AFOSR [illegible]

CALIFORNIA INSTITUTE OF TECHNOLOGY

ANTENNA LABORATORY

Technical Report No. [illegible]

RADIATION OF A HERTZ DIPOLE MOVING AROUND A CONDUCTING SPHERE

by

John Lam

Electronics Division
Air Force Office of Scientific Research
Grant No. AFOSR-68-1500
March 19[illegible]

Best Available Copy

AD 669 370

RADIATION OF A HERTZ DIPOLE MOVING AROUND A CONDUCTING SPHERE

John Lam

California Institute of Technology
Pasadena, California

March 1968

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

Antenna Laboratory

TECHNICAL REPORT NO. 43

RADIATION OF A HERTZ DIPOLE MOVING AROUND A CONDUCTING SPHERE

John Lam

This research was supported by the U. S. Air Force through the Air Force Office of Scientific Research of the Office of Aerospace Research, under Grant No. AFOSR-68-1400. Reproduction in whole or in part is permitted for any purpose of the United States Government.

March 1968

Radiation of a Hertz Dipole Moving around a Conducting Sphere

John Lam

ABSTRACT

An exact expression is obtained for the field of a horizontal Hertz dipole moving around a conducting sphere in a circular orbit. This expression is in the form of a double series which, in the case of a large sphere, is evaluated by the method of Watson transformation. The geometrical optics field is separated out and its properties examined. It is found that the incident and reflected waves are of different frequencies.

1. Introduction

The problem of wave scattering by a spherical object occurs in many branches of science and engineering. In the majority of cases the source of the wave is fixed relative to the sphere. A rigorous solution is in general obtainable by the method of separation of variables, giving the scattered wave as an infinite series of eigenfunctions. If the radius of the sphere is large compared to a wavelength of the incident wave, the convergence of the series is very slow. The method of Watson transformation is then applied to convert this series into a more rapidly convergent series of "residue waves". This method of solution can be carried over, with only minor extensions, to problems involving moving sources. In this work we calculate the radiation field of an oscillating electric dipole revolving around a perfectly conducting sphere. This problem is an idealization of the situation of a transmitting antenna carried by an artificial satellite in an orbit around a planet.

2. The Hertz Potential

We set up a spherical polar coordinate system whose origin coincides with the center of the conducting sphere, the radius of the sphere being b . The oscillating electric dipole revolves in a circular orbit of radius a in the x-y plane, given by $r = a > b$, $\theta = \pi/2$. To simplify the problem we assume it to be polarized in the z-direction. Without further loss of generality we can write down its electric polarization as follows:

$$\underline{P}(\underline{r},t) = \frac{p e^{-i\omega_o t}}{r^2 \sin\theta} \, \delta(r-a) \, \delta(\theta - \tfrac{\pi}{2}) \, \delta(\phi - \Omega t) \, \underline{e}_z \tag{2.1}$$

where p is its electric dipole moment, Ω its frequency of revolution around the sphere, and ω_o its proper frequency of oscillation increased by the dilatation factor $(1 - \Omega^2 a^2/c^2)^{-1/2}$.

To find the field of the moving dipole we first calculate the Hertz potential $\underline{\Pi}(\underline{r},t)$ which satisfies the wave equation

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}) \, \underline{\Pi}(\underline{r},t) = - \frac{\underline{P}(\underline{r},t)}{\varepsilon_o} \tag{2.2}$$

Because of our particular choice of the polarization this is really a scalar equation with

$$\underline{\Pi}(\underline{r},t) = \Pi(\underline{r},t) \, \underline{e}_z \tag{2.3}$$

The angular delta-function $\delta(\phi - \Omega t)$ is a periodic function and has the Fourier series representation

$$\delta(\phi - \Omega t) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi - \Omega t)} \tag{2.4}$$

Putting (2.4) in (2.2) and expanding

$$\Pi(\underline{r},t) = \sum_{m=-\infty}^{\infty} \Pi_m(r,\theta) \, e^{im\phi - i(\omega_o + m\Omega)t} \tag{2.5}$$

we obtain the equation

$$\left(\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{m^2}{r^2\sin^2\theta} + k_m^2 \right) \Pi_m(r,\theta)$$

$$= - \frac{p}{2\pi\varepsilon_o a^2} \, \delta(r-a) \, \delta(\theta - \tfrac{\pi}{2}) \tag{2.6}$$

For convenience we have defined the quantities

$$\omega_m = \omega_o + m\Omega \quad , \qquad k_m = k_o + mK$$
$$k_o = \omega_o/c \quad , \qquad K = \Omega/c \tag{2.7}$$

To solve equation (2.6) we divide the whole space into regions 1 and 2 according to $r < a$ or $r > a$, respectively. In these regions the solutions of (2.6) are

$$\Pi_m^1(r,\theta) = \sum_{\ell=0}^{\infty} A_{\ell m}\, h_\ell^{(1)}(k_m a)\, j_\ell(k_m r)\, P_\ell^m(\cos\theta), \; r < a$$
$$\Pi_m^2(r,\theta) = \sum_{\ell=0}^{\infty} B_{\ell m}\, j_\ell(k_m a)\, h_\ell^{(1)}(k_m r)\, P_\ell^m(\cos\theta), \; r > a \tag{2.8}$$

Π_m^1 is chosen to be finite at the origin, and Π_m^2 satisfies the radiation condition at infinity. The constants $A_{\ell m}$ and $B_{\ell m}$ are determined by the boundary conditions on the sphere $r = a$. First Π^1 and Π^2 are continuous across the boundary, and we get

$$A_{\ell m} = B_{\ell m} \tag{2.9}$$

Second we integrate both sides of (2.2) over the volume of a flat pill-box of infinitesimal height bounded by the surfaces $r = a - \varepsilon$ and $r = a + \varepsilon$. The second term on the left-hand side gives negligible contributions since $\Pi(\underline{r},t)$ is continuous across $r = a$. Using the Gauss divergence theorem on the first term, we get

$$\int \nabla^2 \Pi(\underline{r},t)\, dV$$

$$= \iint a^2 \sin\theta\, d\theta\, d\phi \left[\frac{\partial}{\partial r} \Pi^2(\underline{r},t)\Big|_{r=a+\varepsilon} - \frac{\partial}{\partial r} \Pi^1(\underline{r},t)\Big|_{r=a-\varepsilon} \right] \qquad (2.10)$$

The contribution to the surface integral from the side of the pill-box is negligible. On the other hand the right-hand side of (2.2) gives

$$\int - \frac{P(\underline{r},t)}{\varepsilon_o}\, dV$$

$$= - \iint a^2 \sin\theta\, d\theta\, d\phi \frac{p}{\varepsilon_o\, a^2} e^{-i\omega_o t}\, \delta(\theta - \tfrac{\pi}{2})\, \delta(\phi - \Omega t) \qquad (2.11)$$

Comparing (2.10) and (2.11) we have the boundary condition

$$\frac{\partial}{\partial r} \Pi^2(\underline{r},t)\Big|_{r=a+\varepsilon} - \frac{\partial}{\partial r} \Pi^1(\underline{r},t)\Big|_{r=a-\varepsilon}$$

$$= - \frac{p}{\varepsilon_o\, a^2} e^{-i\omega_o t}\, \delta(\theta - \tfrac{\pi}{2})\, \delta(\phi - \Omega t) \qquad (2.12)$$

The right-hand side can be expanded into a series of spherical harmonics by means of the orthogonality relation

$$\int P_\ell^m(\cos\theta)\, P_{\ell'}^{m'}(\cos\theta)\, e^{i(m-m')\phi}\, d\Omega$$

$$= \frac{4\pi}{2\ell+1} \frac{(\ell+|m|)!}{(\ell-|m|)!}\, \delta_{\ell\ell'}\, \delta_{mm'} \qquad (2.13)$$

The result is as follows:

$$- \frac{p}{\varepsilon_o a^2} e^{-i\omega_o t} \delta(\theta - \frac{\pi}{2}) \delta(\phi - \Omega t)$$

$$= - \frac{p}{\varepsilon_o a^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^m(0)\, P_\ell^m(\cos\theta) e^{im\phi - i\omega_m t} \qquad (2.14)$$

Putting (2.5), (2.8), (2.9) and (2.14) into (2.12) we get

$$A_{\ell m} = \frac{p}{4\pi\varepsilon_o} (2\ell+1) \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^m(0)\, ik_m , \qquad |m| \leq \ell$$

$$= 0 \qquad |m| > \ell \qquad (2.15)$$

Here use has been made of the identity

$$j_\ell(z)\, h_\ell^{(1)'}(z) - j_\ell'(z)\, h_\ell^{(1)}(z) = \frac{i}{z^2} \qquad (2.16)$$

Hence we have

$$\begin{pmatrix} \Pi^1(\underline{r},t) \\ \Pi^2(\underline{r},t) \end{pmatrix} = \frac{p}{4\pi\varepsilon_o} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (2\ell+1) \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^m(0)\, ik_m$$

$$\times \begin{pmatrix} h_\ell^{(1)}(k_m a)\, j_\ell(k_m r) \\ j_\ell(k_m a)\, h_\ell^{(1)}(k_m r) \end{pmatrix} P_\ell^m(\cos\theta)\, e^{im\phi - i\omega_m t} \qquad (2.17)$$

We notice that the frequency spectrum of $\Pi(\underline{r},t)$ consists of discrete lines at $\omega_m = \omega_o + m\Omega$.

3. The Incident Field

The incident electric and magnetic fields are derived from the Hertz potential according to the equations

$$\underline{E}^{inc} = \nabla \times (\nabla \times \underline{\Pi}) \quad , \quad c\underline{B}^{inc} = \frac{1}{c}\frac{\partial}{\partial t} \nabla \times \underline{\Pi} \tag{3.1}$$

In what follows we only need to know the radial components. These are given by

$$cB_r^{inc} = \frac{1}{r}\frac{1}{c}\frac{\partial^2}{\partial t\,\partial \phi}\Pi$$

$$E_r^{inc} = \left[-\frac{\sin\theta}{r}\frac{\partial^2}{\partial r\,\partial\theta} - \frac{2\cos\theta}{r}\frac{\partial}{\partial r} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos\theta}{r^2}\left(-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right]\Pi \tag{3.2}$$

Substituting (2.17) into (3.2) we have

$$cB_r^{inc} = \frac{1}{r}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}\, mk_m \begin{pmatrix} h_\ell^{(1)}(k_m a)\, j_\ell(k_m r) \\ j_\ell(k_m a)\, h_\ell^{(1)}(k_m r)\end{pmatrix} P_\ell^m(\cos\theta)e^{im\phi - i\omega_m t}$$

$$E_r^{inc} = -\frac{2\cos\theta}{r} A_{oo}\, k_o \begin{pmatrix} h_o^{(1)}(k_m a)\, j_o'(k_m r) \\ j_o(k_m a)\, h_o^{(1)'}(k_m r)\end{pmatrix} e^{-i\omega_o t}$$

$$+ \frac{1}{r}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}\frac{k_m}{2\ell+1}\begin{pmatrix} h_\ell^{(1)}(k_m a) \\ j_\ell(k_m a)\end{pmatrix}\left[(\ell-1)(\ell+|m|)\begin{pmatrix} j_{\ell-1}(k_m r) \\ h_{\ell-1}^{(1)}(k_m r)\end{pmatrix} P_{\ell-1}^m(\cos\theta)\right.$$

$$\left. + (\ell+2)(\ell-|m|+1)\begin{pmatrix} j_{\ell+1}(k_m r) \\ h_{\ell+1}^{(1)}(k_m r)\end{pmatrix} P_{\ell+1}^m(\cos\theta)\right] e^{im\phi - i\omega_m t} \tag{3.3}$$

In deriving the expression for E_r^{inc} we have used the identities

$$(1 - z^2)\, P_\ell^{m\prime}(z) = \frac{(\ell+1)(\ell+|m|)}{2\ell+1} P_{\ell-1}^m(z) - \frac{\ell(\ell-|m|+1)}{2\ell+1} P_{\ell+1}^m(z) ,$$

$$z\, P_\ell^m(z) = \frac{\ell+|m|}{2\ell+1} P_{\ell-1}^m(z) + \frac{\ell-|m|+1}{2\ell+1} P_{\ell+1}^m(z) ,$$

$$\begin{pmatrix} j_\ell'(z) \\ h_\ell^{(1)\prime}(z) \end{pmatrix} = \frac{\ell}{2\ell+1} \begin{pmatrix} j_{\ell-1}(z) \\ h_{\ell-1}^{(1)}(z) \end{pmatrix} - \frac{\ell+1}{2\ell+1} \begin{pmatrix} j_{\ell+1}(z) \\ h_{\ell+1}^{(1)}(z) \end{pmatrix} ,$$

$$\frac{1}{z} \begin{pmatrix} j_\ell'(z) \\ h_\ell^{(1)}(z) \end{pmatrix} = \frac{1}{2\ell+1} \begin{pmatrix} j_{\ell-1}(z) \\ h_{\ell-1}^{(1)}(z) \end{pmatrix} + \frac{1}{2\ell+1} \begin{pmatrix} j_{\ell+1}(z) \\ h_{\ell+1}^{(1)}(z) \end{pmatrix} \qquad (3.4)$$

After some rearrangement we can write E_r^{inc} in the simpler form

$$E_r^{inc} = \frac{1}{r} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m} \begin{pmatrix} j_\ell(k_m r) \\ h_\ell^{(1)}(k_m r) \end{pmatrix} P_\ell^m(\cos\theta) e^{im\phi - i\omega_m t} \qquad (3.5)$$

where

$$C_{\ell m} = k_m \Bigg[A_{\ell-1,m} \frac{(\ell+1)(\ell-|m|)}{2\ell-1} \begin{pmatrix} h_{\ell-1}^{(1)}(k_m a) \\ j_{\ell-1}(k_m a) \end{pmatrix}$$

$$+ A_{\ell+1,m} \frac{\ell(\ell+|m|+1)}{2\ell+3} \begin{pmatrix} h_{\ell+1}^{(1)}(k_m a) \\ j_{\ell+1}(k_m a) \end{pmatrix} \Bigg] \qquad (3.6)$$

Substituting (2.15) into (3.6) we get

$$C_{\ell m} = \frac{p}{4\pi\varepsilon_0} \frac{(\ell-|m|)!}{(\ell+|m|)!} ik_m^2 \Bigg[(\ell+1)(\ell+|m|)\, P_{\ell-1}^m(0) \begin{pmatrix} h_{\ell-1}^{(1)}(k_m a) \\ j_{\ell-1}(k_m a) \end{pmatrix} +$$

$$+ \ell(\ell-|m|+1)\, P_{\ell+1}^m(0) \begin{pmatrix} h_{\ell+1}^{(1)}(k_m a) \\ j_{\ell+1}(k_m a) \end{pmatrix} \Bigg] . \qquad (3.7)$$

From the first and second identities of (3.4) we get

$$(\ell+|m|)\; P^{m}_{\ell-1}(0) = -\,(\ell-|m|+1)\; P^{m}_{\ell+1}(0)$$

$$= P^{m'}_{\ell}(0) \qquad (3.8)$$

Also from the third and fourth identities of (3.4) we get

$$(\ell+1)\begin{pmatrix} h^{(1)}_{\ell-1}(k_m a) \\ j_{\ell-1}(k_m a)\end{pmatrix} - \ell \begin{pmatrix} h^{(1)}_{\ell+1}(k_m a) \\ j_{\ell+1}(k_m a)\end{pmatrix}$$

$$= \frac{2\ell+1}{k_m a}\begin{pmatrix} [k_m a h^{(1)}_{\ell}(k_m a]' \\ [k_m a\; j_{\ell}(k_m a)]'\end{pmatrix} \qquad (3.9)$$

Using (3.8) and (3.9) we simplify (3.7) to the form

$$C_{\ell m} = \frac{p}{4\pi\varepsilon_o}\,(2\ell+1)\,\frac{(\ell-|m|)!}{(\ell+|m|)!}\,P^{m'}_{\ell}(0)\,\frac{ik_m}{a}\begin{pmatrix} [k_m a h^{(1)}_{\ell}(k_m a)]' \\ [k_m a j_{\ell}(k_m a)]'\end{pmatrix} \qquad (3.10)$$

To summarize the results of this section we write out in full the expressions for the radial components of the incident fields:

$$cB^{inc}_{r} = \frac{p}{4\pi\varepsilon_o r}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}(2\ell+1)\,\frac{(\ell-|m|)!}{(\ell+|m|)!}\,P^{m}_{\ell}(0)\; mik^{2}_{m}$$

$$\times \begin{pmatrix} h^{(1)}_{\ell}(k_m a)\; j_{\ell}(k_m r) \\ j_{\ell}(k_m a)\; h^{(1)}_{\ell}(k_m r)\end{pmatrix} P^{m}_{\ell}(\cos\Theta)\; e^{im\phi - i\omega_m t}$$

$$E_r^{inc} = \frac{p}{4\pi\varepsilon_o r} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} (2\ell+1) \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^{m'}(0) \frac{ik_m}{a} \times$$

$$\times \begin{pmatrix} [k_m a h_\ell^{(1)}(k_m a)]' \; j_\ell(k_m r) \\ [k_m a j_\ell(k_m a)]' \; h_\ell^{(1)}(k_m r) \end{pmatrix} P_\ell^m(\cos\theta)\, e^{im\phi - i\omega_m t} \qquad (3.11)$$

In the second equation E_r^{inc} appears to be discontinuous on the sphere $r = a$. This discontinuity is only apparent. The difference in the expressions for E_r^{inc} on both sides of $r = a$ is an infinite series which can be shown to converge to zero everywhere except at the source.

4. The Debye Potentials

To describe the scattering of electromagnetic waves by a sphere it is most convenient to work with the Debye potentials u and v, from which the fields are derived according to the formulas

$$\underline{E} = -\nabla\times(\underline{r}\,\frac{1}{c}\frac{\partial u}{\partial t} + \underline{r}\times\nabla v)$$

$$c\underline{B} = -\nabla\times(\underline{r}\times\nabla u - \underline{r}\,\frac{1}{c}\frac{\partial v}{\partial t}) \qquad (4.1)$$

$\underline{E}$ and $\underline{B}$ given by (4.1) are solutions of Maxwell's equations in free space provided that u and v separately satisfy the scalar wave equation

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}) \binom{u}{v} = 0 \qquad (4.2)$$

Writing out (4.1) in component form we get

$$E_r = (\frac{\partial^2}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\, rv$$

$$E_\theta = -\frac{1}{r\sin\theta}\frac{1}{c}\frac{\partial^2(ru)}{\partial t\,\partial\phi} + \frac{1}{r}\frac{\partial^2(rv)}{\partial r\,\partial\theta}$$

$$E_\phi = \frac{1}{r}\frac{1}{c}\frac{\partial^2(ru)}{\partial t\,\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial^2(rv)}{\partial r\,\partial\phi}$$

$$cB_r = \left(\frac{\partial^2}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) ru$$

$$cB_\theta = \frac{1}{r}\frac{\partial^2(ru)}{\partial r\,\partial\theta} + \frac{1}{r\sin\theta}\frac{1}{c}\frac{\partial^2(rv)}{\partial t\,\partial\phi}$$

$$cB_\phi = \frac{1}{r\sin\theta}\frac{\partial^2(ru)}{\partial r\,\partial\phi} - \frac{1}{r}\frac{1}{c}\frac{\partial^2(rv)}{\partial t\,\partial\theta} \qquad (4.3)$$

Thus u generates a magnetic wave and v an electric wave. It is also clear that they are determined by the radial components of the magnetic and electric fields respectively.

Since u and v are solutions of the scalar wave equation, they have the eigenfunction expansions

$$u = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} D_{\ell m}\begin{pmatrix} h_\ell^{(1)}(k_m a)\, j_\ell(k_m r) \\ j_\ell(k_m a)\, h_\ell^{(1)}(k_m r)\end{pmatrix} P_\ell^m(\cos\theta)\, e^{im\phi - i\omega_m t}$$

$$v = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} E_{\ell m}\begin{pmatrix} [k_m a\, h_\ell^{(1)}(k_m a)]'\, j_\ell(k_m r) \\ [k_m a\, j_\ell(k_m a)]'\, h_\ell^{(1)}(k_m r)\end{pmatrix} P_\ell^m(\cos\theta) e^{im\phi - i\omega_m t} \qquad (4.4)$$

To get E_r and cB_r we operate on ru and rv with the operator

$$\frac{\partial^2}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

as indicated in (4.3). Because of the following equation, satisfied by the spherical Bessel and Hankel functions

$$\left(\frac{\partial^2}{\partial z^2} + 1 - \frac{\ell(\ell+1)}{z^2}\right)\begin{pmatrix} z j_\ell(z) \\ z h_\ell^{(1)}(z) \end{pmatrix} = 0 \quad , \tag{4.5}$$

this operation merely brings in a factor $\ell(\ell+1)/r^2$. Thus comparing (3.11) and (4.4) we immediately obtain the expressions for the Debye potentials of the incident field:

$$u^{inc} = \frac{p}{4\pi\varepsilon_o} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^m(0)\; m\, i\, k_m^2$$

$$\times \begin{pmatrix} h_\ell^{(1)}(k_m a)\; j_\ell(k_m r) \\ j_\ell(k_m a)\; h_\ell^{(1)}(k_m r) \end{pmatrix} P_\ell^m(\cos\Theta)\; e^{im\emptyset - i\omega_m t}$$

$$v^{inc} = \frac{p}{4\pi\varepsilon_o} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^{m\prime}(0)\, \frac{ik_m}{a}$$

$$\times \begin{pmatrix} [k_m a\, h_\ell^{(1)}(k_m a)]'\; j_\ell(k_m r) \\ [k_m a\, j_\ell(k_m a)]'\; h_\ell^{(1)}(k_m r) \end{pmatrix} P_\ell^m(\cos\Theta)\; e^{im\emptyset - i\omega_m t} \tag{4.6}$$

5. The Scattered Field

The scattered field is determined by the boundary conditions on the perfectly conducting sphere:

$$B_r^{tot} = B_r^{inc} + B_r^{sc} = 0$$

$$E_\Theta^{tot} = E_\Theta^{inc} + E_\Theta^{sc} = 0$$

$$E_\emptyset^{tot} = E_\emptyset^{inc} + E_\emptyset^{sc} = 0 \;, \quad r = b < a \tag{5.1}$$

The scattered electric and magnetic fields can be derived from a pair of Debye potentials u^{sc} and v^{sc} which have outgoing-wave eigenfunction expansions similar to (4.4) with unknown coefficients. These coefficients are determined by putting u^{inc}, v^{inc} and v^{sc} into (5.1). In this way we easily get

$$u^{sc} = - \frac{p}{4\pi\ \varepsilon_o} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^m(0)\ m\, i\, k_m^2$$

$$\times\ h_\ell^{(1)}(k_m a) \frac{j_\ell(k_m b)}{h_\ell^{(1)}(k_m b)} h_\ell^{(1)}(k_m r)\ P_\ell^m(\cos\theta)\ e^{im\phi - i\omega_m t}$$

$$v^{sc} = - \frac{p}{4\pi\varepsilon_o} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-|m|)!}{(\ell+|m|)!}\ P_\ell^{m'}(0) \frac{ik_m}{a}\ \times$$

$$\times\ [k_m a\ h_\ell^{(1)}(k_m a)]' \frac{[k_m b\ j_\ell(k_m b)]'}{[k_m b\ h_\ell^{(1)}(k_m b)]'} h_\ell^{(1)}(k_m r)\ P_\ell^m(\cos\theta)$$

$$\times\ e^{im\phi - i\omega_m t}, \quad r > b \qquad (5.2)$$

These solutions are exact.

6. The Watson Transformation

If we have in mind the application of the solution of this problem to the case of a dipole antenna travelling around a planet, the expressions for the Debye potentials given by (5.2) are practically useless. The reason is that in this situation $k_o b >> 1$, and the convergence of the series is extremely slow. To get a good estimate of the sums we have to include terms up to values of ℓ of the order of $k_o b$. An alternative to direct summation is the method of Watson transformation[1].

Let us first consider the total Debye potential of the magnetic wave for $r > a$:

$$u^{tot} = u^{inc} + u^{sc}$$

$$= \frac{p}{4\pi\varepsilon_o} \sum_{m=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^m(0)\, m\, i\, k_m^2 \quad \times$$

$$\times \left[j_\ell(k_m a) - \frac{j_\ell(k_m b)}{h_\ell^{(1)}(k_m b)} h_\ell^{(1)}(k_m a) \right] h_\ell^{(1)}(k_m r)\, P_\ell^m(\cos\theta) e^{im\phi - i\omega_m t} \tag{6.1}$$

The order of summation has been inverted so that m is now summed from $-\infty$ to ∞ . This is permissible since $P_\ell^m(\cos\theta) = 0$, $(\ell-|m|)!\, P_\ell^m(0)$ is finite for $|m| > \ell$. Using the addition theorem

$$P_\ell(\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\phi)$$

$$= \sum_{m=-\ell}^{\ell} \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^m(\cos\theta)\, P_\ell^m(\cos\theta')\, e^{im\phi} \tag{6.2}$$

we get

$$\frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^m(0)\, P_\ell^m(\cos\theta) = \frac{1}{2\pi} \int_0^{2\pi} P_\ell(\sin\theta \cos\phi') e^{im\phi'}\, d\phi' \tag{6.3}$$

u^{tot} can now be rewritten as

$$u^{tot} = \frac{p}{4\pi\varepsilon_o} \int_0^{2\pi} d\phi' \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} m\, i\, k_m^2\, e^{im(\phi-\phi') - i\omega_m t} \sqrt{\frac{\pi}{8k_m a}}\, S_m \tag{6.4}$$

where

$$S_m = \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} \left[H^{(2)}_{\ell+\frac{1}{2}}(k_m a) - \frac{H^{(2)}_{\ell+\frac{1}{2}}(k_m b)}{H^{(1)}_{\ell+\frac{1}{2}}(k_m b)} H^{(1)}_{\ell+\frac{1}{2}}(k_m a) \right] \times$$

$$\times \quad h^{(1)}_{\ell}(k_m r)\, P_\ell(\sin\theta \cos\phi') \qquad (6.5)$$

We define

$$\cos\psi = \sin\theta\cos\phi' \qquad (6.6)$$

$$\nu = \ell + \frac{1}{2} \qquad (6.7)$$

Then, since $P_\ell(z) = (-1)^\ell P_\ell(-z)$ for ℓ an integer, S_m becomes

$$S_m = 2 \sum_{\nu=3/2}^{\infty} \frac{\nu}{\nu^2 - \frac{1}{4}} \left[H^{(2)}_\nu(k_m a) - \frac{H^{(2)}_\nu(k_m b)}{H^{(1)}_\nu(k_m b)} H^{(1)}_\nu(k_m a) \right] h^{(1)}_{\nu-\frac{1}{2}}(k_m r) \times$$

$$\times \quad (-1)^{\nu-\frac{1}{2}} P_{\nu-\frac{1}{2}}(-\cos\psi) \qquad (6.8)$$

If each term is considered as a function of the complex variable ν this sum can be converted into a contour integral in the complex ν-plane (see Fig. 1):

$$S_m = -i \oint_C \frac{\nu}{\nu^2 - \frac{1}{4}} \left[H^{(2)}_\nu(k_m a) - \frac{H^{(2)}_\nu(k_m b)}{H^{(1)}_\nu(k_m b)} H^{(1)}_\nu(k_m a) \right] \frac{h^{(1)}_{\nu-\frac{1}{2}}(k_m r)}{\cos\nu\pi} \times$$

$$\times \quad P_{\nu-\frac{1}{2}}(-\cos\psi)\, d\nu \qquad (6.9)$$

The contour C encloses the positive real axis along which are the positive poles of $1/\cos\nu\pi$. All other singularities of the integrand are to be excluded. The residues at these poles give back the sum (6.8), except for the contribution from the double pole at $\nu = 1/2$ which we must subtract off. This is equal to

$$\sigma_m = 2 \frac{\partial}{\partial\nu} \left\{ \frac{\nu}{\nu + \frac{1}{2}} \left[H_\nu^{(2)}(k_m a) - \frac{H_\nu^{(2)}(k_m b)}{H_\nu^{(1)}(k_m b)} H_\nu^{(1)}(k_m a) \right] h^{(1)}_{\nu-\frac{1}{2}}(k_m r) \times \right.$$

$$\left. \times \; P_{\nu-\frac{1}{2}}(-\cos\psi) \right\}_{\nu=\frac{1}{2}} \tag{6.10}$$

We ignore this term for the moment.

Let us consider the symmetry of the integrand in (6.9) with respect to ν. The Legendre polynomial can be defined in terms of the hypergeometric function:

$$P_\ell(z) = {}_2F_1(\ell+1,-\ell;1;\frac{1-z}{2}) \tag{6.11}$$

The hypergeometric function being symmetric with respect to the exchange of its first two parameters, we get

$$P_{\nu-\frac{1}{2}}(z) = P_{-\nu-\frac{1}{2}}(z) \tag{6.12}$$

that is, $P_{\nu-\frac{1}{2}}(-\cos\psi)$ is an even function of ν. From the relations

$$H_{-\nu}^{(1)}(z) = e^{i\nu\pi} H_\nu^{(1)}(z)$$

$$H_{-\nu}^{(2)}(z) = e^{-i\nu\pi} H_\nu^{(2)}(z) \tag{6.13}$$

We see that the factor involving the Hankel functions is also an even function of ν. Thus the entire integrand is an odd function of ν. In this case the integration along the lower branch of the contour C can be replaced by one along its image with respect to the origin in the second quadrant, as shown by the broken line in Fig. 1. The integral is now to be evaluated along a contour C_1 lying just above the real axis:

$$S_m = -i \int_{C_1} \frac{\nu}{\nu^2 - \frac{1}{4}} \left[H_\nu^{(2)}(k_m a) - \frac{H_\nu^{(2)}(k_m b)}{H_\nu^{(1)}(k_m b)} H_\nu^{(1)}(k_m a)\right] \times \frac{h^{(1)}_{\nu-\frac{1}{2}}(k_m r)}{\cos \nu\pi} P_{\nu-\frac{1}{2}}(-\cos\psi)d\nu \quad . \qquad (6.14)$$

In a similar way we consider the total Debye potential of the electric wave for $r > a$:

$$v^{tot} = v^{inc} + v^{sc}$$

$$= \frac{p}{4\pi\varepsilon_0} \sum_{m=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^{m\,'}(0) \frac{ik_m}{a} \times$$

$$\times \left[[k_m a\ j_\ell(k_m a)]' - \frac{[k_m b\ j_\ell(k_m b)]'}{[k_m b\ h_\ell^{(1)}(k_m b)]'} [k_m a\ h_\ell^{(1)}(k_m a)]'\right] h_\ell^{(1)}(k_m r) \times$$

$$\times \quad P_\ell^m(\cos\theta)\ e^{im\phi - i\omega_m t} \qquad (6.15)$$

Using the addition theorem (6.2) we get

$$\frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^{m\,'}(0)\ P_\ell^m(\cos\theta) = \frac{\cos\theta}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial(\cos\psi)} P_\ell(\cos\psi) e^{-im\phi'}\ d\phi' \qquad (6.16)$$

Thus

$$v^{tot} = \frac{p}{4\pi\varepsilon_0} \int_0^{2\pi} d\phi'\ \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \frac{ik_m}{a} e^{im(\phi-\phi') - i\omega_m t} \sqrt{\frac{\pi}{8}} \frac{\partial}{\partial(\cos\psi)} T_m \cos\theta \qquad (6.17)$$

where

$$T_m = 2\sum_{\nu=3/2}^{\infty} \frac{\nu}{\nu^2 - \frac{1}{4}} \left[[\sqrt{k_m a}\, H^{(2)}(k_m a)]' - \frac{[\sqrt{k_m b}\, H_\nu^{(2)}(k_m b)]'}{[\sqrt{k_m b}\, H_\nu^{(1)}(k_m b)]'} \times [\sqrt{k_m a}\, H_\nu^{(1)}(k_m a)]' \right] h^{(1)}_{\nu-\frac{1}{2}}(k_m r)(-1)^{\nu-\frac{1}{2}} P_{\nu-\frac{1}{2}}(-\cos\psi) \tag{6.18}$$

T_m can be converted into a contour integral by the Watson transformation:

$$T_m = -i \oint_C \frac{\nu}{\nu^2 - \frac{1}{4}} \left[[\sqrt{k_m a}\, H_\nu^{(2)}(k_m a)]' - \frac{[\sqrt{k_m b}\, H_\nu^{(2)}(k_m b)]'}{[\sqrt{k_m b}\, H_\nu^{(1)}(k_m b)]'} [\sqrt{k_m a}\, H^{(1)}(k_m a]' \right] \times \frac{h^{(1)}_{\nu-\frac{1}{2}}(k_m r)}{\cos \nu\pi} P_{\nu-\frac{1}{2}}(-\cos\psi)\, d\nu \tag{6.19}$$

The contribution from the double pole at $\nu = \frac{1}{2}$ to be subtracted off is

$$\tau_m = 2\frac{\partial}{\partial\nu}\left\{ \frac{\nu}{\nu + \frac{1}{2}} \left[[\sqrt{k_m a}\, H_\nu^{(2)}(k_m a)]' - \frac{[\sqrt{k_m b}\, H_\nu^{(2)}(k_m b)]'}{[\sqrt{k_m b}\, H_\nu^{(1)}(k_m b)]'} [\sqrt{k_m a}\, H_\nu^{(1)}(k_m a)]' \right] \times h^{(1)}_{\nu-\frac{1}{2}}(k_m r)\, P_{\nu-\frac{1}{2}}(-\cos\psi) \right\}_{\nu=\frac{1}{2}} \tag{6.20}$$

The integrand in (6.19) is again an odd function of ν. The integral can be evaluated along a contour lying just above the real axis:

$$T_m = -i \int_{C_1} \frac{\nu}{\nu^2 - \frac{1}{4}} \left[[\sqrt{k_m a}\, H_\nu^{(2)}(k_m a)]' - \frac{[\sqrt{k_m b}\, H_\nu^{(2)}(k_m b)]'}{[\sqrt{k_m b}\, H_\nu^{(1)}(k_m b)]'} \times \right.$$

$$\left. \times \quad [\sqrt{k_m a}\, H_\nu^{(1)}(k_m a)]' \right] \frac{h^{(1)}_{\nu-\frac{1}{2}}(k_m r)}{\cos \nu\pi} P_{\nu-\frac{1}{2}}(-\cos \psi)\, d\nu \qquad (6.21)$$

We now consider the contribution from the pole at $\nu = 1/2$. After some straightforward calculations we find that the parts of u^{tot} and v^{tot} due to σ_m and τ_m jointly give, upon substitution in (4.3), identically zero contribution to the total field. From now on these parts will be discarded.

7. Separation of the Geometrical Optics Field

We want to evaluate the contour integrals (6.14) and (6.21) asymptotically in the limit of a very large sphere: $k_o b >> 1$. For this purpose it is most elegant to use a method due to Franz[2] to separate the total field into two parts. One part can be identified with the geometrical optics field, and the other is in the form of a series of damped "creeping waves".

As will be seen below, for $k_m b >> 1$, most of the contributions to the integrals (6.14) and (6.21) come from large values of ν of the order of $k_m b$. From the identity

$$P_\nu(-\cos \psi) = e^{i\nu\pi} P_\nu(\cos \psi) - i \sin \nu\pi[P_\nu(\cos \psi) - i \frac{2}{\pi} Q_\nu(\cos \psi)] \qquad (7.1)$$

and the asymptotic formulas

$$P_\nu(\cos\psi) \sim \sqrt{\frac{2}{\pi\nu\sin\psi}}\cos[(\nu+\tfrac{1}{2})\psi - \tfrac{\pi}{4}]$$

$$\frac{2}{\pi} Q_\nu(\cos\psi) \sim \sqrt{\frac{2}{\pi\nu\sin\theta}}\cos[(\nu+\tfrac{1}{2})\psi + \tfrac{\pi}{4}] \;,\; |\nu\sin\psi| >> 1 \quad (7.2)$$

we get

$$P_{\nu-\frac{1}{2}}(-\cos\psi) \sim \frac{2i\cos\nu\pi}{\sqrt{2\pi\nu\sin\psi}}\, e^{i\nu\psi - i\frac{\pi}{4}} + e^{i(\nu-\frac{1}{2})} P_{\nu-\frac{1}{2}}(\cos\psi)$$

$$|\nu\sin\psi| >> 1 \quad (7.3)$$

Thus for $\sin\psi \neq 0$ we can substitute (7.3) into the integrals, each of which is now separated into two parts:

$$S_m = S_m^g + S_m^{cr}$$

$$T_m = T_m^g + T_m^{cr} \quad (7.4)$$

where

$$S_m^g = \sqrt{\frac{2}{\pi\sin\psi}}\, e^{-i\frac{\pi}{4}} \int_{C_1} \frac{\sqrt{\nu}}{\nu^2 - \frac{1}{4}} \left[H_\nu^{(2)}(k_m a) - \frac{H_\nu^{(2)}(k_m b)}{H_\nu^{(1)}(k_m b)} H_\nu^{(1)}(k_m a)\right]$$

$$\times\; h^{(1)}_{\nu-\frac{1}{2}}(k_m r)\, e^{i\nu\psi} d\nu$$

$$T_m^g = \sqrt{\frac{2}{\pi\sin\psi}}\, e^{-i\frac{\pi}{4}} \int_{C_1} \frac{\sqrt{\nu}}{\nu^2 - \frac{1}{4}} \Bigg[[\sqrt{k_m a}\, H_\nu^{(2)}(k_m a)]' - \frac{[\sqrt{k_m b}\, H_\nu^{(2)}(k_m b)]'}{[k_m b\, H_\nu^{(1)}(k_m b)]'}$$

$$\times\; [\sqrt{k_m a}\, H^{(1)}(k_m a)]'\Bigg]\, h^{(1)}_{\nu-\frac{1}{2}}(k_m r)\, e^{i\nu\psi}\, d\nu \quad (7.5)$$

and

$$S_m^{cr} = - \int_{C_1} \frac{\nu}{\nu^2 - \frac{1}{4}} \left[H_\nu^{(2)}(k_m a) - \frac{H_\nu^{(2)}(k_m b)}{H_\nu^{(1)}(k_m b)} H_\nu^{(1)}(k_m a) \right] \times$$

$$\times \frac{h^{(1)}_{\nu-\frac{1}{2}}(k_m r)}{\cos \nu\pi} P_{\nu-\frac{1}{2}}(\cos \psi) e^{i\nu\pi} d\nu \quad ,$$

$$T_m^{cr} = - \int_{C_1} \frac{\nu}{\nu^2 - \frac{1}{4}} \left[[\sqrt{k_m a}\, H_\nu^{(2)}(k_m a)]' - \frac{[\sqrt{k_m b}\, H_\nu^{(2)}(k_m b)]'}{[\sqrt{k_m b}\, H_\nu^{(1)}(k_m b)]'} \times \right.$$

$$\left. \times \quad [\sqrt{k_m a}\, H_\nu^{(1)}(k_m a)]' \right] \frac{h^{(1)}_{\nu-\frac{1}{2}}(k_m r)}{\cos \nu\pi} P_{\nu-\frac{1}{2}}(\cos \psi) e^{i\nu\pi} d\nu \ . \quad (7.6)$$

In the next section it will be seen that S_m^g and T_m^g in (7.5) yield the geometrical optics field.

It ¬ be shown that the integrands in (7.6) tend to zero at infinity in the upper ν-plane for $r > a > b$, except in the neighborhood of the zeros of $H_\nu^{(1)}(k_m b)$ or $[\sqrt{k_m b}\, H_\nu^{(1)}(k_m b)]'$.[3] For $k_m b \gg 1$ these are located approximately at

$$\nu_s \simeq k_m b + A_s (k_m b)^{1/3} e^{i\pi/3} \quad , \quad s=1,2,3,\cdots \quad (7.7)$$

where the A's are real positive constants which are different for the magnetic and electric waves. We observe that these zeros lie in a row in the first quadrant (see Fig. 2). They are simple poles of the integrands and are the only singularities in the upper ν-plane. The straight contour C_1 can now be deformed to the contour C_2 which encloses these poles, as shown in Fig. 2. The residues obtained from these poles form an infinite series of "residue waves" which are rapidly convergent for large $k_m b$. Each term of the series can be interpreted as an exponentially damped "creeping wave". It must be remarked that we have tacitly assumed that the main contributions to u^{tot} and v^{tot} come from terms with small m's such that $k_m b = k_o b + mKb \approx k_o b >> 1$.

In what follows we will confine our attention to the geometrical optics field, since it is more interesting, besides being the dominant part for $k_o b >> 1$.

8. The Geometrical Optics Field

We first consider the integral S_m^g in (7.5). We will evaluate it by the method of stationary phase. This integral in general has one or two stationary points on the real axis in the neighborhood of $\nu = k_m b$. In order to simplify the problem we take the limit $r >> a$ so that we will be calculating the far field. It is now permissible to apply to S_m^g the Hankel asymptotic formula

$$h^{(1)}_{\nu-\frac{1}{2}}(k_m r) \sim \frac{1}{k_m r} e^{i(k_m r - \nu\frac{\pi}{2} - \frac{\pi}{4})} \qquad k_m r >> \nu \tag{8.1}$$

The case where r is of the same order as a should present no special difficulty; but the calculations that follow would be more cumbersome. Substituting (8.1) into S_m^g we get

$$S_m^g = \sqrt{\frac{2}{\pi \sin\psi}} \; \frac{e^{ik_m r - i\frac{\pi}{2}}}{k_m r} \int_{-\infty}^{\infty} \frac{\sqrt{\nu}}{\nu^2 - \frac{1}{4}} \left[H_\nu^{(2)}(k_m a) - \frac{H_\nu^{(2)}(k_m b)}{H_\nu^{(1)}(k_m b)} H_\nu^{(1)}(k_m a) \right] \times e^{i\nu(\psi - \frac{\pi}{2})} d\nu \tag{8.2}$$

To locate the stationary points we apply to (8.2) the Debye asymptotic formulas for the Hankel functions in the limit when ν and z are large and comparable:

$$\begin{pmatrix} H_\nu^{(1)}(z) \\ H_\nu^{(2)}(z) \end{pmatrix} \sim \sqrt{\frac{2}{\pi}} \frac{1}{(z^2 - \nu^2)^{1/4}} e^{\pm i\left(\sqrt{z^2-\nu^2} - \nu \cos^{-1}\frac{\nu}{z} - \frac{\pi}{4}\right)}, \quad \nu < z$$

$$\sim \mp i\sqrt{\frac{2}{\pi}} \frac{1}{(\nu^2 - z^2)^{1/4}} e^{\nu \ln\frac{\nu + \sqrt{\nu^2 - z^2}}{z} - \sqrt{\nu^2 - z^2}}, \quad \nu > z \tag{8.3}$$

with the condition

$$0 < \cos^{-1}\frac{\nu}{z} < \frac{\pi}{2} \tag{8.4}$$

We split S_m^g into three integrals

$$S_m^g = \sqrt{\frac{2}{\pi \sin \psi}} \frac{e^{ik_m r - i\frac{\pi}{2}}}{k_m r} [I_1 + I_2 + I_3] \tag{8.5}$$

where

$$I_1 = \int_{-\infty}^{\infty} \frac{\sqrt{\nu}}{\nu^2 - \frac{1}{4}} H_\nu^{(2)}(k_m a)\, e^{i\nu(\psi - \frac{\pi}{2})} d\nu \tag{8.6}$$

$$I_2 = -\int_{k_m b}^{\infty} \frac{\sqrt{\nu}}{\nu^2 - \frac{1}{4}} \frac{H_\nu^{(2)}(k_m b)}{H_\nu^{(1)}(k_m b)} H_\nu^{(1)}(k_m a)\, e^{i\nu(\psi - \frac{\pi}{2})} d\nu \tag{8.7}$$

$$I_3 = -\int_{-\infty}^{k_m b} \frac{\sqrt{\nu}}{\nu^2 - \frac{1}{4}} \frac{H_\nu^{(2)}(k_m b)}{H_\nu^{(1)}(k_m b)} H_\nu^{(1)}(k_m a)\, e^{i\nu(\psi - \frac{\pi}{2})} d\nu \tag{8.8}$$

Substituting the appropriate Debye asymptotic formulas in these integrals we get

$$I_1 = \sqrt{\frac{2}{\pi}}\, e^{i\frac{\pi}{4}} \int_{-\infty}^{\infty} \frac{\sqrt{\nu}}{\nu^2 - \frac{1}{4}} \frac{1}{(k_m^2 a^2 - \nu^2)^{1/4}}\, e^{i[-\sqrt{k_m^2 a^2 - \nu^2} + \nu \cos^{-1} \frac{\nu}{k_m a} + \nu(\psi - \frac{\pi}{2})]}\, d\nu \tag{8.9}$$

$$I_2 = \sqrt{\frac{2}{\pi}}\, e^{-i\frac{\pi}{4}} \int_{k_m b}^{\infty} \frac{\sqrt{\nu}}{\nu^2 - \frac{1}{4}} \frac{1}{(k_m^2 a^2 - \nu^2)^{1/4}}\, e^{i[\sqrt{k_m^2 a^2 - \nu^2} - \nu \cos^{-1} \frac{\nu}{k_m a} + \nu(\psi - \frac{\pi}{2})]}\, d\nu \tag{8.10}$$

$$I_3 = -\sqrt{\frac{2}{\pi}}\, e^{i\frac{\pi}{4}} \int_{-\infty}^{k_m b} \frac{\sqrt{\nu}}{\nu^2 - \frac{1}{4}} \frac{1}{(k_m^2 a^2 - \nu^2)^{1/4}} \times$$

$$\times\, e^{i[\sqrt{k_m^2 a^2 - \nu^2} - 2\sqrt{k_m^2 b^2 - \nu^2} - \nu \cos^{-1} \frac{\nu}{k_m a} + 2\nu \cos^{-1} \frac{\nu}{k_m b} + \nu(\psi - \frac{\pi}{2})]}\, d\nu$$

$$k_m a > k_m b >> 1\,. \tag{8.11}$$

In writing down the above expressions we have used the fact that I_1 and I_2 have no stationary points to the right of $k_m a$.

Each of the three integrals has at most one stationary point. For the integral I_1 the location of its stationary point is given by the solution of the trigonometric equation

$$\cos^{-1} \frac{\nu}{k_m a} + \psi - \frac{\pi}{2} = 0 \tag{8.12}$$

Because of the condition (8.4), this equation has a solution only when $\psi < \frac{\pi}{2}$. When this is the case we have

$$\nu = k_m a \sin \psi \tag{8.13}$$

Thus evaluating I_1 by the well-known method of stationary phase, we get

$$I_1 = \frac{2}{(k_m a \sin\psi)^{3/2}} e^{-ik_m a \cos\psi} \quad , \quad \psi < \frac{\pi}{2}$$

$$= 0 \quad , \quad \psi > \frac{\pi}{2} \qquad (8.14)$$

In the same way the location of the stationary point of I_2 is given by

$$-\cos^{-1}\frac{\nu}{k_m a} + \psi - \frac{\pi}{2} = 0 \qquad (8.15)$$

Because of (8.4), this equation has a solution only when $\psi > \frac{\pi}{2}$. Then the stationary point is given by

$$\nu = k_m a \sin\psi \qquad (8.16)$$

This must be greater than $k_m b$ or else the point lies outside the lower limit of integration. Hence we have the additional condition for the existence of the stationary point:

$$\sin\psi > \frac{b}{a} \qquad (8.17)$$

Therefore we get

$$I_2 = 0 \quad , \quad \psi < \frac{\pi}{2}$$

$$= \frac{2}{(k_m a \sin\psi)^{3/2}} e^{-ik_m a \cos\psi} \quad , \quad \frac{\pi}{2} < \psi < \pi - \sin^{-1}\frac{b}{a}$$

$$= 0 \quad , \quad \pi - \sin^{-1}\frac{b}{a} < \psi < \pi$$

$$(8.18)$$

The sum $I_1 + I_2$ has a very obvious interpretation in geometrical optics. It represents the incident wave. ψ as defined in (6.6) is the angle between the directions (θ,ϕ') and $(\frac{\pi}{2},0)$. Suppose a point source of frequency ω_m is placed at the point $(a,\frac{\pi}{2},0)$ as shown in Fig. 3. The geometrical shadow region is a cone with apex at the source and angle $\sin^{-1}\frac{b}{a}$. Inside the shadow region $I_1 + I_2$ is zero. Outside this region $I_1 + I_2$ contributes a term to S_m^g proportional to

$$\frac{1}{r}\, e^{ik_m(r - a\cos\psi)}$$

which indeed describes the far field of a point source displaced from the origin by a distance a .

We now want to show that I_3 gives rise to the reflected wave. This integral has a stationary point given by

$$-\cos^{-1}\frac{\nu}{k_m a} + 2\cos^{-1}\frac{\nu}{k_m b} + \psi - \frac{\pi}{2} = 0 \qquad (8.19)$$

We denote the solution by

$$\nu = k_m b \sin\gamma\ , \qquad 0 < \gamma < \frac{\pi}{2} \qquad (8.20)$$

Then γ is the angle of reflection for a ray scattered into a direction inclined at an angle ψ with the x-axis. This can readily be seen from the ray diagram, Fig. 3. Without loss of generality we have taken the z-x plane to be the plane of reflection. From this diagram we get

$$b\sin\gamma = a\sin\alpha \qquad (8.21)$$

where α is the angle the incident ray makes with the x-axis. Using this result in (8.19) we get

$$\alpha - 2\gamma + \psi = 0 \tag{8.22}$$

which is just the law of reflection. Eliminating α from (8.21) and (8.22) we obtain

$$\frac{\sin(2\gamma - \psi)}{\sin \gamma} = \frac{b}{a} \tag{8.23}$$

This is a quartic equation in $\sin \gamma$ which can always be solved. The exact expression for the solution, however, is too complicated. For $\gamma > \frac{\pi}{2}$, that is, inside the geometrical shadow region (8.19) has no solution and there is no stationary point. Thus we have

$$\begin{aligned} I_3 &= -\frac{2}{(k_m b \sin \gamma)^{3/2}} \sqrt{\frac{b \cos \gamma}{2\sqrt{a^2 - b^2 \sin^2\gamma} - b \cos \gamma}} \times \\ &\quad \times e^{ik_m(\sqrt{a^2 - b^2\sin^2\gamma} - 2b \cos \gamma)}, \quad 0 < \psi < \pi - \sin^{-1}\frac{b}{a} \\ &= 0 \quad , \qquad \pi - \sin^{-1}\frac{b}{a} < \psi < \pi \end{aligned} \tag{8.24}$$

Summarizing the foregoing results we have

$$\begin{aligned} S_m^g = -i \frac{e^{ik_m r}}{r} \sqrt{\frac{8}{\pi \sin \psi}} \frac{1}{k_m^{5/2}} \Bigg[& \frac{1}{(a \sin \psi)^{3/2}} e^{-ik_m a \cos \psi} - \\ & - \frac{1}{(b \sin \gamma)^{3/2}} \sqrt{\frac{b \cos \gamma}{2\sqrt{a^2 - b^2\sin^2\gamma} - b \cos \gamma}} \, e^{ik_m(\sqrt{a^2-b^2\sin^2\gamma} - 2b \cos \gamma)} \Bigg] \end{aligned}$$

$$0 < \psi < \pi - \sin^{-1}\frac{b}{a}$$

$$= 0 \quad , \qquad \pi - \sin^{-1}\frac{b}{a} < \psi < \pi \tag{8.25}$$

For the electric wave we get in a similar manner

$$T_m^g = -i\,\frac{e^{ik_m r}}{r}\sqrt{\frac{8}{\pi \sin\psi}}\;\frac{\sqrt{k_m a}}{k_m^{5/2}}\left[\frac{-i\cos\psi}{(a\sin\psi)^{3/2}}\,e^{-ik_m a\cos\psi} + \right.$$

$$\left. + i\,\frac{\sqrt{a^2-b^2\sin^2\gamma}}{a(b\sin\gamma)^{3/2}}\sqrt{\frac{b\cos\gamma}{2\sqrt{a^2-b^2\sin^2\gamma}-b\cos\gamma}}\;e^{ik_m(\sqrt{a^2-b^2\sin^2}\,-2b\cos\gamma)}\right]$$

$$0 < \psi < \pi - \sin^{-1}\frac{b}{a}$$

$$= 0 \quad , \qquad \pi - \sin^{-1}\frac{b}{a} < \psi < \pi \tag{8.26}$$

From this we get

$$\frac{\partial}{\partial(\cos\psi)}T_m^g = -i\,\frac{e^{ik_m r}}{r}\sqrt{\frac{8}{\pi\sin\psi}}\;\frac{\sqrt{k_m a}}{k_m^{5/2}}\left[\frac{-k_m a\cos\psi}{(a\sin\psi)^{3/2}}\,e^{-ik_m a\cos\psi} + \right.$$

$$\left.\frac{k_m\sqrt{a^2-b^2\sin^2\gamma}}{a\sin\psi\,(b\sin\gamma)^{1/2}}\sqrt{\frac{b\cos}{2\sqrt{a^2-b^2\sin^2\gamma}-b\cos\gamma}}\;e^{ik_m(\sqrt{a^2-b^2\sin^2}\,-2b\cos\gamma)}\right]$$

$$0 < \psi < \pi - \sin^{-1}\frac{b}{a}\,,$$

$$= 0 \quad , \qquad \pi - \sin^{-1}\frac{b}{a} < \psi < \pi\,. \tag{8.27}$$

In the above we retained only the highest order terms in $k_m a$.

Substituting (8.25) into (6.4) we get for the incident part

$$u^{inc} = \frac{p}{4\pi\epsilon_o r a^2}\int\frac{d\phi'}{\sin^2\psi}\,\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\frac{m}{k_m}\,e^{i[k_m r-\omega_m t-k_m a\cos\psi+m(\phi-\phi')]} \tag{8.28}$$

where the integral is taken only over the illuminated region, that is, over the range

$$-\cos^{-1}\frac{-\sqrt{a^2-b^2}}{a\sin\theta} < \phi' < \cos^{-1}\frac{-\sqrt{a^2-b^2}}{a\sin\theta}$$

From (8.28) we get

$$\frac{\partial}{\partial r} ru^{inc} = \frac{p}{4\pi\varepsilon_o a^2}\frac{\partial}{\partial\phi}\int d\phi' \, \frac{e^{i(k_o r - \omega_o t - k_o a\cos\psi)}}{\sin^2\psi}$$

$$\times \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(Kr - \Omega t - Ka\cos\psi + \phi - \phi')} \qquad (8.29)$$

By (2.4) we have

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(Kr - \Omega t - Ka\cos\psi + \phi - \phi')}$$

$$= \delta(Kr - \Omega t - Ka\cos\psi + \phi - \phi') \qquad (8.30)$$

The position of the peak of the delta-function is given by the solution of the equation

$$Kr - \Omega t - Ka\sin\theta\cos\phi' + \phi - \phi' = 0 \qquad (8.31)$$

In most situations $Ka = \frac{\Omega a}{c} << 1$. (8.31) can be solved by iteration:

$$\phi' = \phi + Kr - \Omega t - Ka\sin\theta\cos(\phi + Kr - \Omega t) + \cdots \quad . \qquad (8.32)$$

In what follows, for simplicity, we will ignore all terms in Ka. Therefore, (8.29) becomes

$$\frac{\partial}{\partial r} ru^{inc} = \frac{p}{4\pi\varepsilon_o a^2} \frac{\partial}{\partial \phi} \left[\frac{e^{i[k_o r - \omega_o t - k_o a \sin\Theta \cos(\phi + Kr - \Omega t)]}}{1 - \sin^2\Theta \cos^2(\phi + Kr - \Omega t)} \right] \tag{8.33}$$

when $\phi + Kr - \Omega t$ lies in the illuminated region and zero otherwise. (8.33) is equivalent to

$$ru^{inc} = \frac{p}{4\pi\varepsilon_o a} \frac{\sin\Theta \sin(\phi + Kr - \Omega t)}{1 - \sin^2\Theta \cos^2(\phi + Kr - \Omega t)} \times$$

$$\times\ e^{i[k_o r - \omega_o t - k_o a \sin\Theta \cos(\phi + Kr - \Omega t)]} \tag{8.34}$$

Carrying out the same calculations for the reflected part we get

$$ru^{ref} = -\frac{p}{4\pi\varepsilon_o a} \frac{\sin\Theta \sin(\phi + Kr - \Omega t)}{1 - \sin^2\Theta \cos^2(\phi + Kr - \Omega t)} \sqrt{\frac{a \sin\psi}{b \sin\gamma} \frac{b \cos\gamma}{2\sqrt{a^2 - b^2\sin^2\gamma} - b\cos\gamma}}$$

$$\times\ e^{i[k_o r - \omega_o t + k_o(\sqrt{a^2 - b^2\sin^2\gamma} - 2b\cos\gamma)]} \tag{8.35}$$

The angles γ and ψ are to be evaluated at $\phi' = \phi + Kr - \Omega t$. The whole expression is zero in the shadow region. Similarly for the electric wave we get

$$rv^{inc} = -\frac{p}{4\pi\varepsilon_o a} \frac{\cos\Theta \sin\Theta \cos(\phi + Kr - \Omega t)}{1 - \sin^2\Theta \cos^2(\phi + Kr - \Omega t)}$$

$$\times\ e^{i[k_o r - \omega_o t - k_o a \sin\Theta \cos(\phi + Kr - \Omega t)]} \tag{8.36}$$

$$rv^{ref} = \frac{p}{4\pi\varepsilon_o a} \frac{\cos\theta\sqrt{1-\frac{b^2}{a^2}\sin^2\gamma}}{1-\sin^2\theta\cos^2(\phi+Kr-\Omega t)} \sqrt{\frac{a\sin\psi}{b\sin\gamma}\frac{b\cos\gamma}{2\sqrt{a^2-b^2\sin^2\gamma}-b\cos\gamma}}$$

$$\times\; e^{i[k_o r-\omega_o t+k_c(\sqrt{a^2-b^2\sin^2\gamma}-2b\cos\gamma)]} \qquad (8.37)$$

9. Nature of the Solution

We recall that the solution is zero when the direction $(\theta,\phi+Kr-\Omega t)$ lies inside the shadow cone of Fig. 3. $\phi+Kr-\Omega t$ is just the difference between the azimuthal angle ϕ of the direction of observation (θ,ϕ) and the retarded azimuthal angle $\Omega(t-\frac{r}{c})$ of the revolving dipole. Thus we have the simple conclusion that the total field is zero when the observation point lies inside the retarded shadow region of the dipole.

We substitute ru^{inc} and rv^{inc} in (8.34) and (8.36) into (4.3) and obtain the asymptotic form of the incident field in the illuminated region:

$$E_r^{inc} = E_\phi^{inc} = cB_r^{inc} = cB_\theta^{inc} = 0$$
$$E_\theta^{inc} = cB_\phi^{inc} = -\frac{pk_o^2}{4\pi\varepsilon_o r}\sin\theta\, e^{i[k_o r-\omega_o t-k_o a\sin\theta\cos(\phi+Kr-\Omega t)]} \qquad (9.1)$$

When $\Omega = 0$ these field components coincide with those of a Hertz dipole fixed at the point $x = a,\ y = 0,\ z = 0$ and oriented parallel to the z-axis. We can define an instantaneous frequency of the incident wave by differentiating the exponential with respect to t :

$$\omega^{inc} = \omega_o[1+\beta\sin\theta\sin(\Omega t-Kr-\phi)] \qquad (9.2)$$

where

$$\beta = \frac{\Omega a}{c} \tag{9.3}$$

Here Ωa is just the velocity of the dipole, and $\sin\Theta \sin(\Omega t - Kr - \emptyset)$ is the cosine of the angle between the velocity of the dipole and the direction of observation at the retarded time $t - \frac{r}{c}$. (9.2) therefore agrees with the Doppler formula for the frequency shift of a moving source.

Similarly we calculate the far field of the reflected wave in the illuminated region from ru^{ref} and rv^{ref} in (8.35) and (8.37):

$$E_r^{ref} = cB_r^{ref} = 0$$

$$E_\Theta^{ref} = cB_\emptyset^{ref} = \frac{pk_o^2}{4\pi\varepsilon_o r}\sqrt{\frac{b\sin\gamma}{a\sin\psi}\,\frac{b\cos\gamma}{2\sqrt{a^2 - b^2\sin^2\gamma} - b\cos\gamma}} \times$$

$$\times \frac{\sin\Theta\sin^2\emptyset' + \cos^2\Theta\sin\emptyset'\sqrt{1 - \frac{b^2}{a^2}\sin^2\gamma}}{\sin^2\psi} \times$$

$$\times\; e^{i[k_o r - \omega_o t + k_o(a^2 - b^2\sin^2\gamma - 2b\cos\gamma)]}$$

$$E_\emptyset^{ref} = -cB_\Theta^{ref} = \frac{pk_o^2}{4\pi\varepsilon_o r}\sqrt{\frac{b\sin\gamma}{a\sin\psi}\,\frac{b\cos\gamma}{2\sqrt{a^2 - b^2\sin\gamma} - b\cos\gamma}}$$

$$\times \frac{\cos\Theta\sin\emptyset'(\sin\Theta\cos\emptyset' - \sqrt{1 - \frac{b^2}{a^2}\sin^2\gamma})}{\sin^2\psi} \times$$

$$\times\; e^{i[k_o r - \omega_o t + k_o(\sqrt{a^2 - b^2\sin^2\gamma} - 2b\cos\gamma)]} \tag{9.4}$$

The angles ψ, γ and ϕ' are defined in terms of r, θ, ϕ and t by (6.6), (8.23) and (8.32) respectively. The instantaneous frequency of the reflected wave is given by

$$\omega^{ref} = \omega_o[1 + \beta' \sin\theta \sin(\Omega t - Kr - \phi)] \tag{9.5}$$

where

$$\beta' = \frac{\Omega b}{c} \frac{\sin\gamma}{\sin\psi} \quad . \tag{9.6}$$

Thus the reflected wave is not of the same frequency as the incident wave. Unlike β in (9.3) β' is a periodic function of time. The quantity $b \sin\gamma/\sin\psi$ is the intercept of the ray reflected into the direction (θ,ϕ) when projected backwards, on the retarded radius vector of the moving dipole as can be seen from Fig. 3. Despite the apparent similarity of (9.2) and (9.5) it does not seem to be possible to interpret the reflected wave as due to a virtual moving point source.

10. Acknowledgements

The author would like to thank Professor C. H. Papas and Dr. K. S. H. Lee for many helpful discussions and critical reading of the manuscript.

This work was supported by the U. S. Air Force Office of Scientific Research under Grant AFOSR-68-1400.

References

1. G. N. Watson, "The Diffraction of Electric Waves by the Earth", Proc. Roy. Soc., London A95, 83 (1918).

2. W. Franz, "Über die Greenschen Funktionen des Zylinders und der Kugel", Z. Naturforschg 9a, 705 (1954).

3. See an Appendix of Reference 2.

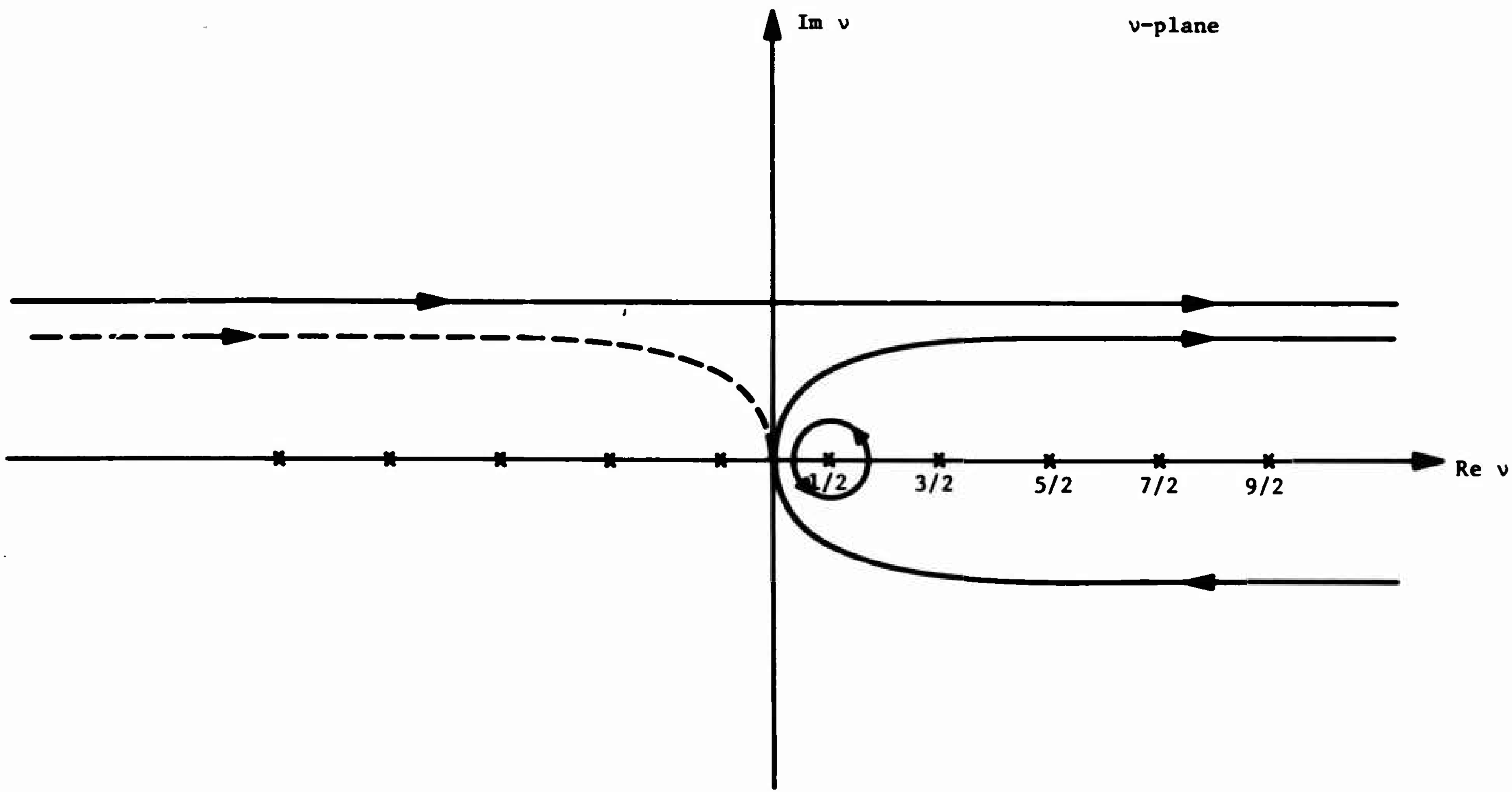

Fig. 1. The Watson Transformation

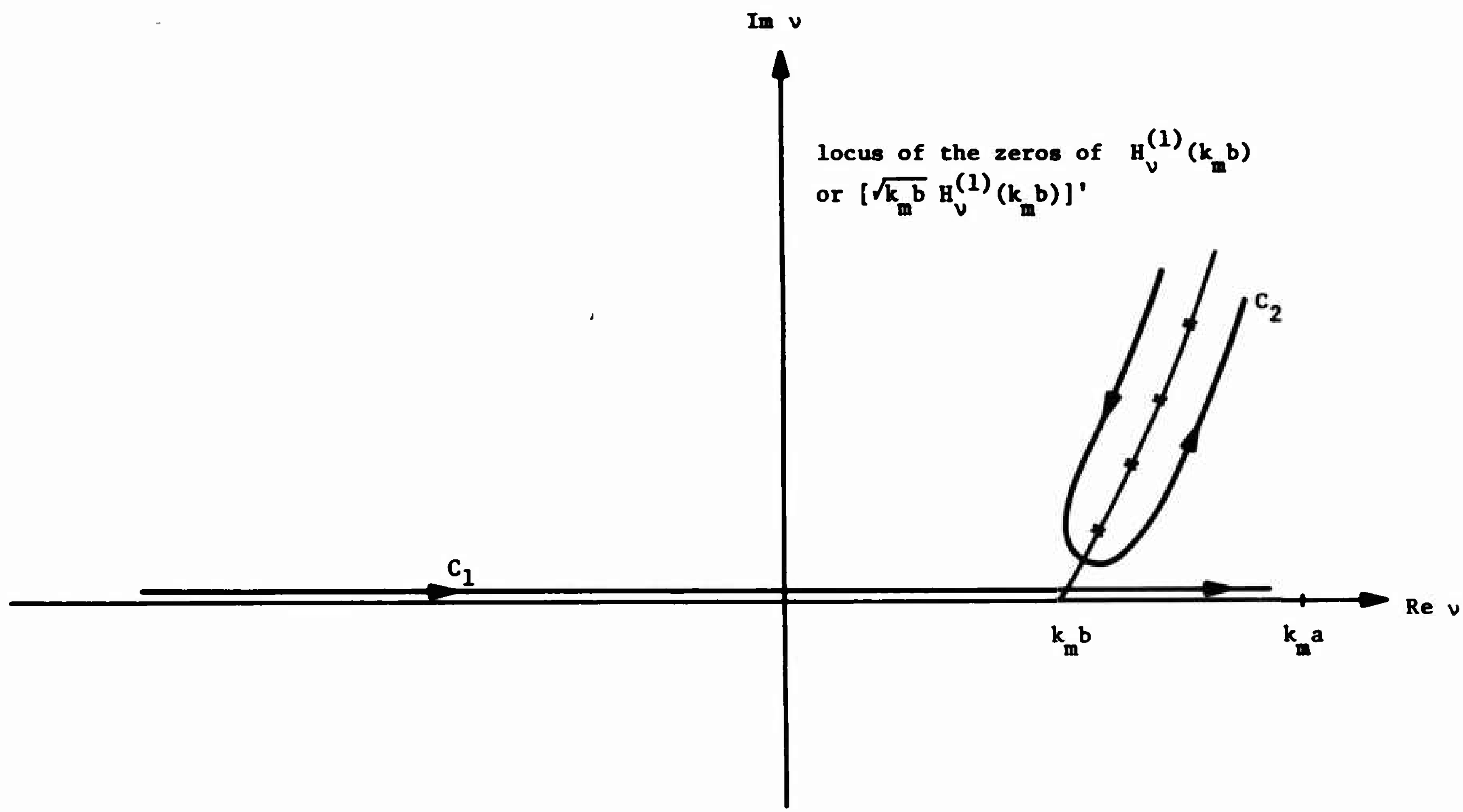

Fig. 2. Contours and singularities in the ν-plane

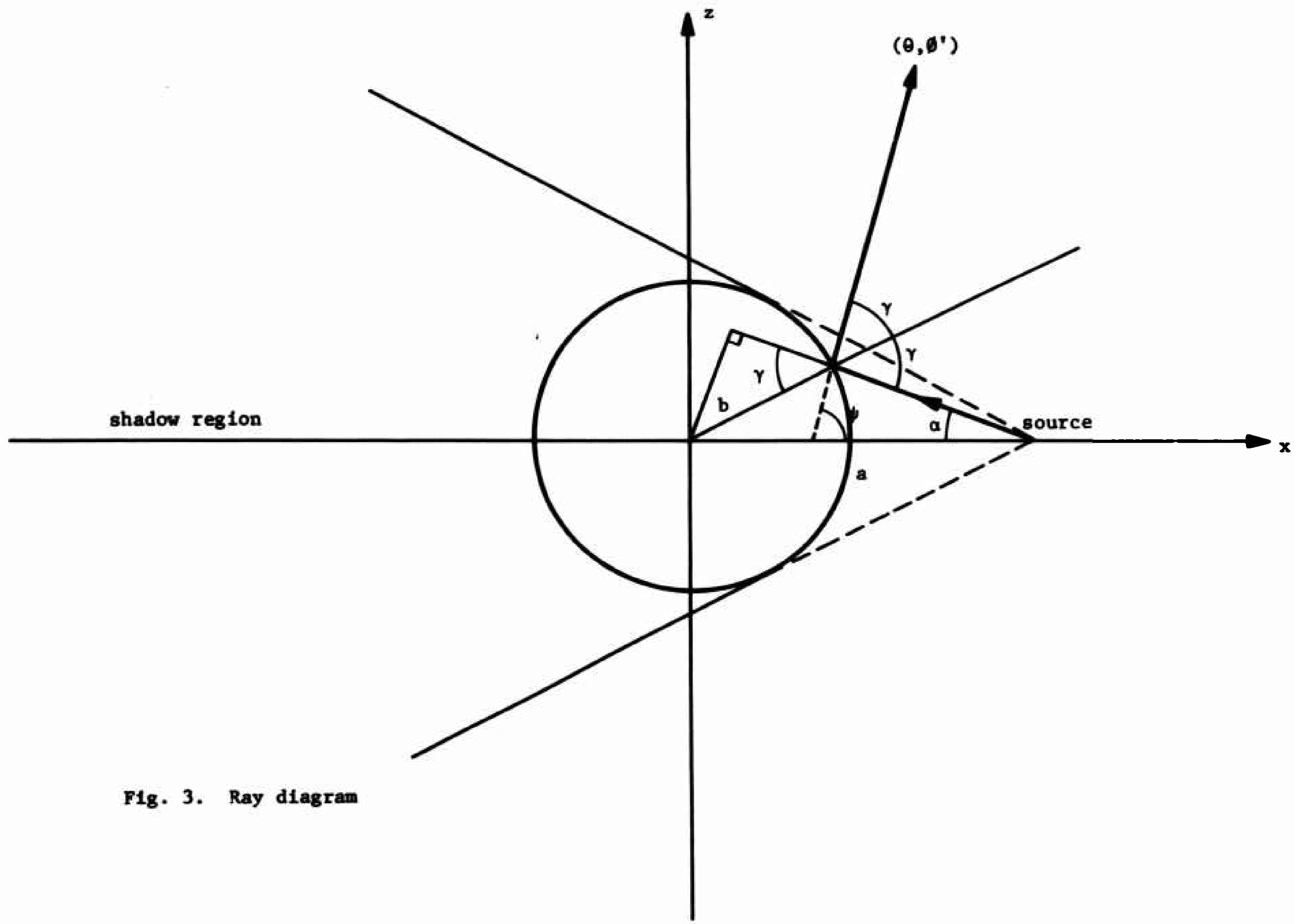

Fig. 3. Ray diagram

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) California Institute of Technology Electrical Engineering Department Pasadena, California 91109	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

RADIATION OF A HERTZ DIPOLE MOVING AROUND A CONDUCTING SPHERE

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Scientific Interim

5. AUTHOR(S) (Last name, first name, initial)

LAM, John

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 1968	35	3

8a. CONTRACT OR GRANT NO. AFOSR-68-1400 b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Antenna Laboratory Report No. 43 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/LIMITATION NOTICES

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

11. SUPPLEMENTARY NOTES TECH, Other	12. SPONSORING MILITARY ACTIVITY AF Office of Scientific Research (SREE) 1400 Wilson Blvd. Arlington, Va. 22209

13. ABSTRACT

An exact expression is obtained for the field of a horizontal Hertz dipole moving around a conducting sphere in a circular orbit. This expression is in the form of a double series which, in the case of a large sphere, is evaluated by the method of Watson transformation. The geometrical optics field is separated out and its properties examined. It is found that the incident and reflected waves are of different frequencies.

DD FORM 1 JAN 64 1473

UNCLASSIFIED

Security Classification

UNCLASSIFIED

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
electromagnetic scattering						
moving source						

INSTRUCTIONS

1. **ORIGINATING ACTIVITY:** Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. **REPORT SECURITY CLASSIFICATION:** Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. **GROUP:** Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. **REPORT TITLE:** Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. **DESCRIPTIVE NOTES:** If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. **AUTHOR(S):** Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. **REPORT DATE:** Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. **TOTAL NUMBER OF PAGES:** The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. **NUMBER OF REFERENCES:** Enter the total number of references cited in the report.

8a. **CONTRACT OR GRANT NUMBER:** If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. **PROJECT NUMBER:** Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. **ORIGINATOR'S REPORT NUMBER(S):** Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. **OTHER REPORT NUMBER(S):** If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. **AVAILABILITY/LIMITATION NOTICES:** Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through

______________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through

______________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through

______________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. **SUPPLEMENTARY NOTES:** Use for additional explanatory notes.

12. **SPONSORING MILITARY ACTIVITY:** Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. **ABSTRACT:** Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. **KEY WORDS:** Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.