CNA

ON SUBOPTIMIZATION: AN EXAMPLE

By J.M. Danskin

CNA Research Contribution No. 22

CONTRACT NONR 3732(00)

Research Contribution

Center for Naval Analyses
THE FRANKLIN INSTITUTE
WASHINGTON 25, D.C.

10

CENTER FOR NAVAL ANALYSES
OF THE
UNIVERSITY OF ROCHESTER

40 W[illegible] Boulevard
Arlington Virginia 22209

Area code 703
JAckson 4-9400

(CNA)131-68
17 MAY 1968

From: Vice President for Administration, CNA
To: Distribution List

Subj: Center for Naval Analyses Research Contribution No. 22; forwarding of

Encl: (1) CNA RC No. 22, "On Suboptimization: An Example"

1. Enclosure (1) is forwarded as a matter of possible interest.

2. In certain allocation problems concerning groups of systems, it is possible to allocate by distributing within each system and then combining the results. This paper shows that while this method is correct for pure maximum problems and for cases in which the overall problem is a game, it is not true for Max-Min problems.

3. Research Contributions represent the opinions of the authors and not necessarily those of the Center for Naval Analyses or the U. S. Navy. They are not reviewed in detail. They are distributed in this form to stimulate thought and facilitate research in given problem areas.

4. The enclosure has been approved for public release.

5. Registered users of the Defense Documentation Center services should request additional copies from that agency.

Carl B. Amthor

CARL B. AMTHOR

DISTRIBUTION:

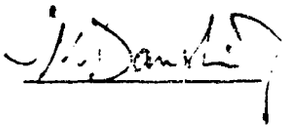
ADMIN, DDC (20)
USNA ANNA (2)
SUPT NAVPGSCOL (2)
PRES NAVWARCOL (2)
IDA
RAND
RAC

CNA RESEARCH CONTRIBUTION NO. 22

Center for Naval Analyses

ON SUBOPTIMIZATION: AN EXAMPLE

By J.M. Danskin



17 June 1966

Work conducted under contract NONR 3732 (00)

ABSTRACT

In certain allocation problems concerning groups of systems, it is possible to allocate by allocating within each system and then combining the results. This paper shows that while this method is correct for pure maximum problems and for cases in which the overall problem is a game, it is not true for Max-Min problems.

(REVERSE BLANK)

Suppose

$$H(x,\xi) = F(x) + G(\xi), \tag{1}$$

where x and ξ are vectors satisfying $\Sigma x_i + \Sigma \xi_j = 1$, $x_i \geq 0$, $\xi_j \geq 0$. Suppose the pair x^0, ξ^0 maximizes (1). Let $\sum x_i^0 = \alpha$. It is a trivial fact that then x_i^0 maximizes $F(x)$ subject to $\Sigma x_i = \alpha$, $x_i \geq 0$. It follows from this in particular that the solution for (1) can be found by solving the problem for $F(x)$ for $\Sigma x_i = \alpha$ and the problem for $G(\xi)$ for $\Sigma \xi_j = 1 - \alpha$, and then turning through all values of α. This process is sometimes called "suboptimization". The idea is that if one solves the problem for the whole, one gets the solution for the parts, and vice-versa.

This "suboptimization principle" also holds when the maximum problem for $H(x,\xi)$ is replaced by a game problem

$$H(x,\xi,y,\eta) = F(x,y) + G(\xi,\eta), \tag{2}$$

where the first player seeks to maximize subject to $\Sigma x_i + \Sigma \xi_j = 1$, $x_i \geq 0$, $\xi_j \geq 0$, and the second player seeks to minimize subject to $\Sigma y_i + \Sigma \eta_j = 1$, $y_i \geq 0$, $\eta_j \geq 0$. Suppose that there is a pair of pure strategy

solutions (x^0,ξ^0) and (y^0,ξ^0) for the two players respectively and suppose that $\sum x_i^0 = \alpha$, $\sum y_i^0 = \beta$. Let the value of the game defined by H be v, and suppose that $G(\xi^0,\eta^0) = \gamma$. Then, for any x satisfying $\Sigma x_i = \alpha$, $x_i \geq 0$,

$$F(x,y^0) = H(x,\xi^0,y^0,\eta^0) - G(\xi^0,\eta^0) \leq v - \gamma ,$$

and for any y satisfying $\Sigma y_i = \beta$, $y_i \geq 0$,

$$F(x^0,y) = H(x^0,\xi^0,y,\eta^0) - G(\xi^0,\eta^0) \geq v - \gamma .$$

Thus the components x^0 and y^0 are optimal strategy solutions for F; and similarly ξ^0 and η^0 for G.

It follows that games can be solved piece-by-piece. One picks pairs α,β and solves the games separately. One then runs through all pairs with $0 \leq \alpha, \beta \leq 1$; this will surely lead to a solution of the overall game.

The object of this paper is to show that this suboptimization principle does not always hold for Max-Min problems which are not games.

Suppose that

$$F(x,y) = x_1 e^{-y_1} + .9x_2 e^{-y_2/x_2}$$

and

$$G(\xi,\eta) = \xi e^{-\eta/\xi} .$$

Here $x_1 + x_2 + \xi = 1$, $y_1 + y_2 + \eta = 1$, and $x_i, \xi, y_i, \eta \geq 0$.

If we solve the problem $\underset{x,\xi}{\text{Max}}\ \underset{y,\eta}{\text{Min}}\ H(x,\xi,y,\eta)$ for

$$H(x,\xi,y,\eta) = x_1 e^{-y_1} + .9x_2 e^{-y_2/x_2} + \xi e^{-\eta/\xi}$$

subject to the above side conditions, we get (for the methods see Chapter V of [1])

$$x_1 = .465 \quad x_2 = 0 \quad \xi = .535$$

$$y_1 = .615 \quad y_2 = 0 \quad \eta = .385 .$$

Now consider the problem

$$\underset{x_1+x_2=.465}{\text{Max}}\ \underset{y_1+y_2=.615}{\text{Min}}\ x_1 e^{-y_1} + .9x_2 e^{-y_2/x_2} . \qquad (3)$$

The solution to this problem turns out to have both x_2 and y_2 positive. In fact, it is

$$x_1 = .434 \quad x_2 = .031$$

$$y_1 = .575 \quad y_2 = .040 .$$

The return to x at this point for the game (3) is .248. At $x_1 = .465$ the value is .226. The reader will easily verify that the y-solution is correct against $x_1 = .434$, $x_2 = .031$ by checking the derivatives with respect to y_1 and y_2, which are both approximately -.248. These calculations are carried out to slide rule accuracy.

It follows that for the above problem one would never have arrived at the solution to the overall problem by grouping the first and the second system together, optimizing, and then bringing in the third. The three variables must be considered simultaneously.

The meaning for operations research or economic analysis is clear; one cannot, in the presence of conflict, be assured of arriving at a solution for the whole by considering the parts one at a time.

[1] Danskin, J. M., The Theory of Max-Min, Springer-Verlag, Berlin - Heidelberg - New York, 1966.

None

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author) Center for Naval Analyses of the Franklin Institute	2a REPORT SECURITY CLASSIFICATION None
	2b GROUP None

3 REPORT TITLE

On Suboptimization: An Example

4 DESCRIPTIVE NOTES (Type of report and inclusive dates)

Final - Research Contribution

5 AUTHOR(S) (Last name, first name, initial)

Danskin, J.M.

6 REPORT DATE	7a TOTAL NO OF PAGES	7b NO OF REFS
17 June 1966	5	1

8a CONTRACT OR GRANT NO	9a ORIGINATOR'S REPORT NUMBER(S)
NONR 3732 (00)	CNA Research Contribution No. 22
b PROJECT NO ----	
c ----	9b OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d ----	

10 AVAILABILITY/LIMITATION NOTICES

Qualified Requestors may obtain this report from DDC. It has been approved for Public Release.

11 SUPPLEMENTARY NOTES	12 SPONSORING MILITARY ACTIVITY
None	Office of Naval Research Department of the Navy Washington 25, D.C.

13 ABSTRACT

In certain allocation problems concerning groups of systems, it is possible to allocate by allocating within each system and then combining the results. This paper shows that while this method is correct for pure maximum problems and for cases in which the overall problem is a game, it is not true for Max-Min problems.

DD FORM 1473

None

Security Classification

None

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
suboptimization principle allocation Max-Min problems						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (*corporate author*) issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year, or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (*either by the originator or by the sponsor*), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through _____."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through _____."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through _____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (*paying for*) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (*TS*), (*S*), (*C*), or (*U*).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

DD FORM 1 JAN 64 1473 (BACK)

None

Security Classification