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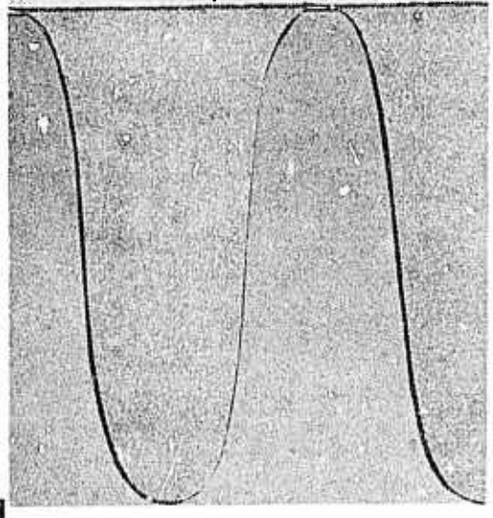

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A GENERALIZATION OF AN INEQUALITY OF VAN DANTZIG

B. Harris

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ABSTRACT

The inequality

$$\int_0^1 f^2(x)\,dx \leq \frac{1}{12}$$

was stated by van Dantzig for all functions of a certain class.

This inequality is generalized.

A GENERALIZATION OF AN INEQUALITY OF VAN DANTZIG

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1. Introduction. In a 1951 paper on the power of the Wilcoxon two sample test, D. van Dantzig stated without proof the following inequality.

For $0 \leq x \leq 1$, let $f(x)$ satisfy

(a) $\int_0^1 f(x)\,dx = 0$

(b) $f(1) \leq f(0)$

(c) $f(x) + x$ is monotone non-decreasing. Then

$$\int_0^1 f^2(x)\,dx \leq \frac{1}{12} .$$

In this note, we show that van Dantzig's inequality is a special case of a fairly general inequality and is in fact a consequence of convexity. We state this inequality for $0 \leq x \leq 1$; however, the argument employed extends immediately to any interval $[a,b]$.

The inequality which is the subject of this note follows.

Theorem. Let μ be any measure on the Borel sets of $[0,1]$ with $0 < \mu[0,1] < \infty$, let $h(x)$ be any μ-integrable function with $h(0) = 0$ and let $\mathfrak{F}$ be the set of functions $f(x)$ on $[0,1]$ with

(a) $\int_0^1 f(x)\,d\mu(x) = 0$

(b) $f(1) \leq f(0)$

(c) $f(x) + h(x)$ is monotone non-decreasing in $[0,1]$.

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Then, if g is any convex function defined on the range of every f satisfying (a), (b), (c),

$$(1)\qquad \sup_{f\in\mathfrak{F}} \int_0^1 g(f(x))\,d\mu(x) = \max\left\{ \sup_{0\le y\le 1} \int_0^1 g(f_{1,y}(x))\,d\mu(x), \sup_{0<y\le 1} \int_0^1 g(f_{2,y}(x))\,d\mu(x) \right\}$$

where

$$f_{1,y}(x) = \begin{cases} \mu^{-1}([0,1])\left[\int_0^1 h(x)\,d\mu(x) - h(1)\mu((y,1])\right] - h(x), & 0\le x\le y \\ h(1) + \mu^{-1}([0,1])\left[\int_0^1 h(x)\,d\mu(x) - h(1)\mu((y,1])\right] - h(x), & y<x\le 1 \end{cases}$$

for $0\le y\le 1$, and

$$f_{2,y}(x) = \begin{cases} \mu^{-1}([0,1])\left[\int_0^1 h(x)\,d\mu(x) - h(1)\mu([y,1])\right] - h(x), & 0\le x<y \\ h(1) + \mu^{-1}([0,1])\left[\int_0^1 h(x)\,d\mu(x) - h(1)\mu([y,1])\right] - h(x), & y\le x\le 1 \end{cases}$$

for $0<y\le 1$.

This reduces the problem from one of maximization over a collection of functions to maximization over a parameter (y). With various specializations of μ, h, and g a variety of interesting special inequalities are obtained. In particular, if μ is Lebesgue measure and $h(x) = x$, the integrals on the right hand side of (1) are independent of y. This is noted in Corollary 3.

2. Proof of Theorem. From (b) and (c) we have

$$(2)\qquad f(0)+h(0) \le f(1)+h(1) \le f(0)+h(1)$$

and hence $h(0) \le h(1)$.

It is easily seen that $\mathfrak{F}$ is a convex set. Then for any $\lambda \in [0,1]$

(3) $$\int_0^1 g(\lambda f_1(x) + (1-\lambda) f_2(x))\, d\mu(x) \le \int_0^1 [\lambda g(f_1(x)) + (1-\lambda) g(f_2(x))]\, d\mu(x)$$

and

(4) $$\int_0^1 [\lambda g(f_1(x)) + (1-\lambda) g(f_2(x))]\, d\mu(x) \le \max\left(\int_0^1 g(f_1(x))\, d\mu(x), \int_0^1 g(f_2(x))\, d\mu(x)\right) .$$

Thus in determining $\max_{f \in \mathfrak{F}} \int_0^1 g(f(x))\, d\mu(x)$, it suffices to restrict attention to the extreme points of $\mathfrak{F}$.

Let Φ be the set of functions $\varphi(x)$ on $[0,1]$ with (i) $\varphi(0) = 0$, (ii) $\varphi(x)$ monotone non-decreasing, and (iii) $\varphi(1) \le h(1)$. In addition, let $T: \mathfrak{F} \to \Phi$ be the mapping defined by

$$T(f(x)) = \varphi_f(x) = f(x) - f(0) + h(x), \quad 0 \le x \le 1 .$$

Then

$$T^{-1}(\varphi) = f_\varphi(x) = \varphi(x) - h(x) + \mu^{-1}([0,1]) \left(\int_0^1 h(x)\, d\mu(x) - \int_0^1 \varphi(x)\, d\mu(x)\right) .$$

Thus T is one-to-one and onto. Further, T and T^{-1} preserve convex combinations so that extreme points of $\mathfrak{F}$ are the images of extreme points of Φ and conversely. The extreme points of Φ are clearly given by

(5a) $$\varphi_{1,y}(x) = \begin{cases} 0 & 0 \le x \le y \\ h(1) & y < x \le 1 \end{cases} \qquad 0 \le y \le 1$$

and

(5b) $$\varphi_{2,y}(x) = \begin{cases} 0 & 0 \le x < y \\ h(1) & y \le x \le 1 \end{cases} \qquad 0 < y \le 1 .$$

Note that $\varphi_{1,1}(x) \equiv 0$. Consequently, the extreme points of $\mathfrak{F}$ are

$$(6a)\quad f_{1,y}(x) = \begin{cases} \mu^{-1}([0,1])\left[\int_0^1 h(x)d\mu(x) - h(1)\mu((y,1])\right] - h(x), & 0\le x\le y \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad & 0\le y\le 1 \\ h(1) + \mu^{-1}([0,1])\left[\int_0^1 h(x)d\mu(x) - h(1)\mu((y,1])\right] - h(x), & y<x\le y, \end{cases}$$

and

$$(6b)\quad f_{2,y}(x) = \begin{cases} \mu^{-1}([0,1])\left[\int_0^1 h(x)d\mu(x) - h(1)\mu([y,1])\right] - h(x), & 0\le x<y \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad & 0<y\le 1 \\ h(1) + \mu^{-1}([0,1])\left[\int_0^1 h(x)d\mu(x) - h(1)\mu([y,1])\right] - h(x), & y\le x\le 1 . \end{cases}$$

Consequently,

$$(7)\quad \sup_{f\in\mathfrak{F}} \int_0^1 g(f(x))d\mu(x) = \max\left(\sup_{0\le y\le 1} \int_0^1 g(f_{1,y}(x)d\mu(x), \sup_{0<y\le 1} \int_0^1 g(f_{2,y}(x))d\mu(x)\right).$$

Corollary 1. If μ is absolutely continuous with respect to Lebesgue measure

$$\sup_{f\in\mathfrak{F}} \int_0^1 g(f(x))d\mu(x) = \max_{0\le y\le 1} \int_0^1 g(f_{1,y}(x))d\mu(x) .$$

The proof is trivial.

Corollary 2. If μ is Lebesgue measure,

$$\sup_{f\in\mathfrak{F}} \int_0^1 g(f(x))dx = \max_{0\le y\le 1} \int_0^1 g(f_{1,y}(x))dx$$

where

$$f_{1,y}'(x) = \begin{cases} \int_0^1 h(x)\,dx - (1-y)h(1) - h(x) & 0\le x\le y \\ \int_0^1 h(x)dx + yh(1) - h(x) & y<x\le 1 . \end{cases}$$

Corollary 3. If μ is Lebesgue measure and $h(x) = x$,

$$\max_{f \in \mathfrak{F}} \int_0^1 g(f(x))dx = \int_0^1 g(\tfrac{1}{2} - x)dx .$$

Proof: Here

$$f_{1,y}(x) = \begin{cases} y - \frac{1}{2} - x & 0 \le x \le y \\ \frac{1}{2} + y - x & y < x \le 1 \end{cases} .$$

Thus,

$$\int_0^1 g(f_{1,y}(x))dx = \int_0^y g(y - \tfrac{1}{2} - x)dx + \int_y^1 g(\tfrac{1}{2} + y - x)dx .$$

In the first integral let $x = z + y - 1$ and in the second integral let $x - y = z$, obtaining

$$\int_0^1 g(f_{1,y}(x))dx = \int_{1-y}^1 g(\tfrac{1}{2} - z)dz + \int_0^{1-y} g(\tfrac{1}{2} - z)dz = \int_0^1 g(\tfrac{1}{2} - x)dx .$$

Thus, in this case, every member of the one parameter family of functions gives the same value. The particular case $g(x) = x^2$ is van Dantzig's Inequality.

REFERENCE

van Dantzig, D., On the Consistency and Power of Wilcoxon's Two Sample Test, Nederlandse Akad. Wetensch. Proc. Ser. A, 54(1951), 1-8.

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