

ALGORITHMS FOR COMPLEMENTARY TREES IN A GRAPH

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TECHNICAL REPORT NO. 89-13

MARCH 1990



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DECEMBER 1969

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ON BOUNDS FOR COMPLEMENTARY TREES IN A GRAPH

By

Ilan Adler

Consider the undirected network $G = (N, \alpha)$ where $|N| = n+1$ ($n \geq 3$), $\alpha = \{a_1, \dots, a_n, \bar{a}_1, \dots, \bar{a}_n\}$ and there are no loops or repeated arcs. The pairs of arcs $a_i, \bar{a}_i$ ($i = 1, \dots, n$) are called complementary arcs.

Let $\mathcal{N}(n)$ denotes the class of all networks that have the form of G .

Definition: Let T be a spanning tree of G ($G \in \mathcal{N}(n)$ for some $n, n \geq 3$) if $a_i \in T$ iff $a_i \notin T$ $i = 1, \dots, n$ then T is called a complementary tree of G .

G. B. Dantzig in [1] proved that if there exists one complementary tree in $G, G \in \mathcal{N}(n)$ ($n \geq 3$), then there exists at least two. The proof is by means of an algorithm which finds a different complementary tree from a given one. This algorithm is described in the first part of this report (Theorem 1). In the second part we show (using basically the same algorithm) that if an arc belongs to one complementary tree then it belongs to at least two (Theorem 2). This in turn implies that if there exists one complementary tree (in a given network which belongs to $\mathcal{N}(n)$ ($n \geq 3$)) there exists at least four (Theorem 3). Let denote by $\bar{\mathcal{N}}(n)$ the class of all networks G such that $G \in \mathcal{N}(n)$ and G has at least one complementary tree; by C_G the number of complementary trees in G ($G \in \mathcal{N}(n)$), and let $C(n) = \min\{C_G | G \in \bar{\mathcal{N}}(n)\}$.

Using these notations Theorem 1 implies $C(n) \geq 2$ while Theorem 3

implies $C(n) \geq 4$. In the last part three examples are provided to show that $C(3) = 4$, $C(4) \leq 6$ and $C(n) \leq 8$ for $n \geq 5$.

Theorem 1: (Dantzig) $C(n) \geq 2$ ($n \geq 3$) i.e., if there exists one complementary tree in G ($G \in \overline{\mathcal{T}}(n)$), there exists at least two.

The general idea in proving this theorem is to pass from one complementary tree to the other by a sequence of "adjacent" or ("neighboring") trees which are "almost" complementary.

Definition: Two trees are said to be adjacent or neighbors if they differ by one arc.

Definition: An almost complementary tree T of G is a spanning tree of G such that for some i_n and i_{n-1} :

- (i) $a_{ij} \in T$ iff $\overline{a_{ij}} \notin T$ $j = 1 \dots n-2$
- (ii) $a_{i_n}, \overline{a_{i_n}} \in T$

less formally, an almost complementary tree is a spanning tree in which each set of arcs $a_i, \overline{a_i}$ furnishes exactly one arc with the exception of one "special" set which furnishes two and one other set which furnishes none.

The procedure for generating a sequence of adjacent almost-complementary trees starting from a given complementary tree and ending in a different one is as follows:

Step 1: Arbitrarily order the nodes in G (e.g., assign, arbitrarily, the first $n+1$ natural numbers to the nodes of G).

Step 2: Orient each arc (i,j) of G as a directed arc from i to j if $i < j$ and from j to i if $j < i$.

Step 3: Assign the (arbitrary) flow values a_{ij} to all arcs (i,j) in G such that $a_{ij} > 0$ if (i,j) belongs to the given complementary tree, $a_{ij} = 0$ otherwise (and $a_{ji} = -a_{ij}$).

The first four steps determine the following network flow problem:

Find $x_{ij} \geq 0$ such that

$$\sum_{i \in U_j} x_{ij} - \sum_{k \in V_j} x_{jk} = b_j \quad i = 1, 2, \dots, n+1$$

where

$$b_j = \sum_i a_{ij} \quad j = 1, \dots, n+1$$

$$U_j = \{j | (i,j) \text{ is directed arc of } G\}$$

$$V_j = \{j | (j,k) \text{ is directed arc of } G\}$$

It is well known that the arcs (i,j) corresponding to basic variables form a tree. If the feasible basic solutions of the network flow problem are non-degenerate and bounded then a new feasible solution from given another one can be obtained by increasing sufficiently the flow x_{ij} on a directed out-of-tree arc (i,j) while adjusting the flows on basic arcs. The arc dropping out of the cycle will then correspond to the unique basic variable whose value decreased to zero. For the network flow problem defined above boundness results from Steps 1 and 2 and non-degeneracy can be guaranteed by assigning (in Step 3) n different powers of $a > 0$ to the n arc flows of the starting complementary tree.

Step 4: Add to the starting complementary tree any out-of-the-tree arc

arc say $\bar{a}_1$ (assume a_1 is in the tree).

Step 5: Drop arc of the current tree and determine the new flow values as in the simplex algorithm. If arc dropped is a_1 or $\bar{a}_1$ then the tree is complementary, terminate; if not go to Step 6.

Step 6: Introduce as incoming arc the complement of the arc dropped.

Go to Step 5.

It is shown in [1] that the algorithm terminates in finite number of steps and that the complementary tree we arrive at is different from the starting one.

Notice that the orientation of arcs and the assignment of flow values are arbitrary. In Theorems 2 and 3 it is shown that choosing the directions and the flows in a more specific way can extend the result of Theorem 1.

Theorem 2: If $T_0 = \{a_1, \dots, a_n\}$ is a C.T. (complementary tree) of G then for every i ($1 \leq i \leq n$) there exists a C.T. (say T_1) such that $a_i \in T_1$ and $T_1 \neq T_0$.

Proof: Given T_0 and a_1 (the arc we want to keep in the second complementary tree) use the same algorithm as in Theorem 1 with the following changes:

Replace Steps 1 and 3 by 1a and 3a respectively.

Step 1a: Index the end nodes of a_1 by n and $n+1$, index all other nodes, arbitrarily, by $1, 2, \dots, n-1$.

Step 3a: Assign to a_1 the flow value M where M is a positive constant which is greater than any finite number with which it will be compared during the steps of the algorithm. Assign arbitrary positive flow values to all other arcs in T_0 and 0 for all out of T_0 arcs.

This modification is within the framework of Steps 1 and 3, it just gives more specific directions to execute these steps, hence it does not effect the proof given in [1], that a different C.T. T^* is obtained in a finite number of iterations. Let us denote the i^{th} A.C.T. (almost complementary tree) in this path by $\bar{T}_i$, thus the path has the form:

$$T_0 \rightarrow \bar{T}_1 \rightarrow \bar{T}_2 \dots \rightarrow \bar{T}_m = T^*$$

Induction assumption: $a_1 \in \bar{T}_k$, the flow value of a_1 is M_k where M_k is greater than any finite number with which it will be compared and all other flow values (of arcs in $\bar{T}_k$) are not a function of M .

The induction assumption is trivially true for $k = 0$. Assume the induction assumption is true for some k ($0 \leq k \leq m-1$). Let us introduce the next out-of-tree arc in accordance with the algorithm. This arc together with some arcs in $\bar{T}_k$ form a cycle. If a_1 is not contained in this cycle then clearly the induction assumption holds for $\bar{T}_{k+1}$.

If a_1 is contained in this cycle then either the flow on a_1 is increasing or decreasing with increasing flow on the out-of-tree arc. In the first case a_1 remains in the next tree and its flow value increases. In the second case Steps 1a and 2 guarantee that there is at least one other arc of $\bar{T}_k$ which its flow value decreases while the

flow value on the out-of-tree arc increases (Steps 1a and 2 imply that all arcs with end point n have the same direction as a_1 , but a cycle which includes a_1 must include at least one other arc with end point in n). By the simplex method the arc with the smallest flow value, among those in the cycle, which have opposite direction from the out-of-tree arc drops. By the induction assumption and the above argument a_1 remains in the next A.C.T., and its flow value changes by a small amount relative to M_k . In both cases the flow value of any arc can increase by at most the flow value of the arc which is dropped hence the induction assumption holds for $k+1$.

$\therefore a_1 \in \bar{T}_k \quad k = 1, 2, \dots, m$ so $a_1 \in \bar{T}_m = T^*$, letting $T_1 = T^*$ completes the proof.

Theorem 3: $c(n) \geq 4$ ($n \geq 3$). i.e., if there exists one complementary tree in G ($G \in \mathcal{T}(n)$), then there exists at least four.

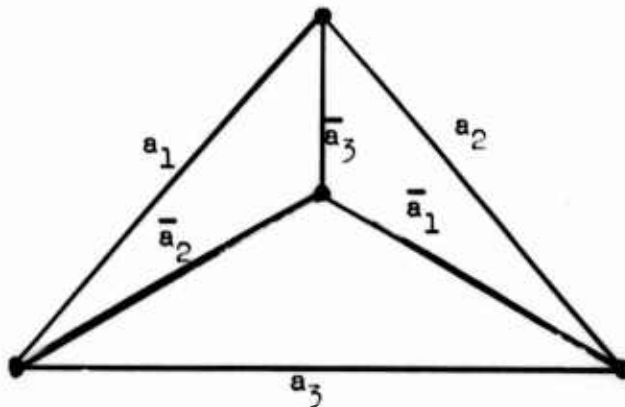
Proof: Let $T_1 = \{a_1, \dots, a_n\}$ be a C.T. of G , then by Theorem 2 there exists a C.T. T_2 such that $a_1 \in T_2$ and $T_2 \neq T_1$. $T_1 \neq T_2$ implies that there is at least one index k ($2 \leq k \leq n$) such that $\bar{a}_k \in T_2$. Starting now with T_1 and a_k as the arc to be kept in the C.T. we get, by using the modified algorithm another C.T. (say T_3) such that $a_k \in T_3$ and $T_3 \neq T_1$. Implying the same technique for T_2 and $\bar{a}_k$ we get a C.T. T_4 such that $\bar{a}_k \in T_4$ and $T_4 \neq T_2$. Clearly also $T_4 \neq T_1$, $T_3 \neq T_2$ and $T_4 \neq T_3$.

$\therefore T_1, T_2, T_3$ and T_4 are C.T.'s and no two of them are identical.

So far we found a lower bound for $c(n)$. Next we try to provide some upper bounds. In the following three examples it is shown that

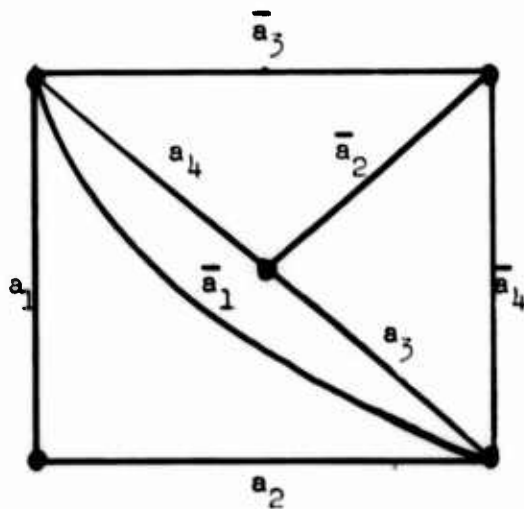
$c(3) = 4$, $c(4) \leq 6$ and $c(n) \leq 8$ for $n \geq 5$ (Examples 1, 2 and 3 respectively).

Example 1:



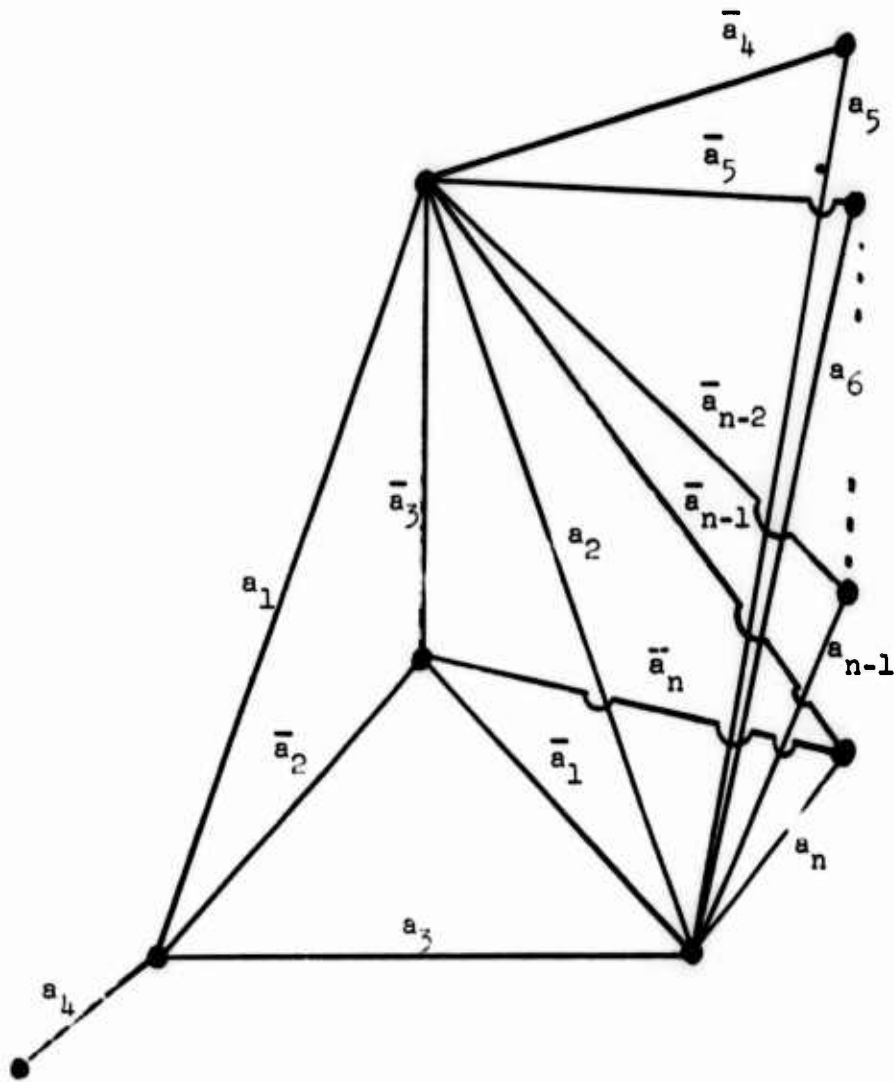
complementary trees: $\{a_1, \bar{a}_2, a_3\}$; $\{\bar{a}_1, a_2, a_3\}$; $\{a_1, a_2, \bar{a}_3\}$; $\{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$.

Example 2:



complementary trees: $\{a_1, a_2, a_3, \bar{a}_4\}$; $\{a_1, a_2, \bar{a}_3, a_4\}$; $\{a_1, \bar{a}_2, a_3, a_4\}$; $\{a_1, \bar{a}_2, \bar{a}_3, \bar{a}_4\}$; $\{\bar{a}_1, a_2, a_3, \bar{a}_4\}$; $\{\bar{a}_1, a_2, \bar{a}_3, a_4\}$.

Example 3:



complementary trees:

$$\begin{aligned}
 &\{a_1, \bar{a}_2, a_3, a_4, a_5, \dots, a_{n-1}, a_n\}; \{a_1, \bar{a}_2, a_3, a_4, a_5, \dots, a_{n-1}, \bar{a}_n\} \\
 &\{\bar{a}_1, a_2, a_3, a_4, a_5, \dots, a_{n-1}, a_n\}; \{\bar{a}_1, a_2, a_3, a_4, a_5, \dots, a_{n-1}, \bar{a}_n\} \\
 &\{a_1, a_2, \bar{a}_3, a_4, a_5, \dots, a_{n-1}, a_n\}; \{a_1, a_2, \bar{a}_3, a_4, a_5, \dots, a_{n-1}, \bar{a}_n\} \\
 &\{\bar{a}_1, \bar{a}_2, \bar{a}_3, a_4, a_5, \dots, a_{n-1}, a_n\}; \{\bar{a}_1, \bar{a}_2, \bar{a}_3, a_4, a_5, \dots, a_{n-1}, \bar{a}_n\}
 \end{aligned}$$

References

1. Dantzig, G. B., Complementary Spanning Trees, Computer Science Department, Stanford University, Technical Report No. 126, March 1969.

Unclassified
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DOCUMENT CONTROL DATA - R&D		
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)		
1. ORIGINATING ACTIVITY (Corporate author) Department of Operations Research Stanford University STANFORD, California 94305		2a. REPORT SECURITY CLASSIFICATION Unclassified
		2b. GROUP
3. REPORT TITLE On Bounds for Complementary Trees in a Graph		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Technical Report		
5. AUTHOR(S) (Last name, first name, initial) ADLER, Ian		
6. REPORT DATE December 1969	7a. TOTAL NO. OF PAGES 9	7b. NO. OF REFS 1
8a. CONTRACT OR GRANT NO. N-00014-67-A-0112-0011	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 69-13	
8b. PROJECT NO. NR-047-064	9b. OTHER REPORT NUM(S) (Any other numbers that may be assigned this report)	
10. AVAILABILITY/LIMITATION NOTICES This document has been approved for public release and sale; its distribution is unlimited.		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Logistics & Mathematical Statistics Branch Mathematical Sciences Division, ONR Washington, D.C. 20306	
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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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