ON AN ATTACK-DEFENSE GAME WITH TARGETS

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1. Summary

This paper considers some aspects of a discrete attack-defense game in which there are targets for both the attacker and the defender. In section two the basic game is described. Following this there is a discussion of the relation of past results to this game.

In section three the discrete game is examined in detail. Expressions are obtained for the probability of victory by either player at any step in the game, time to termination of the game, and the mathematical expectation and variance of time to termination for both fixed and mobile target for the attacker. For the case of fixed target the space of strategies for each player is discussed along with several possible payoff kernels. Some conditions are given which are sufficient to obtain a saddle point and minimax solution to the game.

In section four some examples of strategies are explored. In section five extensions of the discrete game to the continuous game are considered.

2. Description of the Game and Past Research

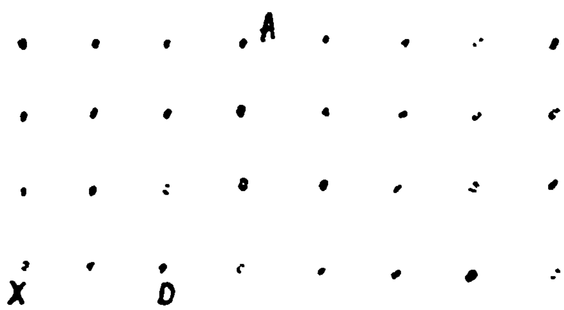
There are two versions of the attack-defense game with targets. Both games will be defined here but only the first version will be considered in Section 3.

The first version will be called the discrete game. It can be described as follows. One is given a finite or infinite grid of lattice points in E^2 or E^3 labeled $0,1,2,\ldots,N$. The two players are A, the attacker, and D, the defender. The target which the attacker wishes to hit is X, which may be assumed to be either stationary or mobile. Also, the initial positions of A, D, and X are given. The initial position of D need not coincide with the initial position of X.

The players, A and D, as well as X, are required to occupy lattice points. Multiple occupancy of a lattice point is allowed. The players move simultaneously at discrete times. The goal of A is to occupy the same lattice point as X does on the same move before the positions of A and D coincide. If this occurs, A wins and the game ends. The goal of D is to occupy the same lattice point as A does before the positions of A and X coincide. If this occurs, D wins and the game ends. If A,D, and X occupy the same lattice point simultaneously, then A wins, since D may destroy A, but X will be destroyed in the process.

A time limit on the game, T_L, is given. The game must end by time T_L. If by time T_L neither A nor D has won, then the game is declared over and neither player wins. The limitation T_L can be thought of as a fuel constraint on one or both of the players.

An example of the initial positions occupied at the beginning of the discrete game in E^2 is shown in the following diagram.



The second version of the game will be called the continuous, or differential game. In this game the players move along continuous, differentiable paths contained in some connected region (finite or infinite) in E^2 or E^3. The goals of the players remain, in general, the same except that, because of the continuous nature of the game, "zones of destruction" for each player need to be introduced. These zones are specified by positive numbers, ε_A and ε_D. ε_A is such that if the Euclidian distance between A and X is less than ε_A, then X is considered to be destroyed and A wins; and if the Euclidian distance between A and D is less than ε_D, then A is destroyed and D wins. As in the discrete game, the convention is made that if both A and X lie within ε_D of D, then A wins and the game is over.

The continuous version can be seen to be quite analogous to the discrete game and some of the results for the discrete game can be carried over to the continuous game.

As noted in the first section certain simple pursuit-evasion games have been analyzed in the existing literature. For example, Karlin [10], Bellman [1], and Dresher [6] have considered some discrete pursuit-evasion games using the method of payoff kernels. It is clear that the discrete game described in the section contains, as special cases, games in which the strategies are matrices. Such pursuit-evasion games have been studied by Cohen [5] and Dubins [7]. Charnes and Schroeder [3] considered some special types of stochastic games relating to antisubmarine warfare. It should be noted that most of these papers do not use the concept of a target for the attacker. The few that do consider targets for the attacker do not allow movement, but instead examine certain allocation problems relating to offensive and defensive units.

There have also been several studies of the classical pursuit-evasion game in the continuous version. Some of these studies are Issacs [9], Ho, *et al* [8], Cockayne [3], Berkovitz and Fleming [2], and Pontryagin [11]. These papers, in addition to not having a target for the attacker, present results based primarily on solutions of certain types of differential equations with various constraints and side conditions.

3. The Discrete Game

3.0. Notation and Basic Definitions

The following notation will be used extensively in this section.

(3.0.1) A = attacker.

(3.0.2) D = defender.

(3.0.3) λ = target for A.

(3.0.4) T_L = time limit for the game.

(3.0.5) N = number of lattice points given.

(3.0.6) INA = position of A after the first move and before the second move.

(3.0.7) IND = position of D after the first move and before the second move.

(3.0.8) INX = position of X after the first move and before the second move.

(3.0.9) T = termination time in the game (a random variable)

(3.0.10) $\mathcal{S}_A$ = set of strategies for A.

(3.0.11) $\mathcal{S}_D$ = set of strategies for D.

(3.0.12) $\mathcal{S}_X$ = set of strategies for X.

(3.0.13) The phrase "ST(A)=U" will mean "A follows strategy U" (similarly for ST(D) and ST(X)).

Moves occur in the game at discrete times 1,2,...,t, t+1,...,T_L.

Define POSA, POSD, and POSX, functions of time, by

(3.0.14) POSA(t) = position of A at time t,

(3.0.15) POSD(t) = position of D at time t,

and

(3.0.16) POSX(t) = position of X at time t.

3.1 Discrete Game with Fixed Target for Attacker.

In this situation X cannot move, POSX(t) = INX for each $t \geq 1$ and $\mathcal{S}_X$ is a singleton strategy. Hence, it is not necessary to consider $\mathcal{S}_X$.

3.1.1 <u>Definition</u>. Define functions TRANSA, TRANSD, PRAD1, PRAD2 as follows:

Let $U \in \mathcal{S}_A$, $V \in \mathcal{S}_D$, and let i,j, and k be lattice points.

For each time t with $t \leq T_L$ define

(3.1.1.1) PRAD1(i,j,t;U,V) = P[POSA(t)=i, POSD(t)=j, and $T \geq t$ | ST(A)=U and ST(D)=V],

(3.1.1.2) PRAD2(i,j,t;U,V) = P[POSA(t)=i, POSD(t)=j, and $T > t$ | ST(A)=U and ST(D)=V],

and for $j \neq k$ and $t < T_L$

(3.1.1.3) TRANSA(i,t+1;j,k,U) = P[POSA(t+1)=i | POSA(t)=j, POSD(t)=k, and ST(A)=U]

and

(3.1.1.4) TRANSD(i,t+1;j,k,V) = P[POSD(t+1)=i | POSA(t)=j, POSD(t)=k, and ST(D)=V].

It should be noted that when the two players select strategies, these strategies will determine the probabilities defined in 3.1.1. For this reason, for any event C, denote P[C | ST(A)=U and ST(D)=V] by P[C;U,V].

The first move of the players is to move onto INA and IND, respectively. This might be termed a LeMans start for the game. From the LeMans start the following lemma is immediately obtained.

3.1.2. __Lemma__. For $U \in \mathcal{S}_A$ and for $V \in \mathcal{S}_D$

$$(3.1.2.1)\quad \mathrm{PRAD1}(i,j,1;U,V) = \begin{cases} 1 & \text{if } i=\mathrm{INA} \text{ and } j=\mathrm{IND} \\ 0 & \text{otherwise} \end{cases}$$

$$(3.1.2.2)\quad \mathrm{PRAD2}(i,j,1;U,V) = \begin{cases} 0 & \text{if } i=\mathrm{INA}=\mathrm{IND}=j \\ \mathrm{PRAD1}(i,j,1;U,V) & \text{otherwise} \end{cases}$$

$$(3.1.2.3)\quad P[T > 1;U,V] = \begin{cases} 0 & \text{if } \mathrm{INA}=\mathrm{IND} \\ 1 & \text{otherwise.} \end{cases}$$

An inductive method can now be used to determine the values of the functions, PRAD1 and PRAD2.

$$(3.1.3.1)\quad PRAD1(i,j,t+1;U,V) = \sum_{\substack{k \\ k\neq m \\ k\neq NX}} \sum_{m} PRAD2(k,m,t;U,V)\,TRANSA(i,t+1;k,m,U)\cdot TRANSD(j,t+1;k,m,V)$$

and

$$(3.1.3.2)\quad PRAD2(i,j,t+1;U,V) = \begin{cases} 0 & \text{if } i=j \text{ or if } i=NX \\ PRAD1(i,j,t+1;U,V) & \text{otherwise.} \end{cases}$$

<u>Proof</u>: (3.1.3.2) follows from (3.1.3.1) and (3.1.1.2). Consider (3.1.3.1). One has, by definition of PRAD1, that

$$(3.1.3.3)\quad PRAD1(i,j,t+1;U,V) = P[POSA(t+1)=i \text{ and } POSD(t+1)=j \mid T > t,\ ST(A)=U, \text{ and } ST(D)=V]\cdot P[T > t \mid ST(A)=U \text{ and } ST(D)=V].$$

The first term on the right hand side of (3.1.3.3) can be decomposed as follows

$$(3.1.3.4)\quad P[POSA(t+1)=i \text{ and } POSD(t+1)=j \mid T > t,\ ST(A)=U, \text{ and } ST(D)=V] =$$

$$= \sum_{\substack{k \\ k\neq m \\ k\neq NX}} \sum_{m} P[POSA(t+1)=i,\ POSD(t+1)=j,\ POSA(t)=k, \text{ and } POSD(t)=m \mid T > t,\ ST(A)=U, \text{ and } ST(D)=V]$$

$$= \sum_{\substack{k \\ k\neq m \\ k\neq NX}} \sum_{m} P[POSA(t+1)=i \text{ and } POSD(t+1)=j \mid POSA(t)=k,\ POSD(t)=m,\ T > t,\ ST(A)=U, \text{ and } ST(D)=V]$$

$$P[POSA(t)=k \text{ and } POSD(t)=m \mid T > t,\ ST(A)=U, \text{ and } ST(D)=V]$$

Substituting (3.1.3.4) into (3.1.3.3) yields

(3.1.3.5) $\text{PRAD1}(i,j,t+1;U,V) = \sum_{k}\sum_{m} \underset{\substack{k\neq m\\ k\neq mx}}{} P[\text{POSA}(t+1)=i \text{ and } \text{POSD}(t+1)=j \mid \text{POSA}(t)=k, \text{POSD}(t)=m,\ T > t, \text{ST}(A)=U, \text{ and } \text{ST}(D)=V] \cdot P[\text{POSA}(t)=k \text{ and } \text{POSD}(t)=m \mid T > t,\ \text{ST}(A)=U \text{ and } \text{ST}(D)=V] \cdot P[T > t \mid \text{ST}(A)=U \text{ and } \text{ST}(D)=V]$

and so,

(3.1.3.6) $\text{PRAD1}(i,j,t+1;U,V) = \sum_{k}\sum_{m} \underset{\substack{k\neq m\\ k\neq mx}}{} P[\text{POSA}(t+1)=i \text{ and } \text{POSD}(t+1)=j \mid \text{POSA}(t)=k, \text{POSD}(t)=m,\ T > t, \text{ST}(A)=U, \text{ST}(D)=V] \cdot P[\text{POSA}(t)=k, \text{POSD}(t)=m, \text{ and } T > t \mid \text{ST}(A)=U, \text{ and } \text{ST}(D) = V].$

Note now that the second probability inside the summations of (3.1.3.6) is just PRAD2(k,m,t;U,V). The first probability inside the summations in (3.1.3.6) can be split into P [POSA(t+1)=i | POSA(t)=k, POSD(t)=m, ST(A)=U, and ST(D)=V] · P[POSD(t+1) = j | POSA(t)=k, POSD(t)=m, ST(A)=U, and ST(D)=V] because the events POSA(t+1)=i and POSD(t+1)=j are conditionally independent given the state of the game and position of the players in the t-th move.

However, this product of probabilities is just TRANSA(i,t+1;k,m,U,V) · TRANSD(i,t+1;k,m,U,V). Substitution of these results into (3.1.3.6) gives (3.1.3.1) and so completes the proof.

The expressions obtained in 3.1.3 can be employed to calculate the probabilities that A(or D) will win in the t-th move, when A and D are using strategies U and V, respectively. Since the probabilities will be obtained inductively, the probabilities for the first move are given in the following lemma, the proof of which is immediate from the LeMans start of the game.

3.1.4 Lemma. For $U \in \mathcal{L}_A$ and $V \in \mathcal{L}_D$

$$(3.1.4.1) \quad P[\text{A wins at step } 1;U,V] = \begin{cases} 0 & \text{if } \text{INA} \neq \text{INX} \\ 1 & \text{if } \text{INA} = \text{INX} \end{cases}$$

$$(3.1.4.2) \quad P[\text{D wins at step } 1;U,V] = \begin{cases} 0 & \text{if IND} \neq \text{INA or if INA} = \text{IND} = \text{INX} \\ 1 & \text{otherwise.} \end{cases}$$

For the general step the following theorem is obtained.

3.1.5 Theorem. For $U \in \mathcal{I}_A$, $V \in \mathcal{I}_D$, and $t \geq 1$

(3.1.5.1) P[A wins in the (t+1)-st move and T > t ;U,V]=

$$\sum_{\substack{i \\ i \neq j \\ i \neq INX}} \sum_{j} \text{TRANSA}(INX,t+1;i,j,U)\ \text{PRAD2}(i,j,t;U,V)$$

and

(3.1.5.2) P[D wins in the (t+1)-st move and T > t ;U,V]=

$$= \sum_{j \neq INX} \text{PRAD1}(j,j,t+1;U,V).$$

Proof: Consider (3.1.5.1). Observing that A wins in the (t+1)-st move if and only if POSA(t+1)=INX=POS(X), one has by decomposition

P[A wins in the (t+1)-st move and T > t ; U,V] =

$$= \sum_{\substack{i \\ i \neq j \\ i \neq INX}} \sum_{j} \text{P[POSA(t+1)=INX, POSA(t)=i, POSD(t)=j, and } T > t ; U,V].$$

Conditioning, one obtains

P[A wins in the (t+1)-st move and T > t ; U,V] =

$$= \sum_{\substack{i \\ i \neq j \\ i \neq INX}} \sum_{j} \text{P[POSA(t+1)=INX} \mid \text{POSA(t)=i, POSD(t)=j, } T > t, \text{ ST(A)=U, and ST(D)=V]} \cdot$$

$$\cdot \text{P[POSA(t)=i, POSD(t)=j, and } T > t \mid \text{ST(A)=U and ST(D)=V]}.$$

Since the first probability inside the summations is TRANSA(INX, t+1;i,j,U) and the second is PRAD2(i,j,t;U,V), (3.1.5.1) is obtained.

To verify (3.1.5.2), notice that D wins in the (t+1)-st move if and only if POSD(t+1)=POSA(t+1), POSA(t+1)≠INX, and T > t. Hence, P[D wins in the (t+1)-st move and T > t ; U,V] = $\sum_{j \neq INX}$ P[POSA(t+1)=j=POSD(t+1) and T > t ; U,V]. Since the term inside the summation is PRAD1(j,j,t+1;U,V), the verification of (3.1.5.2) is complete.

Expressions for the probability of termination at the (t+1)-st move, and not before, can now be derived.

3.1.6. Corollary. For $U \in \mathcal{S}_A$, $V \in \mathcal{S}_D$, and $t \geq 1$

$$(3.1.6.1) \quad P[T=t+1\ ;\ U,V] = \sum_i \sum_{\substack{j \\ (i\neq j \\ i\neq INX)}} TRANSA(INX,t+1;i,j,U) \cdot PRAD2(i,j,t;U,V) + \sum_{j\neq INX} PRAD2(j,j,t+1;U,V).$$

Proof: It is clear that

$$P[T=t+1\ ;U,V] = P[A \text{ wins at the } (t+1)\text{-st move and } T > t;\ U,V] + P[D \text{ wins in the } (t+1)\text{-st move and } T > t;\ U,V].$$

(3.1.6.1) follows then from (3.1.5.1) and (3.1.5.2).

The distribution of the random variable of termination can now be determined whenever $t < T_L$. For $t = T_L$ set

$$P[T=T_L\ ;\ U,V] = 1- \sum_{t<T_L} P[T=t\ ;\ U,V]$$

Also, given strategies $U \in \mathcal{S}_A$ and $V \in \mathcal{S}_D$ the mathematical expectation and variance of termination can be found.

3.2. Discrete Game with Mobile Target for Attacker.

In this situation X, the target, is mobile. The assumption is made that A,D, and X make independent movements at the (t+1)-st move given their positions at the t-th move. The probabilities that will be considered are first defined and then expressions are derived inductively as in the previous part of this section with a fixed target. It might be noted that this version of the game can be viewed as a three person game with coalition and with one of the players in the coalition acting evasively or passively. The terms that are not defined in this section will have the same meaning as they had in 3.1.

3.2.1 **Definition.** Let $U \in \mathcal{S}_A$, $V \in \mathcal{S}_D$, $W \in \mathcal{S}_X$, and t be a time $\leq T_L$. Let i,j,k, and m be lattice points. The functions PRADX1, PRADX2, TRANSA, TRANSD, and TRANSX are defined as follows.

(3.2.1.1) PRADX1(i,j,k,t;U,V,W) = P[POSA(t)=i, POSD(t)=j, POSX(t)=k, and $T \geq t$ | ST(A)=U, ST(D)=V, and ST(X)=W].

(3.2.1.2) PRADX2(i,j,k,t;U,V,W) = P[POSA(t)=i, POSD(t)=j, POSX(t)=k, and $T > t$ | ST(A)=U, ST(D)=V, and ST(X)=W].

(3.2.1.3) TRANSA(i,t+1;j,k,m,U)=P[POSA(t+1)=i | POSA(t)=j, POSD(t)=k, POSX(t)=m, and ST(A)=U].

(3.2.1.4) TRANSD(i,t+1;j,k,m,V)=P[POSD(t+1)=i | POSA(t)=j, POSD(t)=k, POSX(t)=m, and ST(D)=V].

(3.2.1.5) TRANSX(i,t+1;j,k,m,W)=P[POSX(t+1)=i | POSA(t)=j, POSD(t)=k, POSX(t)=m, and ST(X)=W].

The following lemma is an immediate consequence of these definitions.

3.2.2 <u>Lemma</u>. For $U \in \mathcal{L}_A$, $V \in \mathcal{L}_D$, $W \in \mathcal{L}_X$ and for i,j, and k lattice points

(3.2.2.1) $$\text{PRADX1}(i,j,k,1;U,V,W) = \begin{cases} 1 & \text{if } i=\text{INA},\ j=\text{IND}, \text{ and } k=\text{INX} \\ 0 & \text{otherwise,} \end{cases}$$

(3.2.2.2) $$\text{PRADX2}(i,j,k,1;U,V,W) = \begin{cases} 0 & \text{if } i=\text{INA}=\text{IND}=j \text{ or } j=\text{INA}=\text{INX}=k \\ \text{PRADX1}(i,j,k,1;U,V,W) & \text{otherwise,} \end{cases}$$

(3.2.2.3) $$P[\text{A wins in 1st move};U,V,W] = \begin{cases} 1 & \text{if } \text{INA}=\text{INX} \\ 0 & \text{otherwise,} \end{cases}$$

(3.2.2.4) $$P[\text{D wins in 1st move};U,V,W] = \begin{cases} 1 & \text{if } \text{IND}=\text{INA}\neq\text{INX} \\ 0 & \text{otherwise,} \end{cases}$$

(3.2.2.5) $$P[T=1\ ;\ U,V,W] = \begin{cases} 1 & \text{if } \text{INA}=\text{IND} \text{ or } \text{INA}=\text{INX} \\ 0 & \text{otherwise.} \end{cases}$$

An inductive process can now be used to calculate the values of the functions PRADX1 and PRADX2 using the result of the following theorem.

3.2.3. <u>Theorem</u>. Let $U \in \mathcal{S}_A$, $V \in \mathcal{S}_D$, $W \in \mathcal{S}_X$ and suppose i, j, and k are lattice points and that $1 \leq t < T_L$.

(3.2.3.1) $$\text{PRADX1}(i,j,k,t+1;U,V,W) = \sum_{\substack{m \\ m \neq n \\ n \neq q}} \sum_{n} \sum_{q} \text{PRADX2}(m,n,q,t;U,V,W) \cdot$$
$$\cdot \text{TRANSA}(i,t+1;m,n,q,U) \cdot \text{TRANSD}(j,t+1;m,n,q,V) \cdot$$
$$\cdot \text{TRANSX}(k,t+1;m,n,q,W)$$

(3.2.3.2) $$\text{PRADX2}(i,j,k,t+1;U,V,W) = \begin{cases} 0 & \text{if } i=j \text{ or } i=k \\ \text{PRADX1}(i,j,k,t+1;U,V,W) & \text{otherwise} \end{cases}$$

<u>Proof</u>: This proof parallels the proof of theorem 3.1.3 and so is omitted.

3.2.4. <u>Theorem</u>. For $U \in \mathcal{S}_A$, $V \in \mathcal{S}_D$, and $W \in \mathcal{S}_X$

(3.2.4.1) $$P[\text{A wins in the } (t+1)\text{-st move and } T > t\ ;U,V,W] =$$
$$= \sum_{\substack{i \\ j \neq k \\ j \neq m}} \sum_{j} \sum_{k} \sum_{m} \text{TRANSA}(i,t+1;j,k,m,U) \cdot \text{TRANSX}(i,t+1;j,k,m,W) \cdot$$
$$\cdot \text{PRADX2}(j,k,m,t;U,V,W).$$

<u>Proof</u>: The proof of (3.2.4.1) is analogous to that of (3.1.5.1) and is omitted. The probability on the left hand side in (3.2.4.2) can be decomposed over all possible locations A and D can meet and then decomposed over all possible prior positions of A,D, and X so that one has

$$
\begin{aligned}
&P[\text{D wins in the } (t+1)\text{-st move and } T > t; U,V,W] = \\
&= \quad P[POSA(t+1)=i,\ POSD(t+1)=i,\ POSX(t+1)\neq i, \text{ and } T > t;\ U,V,W] \\
&= \sum_{i} \sum_{\substack{j \\ j\neq k \\ j\neq m}} \sum_{k} \sum_{m} P[POSA(t+1)=i,\ POSD(t+1)=i,\ POSX(t+1)\neq i \mid POSA(t)=j, \\
&\qquad POSD(t)=k,\ POSX(t)=m,\ T > t,\ ST(A)=U,\ ST(D)=V, \text{ and} \\
&\qquad ST(X)=W] \cdot P[POSA(t)=j,\ POSD(t)=k,\ POSX(t)=m, \text{ and} \\
&\qquad T > t\ ;\ U,V,W].
\end{aligned}
$$

Using the fact that the events POSA(t+1)=i, POSD(t+1)=i, and POSX(t+1)≠i are conditionally independent given the positions of A,D, and X at time t one obtains (3.2.4.2).

The probability of termination along with the mathematical expectation and variance of termination can be determined from 3.2.4 as was done in 3.1.

3.3. Strategies for the Fixed Target Game.

Having derived expressions for the various probabilities for the case of where X is fixed, it might be useful to consider the spaces of strategies, $\mathcal{S}_A$ and $\mathcal{S}_D$. A partial ordering can be induced on each of $\mathcal{S}_A$ and $\mathcal{S}_D$.

3.3.1. <u>Definition</u>. For U_1 and U_2 in $\mathcal{S}_A$ define $\leq_A$ on $\mathcal{S}_A$ by

$U_1 \leq_A U_2$ if and only if

(3.3.1.1) $P[\text{A wins on the t-th move and } T \geq t;\ U_1, V] \geq$
$P[\text{A wins on the t-th move and } T \geq t;\ U_2, V]$

and

(3.3.1.2) $P[\text{D wins on the t-th move and } T \geq t;\ U_1, V] \leq$
$P[\text{D wins on the t-th move and } T \geq t;\ U_2, V]$

for all $t \leq T_L$ and $V \in \mathcal{S}_D$.

3.3.2. <u>Definition</u>. For V_1 and V_2 in $\mathcal{S}_D$ define $\leq_D$ on $\mathcal{S}_D$ by

$V_1 \leq_D V_2$ if and only if

(3.3.2.1) $P[\text{A wins on the t-th move and } T \geq t;\ U, V_1] \leq$
$P[\text{A wins on the t-th move and } T \geq t;\ U, V_2]$

and

(3.3.2.2) $[\text{D wins on the t-th move and } T \geq t;\ U, V_1] \geq$
$P[\text{D wins on the t-th move and } T \geq t;\ U, V_2]$

for all $t \leq T_L$ and $U \in \mathcal{S}_A$.

A strategy, U, in $\mathcal{J}_A$ for which there is a stategy in $\mathcal{J}_A$ different from U which is $\underset{A}{\geq}$ U will be called <u>strongly inadmissible</u> in $\mathcal{J}_A$. Similarly, a strategy, V, in $\mathcal{J}_D$ for which there is a strategy in $\mathcal{J}_D$ different from V and $\underset{D}{\geq}$ V will be called a <u>strongly indadmissible</u> strategy in $\mathcal{J}_D$.

3.3.3 <u>Lemma</u>. ($\mathcal{J}_A$, $\underset{A}{\leq}$) and ($\mathcal{J}_D$, $\underset{D}{\leq}$) are partially ordered sets.

<u>Proof</u>: This is a direct verification.

The effect of this partial ordering is to eliminate from consideration those strategies for both players which behave "badly" at every time against every strategy of the opponent.

Several payoff kernels will now be presented along with some discussion of their importance. Define $K_{A,1}$, a function from $\mathcal{J}_A \times \mathcal{J}_D \times \{1,2,\ldots,T_L\}$ into [-1,1] by

(3.3.4.1) $K_{A,1}(U,V,t)$ = P[A wins in t-th move and T ≥ t;U,V] − P[D wins in the t-th move and T ≥ t; U,V]

for all $U \in \mathcal{J}_A$ $V \in \mathcal{J}_D$ and t.

Define $K_{A,2}$, a function from $\mathcal{J}_A \times \mathcal{J}_D$ to E^1 by

(3.3.4.2) $K_{A,2}(U,V) = \sum_{t \leq T_L} K_{A,1}(U,V,t)$

for all $U \in \mathcal{J}_A$ and $V \in \mathcal{J}_D$.

Set $K_{D,1} = -K_{A,1}$ and $K_{D,2} = -K_{A,2}$. The game with the payoff kernel, $K_{A,2}$, for the attacker is a two person zero-sum game. This payoff kernel seems intuitively appealing when one considers the goal of the attacker: the attacker seeks a strategy that will maximize the probability that he wins and will minimize the probability that he loses to D. This would be reflected in A's efforts to maximize, for every choice of strategy in $\mathcal{S}_D$, the expression in (3.3.4.2). The defender will attempt to do the opposite.

If the players wish to consider strategies that will do well at each move, they might use the payoff kernel $K_{A,1}$.

Each strategy in either space of strategies is really a sequence and can be represented as a vector, say $U = (U_1, \ldots, U_{T_L}) \in \mathcal{S}_A$ and $V = (V_1, \ldots, V_{T_L}) \in \mathcal{S}_D$. Then the players are really making sequential decisions and at the t-th move A will seek a strategy which maximizes the probability that he wins in this move. Using the payoff kernel $K_{A,1}$, this is reflected in A's efforts to maximin the expression in (3.3.4.1).

If termination at the early stages of the game is important, then the attacker could attach a weighting to the summands in (3.3.4.2). The general payoff kernel, $K_{A,3}$, would be defined by

(3.3.4.3) $K_{A,3}(U,V) = \sum_{t \in T_L} a_t K_{A,1}(U,V,t)$

with $a_t \geq 0$ for all $t \geq 1$ $\sum_{t \leq T_L} a_t < \infty$.

Inequalities for the above payoff kernels can be obtained.

3.3.5. Lemma. For $K_{A,1}$, $K_{A,2}$, and $K_{A,3}$ as defined in (3.3.4.1), (3.3.4.2) and (3.3.4.3) respectively,

(3.3.5.1) $-1 \leq K_{A,1}(U,V,t) \leq 1$

(3.3.5.2) $-T_L < K_{A,2}(U,V) < T_L$

(3.3.5.3) $- \sum_{t \leq T_L} a_t < K_{A,3}(U,V) < \sum_{t \leq T_L} a_t$

for all $U \in \mathcal{J}_A$ $V \in \mathcal{J}_B$ and t.

The following theorem can then be obtained for conditions on the value of the game.

3.3.6. Theorem. If there exists a real number s and strategies $U_o \in \mathcal{J}_A$ and $V_o \in \mathcal{J}_B$ such that

$K_{A,1}(U_o,V,t) \geq s$ for all $V \in \mathcal{J}_B$

and

$K_{A,1}(U,V_o,t) \leq s$ for all $U \in \mathcal{J}_A$,

then $\min_{V \in \mathcal{J}_B} \max_{U \in \mathcal{J}_A} K_{A,1}(U,V,t) = K_{A,1}(U_o,V_o,t) =$

$\max_{U \in \mathcal{J}_A} \min_{V \in \mathcal{J}_B} K_{A,1}(U,V,t)$.

Similar statements hold for $K_{A,2}$ and $K_{A,3}$.

Proof: The proof is straightforward and so is mitted. A proof also appears in Karlin [10].

Let us now consider the payoff kernels considered in other sources. Charnes and Schroeder in [3] considered a payoff function for a pursuit-evasion game that depended not only of the strategies of each player and the time t, but also on the position of the players at time t. This approach also appears in other sources. It would seem that, in the game considered in this paper, the players should seek strategies that maximize their probability of victory and minimize their probability of defeat (perhaps, at each stage). Where the players win or lose should be irrelevant. If position was important, this could be built into the strategies and transition probabilities at the outset of the game.

A few remarks might now be made concerning the version of the discrete game with mobile target for the attacker. The basic payoff kernels would seem at first glance to generali-- to this situation. However, before any such step can be made, explicit statements concerning the amount and nature of interaction between D and X need to be made. With certain assumptions the generalization should be possible.

4. Examples of Strategies in the Discrete Game.

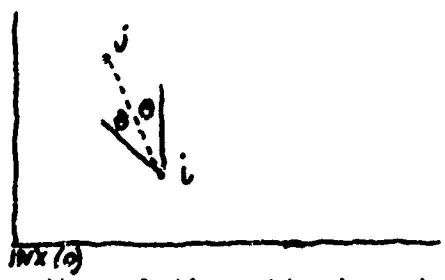
Several examples will be considered in this section. The first will be the case where A acts ballistically and so in a completely nonrandom manner. Another example is the case of completely randomized strategies by one or both players. The thrid example that will be examined is the case where one or both of the players randomize over all lattice points in the sector formed by a certain arc determined by the position of the players.

Suppose that the attacker A acts in a completely nonrandom manner. If A is not employing any evasive tactics, then D will attempt to predict the future position of A and will then attempt to intercept A. This game would involve some interesting problems in information theory. If, as seems more likely, A employs some type of nonrandomized evasive maneuvers, then A's strategy is fixed and D will attempt to select a strategy maximizing the probability of capture before A annihilates X.

A second example of possible strategies is the case of where either or both players act in a completely randomized fashion. The most elementary example of this type occurs when each player simply counts the number of lattice points to which he can move and assigns equal probability to each lattice point. The game then reduces to one where two blindfolded football players are hopping about a large checker board with (A) trying to reach the goal line (X) and the other player (D) trying to tackle A before A reaches X. This case where both players act completely independently of each other is amusing but not very useful.

A slightly less trivial strategy is obtained by having the players randomize as long as they are far enough apart, but by requiring that they act using the knowledge of the position of each other when "close enough" together. Proximity can be measured by the usual Euclidian metric superimposed on the lattice structure.

The third example is one in which either or both players employ(s) "arc randomization" strategies. Suppose one is given the following diagram describing the position of the players at time t.



The defender is at i and the attacker is at j. The position of X is assumed to be fixed at O. D will draw the line connecting i to j. His strategy, determined before the game, will prescribe an arc of say θ°, which he will use to construct the two lines about $\overrightarrow{ij}$ given in the diagram above. D will then randomize according as his strategy dictates, over the points accessible from his position and within the arc.

The attacker will be given two angles, α and β, by his "arc randomization" strategy. A will draw the lines joining j to O and joining j to i and will construct sectors determined by the angles α and β about the lines $\overrightarrow{j,0}$ and $\overrightarrow{j,i}$ respectively, as in the following diagram.

A will randomize with his strategy over all accessible points that lie in the sector determined by β and not in the sector determined by α. If, as is the case when D lies between X and A and β and α, there are no points in this set, A's strategy might be to perform some evasive maneuver directed away from D.

The "arc randomization" seems to be a reasonable strategy in the sense that it takes into account the positions of the players and α the target. To take into account the distance between the players and the target the strategy could make the angle(s) depend on the distances involved.

5. Extensions to the Continuous Game.

Many of the results in section 3, as well as the examples in section 4, can be carried over into the continuous game. However, instead of considering the probabilities that two or more players occupy the same lattice point, one would have to consider the probabilities that two or more players lie in certain regions. This would involve replacing multiple summation over discrete lattice points by multiple Riemann-Stieltjes integrals over regions in E^2 or E^3.

If it should happen that the game is being played on a closed, bounded region in E^2 or E^3, then, using a compactness argument, only a finite number of probabilities would have to be calculated. Restricting the game to such a region might be done through the introduction of reflecting or absorbing barriers.

As in the discrete game, it should be possible to consider different partial orderings on the spaces of strategies and different payoff kernels. In the discrete game the partial orderings involve inequalities among certain probabilities depending on lattice points. The partial orderings in the continuous game would involve inequalities among probabilities of player occupation of spheres in the appropriate space. The payoff kernels would become multiple integrals over certain regions.

The examples of strategies remain virtually intact. The ballistic missile case remains the same. The example where one or both players employ the "arc randomization" also remains about the same in that the distance is now the usual Euclidian distance and the randomization can now be done by imposing a probability distribution function on the region bounded by the arc and a fixed distance from the position of the player(s).

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13. ABSTRACT

This paper deals with an attack-defense game in which there are targets for both attacker and defender. There are two versions of the game, discrete and continuous. After these have been discussed, the relation of past research to the game is described.

The discrete game is considered in detail and is divided into two cases, fixed target and mobile target for the attacker. In the first, the attacker's target is stationary and in the second, the target is mobile. In both cases, expressions for probabilities of victory by either players and termination at each move are derived inductively. The strategies are then examined for the fixed target case along with some payoff kernels.

Some strategies, such as the instance where the attacker is ballistic, are considered for the discrete game.

Finally, the extension of the results obtained earlier in the paper to the continuous version of the game is discussed.

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