p}, t)$ and $G(\underset{\sim}{p}, t)$ be defined by

$$r(\underset{\sim}{p}, t) = e^{-i\underset{\sim}{p}\cdot\underset{\sim}{V}t}R(\underset{\sim}{p}, t), \quad g_0(\underset{\sim}{p}, t) = e^{-i\underset{\sim}{p}\cdot\underset{\sim}{V}t}G(\underset{\sim}{p}, t) . \tag{98}$$

The differential Eqs. (91) and (93) are now differential equations for R and G:

$$\frac{\partial R(\underset{\sim}{p}, t)}{\partial t} - i\rho_0 p G(\underset{\sim}{p}, t) = 0 ,$$

$$\frac{\partial G(\underset{\sim}{p}, t)}{\partial t} + \frac{4}{3}\tau p^2 G(\underset{\sim}{p}, t) - i\frac{c^2}{\rho_0} pR(\underset{\sim}{p}, t) = 0 . \tag{99}$$

The usual techniques are used to solve these coupled differential equations. The second part of Eq. (99) is differentiated with respect to t, the $\partial R(\underset{\sim}{p}, t)/\partial t$ is eliminated through the use of the first part of Eq. (98), and the following second order differential equation is thereby obtained for G:

$$\frac{\partial^2 G(\underset{\sim}{p}, t)}{\partial t^2} + \frac{4}{3}\tau p^2 \frac{\partial G(\underset{\sim}{p}, t)}{\partial t} + c^2 p^2 G(\underset{\sim}{p}, t) = 0 . \tag{100}$$

The general solution of this equation is of the form

$$G(\underset{\sim}{p}, t) = G_+(\underset{\sim}{p})\exp[w_+t] + G_-(\underset{\sim}{p})\exp[w_-t] , \tag{101}$$

where $w_\pm$ are the two roots of the quadratic equation

$$w^2 + \frac{4}{3}\tau p^2 w + c^2 p^2 = 0 , \tag{101a}$$

and $G_\pm(\underset{\sim}{p})$ are constants of integration. The plus and minus signs refer to the signs in front of the discriminant of the quadratic equation. Of course, $w_\pm$ depends on p.

Because of the radically different character of $w_\pm$ for different regions in $\underset{\sim}{p}$-space, it is convenient to divide $\underset{\sim}{p}$-space into two domains. A critical length L_c is introduced and a corresponding critical wave number k by

$$k = \frac{2\pi}{L_c} = (3c/2\tau) . \tag{102}$$

Call P the sphere in $\underset{\sim}{p}$-space for which $p < k$, where the letter P stands for "Propagating". The remainder of $\underset{\sim}{p}$-space, that is all of $\underset{\sim}{p}$-space other than the sphere P, will be denoted by N, which stands for "Non-propagating". Now the results can be combined to give $g_0(\underset{\sim}{p}, t)$ and $r(\underset{\sim}{p}, t)$ in a physically significant manner.

For $\underset{\sim}{p}$ in P:

$$g_0(\underset{\sim}{p}, t) = \exp[-i\underset{\sim}{p}\cdot\underset{\sim}{V}t]\exp[-cp(p/k)t]\Big\{G_+(\underset{\sim}{p})\exp[icp(1-(p/k)^2)^{1/2}t]$$
$$+ G_-(\underset{\sim}{p})\exp[-icp(1-(p/k)^2)^{1/2}t]\Big\} ,$$
$$r(\underset{\sim}{p}, t) = -i\frac{\rho_0}{c}\exp[-i\underset{\sim}{p}\cdot\underset{\sim}{V}t]\exp[-cp(p/k)t]\Big\{G_+(\underset{\sim}{p})$$
$$\times[(p/k)+i(1-(p/k)^2)^{1/2}]\exp[icp(1-(p/k)^2)^{1/2}t]$$
$$+ G_-(\underset{\sim}{p})[(p/k)-i(1-(p/k)^2)^{1/2}]\exp[-icp(1-(p/k)^2)^{1/2}t]\Big\} . \qquad (103)$$

For $\underset{\sim}{p}$ in N:

$$g_0(\underset{\sim}{p}, t) = \exp[-i\underset{\sim}{p}\cdot\underset{\sim}{V}]\Big\{G_+(\underset{\sim}{p})\exp[-cpF_+(p/k)t]$$
$$+ G_-(\underset{\sim}{p})\exp[-cpF_-(p/k)t]\Big\} ,$$
$$r(\underset{\sim}{p}, t) = -i\frac{\rho_0}{c}\exp[-i\underset{\sim}{p}\cdot\underset{\sim}{V}]\Big\{G_+(\underset{\sim}{p})F_-(p/k)\exp[-cpF_+(p/kt]$$
$$+ G_-(\underset{\sim}{p})F_+(p/k)\exp[-cpF_-(p/k)t]\Big\} , \qquad (104)$$

where the function $F_\pm(x)$ which appears in Eq. (104) is defined by

$$F_+(x) = x - (x^2-1)^{1/2}, \quad F_-(x) = x + (x^2-1)^{1/2} . \qquad (104a)$$

On using Eqs. (103) and (104) in Eqs. (95a) and (90) it is seen that $\underset{\sim}{u}_{ir}$ and $\hat{\rho}$ each can be broken up into two parts corresponding to the spaces P and N. It is also seen that the parts of $\underset{\sim}{u}_{ir}$ and $\hat{\rho}$ that correspond to P space are damped moving waves which, or sufficiently small p, are undamped and travel with the velocity c with respect to the constant wind which has the velocity $\underset{\sim}{V}$. The shorter wavelengths damp out faster than the longer wavelengths (with a different damping factor from that for rotational flow), so that the longer wavelengths persist for a longer time. Thus the contribution of the space P may be identified as being sound waves.

The contribution to $\underset{\sim}{u}_{ir}$ and $\hat{\rho}$ from the space N does not at all behave like sound waves. These motions are, in a sense, "overdamped".

Because of the nature of the function F_+ which appears in Eq. (104), it is seen that the shorter wavelength contributions damp out more slowly than the longer wavelength contributions. Thus small scale phenomena persist longer than the large scale phenomena. In turbulent flow, a similar pattern is observed, though usually it is assumed that turbulent flow is rotational, rather than irrotational. Our results suggest that perhaps some of the phenomena of turbulent flow are related to the irrotational components of the flow corresponding to the space N. Perhaps, also, inclusion of the non-linear terms in the equations of flow would induce a similar vehaviour in the rotational flow. However, such speculation is beyond the intent of this report.

Without going into detail, note that all the constants of integration $g_\lambda(\underset{\sim}{p})$ and $G_\pm(\underset{\sim}{p})$ are uniquely determined when $\underset{\sim}{u}$ and $\hat{\rho}$ are given at time t=0. Hence the initial value problem has been solved in all generality for the propagation of small disturbances in a compressible viscous fluid.

5. FURTHER PROPERTIES OF THE VECTORS $\underset{\sim}{Q}_\lambda(\underset{\sim}{\eta})$ AND GENERALIZED SURFACE HARMONICS

Having given some applications of the eigenfunctions of the curl operator, we shall go into further properties of the vectors $\underset{\sim}{Q}_\lambda(\underset{\sim}{\eta})$, primarily to obtain the rotation property of Eq. (9) and to restate the Helmholtz theorem in terms of spherical coordinates. Up to now, it will be noted that the cartesian coordinates are emphasized over other coordinate systems.

It will be necessary to go into some properties of the representation of the rotation group. Indeed, as will be shown in the Appendix A, rotation group considerations have motivated our introduction of the Helmholtz theorem.

The irreducible representations of the generators of the rotation group are three matrices $S_i^{(j)}$ (i=1, 2, 3) which satisfy the commutation rules

$$\left[S_1^{(j)},\ S_2^{(j)}\right] = iS_3^{(j)} \quad \text{(cyc.)} \tag{105}$$

where j characterizes the irreducible representation and is a positive integer or half-odd integer. The matrices have 2j+1 by 2j+1 in dimension. The elements are denoted by $(j, m|S_i|j, m')$ where m, m' take on the values -j, -j+1, . . . , j-1, j.

The usual form for the elements are (for example, see Edmonds, 1957).

$$(j,m|S_3|j,m') = m\delta_{m,m'} \ ,$$

$$(j,m|S_1|j,m') \pm i(j,m|S_2|j,m') = [(j\mp m')(j\pm m'+1)]^{1/2}\delta_{m,m'\pm 1} \ . \tag{106}$$

The column vectors upon which the matrices $S_i^{(j)}$ act will have the components labelled by m, where the top component will be labelled by m=j, the second component by m=j-1 and so on until the bottom component which is labelled by m=-j. Correspondingly, the first row of the matrices $S_i^{(j)}$ will be labelled by m=j, the second by m=j-1, etc. A similar labelling convention is used for the columns of these matrices.

Let $\theta = (\theta_1, \theta_2, \theta_3)$ be any real vector. Then $\exp[i\underset{\sim}{\theta}\cdot\underset{\sim}{S}^{(j)}]$ is an irreducible representation of the rotation matrix $R(\underset{\sim}{\theta})$ of Eqs. (7) and (8). (We use $\underset{\sim}{\theta}\cdot\underset{\sim}{S}^{(j)} = \sum_i \theta_i S_i^{(j)}$). Moses (1966b) has given the matrices $\exp[i\underset{\sim}{\theta}\cdot\underset{\sim}{S}^{(j)}]$ for all $\underset{\sim}{\theta}$. It will be necessary to introduce the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. Its properties are given by Szegö (1959), for example. We define the function S(j, m, m', x) by

$$S(j,m,m',x) = P_{j-m}^{(m-m',m+m')}(x) \ . \tag{107}$$

From Rodrigues' formula for the Jacobi polynomials (Szegö, 1959)

$$S(j,m,m',x) = (-1)^{j-m}\frac{2^{m-j}}{(j-m)!}(1-x)^{-(m-m')}(1+x)^{-(m+m')} \times \frac{d^{j-m}}{dx^{j-m}}\left[(1-x)^{j-m'}(1+x)^{j+m'}\right] . \tag{108}$$

Moses (1965) has given an alternative expression for S(j, m, m', x):

$$S(j,m,m',x) = (-1)^{j+m'}\frac{2^{m-j}}{(j+m)!}\frac{d^{j+m}}{dx^{j+m}}\left[(1-x)^{j+m'}(\ +x)^{j-m'}\right] . \tag{109}$$

In fact, one may use either Eq. (108) or (109) to define S(j, m, m', x). Let us denote the matrix elements of $\exp[i\theta\cdot S^{(j)}]$ by $(j,m|\exp[i\underset{\sim}{\theta}\cdot\underset{\sim}{S}]|j,m')$. Then

$$(j,m|\exp[i\underset{\sim}{\theta}\cdot\underset{\sim}{S}]|j,m') = [(j-m)!/(j-m')!]^{1/2}[(j+m)!/(j+m')!]^{1/2}$$

$$\times(\sin\frac{\theta}{2})^{m-m'}\left(\frac{\theta_1+i\theta_2}{\theta}\right)^{m-m'}\left(\cos\frac{\theta}{2}+i\frac{\theta_3}{\theta}\sin\frac{\theta}{2}\right)^{m+m'} S(j,m,m',z) \ , \tag{110}$$

where

$$\theta = |\underset{\sim}{\theta}|, \; z = \left[1 - \left(\frac{\theta_3}{\theta}\right)^2\right] \cos\theta + \left(\frac{\theta_3}{\theta}\right)^2 . \tag{110a}$$

We have previously introduced "generalized surface harmonics" (Moses, 1967). The generalized surface harmonics $Y_j^{m,m'}(\theta,\phi)$ are defined for $0<\theta<\pi$ and $0<\phi<2\pi$ by

$$Y_j^{m,m'}(\theta,\phi) = (-1)^{m-m'}(1/2)^{m+1}[(2j+1)/\pi]^{1/2}[(j-m)!/(j-m')!]^{1/2}$$
$$\times [(j+m)!/(j+m')!]^{1/2} e^{i(m-m')\phi}[\sin\theta]^{m-m'}$$
$$\times [1+\cos\theta]^{m'} S(j,m,m',\cos\theta]. \tag{111}$$

These functions have the following properties:

$$Y_j^{m,0}(\theta,\phi) = Y_{jm}(\theta,\phi), \tag{112}$$

where $Y_{jm}(\theta,\phi)$ are the usual surface harmonics in the notation of Edmonds.

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, Y_j^{m,n*}(\theta,\phi) Y_{j'}^{m',n}(\theta,\phi) = \delta_{j,j'}\delta_{m,m'}, \tag{113}$$

$$\sum_{j=|n|}^{\infty} \sum_{m=-j}^{j} Y_j^{m,n}(\theta,\phi) Y_j^{m,n*}(\theta',\phi') \sin\theta' = \delta(\theta-\theta')\delta(\phi-\phi'), \tag{114}$$

$$\sum_{m=-j}^{j} Y_j^{m,n}(\theta,\phi) Y_j^{m,n'*}(\theta,\phi) = [(2j+1)/4\pi]\delta_{n,n'}, \tag{115}$$

$$\sum_{n=-j}^{j} Y_j^{m,n}(\theta,\phi) Y_j^{m',n*}(\theta,\phi) = [(2j+1)/4\pi]\delta_{m,m'}, \tag{116}$$

$$Y_j^{m,n}(\theta,\phi) = (-1)^{n-m} Y_j^{n,m*}(\theta,\phi) , \tag{117}$$

$$Y_k^{n,n'}(\theta,\phi) Y_j^{m,m'}(\theta,\phi) = \sum_{J=|j-k|}^{j+k} \left\{ [(2j+1)(2k+1)]/[4\pi(2J+1)] \right\}^{1/2}$$

$$\times (k,n',j,m'|k,j,J,n'+m')(k,n,j,m|k,j,J,n+m)$$

$$\times \; Y_J^{m+n,m'+n'}(\theta,\phi) . \tag{118}$$

In Eq. (118), we have used the Clebsch-Gordan coefficients in the notation used by Edmonds.

The matrix elements of $\exp[i\underset{\sim}{\theta}\cdot\underset{\sim}{S}^{(j)}]$ for the case $\theta_3 = 0$ is conveniently expressed in terms of the generalized surface harmonics. Using polar coordinates to describe $\underset{\sim}{\theta}$ which has only x and y components, θ and ϕ are defined by

$$\theta = |\underset{\sim}{\theta}| \qquad \underset{\sim}{\theta} = \theta(\cos\phi, \sin\phi, 0) . \tag{119}$$

Then

$$(j,m|\exp[i\underset{\sim}{\theta}\cdot\underset{\sim}{S}]|j,m') = (-i)^{m-m'}[4\pi/(2j+1)]^{1/2} Y_j^{m,m'*}(\theta,\phi) . \tag{120}$$

One of the more interesting properties of the generalized surface harmonics is their behaviour under rotations. Let the angles θ, ϕ determine a unit vector $\underset{\sim}{\eta}$ by

$$\underset{\sim}{\eta} = (\sin\theta\cos\phi, \; \sin\theta\sin\phi, \; \cos\theta) . \tag{121}$$

Let $\underset{\sim}{\eta}'$ be the vector obtained from $\underset{\sim}{\eta}$ by means of the rotation $R(-\underset{\sim}{\Omega})$. That is

$$\underset{\sim}{\eta}' = R(-\underset{\sim}{\Omega})\underset{\sim}{\eta} , \tag{122}$$

where the elements of the rotation matrix are given by Eq. (8). Furthermore, let θ' and ϕ' be the angles which give $\underset{\sim}{\eta}'$ through Eq. (121). Then

$$Y_j^{mn}(\theta', \phi') = \exp[-2in\phi(\underset{\sim}{\Omega}, \underset{\sim}{\eta})] \sum_{m'=-j}^{j} Y_j^{m',n}(\theta,\phi)(j, m'|\exp[i\underset{\sim}{\Omega}\cdot\underset{\sim}{S}]|j, m) , \quad (123)$$

where $\phi(\underset{\sim}{\Omega}, \underset{\sim}{\eta})$ is given by Eq. (9b) .

It will now be convenient to express the matrix $R(\underset{\sim}{\theta})$ whose elements are given by Eq. (8) as

$$R(\underset{\sim}{\theta}) = \exp[i\underset{\sim}{\theta}\cdot\underset{\sim}{K}] , \quad (124)$$

where $\underset{\sim}{\theta}\cdot\underset{\sim}{K} = \sum_{i=1}^{3} \theta_i K_i$, where K_i are three matrices given by

$$K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; K_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \; K_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (125)$$

It is a well-known theorem that Eq. (124) is valid, and can be verified by direct expansion of the exponential using the fact that $(\underset{\sim}{\theta}\cdot\underset{\sim}{K})^3 = \theta^2(\underset{\sim}{\theta}\cdot\underset{\sim}{K})$.

Now the matrices K_i are unitarily equivalent to the matrices $S_i^{(1)}$:

$$K_i = V S_i^{(1)} V^{\dagger} \quad (126)$$

where the dagger means hermitian adjoint and the matrix V is given by

$$V = (2)^{-1/2} \begin{pmatrix} 1 & 0 & -1 \\ i & 0 & i \\ 0 & -(2)^{1/2} & 0 \end{pmatrix} . \quad (127)$$

It is clear that

$$V^{-1} = V^{\dagger} . \quad (128)$$

Let us construct the matrix Γ as follows for any unit vector $\underset{\sim}{\eta}$:

$$\Gamma = V \exp[i\underset{\sim}{\omega}\cdot\underset{\sim}{S}^{(1)}] = \exp[i\underset{\sim}{\omega}\cdot K]\, V , \tag{129}$$

where $\underset{\sim}{\omega}$ is related to $\underset{\sim}{\eta}$ by means of

$$\underset{\sim}{\omega} = \theta(\sin\phi,\ -\cos\phi,\ 0) , \tag{130}$$

where $\underset{\sim}{\eta}$ is given in terms of θ and ϕ by Eq. (121).

Then the vector $\underset{\sim}{Q}_1(\underset{\sim}{\eta})$ has its components formed from the components of the first column in Γ, the top component of the column vector being the x-component of $\underset{\sim}{Q}_1(\underset{\sim}{\eta})$, the second component being the y- component, and the third component being the z-component. $\underset{\sim}{Q}_0(\underset{\sim}{\eta})$ and $\underset{\sim}{Q}_{-1}(\underset{\sim}{\eta})$ are formed similarly from the second and third columns of Γ.

The rotation properties of $\underset{\sim}{Q}_\lambda(\underset{\sim}{\eta})$ given by Eq. (9) then follow from Eqs. (120) and (123). Let $\underset{\sim}{V}_1$, $\underset{\sim}{V}_0$, $\underset{\sim}{V}_{-1}$ be vectors constructed from the first, second, and third columns of the matrix V of Eq. (127). That is

$$\begin{aligned} \underset{\sim}{V}_\lambda &= (2)^{-1/2}(\lambda, i, 0) , \text{ for } \lambda = \pm 1 , \\ \underset{\sim}{V}_0 &= (0, 0, -1) . \end{aligned} \tag{131}$$

Then

$$\underset{\sim}{Q}_\lambda(\underset{\sim}{\eta}) = (4\pi/3)^{1/2} \sum_\mu \underset{\sim}{V}_\mu Y_1^{\mu\lambda*}(\theta, \phi) . \tag{132}$$

6. THE HELMHOLTZ THEOREM IN TERMS OF SPHERICAL COORDINATES

This section gives the general vector $\underset{\sim}{u}(\underset{\sim}{x})$ in terms of spherical coordinates in such a way that the Helmholtz decomposition $\underset{\sim}{u}(\underset{\sim}{x}) = \sum_\lambda \underset{\sim}{u}_\lambda(\underset{\sim}{x})$ is preserved. The components of $\underset{\sim}{u}_\lambda(\underset{\sim}{x})$ will also be given in terms of spherical coordinates. Toward this end, it will be useful to introduce the vector spherical harmonics.

6.1 Vector Spherical Harmonics

The vector spherical harmonics are defined in terms of our notation (Blatt and Weisskopf, 1952) by

$$\underset{\sim}{Y}_{JLM}(\theta, \phi) = - \sum_{m,\lambda} \underset{\sim}{V}_\lambda Y_L^{m,0}(\theta, \phi)(L, m, 1, \lambda | L, 1, J, M) . \tag{133}$$

where $\underset{\sim}{V}_\lambda$ is given by Eq. (131) and the summation is over all λ. Instead of giving the properties of the vector spherical harmonics, we refer the reader to Blatt and Weisskopf. We shall be interested in the components of $\underset{\sim}{Y}_{JLM}(\theta, \phi)$ in spherical coordinates. Let $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ be the three unit vectors in the x, y, z directions, respectively. Let $\underset{\sim}{a}_r, \underset{\sim}{a}_\theta, \underset{\sim}{a}_\phi$ be the unit vectors along the r-direction, θ-direction, and ϕ-direction respectively. Then, as is well known,

$$\underset{\sim}{a}_r = \sin\theta\cos\phi\,\underset{\sim}{i} + \sin\theta\sin\phi\,\underset{\sim}{j} + \cos\theta\,\underset{\sim}{k}\,,$$

$$\underset{\sim}{a}_\theta = \cos\theta\cos\phi\,\underset{\sim}{i} + \cos\theta\sin\phi\,\underset{\sim}{j} - \sin\theta\,\underset{\sim}{k}\,,$$

$$\underset{\sim}{a}_\phi = -\sin\phi\,\underset{\sim}{i} + \cos\phi\,\underset{\sim}{j}\,. \tag{134}$$

The following can be shown:

$$\underset{\sim}{a}_r \cdot \underset{\sim}{Y}_{JJM}(\theta, \phi) = 0,$$

$$\underset{\sim}{a}_r \cdot \underset{\sim}{Y}_{J,J-1,M}(\theta, \phi) = [J/(2J+1)]^{1/2} Y_J^{M,0}(\theta, \phi)\,,$$

$$\underset{\sim}{a}_r \cdot \underset{\sim}{Y}_{J,J+1,M}(\theta, \phi) = -[(J+1)/(2J+1)]^{1/2} Y_J^{M,0}(\theta, \phi)\,. \tag{135}$$

$$\sin\theta\,\underset{\sim}{a}_\theta \cdot \underset{\sim}{Y}_{JJM}(\theta, \phi) = -\frac{M}{[J(J+1)]^{1/2}}\,Y_J^{M,0}(\theta, \phi)\,,$$

$$\sin\theta\,\underset{\sim}{a}_\theta \cdot \underset{\sim}{Y}_{J,J-1,M}(\theta, \phi) = \frac{J}{2J+1}[(J-M+1)(J+M+1)/J(2J+3)]^{1/2}$$

$$\times\, Y_{J+1}^{M,0}(\theta, \phi) - \frac{J+1}{2J+1}[(J-M)(J+M)/J(2J-1)]^{1/2} Y_{J-1}^{M,0}(\theta, \phi)\,,$$

$$\sin\theta\,\underset{\sim}{a}_\theta \cdot \underset{\sim}{Y}_{J,J+1,M}(\theta, \phi) = \frac{J}{2J+1}[(J-M+1)(J+M+1)/(J+1)(2J+3)]^{1/2}$$

$$\times\, Y_{J+1}^{M,0}(\theta, \phi) - \frac{J+1}{2J+1}[(J-M)(J+M)/(J+1)(2J-1)]^{1/2} Y_{J-1}^{M,0}(\theta, \phi)\,. \tag{136}$$

$$\sin\theta\,\underset{\sim}{a}_\phi \cdot \underset{\sim}{Y}_{JJM}(\theta, \phi) = -i\Big\{[J(J-M+1)(J+M+1)/(J+1)(2J+1)(2J+3)]^{1/2}$$

$$\times\, Y_{J+1}^{M,0}(\theta, \phi) - [(J+1)(J-M)(J+M)/J(2J-1)(2J+1)]^{1/2} Y_{J-1}^{M,0}(\theta, \phi)\Big\}\,,$$

$$\sin\theta\, \underset{\sim}{a}_\phi \cdot \underset{\sim}{Y}_{J,J-1,M}(\theta,\phi) = i\, \frac{M}{[J(2J+1)]^{1/2}}\, Y_J^{M,0}(\theta,\phi)\ ,$$

$$\sin\theta\, \underset{\sim}{a}_\phi \cdot \underset{\sim}{Y}_{J,J+1,M}(\theta,\phi) = i\, \frac{M}{[(J+1)(2J+1)]^{1/2}}\, Y_J^{M,0}(\theta,\phi)\ . \qquad (137)$$

In Eqs. (135) through (137), one must recollect that $Y_J^{m,0}(\theta,\phi)$ are just the usual surface harmonics. See Eq. (112). The following sketches the derivation of Eqs. (135) through (137). Note that:

$$\underset{\sim}{a}_r \cdot \underset{\sim}{V}_0 = -\cos\theta\ ,$$

$$\underset{\sim}{a}_r \cdot \underset{\sim}{V}_\lambda = (2)^{-1/2} \sin\theta\, e^{i\ \phi}\ , \quad \text{for } \lambda = \pm 1\ . \qquad (138)$$

$$\underset{\sim}{a}_\theta \cdot \underset{\sim}{V}_0 = \sin\theta\ ,$$

$$\underset{\sim}{a}_\theta \cdot \underset{\sim}{V}_\lambda = (2)^{-1/2} \lambda \cos\theta\, e^{i\lambda\phi}\ , \quad \text{for } \lambda = \pm 1\ . \qquad (139)$$

$$\underset{\sim}{a}_\phi \cdot \underset{\sim}{V}_0 = 0,$$

$$\underset{\sim}{a}_\phi \cdot \underset{\sim}{V}_\lambda = i(2)^{-1/2}\, e^{i\lambda\phi}\ , \text{ for } \lambda = \pm 1\ . \qquad (140)$$

Also

$$\cos\theta = (4\pi/3)^{1/2} Y_1^{0,0}(\theta,\phi)\ ,$$

$$\sin\theta\, e^{i\phi} = -(8\pi/3)^{1/2} Y_1^{1,0}(\theta,\phi)\ ,$$

$$\sin\theta\, e^{-i\phi} = (8\pi/3)^{1/2} Y_1^{-1,0}(\theta,\phi)\ . \qquad (141)$$

After multiplying through by the appropriate unit vectors on both sides of Eq. (133) and also multiplying through by $\sin\theta$ to obtain Eqs. (136) and (137), substitute the appropriate expressions from Eqs. (138) through (141) on the right hand side, use Equation (118) to reduce the product of surface harmonics, and finally use the explicit forms for the vector coupling coefficients $(L, m, 1, \lambda | L, 1, J, M)$.

6.2 Expansion of Vector Fields in Terms of the Irreducible Representations of the Rotation Group

Let us now return to the problem of expanding an arbitrary vector $\underset{\sim}{u}(\underset{\sim}{x})$ in terms of spherical coordinates. The starting point is the expansion of Eqs. (13) and (13a). Define the function $g(p, j, m, \lambda)$ by

$$g(p,j,m,\lambda) = \int_0^{2\pi} d\hat{\phi} \int_0^{\pi} \sin\hat{\theta}\, d\hat{\theta}\, Y_j^{m,\lambda *}(\hat{\theta},\hat{\phi}) g_\lambda(\underset{\sim}{p}) . \tag{142}$$

In Eq. (142), $\hat{\theta}, \hat{\phi}$ is used as the polar angles of $\underset{\sim}{p}$; it is desired to reserve θ, ϕ to denote the polar angles of $\underset{\sim}{x}$. Strictly speaking, $g(p,j,m,\lambda)$ is defined for $j \geq |\lambda|$ only, but $g(p,j,m,\lambda)$ will be defined for $j = -1, 0, 1, 2, \ldots$ with the understanding that

$$g(p,0,0,\pm 1) = 0 , \quad g(p,-1,m,\lambda) = 0 \tag{142a}$$

for the sake of minimizing the difference in appearance of future expressions between the cases for which $\lambda = 0$ and $\lambda = \pm 1$.

From the completeness relation for the generalized surface harmonics, Eq. (114), $g_\lambda(\underset{\sim}{p})$ can be found from $g(p,j,m,\lambda)$ through

$$g_\lambda(\underset{\sim}{p}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} Y_j^{m,\lambda}(\hat{\theta},\hat{\phi}) g(p,j,m,\lambda) . \tag{143}$$

Thus one may express the vectors $\underset{\sim}{u}_\lambda(\underset{\sim}{x})$ as

$$\underset{\sim}{u}_\lambda(\underset{\sim}{x}) = (6\pi^2)^{-1/2} \sum_{j,m} \sum_{\mu} \int d\underset{\sim}{p}\; e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} Y_1^{\mu,\lambda *}(\theta,\phi) \underset{\sim}{V}_\mu$$

$$\times\, Y_j^{m,\lambda}(\hat{\theta},\hat{\phi}) g(p,j,m,\lambda) , \tag{144}$$

where Eq. (132) has been used for $\underset{\sim}{Q}_\lambda(\underset{\sim}{\eta})$.

It is clear that from the construction and from Eq. (12) that $\underset{\sim}{\nabla} \times \underset{\sim}{u}_\lambda(\underset{\sim}{x})$ has the same expansion as Eq. (144) except that $g(p,j,m,\lambda)$ is replaced by $\lambda p g(p,j,m,\lambda)$. Furthermore, from Eqs. (12) and (13a), it will be shown that

$$\underset{\sim}{\nabla}\cdot \underset{\sim}{u}_0(\underset{\sim}{x}) = -(2/\pi)^{1/2} i \sum_{k=0}^{\infty} \sum_{m=-k}^{k} Y_k^{m,0}(\theta,\phi)(i)^k$$

$$\times \int_0^\infty dp\; p^3 j_k(pr) g(p,k,m,0) , \tag{145}$$

where $r = |\underset{\sim}{x}|$ and θ, ϕ are the polar angles of $\underset{\sim}{x}$. In Eq. (145), the functions $j_k(x)$ are the usual spherical Bessel functions. To derive Eq. (145), Eq. (12) is used in Eq. (13a) for $\lambda = 0$. But a well-known expansion in terms of our notation for $e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}$ is

$$e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} = 4\pi \sum_{q=0}^{\infty} \sum_{r=-q}^{q} (i)^q j_q(pr) Y_q^{r,0}(\theta,\phi) Y_q^{r,0*}(\hat{\theta},\hat{\phi}) . \tag{146}$$

Equations (145) is obtained by substituting this expression for $e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}$, then integrating over $\hat{\theta}, \hat{\phi}$ and finally using the orthogonality relation for the surface harmonics of Eq. (113).

The functions $g(p, j, m, \lambda)$ have interesting properties under rotations of coordinates. Under the rotation of axis described by the vector $\underset{\sim}{\theta}$ ($\underset{\sim}{\theta}$ is perfectly general and has no connection with the polar angles θ, ϕ), the vector $\underset{\sim}{u}_\lambda(\underset{\sim}{x})$ goes over into $\underset{\sim}{u}_\lambda{}'(\underset{\sim}{x})$ where $\underset{\sim}{u}_\lambda{}'(\underset{\sim}{x})$ is given by Eq. (17). Similarly, the amplitudes $g_\lambda(\underset{\sim}{p})$ go over into $g_\lambda{}'(\underset{\sim}{p})$, where $g_\lambda{}'(\underset{\sim}{p})$ is given by Eq. (18). It has been shown by Moses (1967, 1970) that the relations Eqs. (142) and (143) which relate $g(p, j, m, \lambda)$ and $g_\lambda(\underset{\sim}{p})$ lead to the following relation between the transformed coefficients $g'(p, j, m, \lambda)$ and the coefficients in the original frame $g(p, j, m, \lambda)$:

$$g'(p, j, m, \lambda) = \sum_{m'=-j}^{j} (j, m \,|\, \exp[\, i\underset{\sim}{\theta}\cdot\underset{\sim}{S} \,] \,|\, j, m') g(p, j, m', \lambda) , \tag{147}$$

where $(j, m|\exp[i\underset{\sim}{\theta}\cdot\underset{\sim}{S}]|j, m')$ is given by Eq. (110). Thus the coefficients $g(p, j, m, \lambda)$ transform under the irreducible representations of the rotation group without any mixing of the variables λ. This expansion confirms the rotational invariance of the Helmholtz decomposition of this report.

Equation (144) will now be simplified through the use of vector spherical harmonics. The right hand side of Eq. (146) is substituted for $e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}$, the product of $Y_q^{r,0*}(\hat{\theta},\hat{\phi})$ and $Y_1^{\mu,\lambda*}(\hat{\theta},\hat{\phi})$ is reduced through the use of Eq. (118), the orthogonality relation Eq. (113) is used on integrating over the variables $\hat{\theta}, \hat{\phi}$ of $\underset{\sim}{p}$, the definition of the vector spherical harmonics Eq. (133) is used, and finally the known values for the remaining vector coupling coefficients $(L, 0, 1, \lambda|L, 1, J, \lambda)$ are substituted to obtain the following:

$$\underset{\sim}{u}_\lambda(\underset{\sim}{x}) = -(\pi)^{-1/2} \sum_{k=1}^{\infty} \sum_{m=-k}^{k} (i)^k \Big\{ -\lambda \underset{\sim}{Y}_{k,k,m}(\theta,\phi)\, A_{k,m,\lambda,k}(r)$$

$$+ [k/(2k+1)]^{1/2} \underset{\sim}{Y}_{k,k+1,m}(\theta,\phi) A_{k,m,\lambda,k+1}(r)$$

$$-i[(k+1)/(2k+1)]^{1/2} \underset{\sim}{Y}_{k,k-1,m}(\theta,\phi) A_{k,m,\lambda,k-1}(r) \Big\},$$

for $\lambda = \pm 1$, and

$$\underset{\sim}{u}_0(\underset{\sim}{x}) = (2/\pi)^{1/2} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} (i)^{k+1} \Big\{ [(k+1)/(2k+1)]^{1/2} \underset{\sim}{Y}_{k,k+1,m}(\theta,\phi)$$

$$\times A_{k,m,0,k+1}(r) + [k/(2k+1)]^{1/2} \underset{\sim}{Y}_{k,k-1,m}(\theta,\phi)$$

$$\times A_{k,m,0,k-1}(r) \Big\}, \tag{148}$$

where

$$A_{j,m,\lambda,k}(r) = \int_0^\infty p^2 dp\, j_k(pr) g(p,j,m,\lambda) \ . \tag{149}$$

The vector potential $\underset{\sim}{A}_\lambda(\underset{\sim}{x})$ of Eq. (15) for $\lambda = \pm 1$ has the same expansion as the first of Eq. (148) with $g(p,j,m,\lambda)$ replaced by $(\lambda/p)g(p,j,m,\lambda)$. The scalar potential $V(\underset{\sim}{x})$ of Eq. (15) is given by

$$V(\underset{\sim}{x}) = (2/\pi)^{1/2} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} Y_k^{m,0}(\theta,\phi) \int_0^\infty p\, dp\, j_k(pr)(i)^{k+1} g(p,k,m,0). \tag{150}$$

Though the expansions of Eq. (148) are very useful, for many applications one wishes the components of $\underset{\sim}{u}_\lambda(\underset{\sim}{x})$ in spherical coordinates. Hence $\underset{\sim}{a}_r \cdot \underset{\sim}{u}_\lambda(\underset{\sim}{x})$, $\sin\theta\ \underset{\sim}{a}_\theta \cdot \underset{\sim}{u}_\lambda(\underset{\sim}{x})$ and $\sin\theta\ \underset{\sim}{a}_\phi \cdot \underset{\sim}{u}_\lambda(\underset{\sim}{x})$ will be given in terms of $g(p,j,m,\lambda)$. One uses Eqs. (135) through (137) and sees that the components of $\underset{\sim}{u}_\lambda(\underset{\sim}{x})$ are sums of series in $Y_k^{m,0}$, $Y_{k\pm1}^{m,0}$. In order that the components be a series in $Y_k^{m,0}$ only (that is, a series in surface harmonics), $k\pm1$ is replaced by k in summations and the series is then expressed in terms of the spherical Bessel functions j_k, $j_{k\pm1}$, $j_{k\pm2}$. The functions $j_{k\pm1}$ and $j_{k\pm2}$ can be expressed in terms of j_k and derivatives of j_k by using the well-known relations, recursively, if necessary:

$$\frac{k+1}{x} j_k(x) + \frac{d}{dx} j_k(x) = j_{k-1}(x) \, ,$$

$$\frac{k}{x} j_k(x) - \frac{d}{dx} j_k(x) = j_{k+1}(x) \, . \qquad (151)$$

The differential equation for the spherical Bessel functions is also used, namely

$$x^2 \frac{d^2}{dx^2} j_k(x) + 2x \frac{d}{dx} j_k(x) + [x^2 - k(k+1)] j_k(x) = 0 \, . \qquad (152)$$

Though the details of the calculation are tedious, one can give the final answers in a straightforward manner.

$$\underset{\sim}{a}_r \cdot \underset{\sim}{u}_\lambda(\underset{\sim}{x}) = (\pi)^{-1/2} \frac{1}{r} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} (i)^{k+1} [k(k+1)]^{1/2} Y_k^{m,0}(\theta, \phi)$$

$$\times \int_0^\infty p \, dp \, j_k(pr) g(p, k, m, \lambda) \, , \qquad (153)$$

for $\lambda = \pm 1$, and

$$\underset{\sim}{a}_r \cdot \underset{\sim}{u}_C(\underset{\sim}{x}) = (2/\pi)^{1/2} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} (i)^{k+1} Y_k^{m,0}(\theta, \phi)$$

$$\times \int_0^\infty p^2 dp \, j_k'(pr) g(p, k, m, 0) \, . \qquad (154)$$

In Eq. (154), the prime means derivative with respect to the argument of the function. Similarly, later, the double prime means that a second derivative has been taken with respect to the argument.

$$\sin\theta \, \underset{\sim}{a}_\theta \cdot \underset{\sim}{u}_\lambda(\underset{\sim}{x}) = -(\pi)^{-1/2} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} (i)^k Y_k^{m,0}(\theta, \phi)$$

$$\times \Bigg\{ \int_0^\infty p^2 dp \, j_k(pr) \Bigg[\frac{m\lambda}{[k(k+1)]^{1/2}} g(p, k, m, \lambda)$$

$$+ [(k-1)(k-m)(k+m)/k(2k-1)(2k+1)]^{1/2} [1 - \frac{k(k+1)}{p^2 r^2}] g(p, k-1, m, \lambda)$$

$$- [(k+2)(k-m+1)(k+m+1)/(k+1)(2k+1)(2k+3)]^{1/2}\left(1 - \frac{k(k+1)}{p^2r^2}\right)$$

$$\times g(p, k+1, m, \lambda)\Big]$$

$$- \frac{1}{r}\int_0^\infty p\ dp\ j_k'(pr)\Big[[(k-1)k(k-m)(k+m)/(2k-1)(2k+1)]^{1/2}$$

$$\times g(p, k-1, m, \lambda)$$

$$+ [(k+1)(k+2)(k-m+1)(k+m+1)/(2k+1)(2k+3)]^{1/2}$$

$$\times g(0, k+1, m, \lambda)\Big]\Big\}, \tag{155}$$

for $\lambda = \pm 1$.

In Eq. (155) and later, $g(p, 0, 0, \pm 1) = g(p, -1, m, \lambda) = 0$. An alternative expression for $\sin\theta\ \underset{\sim}{a}_\theta \cdot \underset{\sim}{u}_\lambda(\underset{\sim}{x})$ for $\lambda = \pm 1$ is given by

$$\sin\theta\ \underset{\sim}{a}_\theta \cdot \underset{\sim}{u}_\lambda(\underset{\sim}{x}) = -(\pi)^{-1/2} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} (i)^k Y_k^{m,0}(\theta, \ldots$$

$$\times \Big\{ \int_0^\infty p^2 dp\ j_k(pr)\ \frac{m\lambda}{[k(k+1)]^{1/2}}\ g(p, k, m, \lambda)$$

$$- \int_0^\infty p^2 dp\ j_k''(pr)\Big[[(k-1)(k-m)(k+m)/k(2k-1)(2k+1)]^{1/2}$$

$$\times g(p, k-1, m, \lambda) - [(k+2)(k-m+1)(k+m+1)/(k+1)(2k+1)(2k+3)]^{1/2}$$

$$\times g(p, k+1, m, \lambda)\Big] - \frac{1}{r}\int_0^\infty p\ dp\ j_k'(pr)$$

$$\times \Big[(k+2)[(k-1)(k-m)(k+m)/k(2k-1)(2k+1)]^{1/2}\ g(p, k-1, m, \lambda)$$

$$+ (k-1)[(k+2)(k-m+1)(k+m+1)/(k+1)(2k+1)(2k+3)]^{1/2}$$

$$\times g(p, k+1, m, \lambda)\Big]\Big\} . \tag{156}$$

Also

$$\sin\theta\, \underset{\sim}{a}_\theta \cdot \underset{\sim}{u}_0(\underset{\sim}{x}) = (2/\pi)^{1/2} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} (i)^k Y_k^{m,0}(\theta,\phi)$$

$$\times \Big\{ \frac{1}{r^2} \int_0^\infty dp\, j_k(pr) \Big[(k-1)(k+1)[(k-m)(k+m)/(2k-1)(2k+1)]^{1/2}$$

$$\times\ g(p,k-1,m,0) + k(k+2)[(k-m+1)(k+m+1)/(2k+1)(2k+3)]^{1/2}$$

$$\times\ g(p,k+1,m,0)\Big] + \frac{1}{r} \int_0^\infty dp\, p\, j_k'(pr)$$

$$\times \Big[(k-1)[(k-m)(k+m)/(2k-1)(2k+1)]^{1/2} g(p,k-1,m,0)$$

$$-\ (k+2)[(k-m+1)(k+m+1)/(2k+1)(2k+3)]^{1/2} g(p,k+1,m,0)\Big]\Big\} . \quad (157)$$

Finally,

$$\sin\theta\, \underset{\sim}{a}_\phi \cdot \underset{\sim}{u}_\lambda(\underset{\sim}{x}) = -(\pi)^{-1/2} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} (1)^k [k(k+1)]^{-1/2} Y_k^{m,0}(\theta,\phi)$$

$$\times \Big\{ \frac{1}{r} \int_0^\infty p\, dp\, j_k(pr) \Big[m g(p,k,m,\lambda) + \lambda(k+1)$$

$$\times\ [(k-1)(k+1)(k-m)(k+m)/(2k-1)(2k+1)]^{1/2} g(p,k-1,m,\lambda)$$

$$+\ \lambda k[k(k+2)(k-m+1)(k+m+1)/(2k+1)(2k+3)]^{1/2} g(p,k+1,m,\lambda)\Big]$$

$$+ \int_0^\infty p^2 dp\, j_k'(pr) \Big[m g(p,k,m,\lambda)$$

$$+\ \lambda[(k-1)(k+1)(k-m)(k+m)/(2k-1)(2k+1)]^{1/2} g(p,k-1,m,\lambda)$$

$$-\ \lambda[k(k+2)(k-m+1)(k+m+1)/(2k+1)(2k+3)]^{1/2} g(p,k+1,m,\lambda)\Big]\Big\} . \quad (158)$$

for $\lambda = \pm 1$, and

$$\sin\theta\, \underset{\sim}{a}_\phi \cdot \underset{\sim}{u}_0(\underset{\sim}{x}) = -(2/\pi)^{1/2} \frac{1}{r} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} (i)^k m Y_k^{m,0}(\theta,\phi)$$

$$\times \int_0^\infty p\, dp\, j_k(pr)\, g(p,k,m,0) . \quad (159)$$

In Eq. (158), the ambiguity of the term for $k = 0$ is evaluated in the obvious way by setting $g(p, -1, n, \lambda) = g(p, 0, m, \lambda) = 0$ and then by multiplying through by the factor $[k(k+1)]^{-1/2}$ (so as to cancel the factor $k^{1/2}$) before setting $k = 0$.

Thus our program of exhibiting the components of the rotational and irrotational parts of an arbitrary vector in a spherical coordinate system has been completed. It is interesting to note that while general surface harmonics $Y_j^{m,n}$ had to be introduced to define $g(p, j, m, \lambda)$, the expansions of the components of the vectors involve the ordinary spherical harmonics $Y_j^{m,0}$. The components of the curl of the vector in spherical coordinates have the same expansion except that $g(p, j, m, \lambda)$ is replaced by $\lambda p g(p, j, m, \lambda)$.

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Appendix A

Motivation for the Introduction of the Eigenfunctions of the Curl Operator

It will now be shown how we came to use the vectors $\underset{\sim}{Q}_\lambda(\underset{\sim}{\eta})$ from considerations of properties of representations of the rotation group.

Let us define the vector $\underset{\sim}{G}(\underset{\sim}{p})$ as being the Fourier transform of the vector $\underset{\sim}{u}(\underset{\sim}{x})$:

$$\underset{\sim}{G}(\underset{\sim}{p}) = (2\pi)^{-3/2} \int d\underset{\sim}{x}\; e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, \underset{\sim}{u}(\underset{\sim}{x}) , \tag{A1}$$

$$\underset{\sim}{u}(\underset{\sim}{x}) = (2\pi)^{-3/2} \int d\underset{\sim}{p}\; e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, \underset{\sim}{G}(\underset{\sim}{p}) . \tag{A2}$$

Then

$$\underset{\sim}{\nabla} \times \underset{\sim}{u}(\underset{\sim}{x}) = i(2\pi)^{-3/2} \int d\underset{\sim}{p}\; e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, [\underset{\sim}{p} \times \underset{\sim}{G}(\underset{\sim}{p})] . \tag{A3}$$

Let us now introduce the column vector $G(\underset{\sim}{p})$ whose components from top to bottom are the x-, y-, and z- components respectively of $\underset{\sim}{G}(\underset{\sim}{p})$. It is readily shown that the column vector corresponding to the vector $\underset{\sim}{p}\times\underset{\sim}{G}(\underset{\sim}{p})$ is $-i(\underset{\sim}{p}\cdot\underset{\sim}{K})G(\underset{\sim}{p})$, where $\underset{\sim}{p}\cdot\underset{\sim}{K} = \sum_i p_i K_i$ and the matrices K_i are given by Eq. (125). As is shown in Eq. 24), these matrices are the infinitesimal generators of the rotation matrix. Let us now define the column vector $H(\underset{\sim}{p})$ by

$$G(\underset{\sim}{p}) = VH(\underset{\sim}{p}),$$

$$H(\underset{\sim}{p}) = V^{-1}G(\underset{\sim}{p}), \qquad \text{(A4)}$$

where V is the matrix of Eq. (127). Then the column vector $(\underset{\sim}{p}\cdot\underset{\sim}{K})G(\underset{\sim}{p})$ maps into the column vector $(\underset{\sim}{p}\cdot\underset{\sim}{S}^{(1)})H(\underset{\sim}{p})$, where the matrices $S_i^{(1)}$ have the elements given by (106). This mapping follows from Eq. (126). Because of the way that the rows and columns of the representations of the generators of the rotation group were labelled, the top component of $H(\underset{\sim}{p})$ will be labelled $H_1(\underset{\sim}{p})$, the second component by $H_0(\underset{\sim}{p})$, and the bottom component by $H_{-1}(\underset{\sim}{p})$. Now in Moses (1970) and in others mentioned in that reference, a representation is introduced called the helicity representation, in which $\underset{\sim}{p}\cdot\underset{\sim}{S}^{(1)}$ is diagonal. Thus let us introduce the column vector $g(\underset{\sim}{p})$ whose top, middle, and bottom components are denoted by $g_1(\underset{\sim}{p})$, $g_0(\underset{\sim}{p})$ and $g_{-1}(\underset{\sim}{p})$, respectively by

$$g(\underset{\sim}{p}) = e^{-i\underset{\sim}{\omega}\cdot\underset{\sim}{S}^{(1)}} H(\underset{\sim}{p}), \quad H(\underset{\sim}{p}) = e^{i\underset{\sim}{\omega}\cdot\underset{\sim}{S}^{(1)}} g(\underset{\sim}{p}), \qquad \text{(A5)}$$

where $\underset{\sim}{\omega}$ is a function of $\underset{\sim}{p}$ given by

$$p_1 = -p\frac{\omega_2}{\omega}\sin\omega, \; p_2 = p\frac{\omega_1}{\omega}\sin\omega, \; p_3 = p\cos\omega,$$

$$\omega_3 = 0, \; \omega = |\underset{\sim}{\omega}|. \qquad \text{(A5a)}$$

The relation of $\underset{\sim}{\omega}$ to the polar coordinates of $\underset{\sim}{p}$ is given by Eq. (130).

The column vector $(\underset{\sim}{p}\cdot\underset{\sim}{S}^{(1)})H(\underset{\sim}{p})$ maps into the column vector $g'(\underset{\sim}{p})$ whose components are $\lambda p g_\lambda(\underset{\sim}{p})$. Thus, combining all the transformations, $\underset{\sim}{u}(\underset{\sim}{x})$ maps into the column vector $g(\underset{\sim}{p})$ in such a way that $\underset{\sim}{\nabla}\times\underset{\sim}{u}(\underset{\sim}{x})$ maps into the column vector $g'(p)$. Equations (13) are an explicit form for the mapping and Eqs. (12) and (129) constitute a statement of the mapping of the curl operator.

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13. ABSTRACT

Air Force Requirements, such as the knowledge of the upper atmosphere environment of vehicles and the knowledge of the propagation characteristics of radio and radar signals, require the solutions of the equations of motion of fluid dynamics and of electromagnetic theory which are often very complicated. This report presents a new mathematical approach to the obtaining of such solutions. The vector field is represented in such a form that new techniques may be used to find the appropriate solutions. Some problems of fluid dynamics and electromagnetic theory are solved as an illustration of the new approach. In later papers, new techniques will be used in other problems related to Air Force needs.

In this report, eigenfunctions of the curl operator are introduced. The expansion of vector fields in terms of these eigenfunctions leads to a decomposition of such fields into three modes, one of which corresponds to an irrotational vector field, and two of which correspond to rotational circularly polarized vector fields of opposite signs of polarization. Under a rotation of coordinates, the three modes which are introduced in this fashion remain invariant. Hence the Helmholtz decomposition of vector fields has been introduced in an irreducible, rotationally invariant form.

These expansions enable one to handle the curl and divergence operators simply. As illustrations of the use of the curl eigenfunctions, four problems are solved.

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