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INTERSECTION THEOREMS FOR POSITIVE SETS

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ABSTRACT

In a vector space over an ordered field, a *positive set* is one that is closed under the operation of forming linear combinations with nonnegative coefficients; it may be described alternatively as a convex cone whose apex is the origin. Such sets arise naturally as solutions of systems of homogeneous linear inequalities, and the intersection theorems proved here can be reformulated as consistency theorems for such systems. The main tool used in proving the intersection theorems is a characterization and classification of sets which enjoy a strong independence property with respect to the formation of nonnegative linear combinations.

INTRODUCTION. Helly's intersection theorem [7] asserts that if $\mathcal{C}$ is a finite family of convex sets in R^d with $\cap\mathcal{C} = \emptyset$ then $\mathcal{C}$ admits a subfamily $\mathcal{K}$ with $\cap\mathcal{K} = \emptyset$ and $|\mathcal{K}| \leq d + 1$. The shortest proof, due to Radon [9], is based on the fact that a subset of R^d is affinely independent if and only if it does not contain disjoint sets whose convex hulls intersect. Here a similar approach leads to short proofs of old and new intersection theorems for positive sets.

Throughout this note, E denotes a vector space over an ordered field. When E is said to be d-dimensional it should be understood that d is finite. A subset P of E is called *positive* provided that $\alpha x + \beta y \in P$ whenever $x,y \in P$ and $\alpha,\beta \geq 0$; equivalently, P is a convex cone with apex 0. (When $E = R^d$, the intersections of positive sets with the unit sphere are precisely the sets which are *spherically convex* in one of the common meanings of that term. Thus for real vector spaces our theorems could be stated alternatively in terms of spherically convex sets.) The *positive hull* pos X of a set $X \subset E$ is the intersection of all positive sets containing X; equivalently, it is the set of all points of the form $\Sigma_{x\in X}\lambda_x x$ with $\lambda_x \geq 0$ for all x and $\lambda_x = 0$ for all but finitely many x. Note that lin X = pos X - pos X, where lin X is the linear hull of X.

STRONG POSITIVE INDEPENDENCE. A subset X of $E \sim \{0\}$ is called *strongly positively independent* provided that pos Y $\cap$ pos Z $\subset \{0\}$ whenever Y and Z are disjoint subsets of X. This notation was introduced by McKinney [8] and characterized in various ways by him, Bonnice and Klee [1], and Reay [10]. The most useful characterization is the following, proved by McKinney when pos X = lin X. Our proof is considerably shorter than his.

THEOREM (McKinney). *A subset* X *of* E *is strongly positively independent if and only if* E *can be expressed as a direct sum of linear subspaces,* $E = E_o \oplus \bigoplus_{\alpha\in A} E_\alpha$, *in such a way that*

(a) $X \subset E_o \cup \bigcup_{\alpha\in A} E_\alpha$,

(b) $X \cap E_o$ *is linearly independent,*

(c) *for each* $\alpha \in A$ *the subspace* E_α *is finite-dimensional and* $X \cap E_\alpha$ *consists of the points of a linear basis for* E_α *together with a sum of negative multiples of these points.*

Proof. For the "if" part it suffices to note that each of the intersections $X \cap E_\alpha$ is strongly positively independent. For the "only if" part, consider a strongly positively independent subset X of E and let B be a linear basis for X--that is, B is linearly independent and $B \subset X \subset$ lin B. Let $A = X \sim B$. For each point x of A there is a unique scalar function λ^x on B such that $\lambda^x_b = 0$ for all but finitely many $b \in B$ and $x = -\Sigma_{b\in B}\lambda^x_b b$. Let $B_x = \{b \in B: \lambda^x_b > 0\}$. Then

$$x + \Sigma_{b\in B_x}\lambda^x_b b = \Sigma_{b\in B\sim B_x}(-\lambda^x_b)b \qquad (1)$$

and it follows from strong positive independence that both sides of (1) are equal to 0. Note that $B_x \cap B_y = \emptyset$ whenever $x, y \in A$ with $x \neq y$. For suppose the contrary and let $\mu = \max\{\lambda^x_b/\lambda^y_b: b \in B_y\} > 0$. Then

$$x + \Sigma_{b\in B_x\sim B_y}\lambda^x_b b = \mu y + \Sigma_{b\in B_x\cap B_y}(\mu\lambda^y_b - \lambda^x_b)b + \Sigma_{b\in B_y\sim B_x}\lambda^x_b b$$

and it follows from strong positive independence that both sides are 0. Referring to (1), we conclude that $\Sigma_{b\epsilon B_x \cap B_y} \lambda_b x_b = 0$, a contradiction implying $B_x \cap B_y = \emptyset$.

Now let B' be a linear basis for E containing B, let $E_o = \mathrm{lin}(B' \sim \bigcup_{x\epsilon A} B_x)$, and for each $x \epsilon A$ let $E_x = \mathrm{lin}\, B_x$. Then $E = E_o \oplus \bigoplus_{x\epsilon A} E_x$ and conditions (a), (b) and (c) are satisfied. □

The subspaces E_α in the above decomposition are uniquely determined by X, for they are exactly those finite-dimensional subspaces L of E such that $L = \mathrm{pos}(X \cap L)$ and $1 + \dim L = |X \cap L|$. (By Davis [3], McKinney [8] and others they have been called the *minimal subspaces* associated with X.) The set $X \cap E_o$ is also determined by X, as is E_o itself when $\mathrm{lin}\, X = E$. When E is finite-dimensional the cardinalities $|X \cap E_o|$ and $|X \cap E_\alpha|$ can be arranged in a finite sequence which starts with $|X \cap E_o|$ and thereafter lists the numbers $|X \cap E_\alpha|$ in increasing order and with proper multiplicity. This sequence will be called the *invariant* of X in E. For example, the sequence (1;2,2,3) is the invariant in R^5 of the eight-pointed strongly positively independent set represented by the columns of the following matrix:

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{matrix}$$

The term *invariant* is justified by the first part of the following theorem, whose straightforward proof is left to the reader.

THEOREM. *Suppose that* E *is* d*-dimensional. Two strongly positively independent subsets* X *and* Y *of* E *have the same invariant in* E *if and only if* E *admits a linear automorphism carrying the rays from the origin through the points of* X *onto the rays from the origin through the points of* Y. *A sequence* $(t_0;t_1,\ldots,t_r)$ *of integers is the invariant in* E *of some strongly positively independent set of cardinality* n *if and only if the following conditions are all satisfied:*
$0 \leq t_0;\ 2 \leq t_1 \leq \cdots \leq t_r;\ n = \Sigma_0^r t_i \leq d + r.$

With slight modifications the above theorem can be extended to the infinite-dimensional case. From the theorem's second assertion it follows readily that $d + [d/j]$ is the maximum cardinality of strongly positively independent subsets of E in which each j-pointed set is linearly independent.

A subset C of E is called a *cross basis* for E (called a *maximal positive basis* by Davis [3] and McKinney [8]) provided that C consists of the points of a linear basis for E together with a negative multiple of each of these points. The following is an immediate consequence of the preceding theorems.

COROLLARY. *Suppose that* E *is* d*-dimensional and* $X \subset E \sim \{0\}$. *If* $|X| > 2d$ *then* X *contains two disjoint subsets whose positive hulls have a common nonzero point. The same is true when* $|X| = 2d$ *unless* X *is a cross basis for* E.

INTERSECTION THEOREMS. When $X \subset E \sim \{0\}$ we will say that the sets of the form $\mathrm{pos}(X \sim \{x\})$, for $x \in X$, are *associated with* X. Note that all the sets associated with a cross basis are closed halfspaces. Before proving the main intersection theorems, we illustrate the method to be employed by proving the following result of Robinson [11].

COROLLARY (Robinson). *Suppose that* E *is* d-*dimensional and* $\mathcal{P}$ *is a finite family of positive sets in* E *with* $\cap\mathcal{P} \subset \{0\}$. *Then* $\mathcal{P}$ *admits a subfamily* $\mathcal{Q}$ *with* $\cap\mathcal{Q} \subset \{0\}$ *and* $|\mathcal{Q}| \le 2d$. *Indeed, there is such a* $\mathcal{Q}$ *with* $|\mathcal{Q}| < 2d$ *unless* $\mathcal{P}$ *consists of the* 2d *halfspaces associated with a cross basis for* E *or of such halfspaces together with* E *itself.*

Proof. Let $Q_1,\ldots,Q_n$ be distinct members of $\mathcal{P}$ forming a subfamily $\mathcal{Q}$ with $\cap\mathcal{Q} \subset \{0\}$ and $|\mathcal{Q}|$ a minimum. For each i there is a nonzero point $x_i \in \cap_{j\neq i} Q_i$. If $x_i = x_k$ with $i \neq k$ then $x_i \in \cap\mathcal{Q}$. We may assume, therefore, that the x_i's are all distinct and let X denote the n-pointed set $\{x_1,\ldots,x_n\}$. If X contains two disjoint sets whose positive hulls have a common nonzero point v then $v \in \cap\mathcal{Q}$, an impossibility since $\cap\mathcal{Q} \subset \{0\}$. From the preceding corollary it follows that $n < 2d$ or $n = 2d$ and X is a cross basis. It remains to examine the nature of $\mathcal{P}$ in the latter instance. Plainly Q_i is a halfspace associated with the cross basis X, for Q_i is positive and $X \sim \{x_i\} \subset Q_i \neq E$. Consider an arbitrary member P of $\mathcal{P} \sim \mathcal{Q}$. If $P \supset X$ then of course $P = E$. If there is an i for which $x_i \notin P$, then $\{P\} \cup (\mathcal{Q} \sim \{Q_i\})$ is a subfamily of $\mathcal{P}$ with intersection $\subset \{0\}$ and by the earlier reasoning is the set of all halfspaces associated with a cross basis for E. It then follows that $P = Q_i$. □

The special case of the above result in which $\mathcal{P}$ consists of closed halfspaces has (or its polar equivalent has) been proved by Steinitz [12], Dines and McCoy [4], Robinson [11], Gustin [6], Gale [5] and others. The polar equivalent asserts that if pos $X = E$ then pos $Y = E$ for some $Y \subset X$ with $|Y| \leq 2d$; further, there is such a Y with $|Y| < 2d$ unless X is a cross basis for E. See Danzer, Grünbaum, and Klee [2] for references to related results.

The statements of our main theorems will require some more definitions. For any set Z let $d_L(Z)$ denote the maximum of the dimensions of the linear subspaces contained in Z. For any family $\mathcal{Z}$ of sets let

$$\underline{k}(\mathcal{Z}) = \min_{Z \in \mathcal{Z}} d_L(Z), \quad \overline{k}(\mathcal{Z}) = \max_{Z \in \mathcal{Z}} d_L(Z),$$

and

$$\ell(\mathcal{Z}) = d_L(\cup \mathcal{Z}).$$

The family $\mathcal{P}$ is said to be *compatible with* the invariant $(t_0;t_1,\ldots,t_r)$ provided that there exists a strongly positively independent set X in E with this invariant such that each member of $\mathcal{P}$ contains a member of $\mathcal{A}_X$, the family of all sets associated with X.

THEOREM. *Suppose that* E *is a* d*-dimensional space*, $0 \leq k \leq \ell \leq d$, *and* $m = \min(k+1,\ell)$. *Let* $\mathcal{Q}$ *be a finite family of positive sets in* E *that is minimal with respect to having* $\cap \mathcal{Q} \subset \{0\}$. *If* $\underline{k}(\mathcal{Q}) \leq k$ *and* $\ell(\mathcal{Q}) \leq \ell$ *then*

$$|\mathcal{Q}| \leq d + m,$$

with equality if and only if $\mathcal{Q}$ *is compatible with*

$$(d - m - s;\ \underbrace{2, \ldots, 2}_{m-1}, 2 + s)$$

for some $0 \leq s \leq \ell - m$.

Proof. Let $Q_1, \ldots, Q_n$ be the n members of $\mathcal{Q}$ and let $X = \{x_1, \ldots, x_n\}$, where for each i the point x_i is such that $0 \neq x_i \in \bigcap_{j \neq i} Q_i$. Let $(t_0; t_1, \ldots, t_r)$ be the invariant of the strongly positively independent set X and let

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_r$$

be the direct sum decomposition of E described in the first theorem. Since $X \sim \{x_i\} \subset Q_i$ for all i, it follows that

$$r - 1 \leq \Sigma_1^{r-1}(t_i - 1) = \underline{k}(\mathcal{A}_X) \leq \underline{k}(\mathcal{Q}) \leq k.$$

And since, for $j > 0$, each point of E_j is a positive combination of proper subset of $X \cap E_j$, it follows that

$$E_1 \oplus \cdots \oplus E_r \subset \cup\mathcal{A}_X \subset \cup\mathcal{Q}$$

and

$$r \leq \Sigma_1^{r}(t_i - 1) = \ell(\mathcal{A}_X) \leq \ell(\mathcal{Q}) \leq \ell.$$

We conclude, therefore, that

$$r \leq \min(k+1, \ell) \quad \text{and} \quad n \leq d + r \leq d + m.$$

Note that the inequality $n \leq d + m$ is all that is required for the corollary below.

Suppose now that $n = d + m$, whence $r = m$. If $t_{r-1} > 2$ then

$$k \geq \underline{k}(\mathcal{Q}) \geq r = \min(k+1,\ell),$$

whence $k = \ell$ and

$$\ell \geq \ell(\mathcal{Q}) \geq r + 1 = \ell + 1,$$

a contradiction. It follows that $t_1 = \ldots = t_{r-1} = 2$. Furthermore,

$$t_r \leq 1 + \ell - \Sigma_1^{r-1}(t_i - 1) = 1 + \ell - (m-1) = 2 + \ell - m.$$

Let $s = t_r - 2$. Then $0 \leq s \leq \ell - m$ and

$$t_0 = n - \Sigma_1^r t_i = d + m - 2(m-1) - (2+s) = d - m - s.$$

Hence the invariant of X is

$$(d - m - s, \underbrace{2, \ldots, 2}_{m-1}, 2 + s)$$

or simply (d) if $m = 0$. Plainly $|\mathcal{Q}| = d + m$ if Q is compatible with such an invariant for some $0 \leq s \leq \ell - m$, and that completes the proof. □

COROLLARY (Robinson). *Suppose that* E *is* d-*dimensional and* $\mathcal{P}$ *is a finite family of positive sets in* E *with* $\cap\mathcal{P} \subset \{0\}$ *and* $\underline{k}(\mathcal{P}) \leq k$. *Then* $\mathcal{P}$ *admits a subfamily* $\mathcal{Q}$ *with* $\cap\mathcal{Q} \subset \{0\}$ *and* $|\mathcal{Q}| \leq d + k + 1$.

Proof. Choose $P_0 \in \mathcal{P}$ with $d_L(P_0) \leq k$ and let $\mathcal{S}$ be a subfamily of $\mathcal{P} \sim \{P_0\}$ that is minimal with respect to having

$P_0 \cap (\cap \mathcal{S}) \subset \{0\}$. Let $\mathcal{M} = \{S \cap P_0 : S \in \mathcal{S}\}$. Then $\ell(\mathcal{M}) \leq k$ and $\mathcal{M}$ is minimal with respect to having $\cap \mathcal{M} \subset \{0\}$, so it follows from the preceding theorem that $|\mathcal{M}| \leq d + k$. But then $|\{P_0\} \cup \mathcal{S}| \leq d + k + 1$. □

THEOREM. *Let the hypotheses of the preceding theorem be strengthened by requiring* $\bar{k}(\mathcal{Q}) \leq k$. *Then*

(a) *when* $m = d$ *or* $k = \ell$,

$|\mathcal{Q}| = d + m$ *if and only if* $\mathcal{Q}$ *is compatible with* $(d - m; \underbrace{2, \ldots, 2}_{m})$;

(b) *when* $k = 0$ *and* $\ell = d$,

$|\mathcal{Q}| = d + m$ *if and only if* $\mathcal{Q}$ *is compatible with* $(0; d + 1)$;

(c) *when* $m < d$ *and* $0 < \ell - k < d$,

$|\mathcal{Q}| \leq d + m - 1 = d + k$, *with equality if and only if* $\mathcal{Q}$ *is compatible with* $(d - k; \underbrace{2, \ldots, 2}_{k})$ *or* $k = d - 2$ *and* $\mathcal{Q}$ *is compatible with* $(0; \underbrace{2, \ldots, 2}_{d-1})$.

Proof. If $|\mathcal{Q}| = d + m$ then $\mathcal{Q}$ is compatible with

$$(d - m - s; \underbrace{2, \ldots, 2}_{m-1}, 2 + s)$$

for some $0 \leq s \leq \ell - m$. There are the following cases to consider:

(i) $m + s < d$. Then

$$k \geq \bar{k}(\mathcal{Q}_X) = \Sigma_1^r (t_i - 1) = m + s,$$

whence $m \leq k$, $k = \ell$, and $s = 0$.

(ii) $m = d$ and $s = d - m = 0$.

(iii) $1 = m$ and $s = d - 1 > 0$. Then $s \leq \ell - m$ implies $\ell = d$ and $k = 0$.

(iv) $1 < m < d$ and $s = d - m$ is impossible, for it implies

$$d = \ell \quad \text{and} \quad k \geq \bar{k}(\mathcal{O}\!\iota_X) = m - 1 + s = d - 1$$

whence $m = d$.

That settles the "only if" parts of (a) and (b), the first part of (c), and supplies all the information needed for the corollary below. The "if" parts of (a) and (b) are obvious.

For the remainder of the proof we assume $m < d$, $0 < \ell - k < d$, and $|Q| = d + k$. Note first that

$$k \leq r \leq k + 1$$

where the left-hand inequality follows from $n \leq d + r$ and the right-hand inequality from

$$k \geq \bar{k}(\mathcal{O}\!\iota_X) \geq \Sigma_2^r(t_i - 1) \geq r - 1.$$

Now suppose first that $t_0 > 0$. Then

$$k \geq \bar{k}(\mathcal{O}\!\iota_X) = \Sigma_1^r(t_i - 1) \geq r + t_r - 2,$$

whence $r = k$ and $t_i = 2$ for all $i > 0$. Hence X's invariant is $(d - k;\ \underbrace{2, \ldots, 2}_{k})$.

Suppose next that $t_0 = 0$ and $r = k + 1$. Then

$$k \geq \bar{k}(\mathcal{O}\!\iota_X) = \Sigma_2^r(t_i - 1) \geq k + t_r - 2,$$

whence $t_i = 2$ for all $i > 0$ and

$$d + k = \Sigma_1^r t_i = 2(k+1).$$

Hence $k = d - 2$ and X has invariant $(0;\ \underbrace{2, \ldots, 2}_{d-1})$.

Suppose, finally, that $t_0 = 0$ and $r = k$. As $t_{r-1} > 2$ or $t_r > 3$ would imply

$$k \geq \bar{k}(\mathcal{A}_X) \geq \Sigma_2^r(t_i - 1) \geq k + 1,$$

it follows that $t_1 = \ldots = t_{r-1} = 2$ and $t_r = 2 + s$ with $0 \leq s \leq 1$. This implies

$$d + k = \Sigma_1^r t_i = 2(k-1) + 2 + s \leq 2k + 1$$

and hence $k \geq d - 1$, contradicting the fact that $k < \ell$ and $m < d$. Thus it cannot happen that $t_0 = 0$ and $r = k$, and the discussion of (c)'s "only if" part is complete. Again, the "if" part is obvious. □

COROLLARY. *Let the hypotheses of the preceding corollary be strengthened by requiring that* $\bar{k}(\mathcal{P}) \leq k$ *and that* $0 < k < d - 1$ *or* $\ell(\mathcal{P}) < d$. *Then* $\mathcal{P}$ *admits a subfamily* $\mathcal{Q}$ *with* $\cap \mathcal{Q} \subset \{0\}$ *and* $|\mathcal{Q}| \leq d + k$.

Proof. Let $\mathcal{Q}$ be a subfamily of $\mathcal{P}$ that is minimal with respect to having $\cap \mathcal{Q} \subset \{0\}$. Apply the theorem just proved.

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