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APPROXIMATE ELASTICITY SOLUTION FOR ORTHOTROPIC CYLINDER UNDER HYDROSTATIC PRESSURE AND BAND LOADS

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ABSTRACT

An approximate solution to the Navier equations of the three-dimensional theory of elasticity for an axisymmetric orthotropic circular cylinder subjected to internal and external pressure, axial loads, and closely spaced periodic radial loads is introduced. Numerical comparison with the exact solution for a transversely isotropic cylinder subjected to periodic band loads shows that very good accuracy is obtainable.

When the results of this approximate solution are compared with previously obtained results of a Flügge-type shell solution of a ring-reinforced orthotropic cylinder, it is found that the shell theory gives a fairly accurate representation of the deformation except in the neighborhood of discontinuous loads. The addition of transverse shear deformations does not improve the accuracy of the shell solution.

When Hill's orthotropic yield criterion is applied, it is found that yielding could begin rather early at the inner surface of the shell adjacent to the frame. It is noted that the transverse shearing stress has no great effect on the initial yield pressure.

LIST OF SYMBOLS

A_{ij}	elastic coefficients in stress-strain relations, Eqs. (3.3b)
$A_{1,2}$	arbitrary constants of integration in solution to generalized plane strain problem, Eqs. (5.8a,b)
B_c	band load, Eq. (2.1a)
B_{c1}	axially varying portion of band load (load for first problem), Eq. (2.2)
B_{ij}	elastic constants in generalized plane strain elastic law, Eqs. (5.2c)
C^n_{ij}	constants, Eqs. (4.13)
D	half width of band load, Fig. 1
D_i	nondimensional elastic constants appearing in Navier equations, Eqs. (3.5c)
E_r, E_θ, E_z, G	radial, circumferential, ial and transverse shear elastic moduli, respectively, Eqs. (3.2)
E	elastic modulus of an isotropic material, Eq. (3.7)
E^n_{ij}	exponential functions of shell geometry and Fourier index, Eqs. (4.13)
G^n_{ij}	arbitrary constants of integration of asymptotic solution to first shell problem, Eqs. (4.12)
H_{ii}	nondimensional elastic constants relating asymptotic displacement solutions, Eqs. (4.12g)

H_j^n, H_n	nondimensional Fourier coefficients of externally applied loads and radial band load, respectively, Eqs. (4.13) and (2.1b)
J_i	constants appearing in plane-strain solutions, Eqs. (5.8)
2ℓ	center-to-center distance between successive band loads, Fig. 1
S_{r_n}, S_{θ_n}, S_{z_n}, S_{rz_n}	respective radial, circumferential, axial and transverse shear stress coefficients in trigonometric series
U, W	nondimensional axial and radial displacements, respectively, Eqs. (4.4)
X, Θ, Z, S	radial, circumferential, axial and transverse shear yield stresses, respectively, Eq. (6.1)
a	radius of datum surface (usually mean radius), Eqs. (4.2)
a_{12}, a_{13}, a_{22}, a_{23}, a_{33}	nondimensional elastic coefficients, Eqs. (4.8e)
e_r, e_θ, e_z	classical radial, circumferential and axial strain parameters, respectively, Eqs. (3.1)
e	Naperian logarithmic constant, Eq. (4.2c)

f	plastic potential function of the stresses, Eq. (6.1)
f_n, g_n	radially varying functions in the solution for the displacements in the first problem, Eqs. (4.4)
f_{ni}, g_{ni}	asymptotic approximations to f_n and g_{n1}, respectively, Eqs. (4.10)
k_n	$= n\pi a/\ell$, Eq. (4.4c)
n	Fourier series index integer, Eqs. (2.1)
p_o, p_i	uniform external and internal pressures, respectively, Fig. 1
p_c	magnitude of band load, Fig. 1
p_1	inner lateral surface pressure for plane strain problem, Eq. (5.7b)
$p_{oy}(r,z)$	pressure at which yielding begins at the position (r,z), Eq. (6.2)
r	radial coordinate, Fig. 1
r_i, r_o	internal and external surface radii, respectively, Fig. 1
u, w	axial and outward radial displacements, respectively, Fig. 1
x, y	nondimensional radial and axial transformation coordinates, Eqs. (4.2)

x_o, x_1	values of x at outer and inner shell surfaces, respectively, Eqs. (4.2)
z	axial coordinate, Fig. 1
$\alpha_{1,2,3,4}$	roots of characteristic equation of asymptotic Navier equations, Eqs. (4.12)
$\alpha_o^*, \beta_1, \beta_2^*, \beta_3$	constants appearing in generalized plane strain solution, Eqs. (5.6)
$\beta^n_{1,2,3,4}$	normalized exponents appearing in asymptotic solution to first problem, Eqs. (4.12)
γ	external-to-internal radius ratio, Eq. (5.8i)
$\gamma_{rz}, \gamma_{r\theta}, \gamma_{\theta z}$	classical shearing strains, Eqs. (3.1)
δ_n	$= \ell/an\pi$, Eq. (4.7c)
θ	cylindrical coordinate, Eqs. (3.1)
λ	nondimensional exponent appearing in orthotropic generalized plane strain solution, Eq. (5.6f)
$\nu_{rz}, \nu_{r\theta}, \nu_{\theta r}$ $\nu_{zr}, \nu_{z\theta}, \nu_{\theta z}, \nu$	Poisson ratios, Eqs. (3.2a) and (3.7)
ξ	nondimensional radial transformation coordinate, Eq. (4.6)

ξ_o, ξ_i	values of ξ at outer and inner shell surfaces, respectively
σ_r, σ_θ, σ_z, τ_{rz}	radial, circumferential, axial normal stresses and transverse shear stresses, respectively

1. INTRODUCTION

To date, investigators of pressurized ring-reinforced circular cylindrical shells have employed shell theory in order to describe the behavior (see, e.g., Ref. 1). For thin-walled, isotropic shells, in which the collapse mechanism is primarily a buckling instability, the results of shell theory are generally quite satisfactory. However, in short, thick-walled, filament wound composite cylinders, the collapse mechanism is quite complicated (as yet no single description has been agreed upon). Examination of the ruptured composite test models indicate that failure is probably due to a transverse shear build up at the frame (Ref. 2), a phenomena not predicted by shell theory.

Because of the complicated failure mechanism of composite shells, the need is obvious for an analysis which would yield a more accurate description of the stress distribution through the thickness than is presently available through existing shell theories. The work presented by Klosner and Levine (Ref. 3) on isotropic shells, and later extended by them (Ref. 4) to include transversally isotropic shells, consisted of an elasticity analysis in which the stress functions suggested by Leknitskii (Ref. 5) were used to satisfy exactly the classical equations of elasticity. Their results demonstrated clearly that, although the commonly used Donnell shell theory gave excellent results throughout most of the shell, it could not accurately predict the stress distributions at the frame. However, the Leknitskii stress functions are useful only for the case of a transversally isotropic (or isotropic) material. The cylindrically orthotropic nature of the filament wound composite shells can only be accounted for by seeking a solution to the more complicated equations of elasticity for such a material. In the present study an approximate solution for the displacements is used to solve the axisymmetric Navier equations of equilibrium for an infinitely long, orthotropic, hollow cylinder under external uniform pressure and closely spaced periodically varying internal band loads. It is indicated how more accuracy may be obtained by use of either a perturbation or iteration method.

It is found that the asympto 'c solution is quite accurate and that the shell solution presented in Ref. I is in very good agreement with the asymptotic solution, except in the vicinity of the frame. When transverse shear deformations are considered in the shell theory no better agreement is obtained.

2. FORMULATION

The shell of infinite length is subjected to an external hydrostatic pressure load p_o and internal, axisymmetric, periodically spaced band loads. These band loads are described as (e.g., see Ref. 4)

$$B_c = p_c[\sum_{n=1}^{\infty} H_n \cos \frac{n\pi z}{\ell} - \Delta] \quad (2.1a)$$

with

$$H_n = \frac{2}{n\pi} (-1)^{n+1} \sin n\pi\Delta \quad (2.1b)$$

and

$$\Delta = D/\ell \quad (2.1c)$$

where p_c is the magnitude of the band load, D is half the width of the band load, ℓ is half the distance between the centers of two successive band loads; and z is the axial coordinate (see Fig. 1).

When the classical linear equations of three-dimensional elasticity are used to describe the shell deformation, the solution may be taken as the superposition of the following two solutions:

1. The solution of the shell subjected to an internal load described by [see Eqs. (2.1)]

$$B_{c1} = p_c \sum_{n=1}^{\infty} H_n \cos \frac{n\pi z}{\ell} \quad (2.2)$$

corresponding to which the displacements can be taken as trigonometric series in z. In the ensuing analysis the classical elasticity equations are used to find the coefficients in the latter series as functions of the thickness coordinate r.

2. A generalized plane strain solution which includes uniform lateral pressure loadings on the inner and outer surfaces [the inner surface load must, of course, include the term $- p_c\Delta$ appearing in Eq. (2.1a)] as well as a constant axial force.

3. BASIC EQUATIONS

The classical strain-displacement equations used to describe the axisymmetric deformations of a circular cylindrical shell are

$$e_r = w_{,r} \quad , \quad e_\theta = w/r \qquad (3.1a,b)$$

$$e_z = u_{,z} \quad , \quad \gamma_{r\theta} = 0 \qquad (3.1c,d)$$

$$\gamma_{rz} = u_{,r} + w_{,z} \quad , \quad \gamma_{\theta z} = 0 \qquad (3.1e,f)$$

in which a comma followed by a subscripted variable denotes differentiation with respect to that variable. r, θ and z are the radial, circumferential and axial coordinates, respectively; e_r, e_θ and e_z are the corresponding normal strains; $\gamma_{r\theta}$, γ_{rz} and $\gamma_{\theta z}$ are the shear strains. w and u are the displacements in the radial and axial directions, respectively. It should be noted that both r and w are taken positive outward (Fig. 1).

The generalized Hooke's law for a cylindrically orthotropic, homogeneous material is taken as

$$\begin{Bmatrix} E_r e_r \\ E_\theta e_\theta \\ E_z e_z \end{Bmatrix} = \begin{pmatrix} 1 & , & -\nu_{r\theta} & , & -\nu_{rz} \\ -\nu_{\theta r} & , & 1 & , & -\nu_{\theta z} \\ -\nu_{zr} & , & -\nu_{z\theta} & , & 1 \end{pmatrix} \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{Bmatrix} \qquad (3.2a)$$

$$\tau_{rz} = G\gamma_{rz} \quad , \quad \tau_{r\theta} = 0 \quad , \quad \tau_{\theta z} = 0 \qquad (3.2b,c,d)$$

where

$$\frac{\nu_{r\theta}}{E_r} = \frac{\nu_{\theta r}}{E_\theta} \quad , \quad \frac{\nu_{rz}}{E_r} = \frac{\nu_{zr}}{E_z} \quad , \quad \frac{\nu_{\theta z}}{E_\theta} = \frac{\nu_{z\theta}}{E_z} \qquad (3.2e)$$

follow from symmetry of the stress and strain tensors. { } denotes a column matrix and ( ) denotes a square matrix. σ_r, σ_θ and σ_z are the normal stresses in the radial, circumferential and axial directions, respectively; τ_{rz} is the transverse shear stress. E_r, E_θ and E_z are the elastic moduli in the radial, circumferential and axial directions; G is the transverse shear modulus and $\nu_{r\theta}$, $\nu_{\theta r}$, ν_{rz}, ν_{zr}, $\nu_{\theta z}$ and $\nu_{z\theta}$ are Poisson ratios. It can be seen that Eqs. (3.2) contain 7 independent elastic constants.

The inverse relation is given by

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{Bmatrix} = \begin{pmatrix} A_{11} , & -A_{12} , & -A_{13} \\ -A_{21} , & A_{22} , & -A_{23} \\ -A_{31} , & -A_{32} , & A_{33} \end{pmatrix} \begin{Bmatrix} e_r \\ e_\theta \\ e_z \end{Bmatrix} \qquad (3.3a)$$

where

$$A_{11} = (1-\nu_{z\theta}\nu_{\theta z})(E_r/N) \quad , \quad A_{22} = (1-\nu_{rz}\nu_{zr})(E_\theta/N) \quad , \quad A_{33} = (1-\nu_{r\theta}\nu_{\theta r})(E_z/N)$$

(continued on next page)

$$A_{12} = A_{21} = -(\nu_{r\theta} + \nu_{rz}\nu_{z\theta})(E_\theta/N) \quad , \quad A_{13} = A_{31} = -(\nu_{rz} + \nu_{r\theta}\nu_{\theta z})(E_z/N)$$

$$A_{23} = A_{32} = -(\nu_{\theta z} + \nu_{\theta r}\nu_{rz})(E_z/N)$$

$$N = 1 - \nu_{z\theta}\nu_{\theta z} - \nu_{\theta r}(\nu_{r\theta} + \nu_{rz}\nu_{z\theta}) - \nu_{zr}(\nu_{r\theta}\nu_{\theta z} + \nu_{rz}) \tag{3.3b}$$

The axisymmetric equilibrium equations are

$$(r\sigma_r)_{,r} - \sigma_\theta + r\tau_{rz,z} = 0 \tag{3.4a}$$

$$r\sigma_{z,z} + (r\tau_{zr})_{,r} = 0 \quad . \tag{3.4b}$$

Equations (3.1), (3.3) and (3.4) are combined to yield the well-known Navier equations for an orthotropic cylinder undergoing axisymmetric deformations (Ref. 5).

$$r^2 w_{,rr} + r w_{,r} + D_1 r^2 w_{,zz} - D_2 w + D_3 r^2 u_{,rz} + D_4 r u_{,z} = 0 \tag{3.5a}$$

and

$$D_5 r^2 w_{,rz} - D_6 r w_{,z} + D_7 r^2 u_{,zz} + r^2 u_{,rr} + r u_{,r} = 0 \tag{3.5b}$$

where

$$D_1 = G/A_{11} \quad , \quad D_4 = (A_{23} - A_{13})/A_{11} \quad , \quad D_7 = A_{33}/G$$

$$D_2 = A_{22}/A_{11} \quad , \quad D_5 = (G - A_{31})/G$$

$$D_3 = (G - A_{13})/A_{11} \quad , \quad D_6 = (A_{32} - G)/G \tag{3.5c}$$

It is noted that only five of the constants $D_1, \cdots, D_7$ are independent, since

$$D_3 = D_1 D_5 \quad \text{and} \quad D_4 = \left(D_5 + D_6\right) D_1 \tag{3.5d}$$

Equations (3.5a) and (3.5b) are a set of second order, linear, homogeneous partial differential equations with variable coefficients.

If the material is transversely isotropic,

$$E_r = E_\theta \quad , \quad \nu_{r\theta} = \nu_{\theta r} \quad , \quad \nu_{rz} = \nu_{\theta z} \quad , \quad \nu_{zr} = \nu_{z\theta}$$

$$A_{13} = A_{31} = A_{32} \quad , \quad A_{11} = A_{22}$$

$$D_2 = 1 \quad , \quad D_5 = - D_6 \quad , \quad D_4 = 0 \tag{3.6}$$

It follows that the number of independent elastic constants reduces to five.

If the material is isotropic,

$$E_r = E_\theta = E_z = E \quad , \quad \nu_{r\theta} = \nu_{\theta r} = \nu_{rz} = \nu_{\theta z} = \nu_{zr} = \nu_{z\theta} = \nu$$

(continued on next page)

$$G = \frac{E}{2(1+\nu)} \quad , \qquad A_{ij} = A_{ji}$$

$$D_2 = 1 \quad , \qquad D_5 = -D_6 = \frac{1}{1-2\nu}$$

$$D_4 = 0 \quad , \qquad D_1 D_7 = 1$$

$$D_1 = \frac{1}{2}\left(\frac{1-2\nu}{1-\nu}\right) \tag{3.7}$$

4. AXIALLY VARYING LOADS

The first problem to be investigated is that of an infinitely long circular cylinder with an internal load described by Eq. (2.2). For such a loading the solution is any solution to the Navier equations, Eqs. (3.5), which satisfy the conditions that:

on the outer surface of the shell

$$\sigma_r = 0 \tag{4.1a}$$

$$\tau_{rz} = 0 \tag{4.1b}$$

and on the inner shell surface

$$\sigma_r = p_c \sum_{n=1}^{\infty} H_n \cos \frac{n\pi z}{L} \tag{4.1c}$$

$$\tau_{rz} = 0 \tag{4.1d}$$

The major difficulties in finding such a solution are caused by the fact that the coefficients in the Navier equations are variable in r. In order to reduce the number of variable coefficients encountered (and at the same time nondimensionalize the equations) the following well-known transformation is introduced:

$$x = \ln(r/a) \tag{4.2a}$$

$$- z/a \tag{4.2b}$$

with the inverse relation

$$r = ae^{x} \tag{4.2c}$$

$$z = ay \tag{4.2d}$$

in which a is the radius to any selected datum surface such that $r_i \leq a \leq r_o$. Hence, for relatively thin-walled cylinders x = (r/a) - 1, which is small compared to unity. The Navier equations now become

$$w_{,xx} + D_1 e^{2x} w_{,yy} - D_2 w + D_3 e^{x} u_{,xy} + D_4 e^{x} u_{,y} = 0 \tag{4.3a}$$

$$D_5 e^{x} w_{,xy} - D_6 e^{x} w_{,y} + D_7 e^{2x} u_{,yy} + u_{,xx} = 0 \tag{4.3b}$$

The separation of variables method is used to find a solution. The assumed from of the displacements must reflect the symmetry of the loading (Fig. 1). Thus,

$$W = \frac{A_{11} w}{a p_c} = \sum_{n=1}^{\infty} g_n(x) \cos k_n y \tag{4.4a}$$

$$U = \frac{A_{11} u}{a p_c} = \sum_{n=1}^{\infty} f_n(x) \sin k_n y \tag{4.4b}$$

where

$$k_n = \frac{n\pi a}{\ell} \tag{4.4c}$$

and W and U are nondimensional displacements. Hence, the Navier equations yield

$$g_{n,xx} - (k^2 D_1 e^{2x} + D_2) g_n + k_n D_3 e^x f_{n,x} + k_n D_4 e^x f_n = 0 \tag{4.5a}$$

$$- k_n D_5 e^x g_{n,x} + k_n D_6 e^x g_n - k_n D_7 e^{2x} f_n + f_{n,xx} = 0 \quad . \tag{4.5b}$$

A significant insight is gained into the nature of the functions g_n and f_n upon the stretching of the x-coordinate through the introduction of the transformation

$$\xi = k_n x = (n\pi a/\ell) x \tag{4.6}$$

The Navier equations finally yield

$$g_{n,\xi\xi} - (D_1 e^{2\delta_n \xi} + \delta_n{}^2 D_2) g_n + D_3 e^{\delta_n \xi} f_{n,\xi} + \delta_n D_4 e^{\delta_n \xi} f_n = 0 \tag{4.7a}$$

$$f_{n,\xi\xi} - D_5 e^{\delta_n \xi} g_{n,\xi} + \delta_n D_6 e^{\delta_n \xi} g_n - D_7 e^{2\delta_n \xi} f_n = 0 \tag{4.7b}$$

where

$$\delta_n = 1/k_n = \ell/n\pi a \tag{4.7c}$$

Equations (4.7a,b) are second order, linear, simultaneous, ordinary differential equations with variable coefficients. These equations can be uncoupled with the aid of integrating factors. However, the resulting relations have complicated variable coefficients and are singular for the not uncommon case of a transversely isotropic cylinder. The solution obtained for transversely isotropic cylinders in Ref. 4 (which, as mentioned previously, is formulated in terms of stresses and solved with the Lehknitskii functions) is in terms of Bessel functions. Accordingly, if Eqs. (3.6) are introduced into Eqs. (4.7a,b) and the resulting relations are uncoupled, the displacement solution is also in terms of Bessel functions, which correspond to those found in Ref. 4.

The success of Klosner and Levine (Ref. 4) in using Bessel functions for the transversely isotropic case tempts one to use a Frobenius or other type of power series solution for the more complicated orthotropic equations. However, ξ grows in proportion to n [Eq. (4.6)] and unless (as in classical shell theory) the thickness-to-length ratio is very small and the load converges quickly (i.e., is continuous and varies slowly with y) ξ becomes large as n increases. The large values of k_n (and hence of ξ) which evolve from the short shells of interest here make a power series solution impractical in that many terms are necessary for convergence. Also, such solutions may give rise to small-difference terms, unless appropriate asymptotic expansions can be obtained.

On the other hand, when k_n is large, δ_n is small [Eq. (4.7c)]. This suggests that a perturbation or iteration scheme may yield a reasonable approximate solution. Any solution which capitalizes on the size of δ_n might also be thought of as an asymptotic expansion.

Once g_n and f_n have been found, the stresses can be determined by combining Eqs. (4.4) and (3.3). Hence,

$$\frac{\sigma_r}{p_c} = \sum_{n=1}^{\infty} \frac{1}{\delta_n} [e^{-\delta_n \xi}(g_{n,\xi} - a_{12}\delta_n g_n) - a_{13} f_n] \cos k_n y \tag{4.8a}$$

$$\frac{\sigma_\theta}{p_c} = \sum_{n=1}^{\infty} \frac{1}{\delta_n} [-e^{-\delta_n \xi}(a_{12} g_{n,\xi} - a_{22}\delta g_n) - a_{23} f_n] \cos k_n y \tag{4.8b}$$

$$\frac{\sigma_z}{p_c} = \sum_{n=1}^{\infty} \frac{1}{\delta_n} [- e^{-\delta_n \xi}(a_{13} g_{n,\xi} + a_{23}\delta_n g_n) + a_{33} f_n] \cos k_n y \tag{4.8c}$$

$$\frac{\tau_{rz}}{p_c} = \sum_{n=1}^{\infty} \frac{1}{\delta_n} [e^{-\delta_n \xi} f_{n,\xi} - g_n] \sin k_n y \tag{4.8d}$$

where

$$a_{12} = A_{12}/A_{11} \quad , \quad a_{13} = D_1 - D_3 \quad , \quad a_{22} = D_2$$

$$a_{23} = D_4 + D_1 - D_3 \quad , \quad a_{33} = D_7 D_1 \tag{4.8e}$$

The boundary conditions are satisfied by [see Eqs. (4.1) and Eqs. (4.8a,d)]

$$e^{-x_o}(g_{n,\xi} - a_{12}\delta_n g_n) - a_{13}f_n = 0 \tag{4.9a}$$

$$e^{-x_o}f_{n,\xi} - g_n = 0 \tag{4.9b}$$

at the shell outer surface $x = x_o$, $\xi = \xi_o \geq 0$

and

$$e^{-x_1}f_{n,\xi} - g_n = 0 \tag{4.9c}$$

$$e^{-x_1}(g_{n,\xi} - a_{12}\delta_n g_n) - a_{13}f_n = H_n \tag{4.9d}$$

at the shell inner surface $x = x_1$, $\xi = \xi_1 \leq 0$

If a perturbation solution is used in which the functions g_n and f_n are considered to be perturbed about about the solution of Eqs. (4.7) with δ_n set to zero, g_n and f_n may be expressed as

$$g_n = \sum_{i=1}^{\infty} g_{ni}(\xi)\delta_n^i \tag{4.10a}$$

$$f_n = \sum_{i=1}^{\infty} f_{ni}(\xi)\delta_n^i \tag{4.10b}$$

For $i = 1$ (zeroth perturbation) the differential equations are

$$g_{n1,\xi\xi} - D_1 g_{n1} + D_3 f_{n1,\xi} = 0 \tag{4.11a}$$

and

$$- D_5 g_{n1,\xi} + f_{n1,\xi\xi} - D_7 f_{n1} = 0 \tag{4.11b}$$

Equations (4.11), in which $\delta_n = \ell/n\pi a = 0$, correspond to the equations of a doubly infinite flat plate (for which $a \to \infty$) with a thickness coordinate $\zeta = ax$. The solution is written as

$$g_{n1} = \sum_{i=1}^{4} G^n_{1i} e^{\beta^n_i} \tag{4.12a}$$

$$f_{n1} = \sum_{i=1}^{4} H_{1i} G^n_{1i} e^{\beta^n_i} \tag{4.12b}$$

where

$$\beta^n_{1,2} = \alpha_{1,2}(\xi - \xi_o) \tag{4.12c}$$

$$\beta^n_{3,4} = - \alpha_{2,1}(\xi - \xi_1) \tag{4.12d}$$

and α_1 and α_2 are those roots of the characteristic equation

$$\alpha^4_i + (D_3 D_5 - D_7 - D_1)\alpha^2_i + D_1 D_7 = 0 \tag{4.12e}$$

which are defined by

$$\alpha_{1,2} = - \frac{1}{2} (D_3 D_5 - D_7 - D_1) \pm \frac{1}{2} [(D_3 D_5 - D_7 - D_1) - 4D_1 D_7]^{1/2} \tag{4.12f}$$

Furthermore,

$$H_{1i} = -\left(\frac{\alpha_i^2 - D_1}{D_3\alpha_i}\right) = \left(\frac{D_5\alpha_i}{\alpha_i^2 - D_7}\right) \tag{4.12g}$$

$$\alpha_{3,4} = -\alpha_{2,1} \tag{4.12h}$$

For the specific material properties considered subsequently, α_i is real; moreover $\alpha_{1,2} > 0$ and $\alpha_{i+1} \leq \alpha_i$. In general, the nature of α_i is dependent only upon the elastic properties of the shell material. For example, if the shell were isotropic, equal roots for the characteristic equation would be found (i.e., $\alpha_{1,2} = -1$). It should also be noted that the roots of the characteristic equation do not change with n. The exponents β_i^n were selected to modify G_{1i}^n in order to avoid numbers of excessive magnitude in the computations. The arbitrary constants G_{1i}^n are obtained by satisfying the boundary condition Eqs. (4.9) which, when combined with Eqs. (4.12) yields the matrix equation

$$(C_{ij}^n)\{G_{1i}^n\} = \{H_j^n\} \quad , \quad i,j = 1,2,3,4 \tag{4.13a}$$

where

$$C_{1i}^n = [e^{-x_o}(\alpha_i - \delta_n a_{12}) - a_{13}H_{1i}]E_{1i}^n \tag{4.13b}$$

$$C_{2i}^n = (e^{-x_o}\alpha_i H_{1i} - 1)E_{1i}^n \tag{4.13c}$$

$$C^n_{3i} = (e^{-x_1}\alpha_i H_{1i} - 1)E^n_{2i} \qquad (4.13d)$$

$$C^n_{4i} = [e^{-x_1}(\alpha_i - a_{12}\delta_n) - a_{13}H_{1i}]E^n_{2i} \qquad (4.13e)$$

also

$$E^n_{11} = E^n_{12} = E^n_{23} = E^n_{24} = 1 \qquad (4.13f)$$

$$E^n_{13} = E^n_{22} = e^{\alpha_2(\xi_1 - \xi_o)} = e^{\alpha_2 k_n (x - x_o)} \qquad (4.13g)$$

$$E^n_{14} = E^n_{21} = e^{\alpha_1(\xi_1 - \xi_o)} = e^{\alpha_1 k_n (x_1 - x_o)} \qquad (4.13h)$$

For the loading of interest here [see Eqs. (4.9) and (4.13)]

$$H^n_1 = H^n_2 = H^n_3 = 0 \qquad (4.13i)$$

$$H^n_4 = H_n \qquad (4.13j)$$

Of course, these conditions may be easily altered to include other loading conditions; a more detailed discussion on this and a means to investigate layered shells is offered in the Appendix.

Once the constants G^n_{1i} have been determined from Eq. (4.13a), the stresses may be obtained by substituting Eqs. (4.12) and (4.10) into Eqs. (4.8); this yields

$$\frac{\sigma_r}{p_c} = \sum_{n=1}^{\infty} \sum_{i=1}^{4} [e^{-x}(\alpha_i - a_{12}\delta_n) - a_{13}H_{1i}]G^n_{1i} e^{\beta^n_i} \cos k_n y = \sum_{n=1}^{\infty} S_{r_n} \cos k_n y \qquad (4.14a)$$

$$\frac{\sigma_\theta}{p_c} = \sum_{n=1}^{\infty} \sum_{i=1}^{4} [-e^{-x}(a_{12}\alpha_i - a_{22}\delta_n) - a_{23}H_{1i}]G_{1i}^n e^{\beta_i^n} \cos k_n y = \sum_{n=1}^{\infty} S_{\theta_n} \cos k_n y \qquad (4.14b)$$

$$\frac{\sigma_z}{p_c} = \sum_{n=1}^{\infty} \sum_{i=1}^{4} [-e^{-x}(a_{13}\alpha_i + a_{23}\delta_n) + a_{33}H_{1i}]G_{1i}^n e^{\beta_i^n} \cos k_n y = \sum_{n=1}^{\infty} S_{z_n} \cos k_n y \qquad (4.14c)$$

$$\frac{\tau_{rz}}{p_c} = D_1 \sum_{n=1}^{\infty} \sum_{i=1}^{4} (e^{-x}\alpha_i H_{1i} - 1)G_{1i}^n e^{\beta_i^n} \sin k_n y = \sum_{n=1}^{\infty} S_{rz_n} \sin k_n y \qquad (4.14d)$$

If more accuracy is desired, the first perturbation can be applied [$i = 2$ in Eqs. (4.10) and (4.7)]. This yields

$$g_{n2,\xi\xi} - D_1 g_{n2} + D_3 f_{n2,\xi} = (2D_1 g_{n1} - D_3 f_{n1,\xi})\xi - D_4 f_{n1} \qquad (4.15a)$$

$$-D_5 g_{n2,\xi} + f_{n2,\xi\xi} - D_7 f_{n2} = (D_5 g_{1,\xi} + 2D_7 f_1)\xi - D_6 g_{n1} \qquad (4.15b)$$

An iteration technique may also be used. The first iteration consists of Eqs. (4.12), which corresponds to $\delta_n = 0$. Any further iteration (g_{ni}, f_{ni}) may be obtained by successively solving

$$g_{ni,\xi\xi} - D_1 g_{ni} + D_3 f_{ni,\xi} = [D_1(e^{2\delta_n\xi} - 1) + \delta_n^2 D_2]g_{ni-1} + D_3(1 - e^{\delta_n\xi})f_{ni-1,\xi} - e^{\delta_n\xi}\delta_n D_4 f_{ni-1} \qquad (4.16a)$$

$$-D_5 g_{ni,\xi} + f_{ni,\xi\xi} - D_7 f_{ni} = D_5(e^{\delta_n\xi} - 1)g_{ni-1,\xi} - \delta_n D_6 e^{\delta_n\xi} g_{ni-1} + D_7(e^{2\delta_n\xi} - 1)f_{ni-1} \qquad (4.16b)$$

where the right hand sides are obtained by inserting results of prior iterations.

In either the perturbation or iteration technique the stress components corresponding to perturbations or iterations beyond the zeroth must vanish on the boundaries. In either approach, a correction term is only necessary in the early terms of the trigonometric series given in Eqs. (4.4), (i.e., when n is relatively small), since δ_n becomes smaller as n increases and, therefore, the approximate solution represented by Eqs. (4.12) becomes more accurate.

The selection of β_i^n to modify the constants is indeed appropriate, for as n grows large the coefficients may be closely approximated by simple formulae which reveal the dependent of the trigonometric series convergence on location through the shell thickness. Hence, as n becomes large, k_n becomes large and [see Eqs. (4.13g) and (4.13h)]

$$E_{13},\ E_{22},\ E_{14},\ E_{21} \rightarrow 0 \tag{4.17a}$$

Therefore,

$$C^n_{13},\ C^n_{14},\ C^n_{23},\ C^n_{24},\ C^n_{31},\ C^n_{32},\ C^n_{41},\ C^n_{42} \rightarrow 0 \tag{4.17b}$$

and Eqs. (4.13) may be replaced by

$$\begin{pmatrix} C^n_{11} & C^n_{12} & 0 & 0 \\ C^n_{21} & C^n_{22} & 0 & 0 \\ 0 & 0 & C^n_{33} & C^n_{34} \\ 0 & 0 & C^n_{43} & C^n_{44} \end{pmatrix} \begin{Bmatrix} G^n_{11} \\ G^n_{12} \\ G^n_{13} \\ G^n_{14} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ H_n \end{Bmatrix} \tag{4.18a}$$

Thus, for large n (n > N say)

$$G^n_{11} = G^n_{12} = 0 \tag{4.18b}$$

$$G^n_{13} = -\frac{C^n_{34}H_n}{C^n_{33}C^n_{44}-C^n_{43}C^n_{44}} \tag{4.18c}$$

$$G^n_{14} = \frac{C^n_{33}H_n}{C^n_{33}C^n_{44}-C^n_{43}C^n_{44}} \tag{4.18d}$$

and

$$g_{n1} = [-C^n_{34}e^{-\alpha_2(\xi-\xi_1)} + C^n_{33}e^{-\alpha_1(\xi-\xi_1)}]\left(\frac{H_n}{C^n_{33}C^n_{44}-C^n_{43}C^n_{34}}\right) \tag{4.19a}$$

$$f_{n1} = [C^n_{34}H_{12}e^{-\alpha_2(\xi-\xi_1)} - C^n_{33}H_{11}e^{-\alpha_1(\xi-\xi_1)}]\left(\frac{H_n}{C^n_{33}C^n_{44}-C^n_{43}C^n_{34}}\right) \tag{4.19b}$$

$$\begin{aligned} S_{r_n} = {} & [-C^n_{34}(-\alpha_2 e^{-x}+a_{13}H_{12})e^{-\alpha_2(\xi-\xi_1)} \\ & + C^n_{33}(-\alpha_1 e^{-x}+a_{13}H_{11})e^{-\alpha_1(\xi-\xi_1)}]\left(\frac{H_n}{C^n_{33}C^n_{44}-C^n_{43}C^n_{34}}\right) \end{aligned} \tag{4.19c}$$

$$\begin{aligned} S_{\theta_n} = {} & [-C^n_{34}(a_{12}\alpha_2 e^{-x}+a_{23}H_{12})e^{-\alpha_2(\xi-\xi_1)} \\ & + C^n_{33}(a_{12}\alpha_1 e^{-x}+a_{23}H_{11})e^{-\alpha_1(\xi-\xi_1)}]\left(\frac{H_n}{C^n_{33}C^n_{44}-C^n_{43}C^n_{34}}\right) \end{aligned} \tag{4.19d}$$

$$\begin{aligned} S_{z_n} = {} & [-C^n_{34}(a_{13}\alpha_2 e^{-x}-a_{33}H_{12})e^{-\alpha_2(\xi-\xi_1)} \\ & + C^n_{33}(a_{13}\alpha_1 e^{-x}-a_{33}H_{11})e^{-\alpha_1(\xi-\xi_1)}]\left(\frac{H_n}{C^n_{33}C^n_{44}-C^n_{43}C^n_{34}}\right) \end{aligned} \tag{4.19e}$$

$$S_{rz_n} = [-C_{34}^{n}(\alpha_2 H_{12} e^{-X} - 1) e^{-\alpha_2(\xi-\xi_1)} + C_{33}^{n}(\alpha_1 H_{11} e^{-X} - 1) e^{-\alpha_1(\xi-\xi_1)}] \left(\frac{H_n}{C_{33}^{n} C_{44}^{n} - C_{43}^{n} C_{44}^{n}}\right) \tag{4.19f}$$

For the problems under investigation in this report the stresses converge as

$$e^{-\alpha_2(\xi-\xi_1)} H_n \cos k_n y$$

The convergence is fastest on the shell outer surface ($\xi = \xi_o \geq 0$) and slowest on the shell inner surface ($\xi = \xi_1 \leq 0$), where i. matches the convergence of the applied load [Eq. (2.2)].

5. SECOND (GENERALIZED PLANE STRAIN) PROBLEM

The second problem to be considered is that of an infinite, unsupported shell subjected to internal and external pressure and an applied external axial force. If the radial displacement is assumed to be only a function of r, and if the axial strain is assumed to be constant, the strain displacement equations are given by Eqs. (3.1) in which now

$$e_z = u_{,z} = \text{const} \tag{5.1a}$$

$$\gamma_{zr} = \gamma_{z\theta} = \gamma_{\theta r} = 0 \tag{5.1b}$$

Hooke's law [Eq. (3.2a)] reduces to

$$e_r = B_{11}\sigma_r - B_{12}\sigma_\theta - B_{13}e_z \tag{5.2a}$$

$$e_\theta = -B_{21}\sigma_r + B_{22}\sigma_\theta - B_{23}e_z \tag{5.2b}$$

where

$$B_{11} = \frac{1}{E_r}(1-\nu_{rz}\nu_{zr}) \quad , \quad B_{12} = B_{21} = \frac{1}{E_r}(\nu_{r\theta} + \nu_{rz}\nu_{z\theta})$$

$$B_{13} = \nu_{zr} \quad , \quad B_{22} = \frac{1}{E_\theta}(1-\nu_{\theta z}\nu_{z\theta}) \quad , \quad B_{23} = \nu_{z\theta} \tag{5.2c}$$

and the equilibrium equations reduce to

$$(r\sigma_r)_{,r} - \sigma_\theta = 0 \qquad (5.3)$$

The governing differential equation is obtained from Eqs. (3.1) and (5.1) to (5.3); thus,

$$r[r(r\sigma_r)_{,r}]_{,r} - \frac{B_{11}}{B_{22}}(r\sigma_r) = \left(\frac{B_{23}-B_{13}}{B_{22}}\right) re_z \qquad (5.4)$$

Also, the displacement equation may be obtained by using the plane strain assumptions in Eqs. (3.5); hence,

$$r^2 w_{,rr} + rw_{,r} - D_2 w = - D_4 e_z r \qquad (5.5)$$

It should be noted that no matter which of Eqs. (5.4) or (5.5) is utilized, the previously mentioned singularity, which occurs for either a transversely isotropic or an isotropic material, persists in the generalized plane strain equations (i.e., when $D_4 = 0$ and $D_2 = 1$ or $B_{23} = B_{13}$ and $B_{11} = B_{22}$). This singularity, which determines whether or not the differential equation is homogeneous, is reflected in the solution. The solution of the present problem is

$$\sigma_r = A_1 r^{\lambda-1} + A_2 r^{-\lambda-1} - \alpha_o^* e_z \qquad (5.6a)$$

$$\sigma_\theta = \lambda(A_1 r^{\lambda-1} - A_2 r^{-\lambda-1}) - \alpha_o^* e_z \qquad (5.6b)$$

$$\sigma_z = (B_{13}+\lambda B_{23})A_1 r^{\lambda-1} + (B_{13}-\lambda B_{23})A_2 r^{-\lambda-1} + [E_z - \alpha_o^*(B_{13}+B_{23})]e_z \qquad (5.6c)$$

$$w = (\lambda B_{22}-B_{21})A_1 r^{\lambda}-(\lambda B_{22}+B_{21})A_2 r^{-\lambda}-\left(\frac{D_4}{1-D_2}\right)^{*} re_z \tag{5.6d}$$

$$u = ze_z \tag{5.6e}$$

in which

$$\lambda = \sqrt{B_{11}/B_{22}} = \sqrt{D_2} \tag{5.6f}$$

$$\alpha_o^{*} = \begin{cases} (B_{13}-B_{23})/(1-\lambda^2)B_{22} \quad , & \text{when } \lambda \neq 1 \\ 0, \text{ when } \lambda = 1 & \text{(transversely isotropic or isotropic)} \end{cases} \tag{5.6g}$$

$$\left(\frac{D_4}{1-D_2}\right)^{*} = \begin{cases} D_4/(1-D_2), \text{ when } \lambda \neq 1 \\ 0, \text{ when } \lambda = 1 \end{cases} \tag{5.6h}$$

The three conditions to determine A_1, A_2 and e_z are the boundary conditions

$$\sigma_r = -p_o \quad \text{at } r = r_o \text{ (outer surface),} \tag{5.7a}$$

$$\sigma_r = -p_1 = -(p_i + \Delta p_c) \quad \text{at } r = r_1 \text{ (inner surface)} \tag{5.7b}$$

and the prescribed axial force condition

$$\int_{r_1}^{r_o} \sigma_z r dr = \frac{1}{2}(r_1^{\,2} p_i - r_o^{\,2} p_o) \tag{5.7c}$$

Substitution of Eqs. (5.6) into (5.7) results in three linear algebraic equations which may be solved to yield

$$A_1 = J_1/J \quad , \quad A_2 = J_2/J \quad , \quad E_z = J_3/J \qquad (5.8a,b,c)$$

where

$$J_1 = \{\frac{M_p}{r_1^2}(\gamma^{-\lambda-1}-1)\alpha_o^* - p_1[\alpha_o^*\beta_2^*(\gamma^{1-\lambda}-1)+\beta_3(\gamma^2-1)\gamma^{-\lambda-1}] + p_o[\alpha_o^*\beta_2^*(\gamma^{1-\lambda}-1)+\beta_3(\gamma^2-1)]\}r_i^{1-\lambda} \qquad (5.8d)$$

$$J_2 = \{\frac{M_p}{r_1^2}(1-\gamma^{\lambda-1})\alpha_o^* + p_1[\alpha_o^*\beta_1(\gamma^{\lambda+1}-1)+\beta_3\gamma^{\lambda-1}(\gamma^2-1)] - p_o[\beta_1\alpha_o^*(\gamma^{\lambda+1}-1)+\beta_3(\gamma^2-1)]\}r_1^{1+\lambda} \qquad (5.8e)$$

$$J_3 = \frac{M_p}{r_1^2}(\gamma^{-\lambda-1}-\gamma^{\lambda-1}) + p_1[\beta_1\gamma^{-\lambda-1}(\gamma^{\lambda+1}-1)-\beta_2^*\gamma^{\lambda-1}(\gamma^{1-\lambda}-1)] - p_o[\beta_1(\gamma^{\lambda+1}-1)-\beta_2^*(\gamma^{1-\lambda}-1)] \qquad (5.8f)$$

$$J = \alpha_o^*\beta_1(\gamma^{\lambda+1}-1)(\gamma^{-\lambda-1}-1) - \alpha_o^*\beta_2^*(\gamma^{1-\lambda}-1)(\gamma^{\lambda-1}-1) + \beta_3(\gamma^2-1)(\gamma^{-\lambda-1}-\gamma^{\lambda-1}) \qquad (5.8g)$$

$$M_p = \frac{1}{2}(r_1^2 p_i - r_o^2 p_o) \qquad (5.8h)$$

$$\gamma = r_o/r_1 \qquad (5.8i)$$

$$\beta_1 = \frac{B_{13} + \lambda B_{23}}{1+\lambda} \qquad (5.8j)$$

$$\beta_2^* = \begin{cases} (B_{13}-\lambda B_{23})/(1-\lambda) \ , & \text{when } \lambda \neq 1 \\ 0 \ , & \text{when } \lambda = 1 \end{cases} \tag{5.8k}$$

$$\beta_3 = \frac{1}{2}[E_z - \alpha_o^*(B_{13} + B_{23})] \tag{5.8ℓ}$$

The stresses and displacements of a pressurized cylinder subjected to periodically spaced band loads may now be obtained by superposing the individually developed solutions [see Eqs. (4.4), (4.8), and (5.6)]; this results in

$$w = (\lambda B_{22}-B_{21})A_1 r^{\lambda} - (\lambda B_{22}+B_{21})A_2 r^{-\lambda} - \left(\frac{D_4}{1-D_2}\right)^* r e_z + \frac{ap_c}{A_{11}} \sum_{n=1}^{\infty} \delta_n \sum_{i=1}^{4} G_{1i}^n e^{B_i^n} \cos k_n y \tag{5.9a}$$

$$u = e_z a y + \frac{ap_c}{A_{11}} \sum_{n=1}^{\infty} \delta_n \sum_{i=1}^{4} H_{1i} G_{1i}^n e^{\beta_i^n} \sin k_n y \tag{5.9b}$$

$$\sigma_r = A_1 r^{\lambda-1} + A_2 r^{-\lambda-1} - \alpha_o^* e_z + p_c \sum_{n=1}^{\infty} \sum_{i=1}^{4} [e^{-x}(\alpha_i - a_{12}\delta_n) - a_{13}H_{1i}] G_{1i}^n e^{\beta_i^n} \cos k_n y \tag{5.9c}$$

$$\sigma_\theta = \lambda(A_1 r^{\lambda-1} - A_2 r^{-\lambda-1}) - \alpha_o^* e_z + p_c \sum_{n=1}^{\infty} \sum_{i=1}^{4} [-e^{-x}(a_{12}\alpha_i - a_{22}\delta_n) - a_{23}H_{1i}] G_{1i}^n e^{\beta_i^n} \cos k_n y \tag{5.9d}$$

$$\sigma_z = (B_{13}+\lambda B_{23})A_1 r^{\lambda-1} + (B_{13}-\lambda B_{23})A_2 r^{-\lambda-1} + 2\beta_3 e_z$$
$$+ p_c \sum_{n=1}^{\infty} \sum_{i=1}^{4} [-e^{-x}(a_{13}\alpha_i + a_{23}\delta_n) + a_{33}H_{1i}] G_{1i}^n e^{\beta_i^n} \cos k_n y \tag{5.9e}$$

$$\tau_{rz} = p_c D_1 \sum_{n=1}^{\infty} \sum_{i=1}^{4} (e^{-x}\alpha_i H_{1i} - 1) G_{1i}^n e^{\beta_i^n} \sin k_n y \tag{5.9f}$$

6. YIELD CRITERION

Any elasticity solution is valid only up to the pressure at which yielding begins at some point on the shell. It is therefore of great interest to attempt to find this initial yeild pressure.

R. Hill, Ref. 9, developed a plastic potential function f for the determination of that combination of stresses for which an orthotropic material would begin to yield. This function of the stresses is defined to be

$$f = \frac{1}{2}\{(\frac{1}{\Theta^2} + \frac{1}{Z^2} - \frac{1}{X^2})(\sigma_\theta - \sigma_z)^2 + (\frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{\Theta^2})(\sigma_z - \sigma_r)^2$$

$$+ (\frac{1}{X^2} + \frac{1}{\Theta^2} - \frac{1}{Z^2})(\sigma_r - \sigma_\theta)^2 + \frac{\tau_{\theta z}^2}{R^2} + \frac{\tau_{zr}^2}{S^2} + \frac{\tau_{r\theta}^2}{T^2}\} \qquad (6.1)$$

in which X, Θ, Z are the radial, circumferential and axial yield stresses, respectively, and R, S, T are the appropriate shear yield stresses. The criterion for yielding to begin is that f = 1.

The stresses in Eqs. (5.9) could be substituted to leave an expression for the pressure $p_{oy}(r,z)$ for which yielding would begin at the position (r,z)

$$p_{oy}(r,z) = \frac{p_{oy}}{p_c}\sqrt{2}\{(\frac{1}{\Theta^2} + \frac{1}{Z^2} - \frac{1}{X^2})(\frac{\sigma_\theta}{p_c} - \frac{\sigma_z}{p_c})^2 + (\frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{\Theta^2})(\frac{\sigma_z}{p_c} - \frac{\sigma_r}{p_c})^2$$

$$+ (\frac{1}{X^2} + \frac{1}{\Theta^2} - \frac{1}{Z^2})(\frac{\sigma_r}{p_c} - \frac{\sigma_\theta}{p_c})^2 + \frac{\tau_{zr}^2}{p_c^2 S^2}\}^{-1/2} \qquad (6.2)$$

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in which X, Θ, Z are the radial, circumferential and axial yield stresses, respectively, and R, S, T are the appropriate shear yield stresses. The criterion for yielding to begin is that f = 1.

The stresses in Eqs. (5.9) could be substituted to leave an expression for the pressure $p_{oy}(r,z)$ for which yielding would begin at the position (r,z)

$$p_{oy}(r,z) = \frac{p_{oy}}{p_c}\sqrt{2}\left\{\left(\frac{1}{\Theta^2} + \frac{1}{Z^2} - \frac{1}{X^2}\right)\left(\frac{\sigma_\theta}{p_c} - \frac{\sigma_z}{p_c}\right)^2 + \left(\frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{\Theta^2}\right)\left(\frac{\sigma_z}{p_c} - \frac{\sigma_r}{p_c}\right)^2 + \left(\frac{1}{X^2} + \frac{1}{\Theta^2} - \frac{1}{Z^2}\right)\left(\frac{\sigma_r}{p_c} - \frac{\sigma_\theta}{p_c}\right)^2 + \frac{\tau_{zr}^2}{p_c^2 S^2}\right\}^{-1/2} \qquad (6.2)$$

In the filament wound composite, the radial yield stress X is essentially that of the resin and is small compared with the axial and circumferential yield stresses. This enables the approximation

$$(1/\theta^2) << (1/X^2) \quad , \quad (1/Z^2) << (1/X^2) \qquad (6.3a,b)$$

$$p_{oy}(r,z) = X(p_{oy}/p_c)/[(\frac{\sigma_x}{p_c})(\frac{\sigma_\theta}{p_c}) + \frac{1}{2}(\frac{X\tau_{zr}}{sp_c})^2 - \frac{\sigma_r}{p_c}(\frac{\sigma_\theta}{p_c} + \frac{\sigma_z}{p_c})]^{1/2} \qquad (6.3c)$$

From either of Eqs. (6.2) or (6.3) it can be clearly seen that the transverse shear term can serve to decrease the yield pressure.

7. NUMERICAL COMPUTATIONS AND DISCUSSION

The approximate solution to the three-dimensional Navier equations of an orthotropic, infinite circular cylinder requires numerical verification before it can be used. For this reason the first set of numerical computations (which were all performed on the IBM 360/50 computer located at the Polytechnic Institute of Brooklyn) were devoted to comparing the approximate results developed here with the exact solution to a transversely isotropic cylinder subjected only to periodically spaced band loads. The exact results were given to the author by H. Levine, who used the exact analysis described in Ref. 4. The cylinder constants are

$$E_r = E_\theta = 2E_z \quad , \quad G_{rz} = E_r/5.28 \quad , \quad \nu_{zr} = 0.16 \quad , \quad \nu_{r\theta} = 0.30$$

$$\frac{D}{L} = \Delta = 0.2 \quad , \quad \frac{L}{r_o} = 0.2 \quad , \quad \frac{r_1}{r_o} = 0.8 \quad , \quad \delta_1 = 0.0616 \tag{7.1}$$

The comparison is shown in Figures 2 and 3. The results indicate that although a correction term [of the type obtained from the solution of either of Eqs. (4.15) and (4.16)] is desirable, the zeroth-approximation solution gives results which would be satisfactory to the designer. This solution gives a fairly accurate representation of the stress distributions through the thickness in that it demonstrates the nonlinear variation of the normal stresses. The tendency of the shear stress to "peak" in the vicinity of the load discontinuity is predicted.

The second set of computations utilizes the approximate three-dimensional elasticity solution of an orthotropic shell, subjected to external pressure and an axial force in addition to prescribed periodically spaced band loads. The results offered by the shell theory developed in Ref. 1 are compared to the more exact results obtained here.

In Ref. 1, the band loads were due to periodically spaced elastic ring supports. The radial displacement of a ring was found as a function of the magnitude of the band load (p_c) in a separate analysis (either ring theory or an orthotropic Lame analysis). The ratio p_o/p_c was then obtained by matching the radial displacements of the ring and the shell at the ring-shell interface. In the present analysis, the ratio p_o/p_c is assumed to have the same numerical values as those determined in Ref. 1.

When Flügge-type shell theory was applied to a corresponding ring-supported orthotropic shell it was found that $p_o/p_c = 1.11$. The addition of transverse shear deformation to this analysis resulted in $p_o/p_c = 1.03$. Obviously, the results of the present three-dimensional elasticity solution do not reflect the restraining effects of the ring. In order to correct this it would be necessary to include an analysis of the ring and then match displacements at the ring-shell interface.

The cylinder constants are*

$E_r = 2.49 \times 10^6$ psi , $E_\theta = 6.14 \times 10^6$ psi , $E_z = 4.74 \times 10^6$ psi

$G_{rz} = 0.7 \times 10^6$ psi , $\nu_{z\theta} = 0.136$, $\nu_{\theta z} = 0.176$

$\nu_{zr} = 0.5$

*Letter dated March 24, 1964 from Mr. W.P. Couch, David Taylor Model Basin

$\frac{D}{L} = 0.1755$, $\frac{L}{a} = 0.2417$, $r_o = 3.388$ in. , $r_1 = 3$ in.

$\delta_1 = 0.0769$

$X = 0.22 \times 10^5$psi , $\Theta = 1.7 \times 10^5$psi , $Z = 1.0 \times 10^5$psi

$$S = 0.09 \times 10^5 \text{psi} \qquad (7.2)$$

The transverse constants E_r, G_{rz} and ν_{zr} were not available, as they are extremely difficult to obtain (Ref. 6) and were simply given values that were felt to be representative of this type of material (e.g., see Ref. 7).

The results are presented in Figures 4 to 10. Shell theory gives excellent results at midbay, but can only approximate the actual state of stress at the frame. The yield pressures predicted by shell theory [using the criterion in Eqs. (6.1) with f = 1] are fairly accurate. Although the present results given more accurate stress distributions for the band load problem, it does not necessarily follow that the band load problem exactly reflects a ring-shell interaction problem.

Klosner and Levine (Ref. 4) found that transverse-shear-deformation shell theory did not lead to improvement over classical shell theory. Such observations can also be made from the present calculations (see Figures 4 and 5). Furthermore, these results indicate that, at a sufficient distance away from the band load, the shear stress can be approximated with a parabolic function. However, the results displayed in Figs. 4 and 8 show that the axial displacement and axial stress vary cubically (at least) through the shell thickness. Hence, in the vicinity of the band load, where the axial displacement reaches its largest value, plane sections do not remain plane and the transverse shear stress can not be assumed to vary parabolically through the thickness. The cubic and higher order thickness terms combine to result in the "peaking" shown in Figure 6.

The Hill yield criterion Eq. (6.2) predicts that yielding begins at a fairly low pressure (3500 psi) near the inner surface at the load discontinuity. Actual tests (Ref. 2) indicate that the shells do fail at the frame near the inner surface; but at a significantly higher pressure (12200 psi). The low theoretical yield pressure results from the low value of the resin yield stress. When this yield stress was increased (i.e., X was set equal to 1.0×10^5 psi) the lowest value of the theoretical yield pressure was found to be $p_{oy}(r_1, L/2) = 16323$ psi.

One might speculate that the resin yields at a very low pressure after which the load is resisted by the glass fibers. If this is the case, the work done here might be extended to a layer analysis (see Appendix), which could be used to approximate the stresses after yielding has begun.

It appears from these results that the transverse shear stress distribution has very little effect on the low pressure at which the resin begins to yield (which was also predicted by the shell theory). However, as is pointed out in Ref. 1, the actual stresses in the individual constituents of the nonhomogeneous filament wound composite shells must be obtained by multiplying the stresses obtained here by suitable stress-concentration factors. This procedure gives rise to substantial changes in the magnitude of the initial yield pressure (e.g., see Ref. 1).

It is interesting to note that the shell wall gets thicker under pressure (see Fig. 9). This can be traced to the Poisson effect of the rather large compressive axial and circumferential stresses.

In general, the numerical results obtained here indicate that the approximate elasticity solution developed here to satisfy the orthotropic Navier equations gives a satisfactory description of the stress distribution through the shell thickness. However, before this technique can be successfully applied, the transverse elastic constants must be experimentally determined. The low yield pressures obtained here may correspond to the "initial yielding" state described by Tsai (Ref. 8) in which the load deformation curve of a similar shell subjected to internal pressure was observed to have a sudden change in slope at a low pressure. If this is the case, the layer analysis suggested in the Appendix of this work would certainly be appropriate.

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APPENDIX

It may be desirable to prescribe a variety of combinations of radial stress, shear stress, axial displacement and radial displacement as periodic functions of z on either of the shell surfaces (for example, the ring supported shell may be analysed by matching ring and shell displacements). Therefore a more general discussion of possible boundary conditions is offered.

On each shell surface (outer and inner), any two of the following four quantities must be prescribed:

$$\sum_{i=1}^{4} [e^{\substack{-x_o \\ 1}}(\alpha_i-\delta_n a_{12}) - a_{13}H_{1i}]G^n_{1i}E_{\substack{1i \\ 2}} = H^n_{\substack{1 \\ 4}} \qquad \text{(radial stress)} \qquad \text{(A1a)}$$

$$\sum_{i=1}^{4} (e^{\substack{-x_o \\ 1}}\alpha_i H_{1i}-1)G^n_{1i}E_{\substack{1i \\ 2}} = H^n_{\substack{2 \\ 3}} \qquad \text{(transverse shear stress)} \qquad \text{(A1b)}$$

$$\sum_{i=1}^{4} G^n_{1i} = G^n \qquad \text{(radial displacement)} \qquad \text{(A1c)}$$

$$\sum_{i=1}^{4} H_{1i}G^n_{1i} = F^n \qquad \text{(axial displacement)} \qquad \text{(A1d)}$$

where H^n_j, G^n and F^n are Fourier series coefficients obtained by expanding the prescribed functions of z.

It is possible to perform a layer analysis by matching the four quantities listed in Eqs. (A1) across each interface. If a shell consists of m layers, 4m arbitrary constants would have to be determined The interaction equations, which would be 4m-4 in number, could be obtained from

$$(H_1^n)_{j-1} = (H_4^n)_j \tag{A2a}$$

$$(H_2^n)_{j-1} = (H_3^n)_j \tag{A2b}$$

$$G_{j-1}^n = G_j^n \tag{A2c}$$

$$F_{j-1}^n = F_j^n \tag{A2d}$$

where $j = 2, \cdots, m$. The subscripts 1 and m correspond to the inner and outer shell layers, respectively. The remaining 4 equations are found by applying two of Eqs. (A1) on layers 1 and m. The corresponding generalized plane strain problem must also be solved layer by layer with the radial stress and both displacements being matched at each interface.

If the layers were permitted an axial motion relative to each other (as might occur after initial yielding) Eqs. (A2b) and (A2d) could be replaced by

$$(H_2^n)_{j-1} = T^n \tag{A2'b}$$

$$(H_3^n)_j = T^n \tag{A2'd}$$

where T^n would be a constant corresponding to a maximum shear stress at resin yield.

ACKNOWLEDGEMENT

The authors are grateful to Mr. Eugene Golub, who was responsible for the success of the numerical computations.

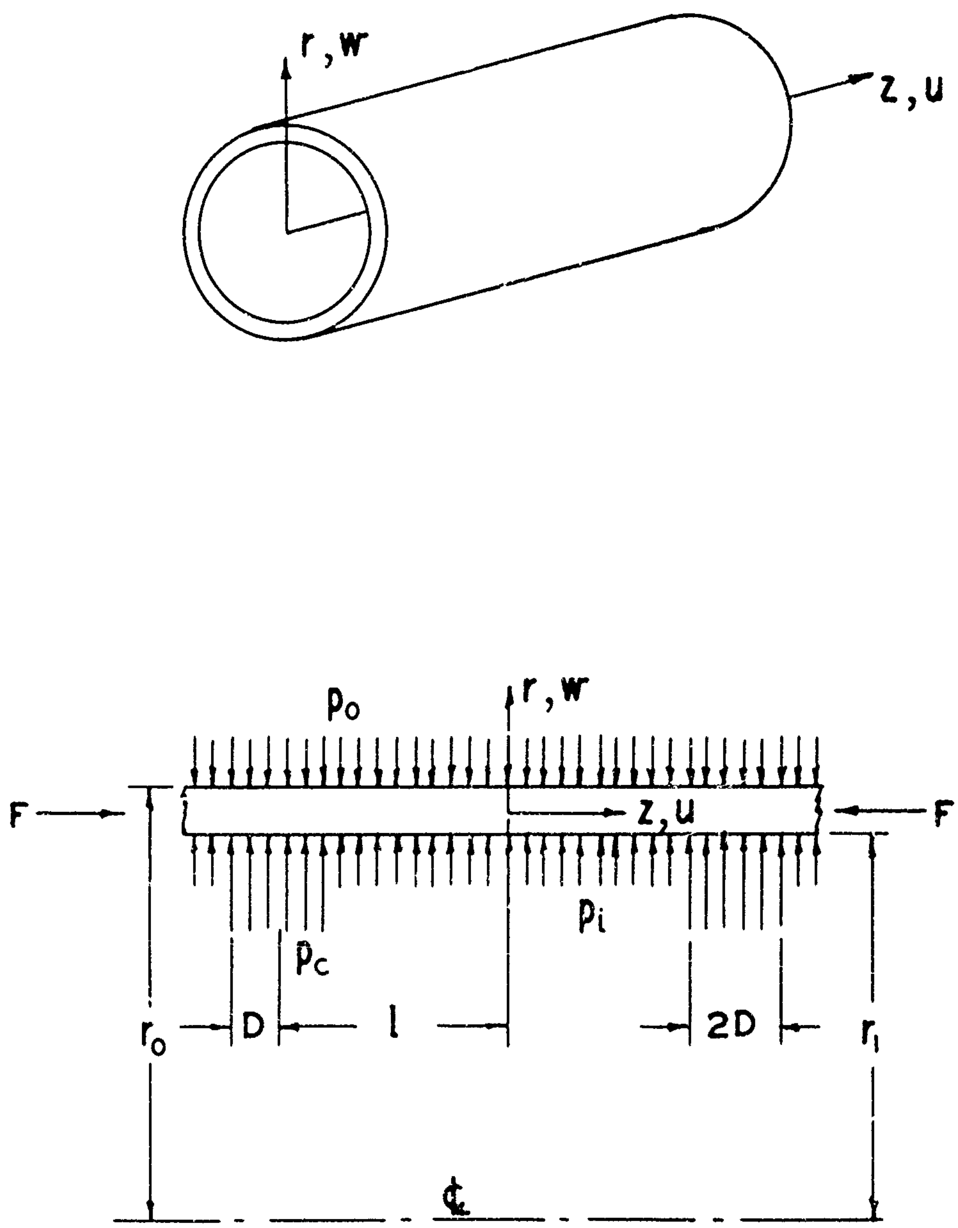


FIG. 1
SHELL GEOMETRY AND LOADING

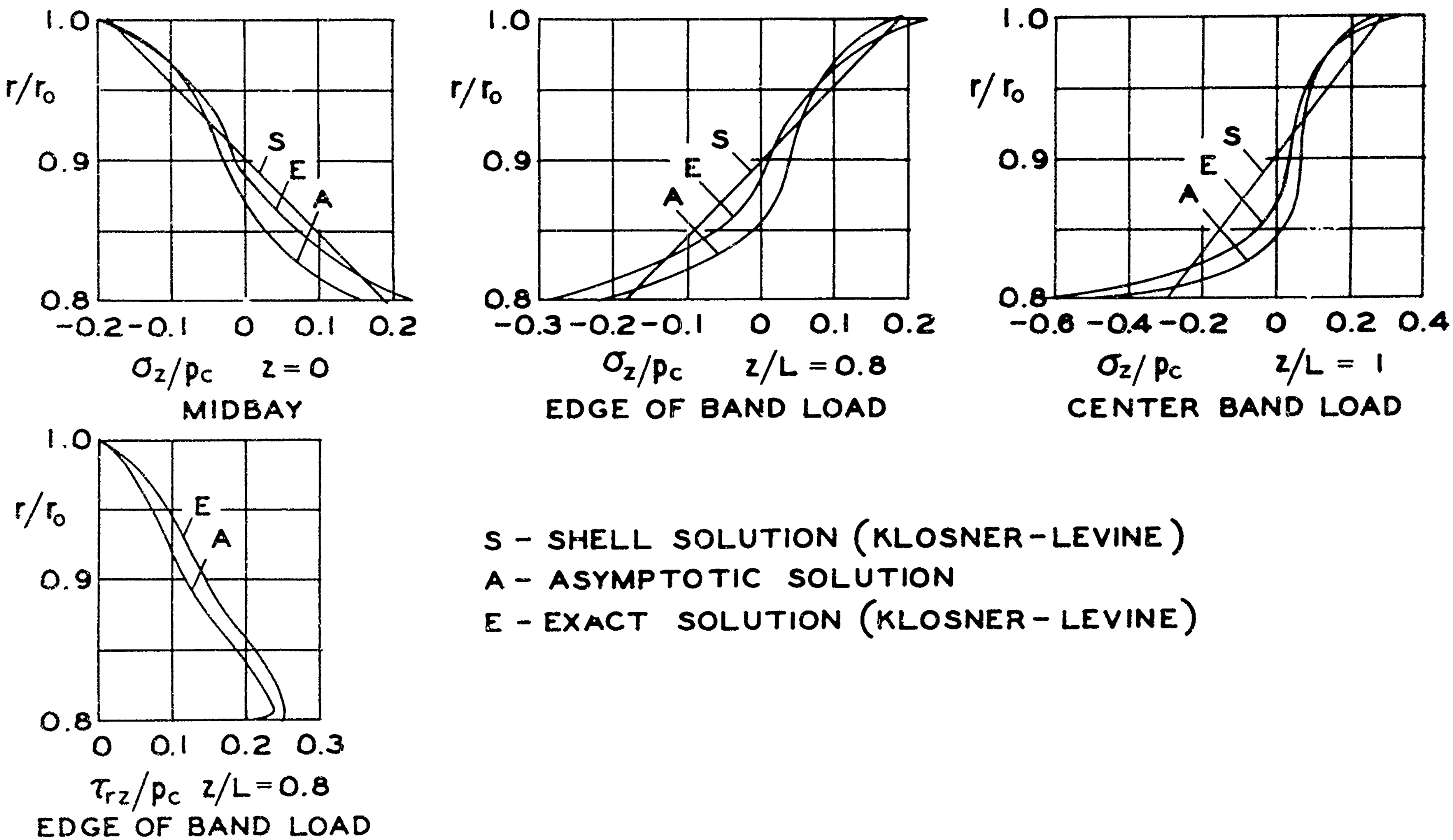


FIG. 2 SOLUTIONS OF A TRANSVERSALLY ISOTROPIC CYLINDER (AXIAL AND TRANSVERSE SHEAR STRESS)

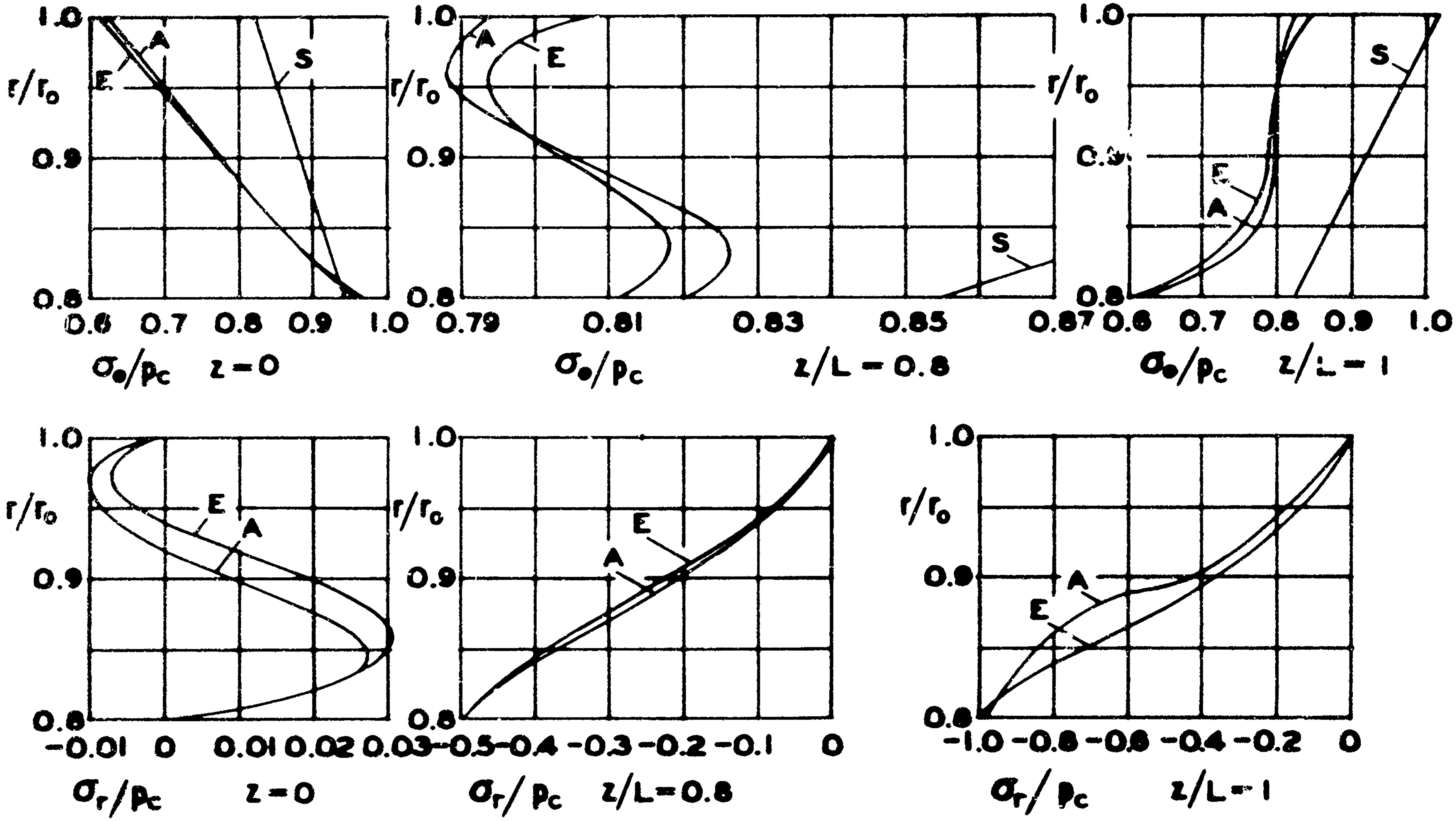
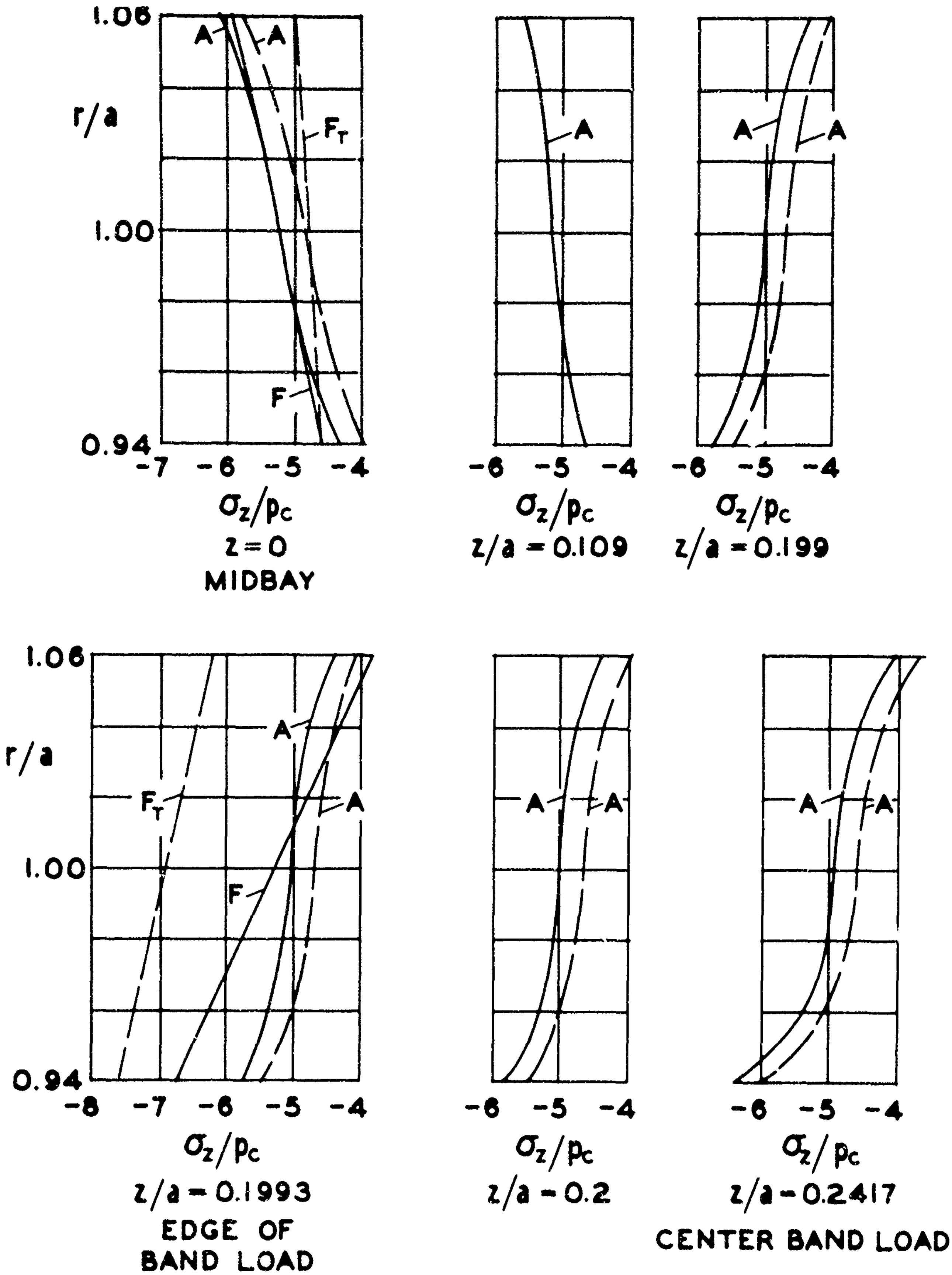


FIG. 3 SOLUTIONS OF A TRANSVERSALLY ISOTROPIC CYLINDER (RADIAL AND CIRCUMFERENTIAL STRESS)



F – FLÜGGE SHELL THEORY

F_T – FLÜGGE SHELL THEORY WITH TRANSVERSE SHEAR

A – APPROXIMATE ELASTICITY SOLUTION

———— $p_o/p_c = 1.111$ — — — $p_o/p_c = 1.034$

FIG. 4

AXIAL STRESSES IN ORTHOTROPIC CYLINDER

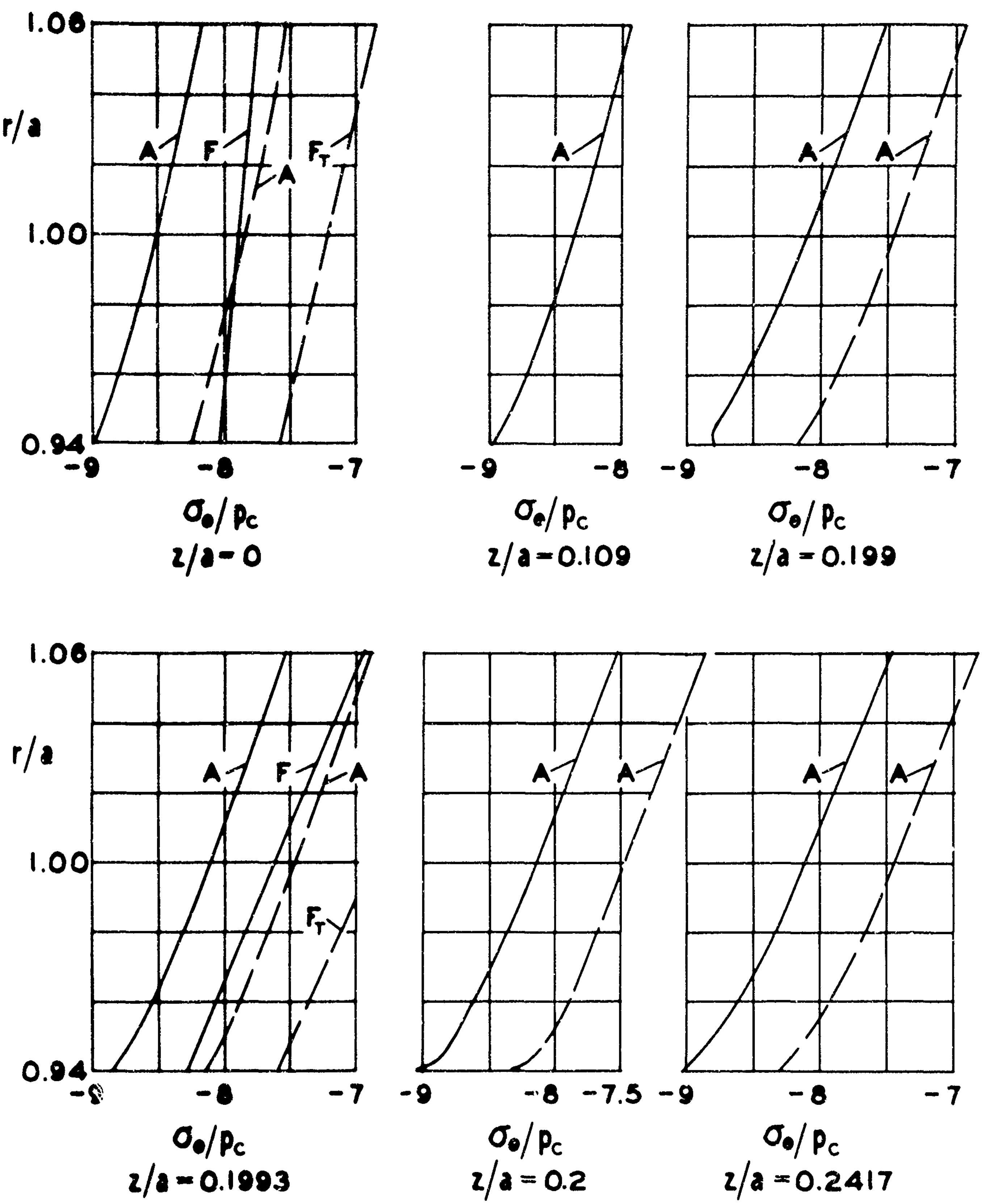


FIG. 5 CIRCUMFERENTIAL STRESSES IN ORTHOTROPIC CYLINDER

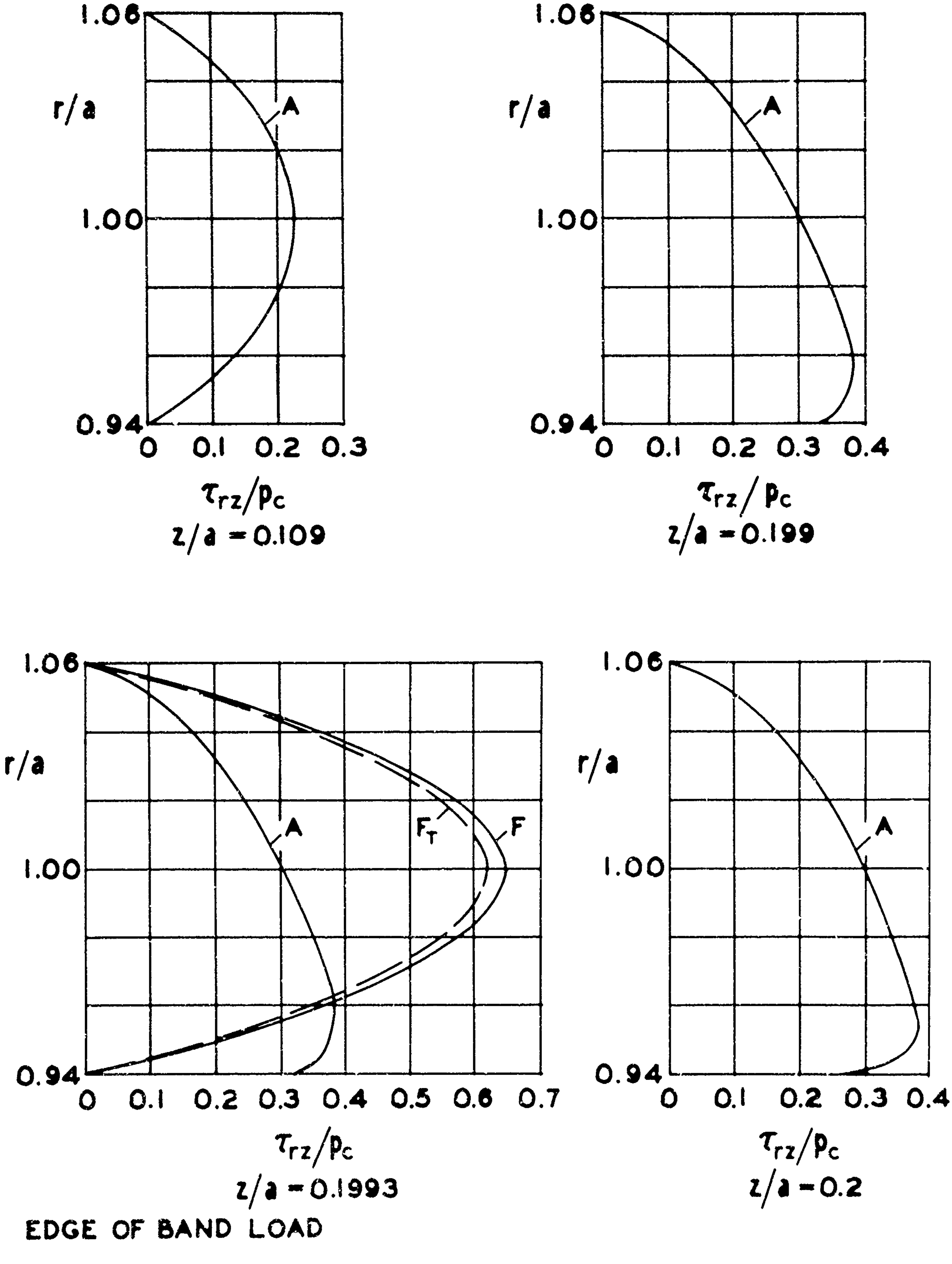


FIG 6 TRANSVERSE SHEAR STRESSES IN ORTHOTROPIC CYLINDER

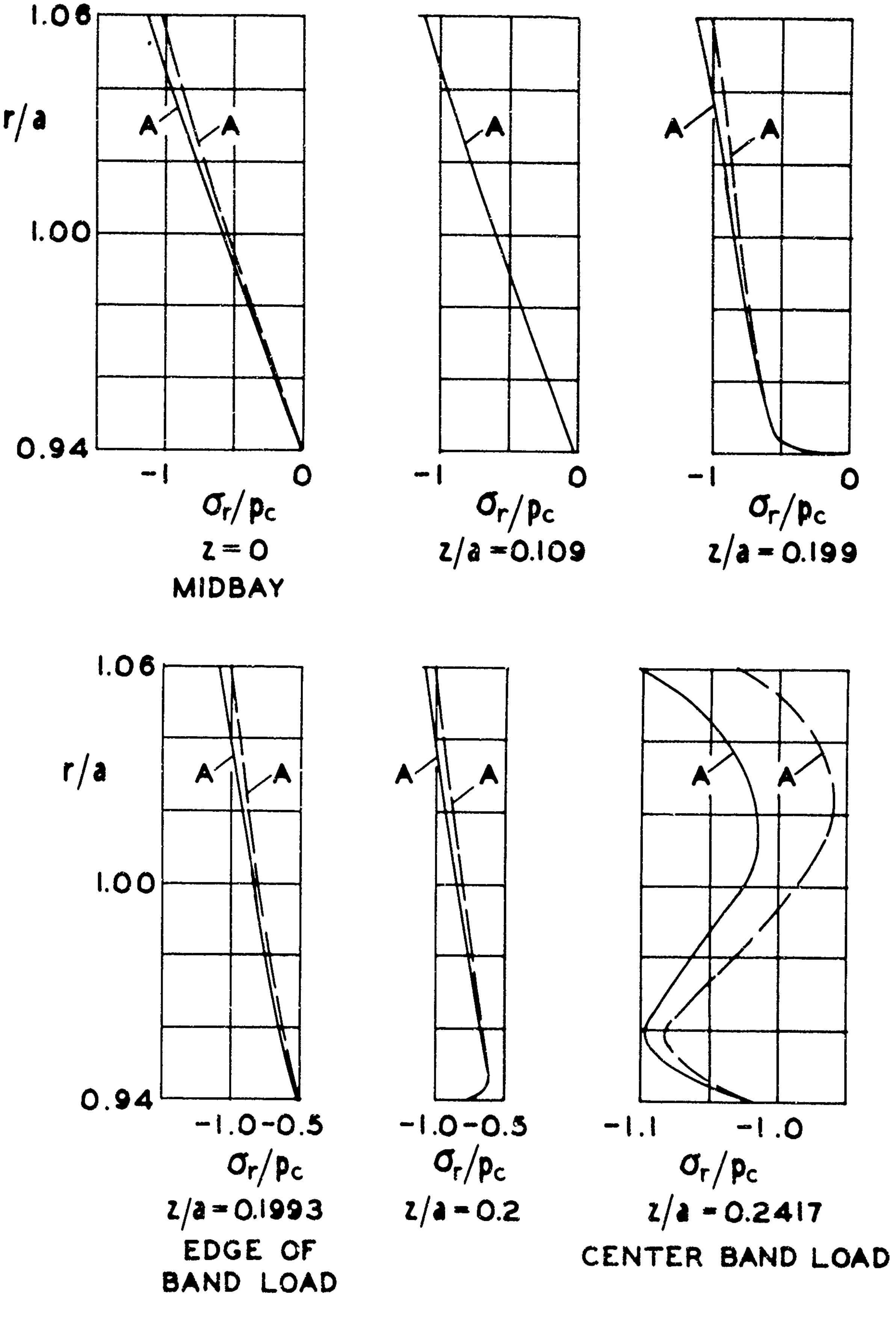


FIG. 7

RADIAL STRESSES IN ORTHOTROPIC CYLINDER

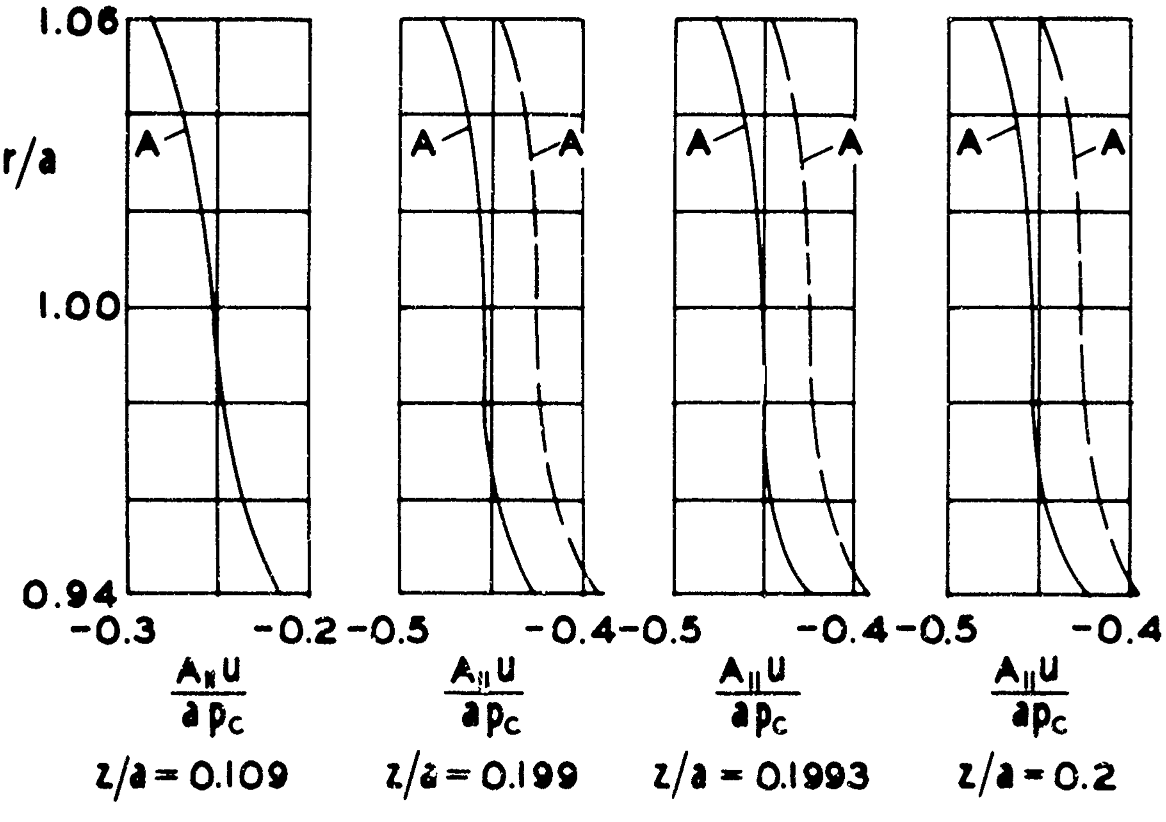


FIG. 8 AXIAL DISPLACEMENTS OF ORTHOTROPIC CYLINDER

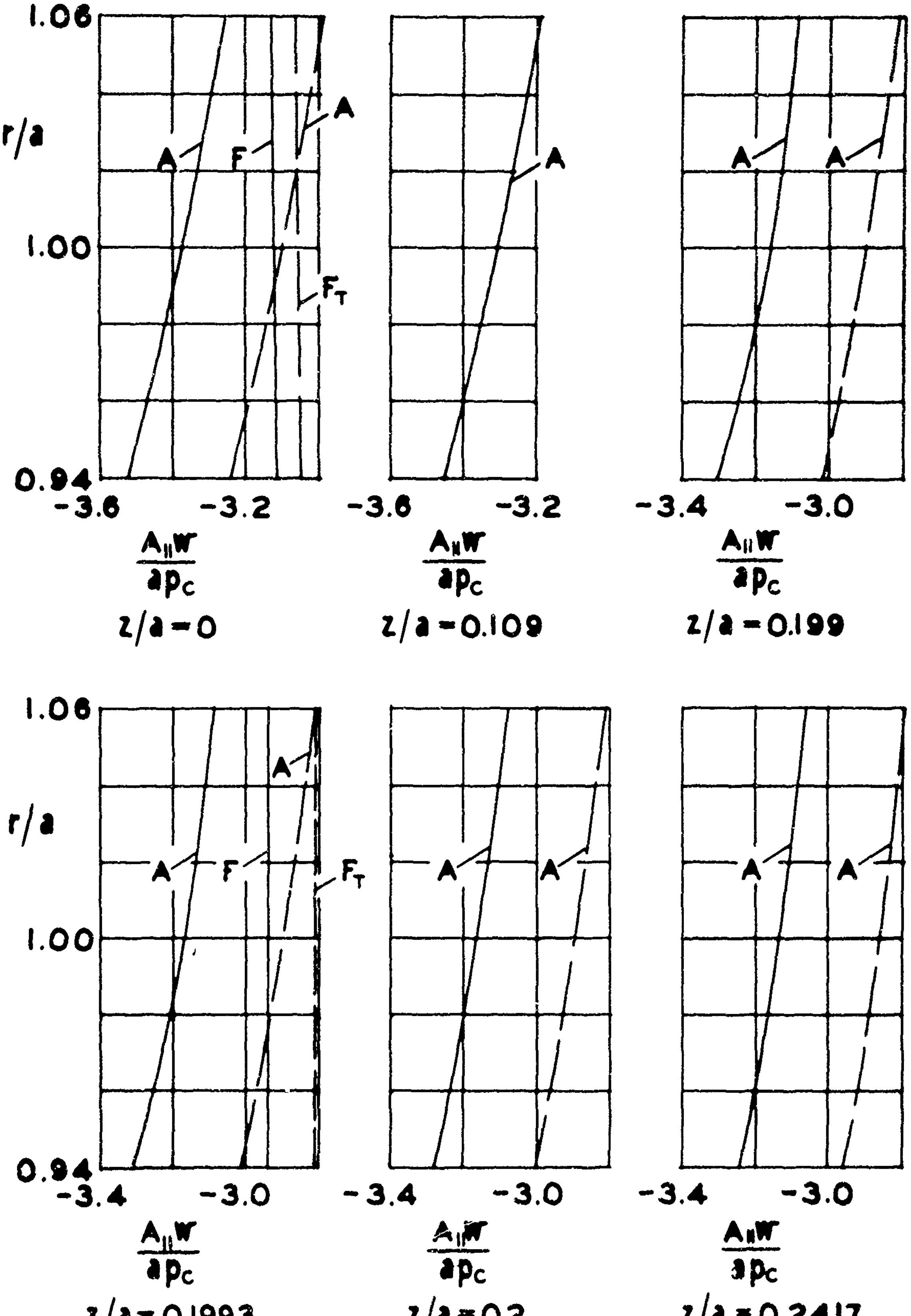


FIG. 9 RADIAL DISPLACEMENTS OF ORTHOTROPIC CYLINDER

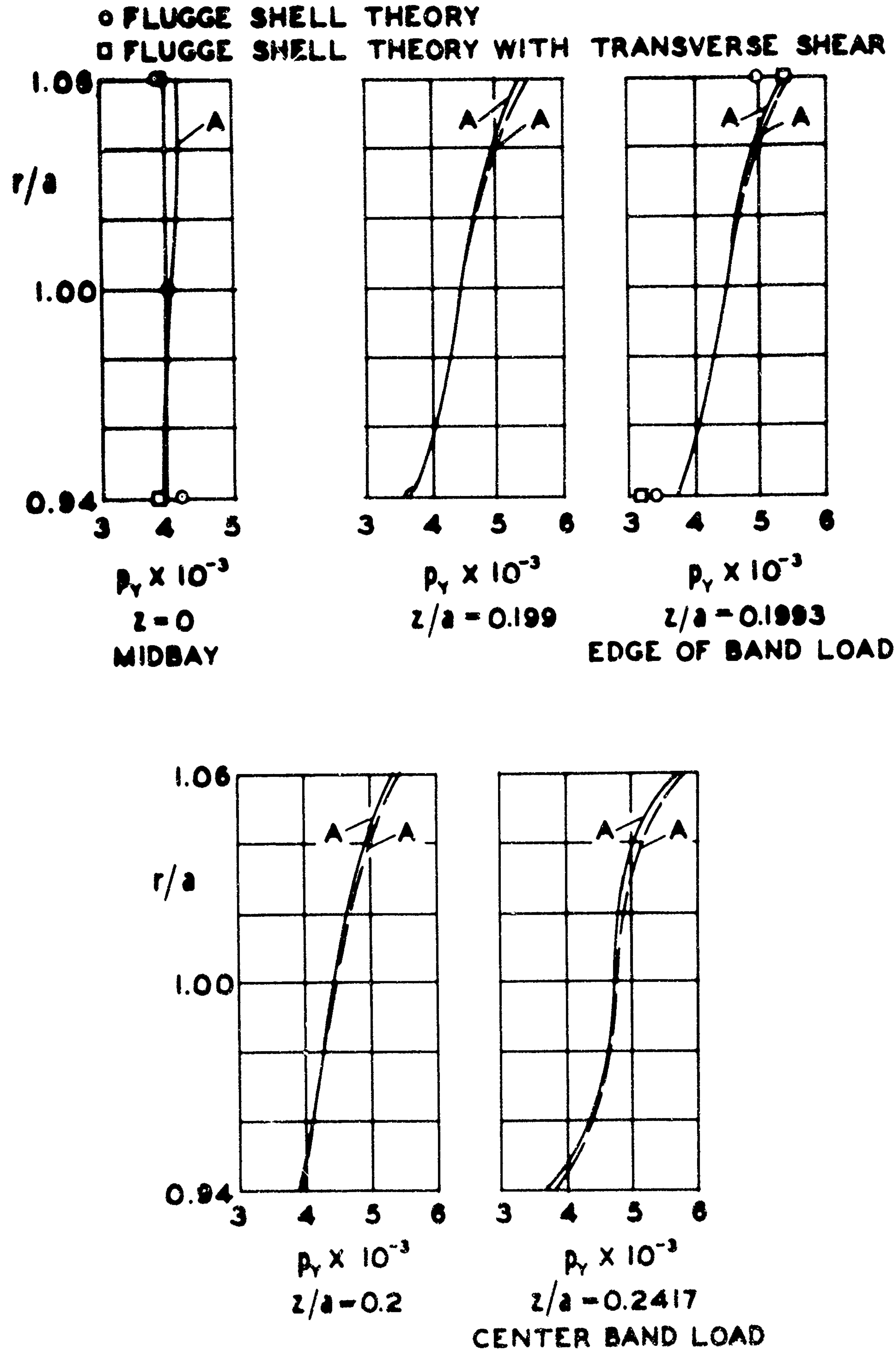


FIG. 10 YIELD PRESSURES

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4 DESCRIPTIVE NOTES (Type of report and inclusive dates)

Research Report

5 AUTHOR(S) (Last name, first name, initial)

Andrew P. Misovec and Joseph Kempner

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When the results of this approximate solution are compared with previously obtained results of a Flugge-type shell solution of a ring-reinforced orthotropic cylinder, it is found that the shell theory gives a fairly accurate representation of the deformation except in the neighborhood of discontinuous loads. The addition of transverse shear deformations does not improve the accuracy of the shell solution.

When Hill's orthotropic yield criterion is applied, it is found that yielding could begin rather early at the inner surface of the shell adjacent to the frame. It is noted that the transverse shearing stress has no great effect on the initial yield pressure.

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band loaded cylinder						
composite cylinder						
orthotropic elasticity solution						
perturbation solution						

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ERRATA

APPROXIMATE ELASTICITY SOLUTION FOR ORTHOTROPIC CYLINDER UNDER HYDROSTATIC PRESSURE AND BAND LOADS

by

A. P. Misovec and Joseph Kempner

May 1968

Replace Figs. 4 to 10 by the enclosed

p. 31 The first two sentences in the third paragraph should read as follows:

When Flügge-type shell theory was applied to a corresponding ring-supported orthotropic shell it was found that $p_o/p_c = 0.960$. The addition of transverse shear deformation to this analysis resulted in $p_o/p_c = 0.972$.

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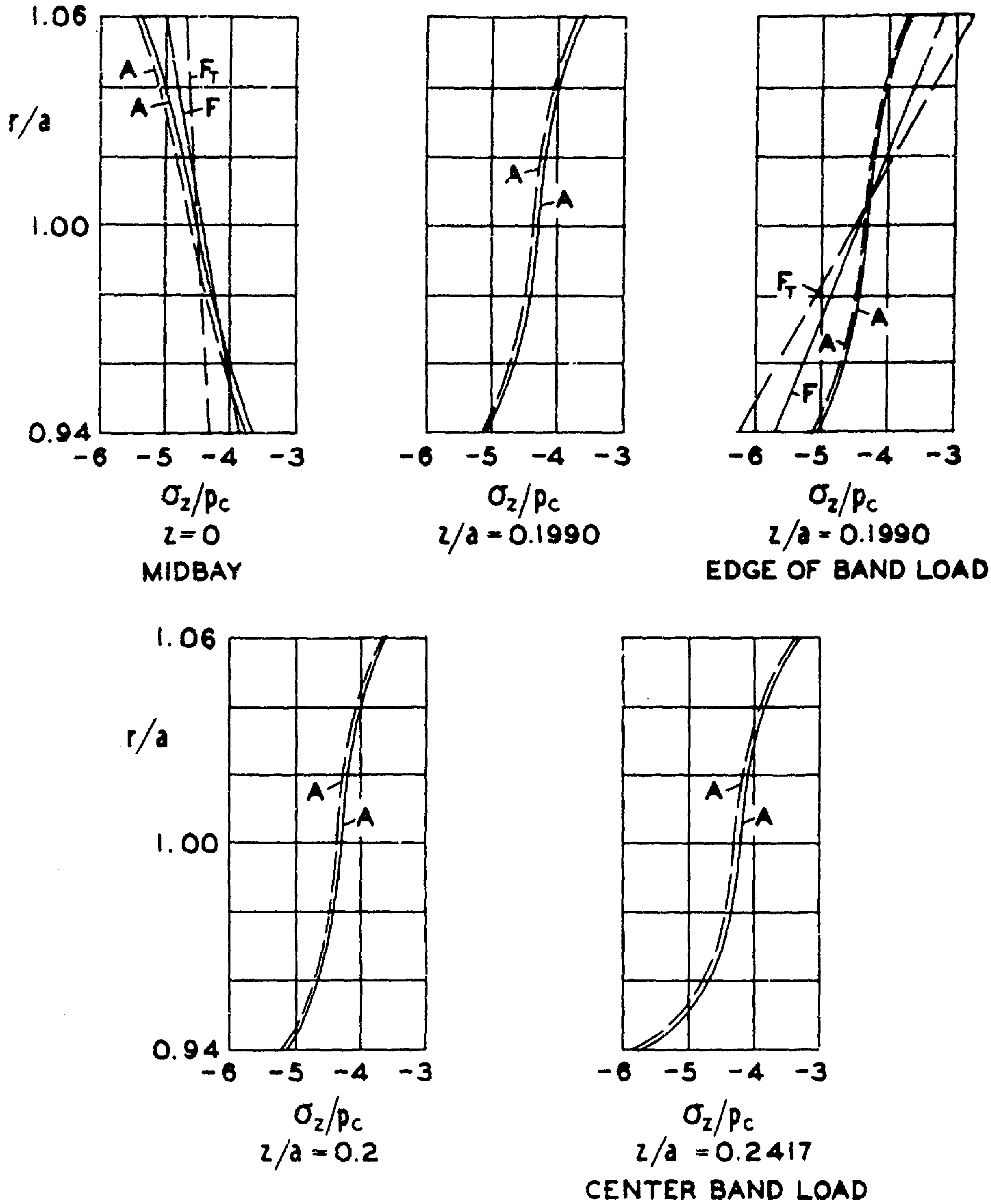

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F_T – FLÜGGE SHELL THEORY WITH TRANSVERSE SHEAR

A – APPROXIMATE ELASTICITY SOLUTION

———— $p_o/p_c = 0.960$ — — — $p_o/p_c = 0.972$

FIG. 4

AXIAL STRESSES IN ORTHOTROPIC CYLINDER

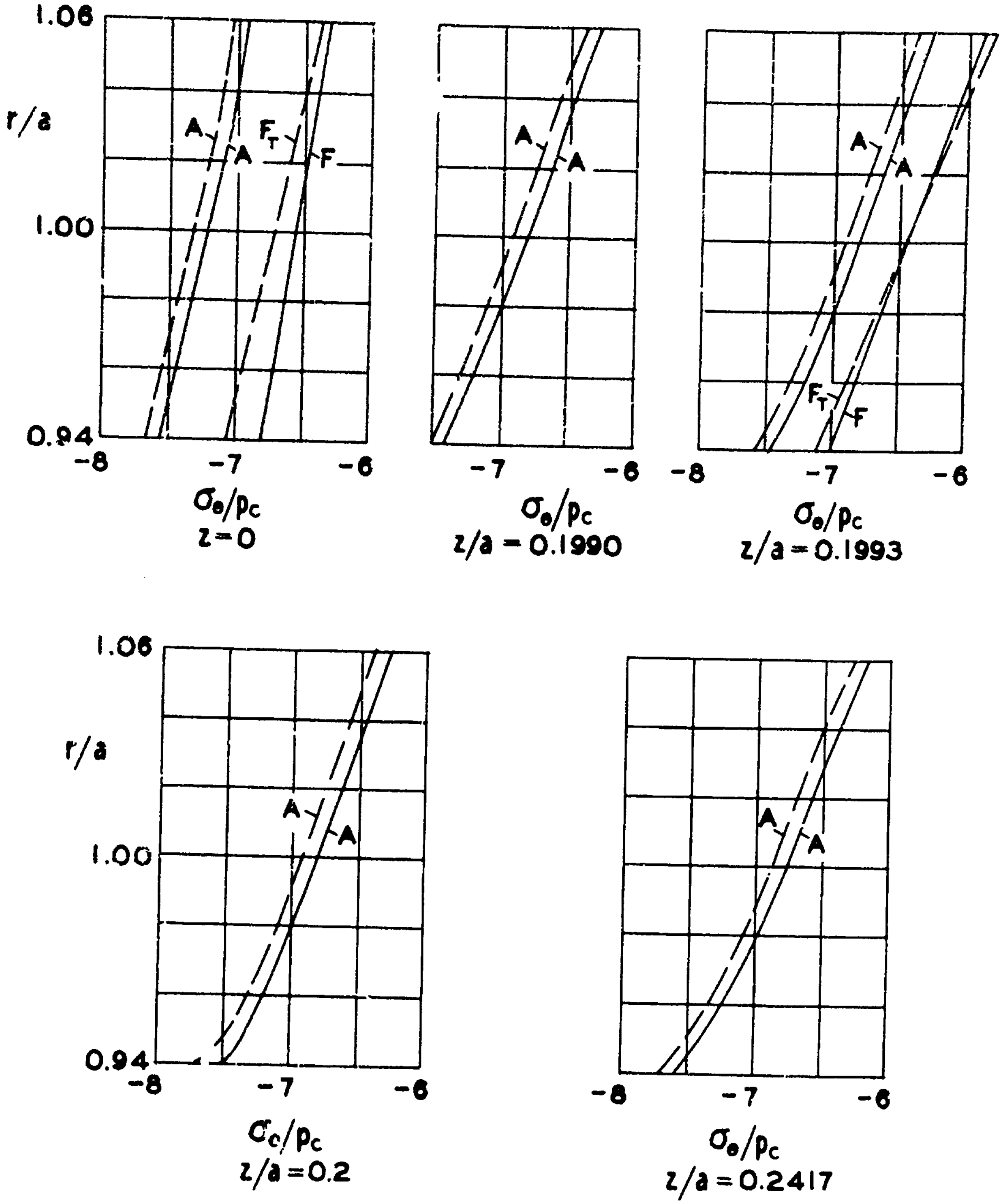

FIG. 5

CIRCUMFERENTIAL STRESSES IN ORTHOTROPIC CYLINDER

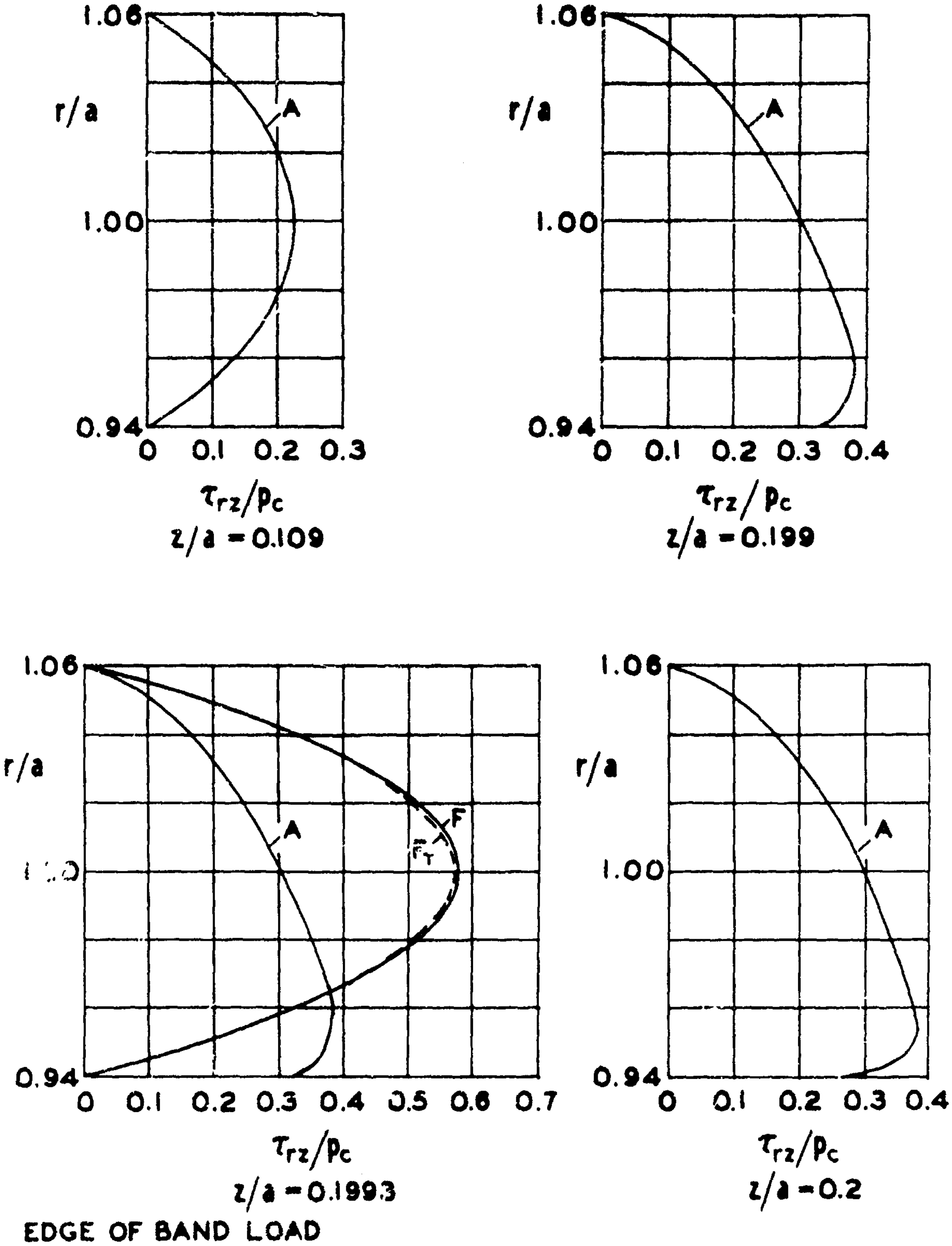

FIG. 6 TRANSVERSE SHEAR STRESSES IN ORTHOTROPIC CYLINDER

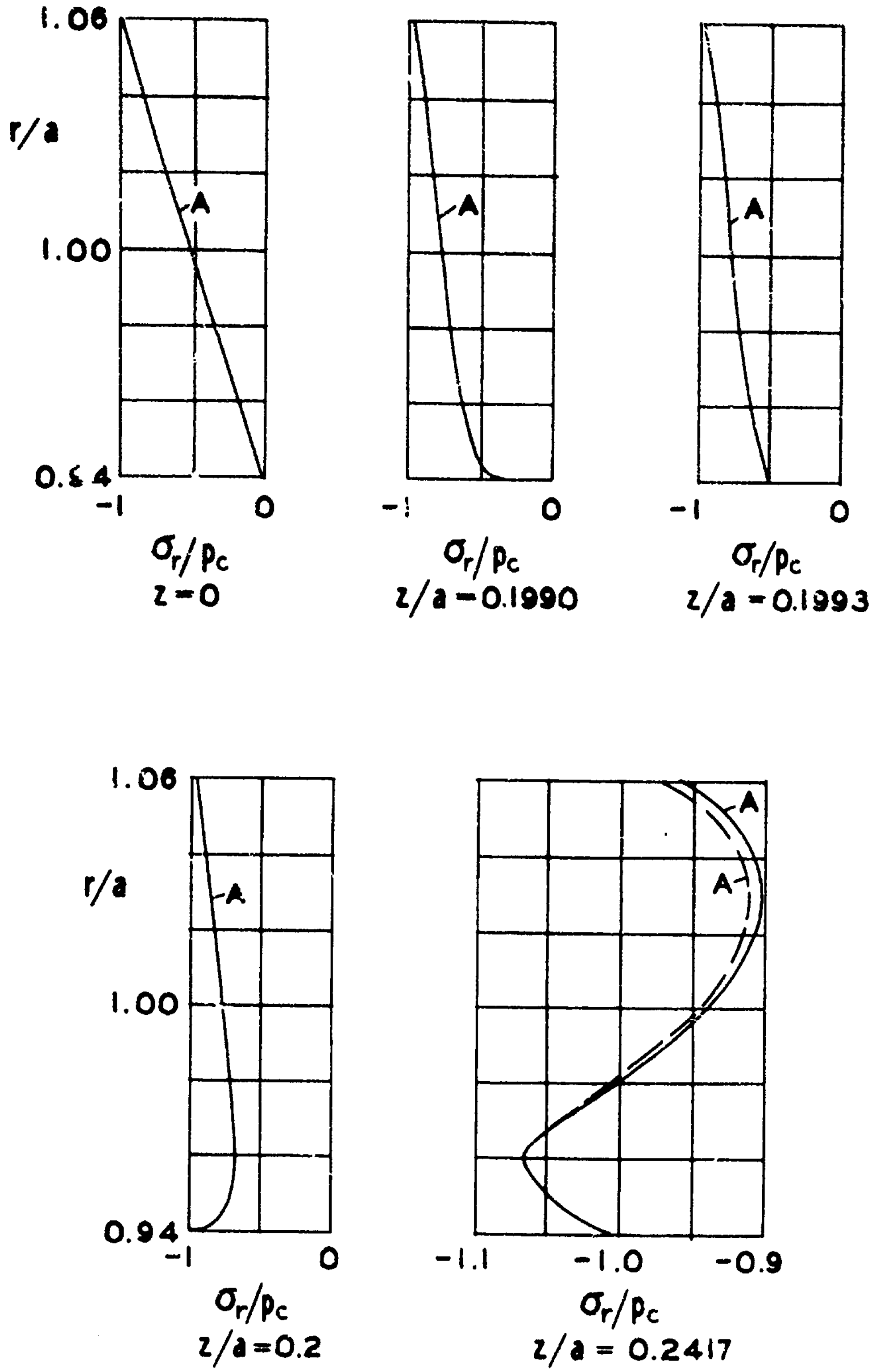

FIG. 7
RADIAL STRESSES IN ORTHOTROPIC CYLINDER

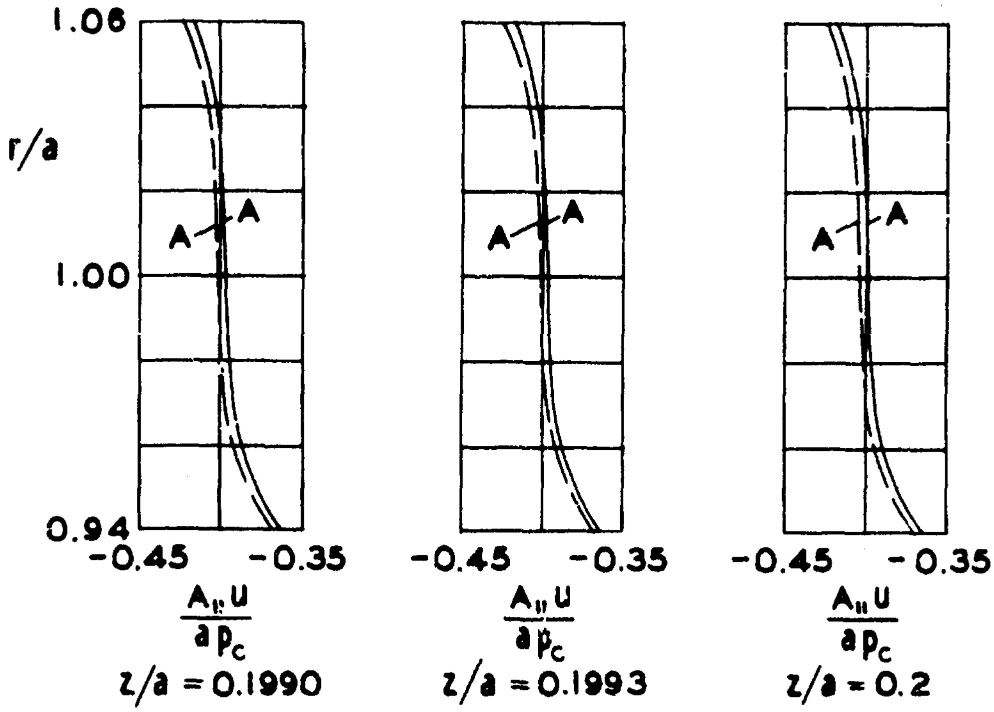

FIG. 8
AXIAL DISPLACEMENTS OF ORTHOTROPIC CYLINDER

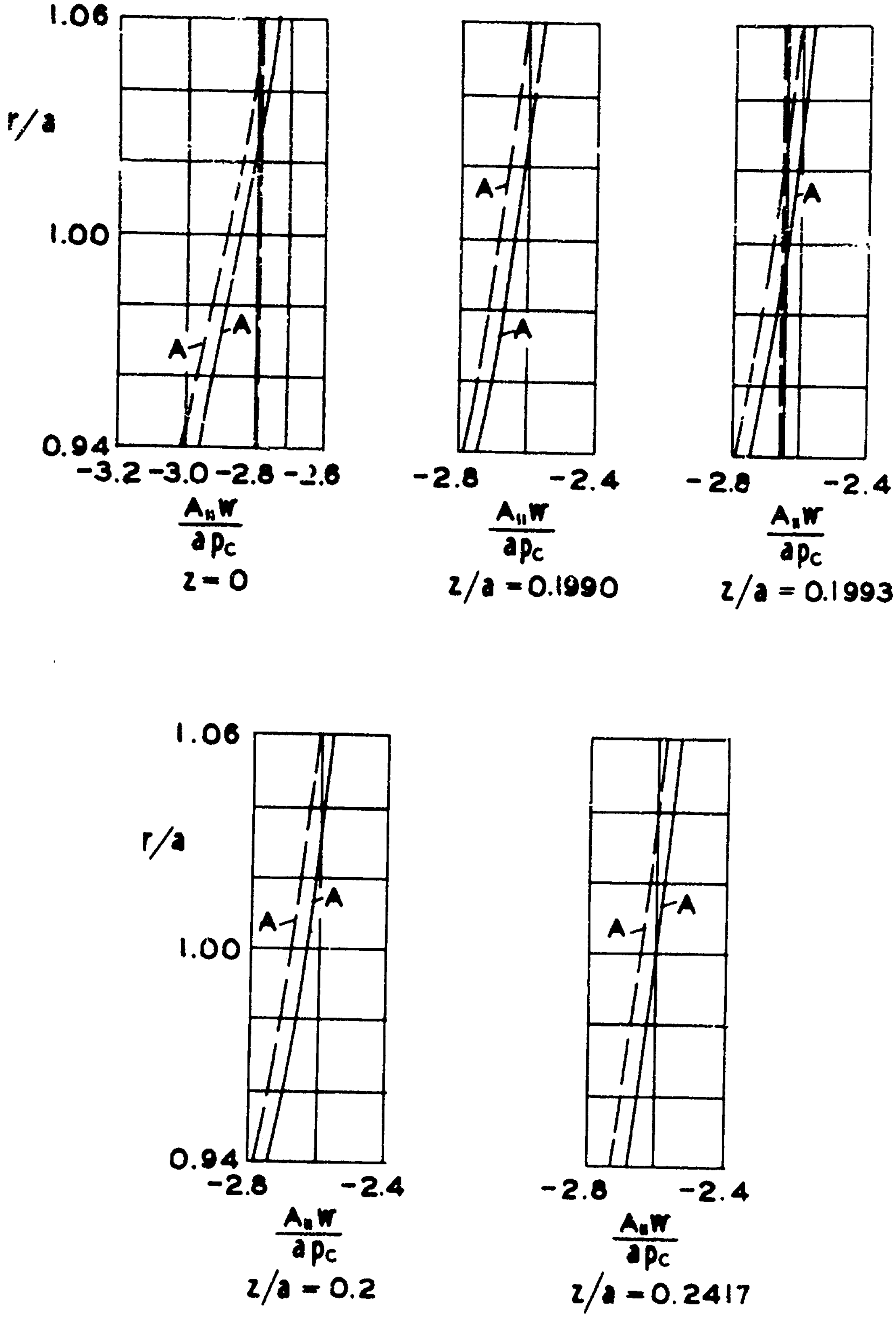

FIG. 9
RADIAL DISPLACEMENTS OF ORTHOTROPIC CYLINDER

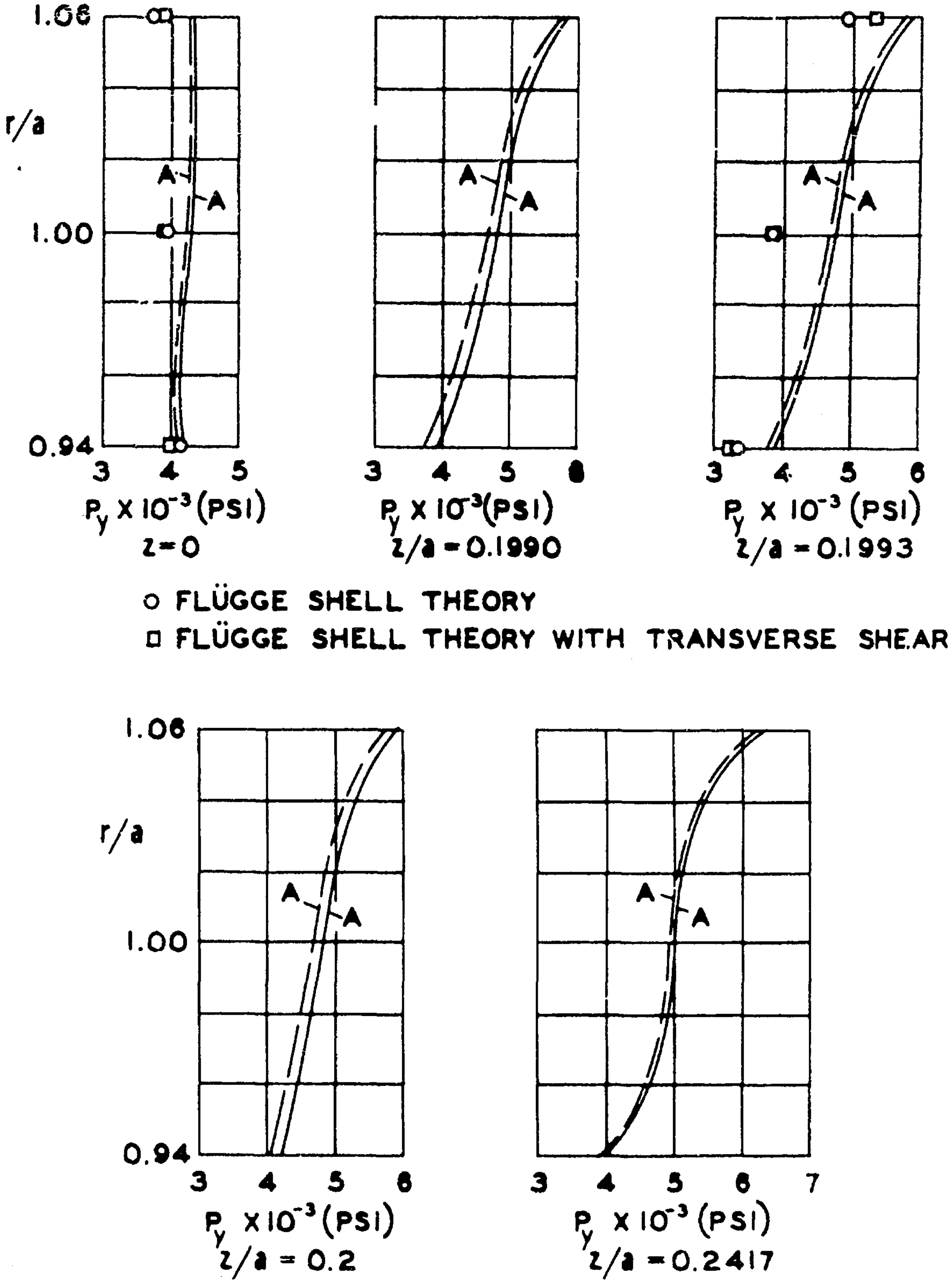

FIG. 10 YIELD PRESSURES