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INVARIANT PROPERTIES OF COMPOSITE MATERIALS

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FOREWORD

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This technical report has been reviewed and is approved.

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ABSTRACT

Invariant properties of the elastic coefficient matrices of laminated composite plates are presented. The use of these invariants in materials evaluation and design optimization is discussed. Simple formulas, based upon micromechanics results, are derived for the invariants in terms of constituent material properties.

TABLE OF CONTENTS

SYMBOLS

A_{ij}, B_{ij}, D_{ij}	elastic constants of laminated composites, Equation 9
C_{ij}	elastic stiffness matrix of 3-dimensional bodies
Q_{ij}	reduced stiffness matrix for bodies under plane stress
L_1, L_2	invariants of Q_{ij}
$U_1, \ldots U_7$	coefficients of transformation equations in Table I, defined by Equation 14
$V_0 [A,B,D]$	Constant terms in A_{ij}, B_{ij}, and D_{ij} defined by Equation 22
$V_1, \ldots V_4 [A,B,D]$	integrals defined by Equation 23
$P_1 \ldots P_6$	invariants of A_{ij}, B_{ij}, and D_{ij}, defined in Equations 27 and 28
N_i	stress resultants
M_i	stress couples
k_i	components of curvature
m	cos θ, or cross-ply ratio which is the ratio of the thickness of 0° to 90° layers in a cross-ply composite
n	sin θ, or number of layers in Equations 38, 39, and 40
h	total thickness of laminated composites
E	Young's modulus
G	shear modulus
F_1, F_2, β	functions defined by Equations 48, 49, and 50
T	temperature
v	volume fraction
α_i	thermal expansion coefficients
ϵ_i	strain components
σ_i	stress components

SYMBOLS (CONTD)

ν	Poisson's ratio
θ, ϕ	angles of rotation
subscript f	pertaining to fibers
subscript m	pertaining to matrix
bar	average quantities or isotropic constants
prime	transformed component

SECTION I

INTRODUCTION

The superior performance of composite materials over lightweight metals has been well publicized in recent years. Composites have been claimed to possess improvements in stiffness and in strength of severalfold over ordinary materials. The claim, however, is based on either the properties of the fibers alone or the longitudinal properties of a unidirectional composite against those of the metals. Since composites are normally used in laminated forms which consist of unidirectional layers, a more realistic measure of the performance of the composites than that based on the fiber or longitudinal properties is needed.

In this work, the transformation properties of unidirectional and laminated composites are derived in terms of multiple angles, instead of the classical relations using powers of sines and cosines. The effect of lamina orientation is then examined. It is shown that the invariant properties of both the unidirectional and laminated composites have the same components, and can be used as an effective measure of the performance of the composites. Simple formulas are derived, from which the invariant properties of composites, irrespective of the lamina orientation, can be determined from the properties of the constituents. This work should be of value to system analysts who must evaluate the performance of composites, to structural designers who must establish a rational design procedure, and to materials engineers who may need guidance in the selection and fabrication of composite materials.

The elastic moduli of laminated composites have been reported by many investigators in recent years, examples of which include Reissner and Stavsky (Reference 1), Dong, et al. (Reference 2), and Tsai, (References 3 and 4). The usual assumptions in all these studies are:

(a) All layers are in a state of plane stress relative to the x-y or 1-2 plane, so that

$$\sigma_3 = \sigma_4 = \sigma_5 = 0 \tag{1}$$

(b) All layers are bonded together and the strain components in the 1-2 plane are linear functions of z,

$$\epsilon_i = \epsilon_i^0 + z\kappa_i \tag{2}$$

where i = 1, 2 refers to the normal components; i = 6, the engineering shear strain component.

(c) All layers obey generalized Hooke's law,

$$\sigma_i = C_{ij}\epsilon_j \tag{3}$$

With these assumptions, the constitutive equations for a laminated composite can be derived. The stress-strain relation for the assumed plane stress condition including the thermal effect for each layer is

$$\sigma_i = Q_{ij}\epsilon_j - Q_{ij}\alpha_j T \tag{4}$$

where

$$Q_{ij} = C_{ij} - \frac{C_{i3}\,C_{j3}}{C_{33}} = \text{reduced stiffness matrix} \tag{5}$$

α_i = anisotropic thermal expansion matrix

T = temperature increase from a reference (stress-free) temperature.

Stress resultants (N_i) and stress couples (M_i) can be defined as:

$$\left[N_i , M_i\right] = \int_{-h/2}^{h/2} \sigma_i \left[1, z\right] dz \tag{6}$$

Substituting Equations 2 and 4 into 6 renders

$$\begin{aligned} \bar{N}_i &= N_i + N_i^T = A_{ij}\,\epsilon_j^o + B_{ij}\,k_j \\ \bar{M}_i &= M_i + M_i^T = B_{ij}\,\epsilon_j^o + D_{ij}\,k_j \end{aligned} \tag{7}$$

where

$$\left[N_i^T , M_i^T\right] = \int_{-h/2}^{h/2} Q_{ij}\,\alpha_j\,T\left[1, z\right] dz \tag{8}$$

$$\left[A_{ij}, B_{ij}, D_{ij}\right] = \int_{-h/2}^{h/2} Q_{ij}\left[1, z, z^2\right] dz \tag{9}$$

The brackets above and for the remaining part of this report are symbolic rather than operational; the equality applies to the corresponding terms in the bracket. The limits of integration are from -h/2 to h/2, and remain the same unless otherwise specified.

The constitutive equations of laminated composites are given by Equation 7, and the material coefficients are expressed by the A, B, and D matrices. Our present work is concerned with the nature of the Q, A, B, and D matrices.

SECTION II

TRANSFORMATION OF Q_{ij}

We would like to establish the transformation equations of the reduced stiffness matrix Q_{ij}. This can be done by the use of Equation 5 and the transformation equations for C_{ij} tabulated by Hearmon (Reference 5), and Tsai (Reference 4). A typical example is as follows:

$$
\begin{aligned}
Q'_{11} &= C'_{11} - \frac{C'^{2}_{13}}{C'_{33}} \\
&= m^4 C_{11} + 2m^2n^2 C_{12} + n^4 C_{22} + 4m^2n^2 C_{66} \\
&\quad + 4m^3 n C_{16} + 4mn^3 C_{26} - \frac{1}{C_{33}} (m^2 C_{13} + n^2 C_{23} + 2mnC_{36})^2 \\
&= m^4 (C_{11} - \frac{C_{13}^2}{C_{33}}) + 2m^2n^2 (C_{12} - \frac{C_{13}C_{23}}{C_{33}}) \\
&\quad + n^4 (C_{22} - \frac{C_{23}^2}{C_{33}}) + 4m^2n^2 (C_{66} - \frac{C_{36}^2}{C_{33}}) \qquad (10) \\
&\quad + 4m^3 n (C_{16} - \frac{C_{13}C_{63}}{C_{33}}) + 4mn^3 (C_{26} - \frac{C_{23}C_{63}}{C_{33}}) \\
&= m^4 Q_{11} + 2m^2n^2 Q_{12} + n^4 Q_{22} + 4m^2n^2 Q_{66} \\
&\quad + 4m^3 n Q_{16} + 4mn^3 Q_{26}
\end{aligned}
$$

The transformation of the other components of Q_{ij} can also be shown. The transformation is a rotation through an angle θ about the 3-axis, for which $C_{33} = C'_{33}$ = invariant, and $m = \cos\theta$ and $n = \sin\theta$. It is assumed that a plane of symmetry exists in the 1-2 plane. Based on Equation 10 and similar results for the other components of Q_{ij}, we conclude that Q_{ij} transforms the same as C_{ij}. Having established the transformation equations, we can apply the usual material symmetries like orthotropy, isotropy, etc., and the invariants of the transformation can be determined.

For our present study, it is more convenient to express the transformation equations in terms of multiple angles than the conventional powers of sines and cosines. The following trigonometric identities shown by Cox (Reference 6) can be used:

$$
\begin{aligned}
m^4 &= (3 + 4\cos 2\theta + \cos 4\theta)/8 \\
m^3 n &= (2\sin 2\theta + \sin 4\theta)/8 \\
m^2 n^2 &= (1 - \cos 4\theta)/8 \\
m n^3 &= (2\sin 2\theta - \sin 4\theta)/8 \\
n^4 &= (3 - 4\cos\theta + \cos 4\theta)/8
\end{aligned}
\qquad (11)
$$

By direct substitution of Equation 1 into the conventional transformation equations, a new form of the transformation equations for Q_{ij} (also C_{ij}) can be derived with the results shown in Table I.

TABLE I

TRANSFORMATION EQUATIONS OF Q_{ij}

	Constant	$\cos 2\theta$	$\sin 2\theta$	$\cos 4\theta$	$\sin 4\theta$
Q'_{11}	U_1	U_2	$2U_6$	U_3	U_7
Q'_{22}	U_1	$-U_2$	$-2U_6$	U_3	U_7
Q'_{12}	U_4	0	0	$-U_3$	$-U_7$
Q'_{66}	U_5	0	0	$-U_3$	$-U_7$
$2Q'_{16}$	0	$2U_6$	$-U_2$	$2U_7$	$-2U_3$
$2Q'_{26}$	0	$2U_6$	$-U_2$	$-2U_7$	$2U_3$

where

$$\begin{aligned}
U_1 &= (3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})/8 \\
U_2 &= (Q_{11} - Q_{22})/2 \\
U_3 &= (Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})/8 \\
U_4 &= (Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66})/8 \\
U_5 &= (Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66})/8 \\
U_6 &= (Q_{16} + Q_{26})/2 \\
U_7 &= (Q_{16} - Q_{26})/2
\end{aligned} \tag{12}$$

From Table I and Equation 12, the following two invariants can be established by observation:

$$\begin{aligned}
L_1 &= Q'_{11} + Q'_{22} + 2Q'_{12} \\
&= 2(U_1 + U_4) \\
&= Q_{11} + Q_{22} + 2Q_{12} \\
L_2 &= Q'_{66} - Q'_{12} \\
&= U_5 - U_4 \\
&= Q_{66} - Q_{12}
\end{aligned} \tag{13}$$

By combining Equations 12 and 13, we can show that among the U's:

$$\begin{aligned}
U_1 &= (3L_1 + 4L_2)/8 \\
U_4 &= (L_1 - 4L_2)/8 \\
U_5 &= (L_1 + 4L_2)/8
\end{aligned} \tag{14}$$

are invariant as expected because they are the constant terms in Table I. But only two of them are independent because

$$U_5 = (U_1 - U_4)/2 \tag{15}$$

U_2, U_3, U_6, and U_7, on the other hand, are not invariant.

If Q_{ij} is orthotropic,

$$Q_{16} = Q_{26} = 0$$

from Equation 12, we see that

$$U_6 = U_7 = 0.$$

If Q_{ij} is isotropic,

$$\begin{aligned} Q_{11} &= Q_{22} \\ Q_{66} &= (Q_{11} - Q_{12})/2 \\ Q_{16} &= Q_{26} = 0. \end{aligned} \tag{16}$$

From Equation 12, we see that

$$\begin{aligned} U_1 &= Q_{11} \\ U_4 &= Q_{12} \\ U_5 &= (Q_{11} - Q_{12})/2 = Q_{66} \\ U_2 &= U_3 = U_6 = U_7 = 0. \end{aligned} \tag{17}$$

The components of Q_{ij} can be expressed in terms of engineering constants if and only if Q_{ij} is orthotropic:

$$\begin{aligned} Q_{11} &= E_{11}/(1 - \nu_{12}\nu_{21}) \\ Q_{22} &= E_{22}/(1 - \nu_{12}\nu_{21}) \\ Q_{12} &= \nu_{12} Q_{22} = \nu_{21} Q_{11} \\ Q_{66} &= G_{12} \end{aligned} \tag{18}$$

We have shown in this section the transformation equations of Q_{ij} in terms of multiple angles, and the meaning of the coefficients U_i.

SECTION III

TRANSFORMATION OF A, B, D MATRICES

If a laminated composite consists of constituent layers of the same orthotropic material ($U_6 = U_7 = 0$) with arbitrary lamina orientations and thicknesses, the elastic moduli of the laminated composite, A_{ij}, B_{ij}, and D_{ij}, can be expressed from Equation 9, for example, by

$$\left[A_{11}, B_{11}, D_{11}\right] = \int Q_{11}\left[1, z, z^2\right] dz \tag{19}$$

where Q_{11} is a function of z, i.e., it varies from layer to layer because of the varying lamina orientations. From Table I (where Q_{11} in Equation 19 is actually Q_{11}'):

$$Q_{11}' = U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta \tag{20}$$

The transformation Equation 20 consists of one constant plus two cyclic terms. If the same material is used in a laminated composite, say, boron-epoxy composite, U_1, U_2, and U_3 remain constant for all the layers and Equation 19 can be expanded in terms of multiple angles as follows:

$$\begin{aligned}\left[A_{11}, B_{11}, D_{11}\right] &= \int \left(U_1\left[1, z, z^2\right] + U_2 \cos 2\theta \left[1, z, z^2\right]\right. \\ &\quad \left. + U_3 \cos 4\theta \left[1, z, z^2\right]\right) dz \\ &= U_1\left[h, 0, h^3/12\right] + U_2 \int \cos 2\theta \left[1, z, z^2\right] dz \\ &\quad + U_3 \int \cos 4\theta \left[1, z, z^2\right] dz\end{aligned} \tag{21}$$

The same derivation can be applied to the other components of A_{ij}, B_{ij}, and D_{ij}, and the final relations are summarized in Table II.

TABLE II

A, B, D MATRICES IN TERMS OF LAMINA PROPERTIES

	$V_0[A,B,D]$	$V_1[A,B,D]$	$V_2[A,B,D]$	$V_3[A,B,D]$	$V_4[A,B,D]$
$[A_{11}, B_{11}, D_{11}]$	U_1	U_2	0	U_3	0
$[A_{22}, B_{22}, D_{22}]$	U_1	$-U_2$	0	U_3	0
$[A_{12}, B_{12}, D_{12}]$	U_4	0	0	$-U_3$	0
$[A_{66}, B_{66}, D_{66}]$	U_5	0	0	$-U_3$	0
$2[A_{16}, B_{16}, D_{16}]$	0	0	$-U_2$	0	$-2U_3$
$2[A_{26}, B_{26}, D_{26}]$	0	0	$-U_2$	0	$2U_3$

where the U_i are the same as those in Equation 18, and the $V_{i[A,B,D]}$ are defined as follows:

$$V_0[A,B,D] = [h, 0, h^3/12] \tag{22}$$

$$V_1[A,B,D] = \int \cos 2\theta \, [1, z, z^2] \, dz$$

$$V_2[A,B,D] = \int \sin 2\theta \, [1, z, z^2] \, dz$$

$$V_3[A,B,D] = \int \cos 4\theta \, [1, z, z^2] \, dz \tag{23}$$

$$V_4[A,B,D] = \int \sin 4\theta \, [1, z, z^2] \, dz$$

Since the constituent layers are assumed to be (macroscopically) homogeneous, the integrals above can be replaced by the following summations:

$$V_{iA} = \sum_{k=1}^{n} W_k (h_{k+1} - h_k)$$

$$V_{iB} = \frac{1}{2} \sum_{k=1}^{n} W_k (h_{k+1}^2 - h_k^2) \tag{24}$$

$$V_{iD} = \frac{1}{3} \sum_{k=1}^{n} W_k (h_{k+1}^3 - h_k^3)$$

such that, when

$$i = 1, \quad W_k = \cos 2\theta_k$$
$$= 2, \quad = \sin 2\theta_k$$
$$= 3, \quad = \cos 4\theta_k$$
$$= 4, \quad = \sin 4\theta_k$$

where k is the index of summation and n, the number of layers. Table II is not a transformation relation as in Table I, although the appearance is very similar. Table II is an expression of Equation 9 in terms of multiple angles and is valid for a laminated composite consisting of layers of the same material, otherwise the U's cannot be taken out of the integral signs. The purpose of expressing A_{ij}, B_{ij}, and D_{ij} in this format is to aid the understanding of laminated composites which may not be as apparent by use of Equation 9. The derivation for the case of an anisotropic material (U_6, $U_7 \neq 0$) can be carried out in a similar fashion.

The transformation equations of A_{ij}, B_{ij}, and D_{ij} can be derived by using the expressions in Table II. For example, A'_{11} can be obtained by rotating the entire laminated composite through an angle ϕ. This is accomplished by substituting $(\theta - \phi)$ for θ. Thus

$$A'_{11} = U_1 h + U_2 \int \cos 2(\theta - \phi)\, dz + U_3 \int \cos 4(\theta - \phi)\, dz \tag{25}$$

Since ϕ is constant for the entire laminated composite, thus, independent of z we get

$$\begin{aligned} A'_{11} &= U_1 h + U_2 \cos 2\phi \int \cos 2\theta\, dz + U_2 \sin 2\phi \int \sin 2\theta\, dz \\ &\quad + U_3 \cos 4\phi \int \cos 4\theta\, dz + U_3 \sin 4\phi \int \sin 4\theta\, dz \\ &= U_1 h + U_2 V_{1A} \cos 2\phi + U_2 V_{2A} \sin 2\phi \\ &\quad + U_3 V_{3A} \cos 4\phi + U_3 V_{4A} \sin 4\phi \end{aligned} \tag{26}$$

where $V_{1A}, \ldots V_{4A}$ represent the integrals defined in Equation 23 or the summations in Equation 24, and the subscript A signifies that a component of A_{ij} is being evaluated. Similar results can be obtained for the other components of A'_{ij}. The final transformation equations for A_{ij} can be shown in tabular form (Table III).

TABLE III

TRANSFORMATION EQUATIONS OF A_{ij}

	Constant	cos 2 ϕ	sin 2 ϕ	cos 4 ϕ	sin 4 ϕ
A'_{11}	$U_1 V_{0A}$	$U_2 V_{1A}$	$U_2 V_{2A}$	$U_3 V_{3A}$	$U_3 V_{4A}$
A'_{22}	$U_1 V_{0A}$	$-U_2 V_{1A}$	$-U_2 V_{2A}$	$U_3 V_{3A}$	$U_3 V_{4A}$
A'_{12}	$U_4 V_{0A}$	0	0	$-U_3 V_{3A}$	$-U_3 V_{4A}$
A'_{66}	$U_5 V_{0A}$	0	0	$-U_3 V_{3A}$	$-U_3 V_{4A}$
$2A'_{16}$	0	$U_2 V_{2A}$	$-U_2 V_{1A}$	$2U_3 V_{4A}$	$-2U_3 V_{3A}$
$2A'_{26}$	0	$U_2 V_{2A}$	$-U_2 V_{1A}$	$-2U_3 V_{4A}$	$2U_3 V_{3A}$

The transformation equations for B_{ij} and D_{ij} are the same as those shown in Table III except the V_{iA} must be replaced by V_{iB} and V_{iD}, respectively, where i = 0, 1, 2, 3, 4. Comparing Tables I and III, in conjunction with Table II, the corresponding transformation relations are identical. U_6 and U_7 do not appear in Table III because we are investigating the case of Q_{ij} being orthotropic. Thus A_{ij} transforms the same as Q_{ij}. Similarly, it can be shown that B_{ij} and D_{ij} also transform like Q_{ij}.* The transformation is needed for establishing the material symmetries like orthotropy, isotropy, etc., and the invariants of this transformation. From Tables II and III and Equation 14, the following invariants exist:

$$P_1 = A_{11} + A_{22} + 2A_{12} = L_1 h = (Q_{11} + Q_{22} + 2Q_{12}) h$$

$$P_2 = A_{66} - A_{12} = L_2 h = (Q_{66} - Q_{12}) h \qquad (27)$$

*These relations can be shown in general by appropriate integrations of the tensor transformation equation of the elastic stiffness tensor.

Similarly, invariants for B_{ij} and D_{ij} are:

$$
\begin{aligned}
P_3 &= B_{11} + B_{22} + 2B_{12} = 0 \\
P_4 &= B_{66} - B_{12} = 0 \\
P_5 &= D_{11} + D_{22} + 2D_{12} = L_1 h^3/12 = P_1 h^2/12 \\
P_6 &= D_{66} - D_{12} = L_2 h^3/12 = P_2 h^2/12
\end{aligned} \tag{28}
$$

There are several features of the invariants above.

(a) The invariants of the A and D matrices are the same as those for the Q matrix except for correction factors involving the thickness h.

(b) The invariants of the A, B, and D matrices impose definite limits on the variability of their components. If A_{11} and A_{22} are selected to meet certain loading requirements, we no longer have any freedom in specifying A_{12} and A_{66} because of invariants P_1 and P_2. A similar statement can be made about the D matrix.

(c) When $B_{11} + B_{22} = 0$, which occurs in a cross-ply composite (Reference 3), we know immediately from invariants P_3 and P_4 that

$$B_{12} = B_{66} = 0.$$

Thus, in a lamina optimization procedure of a given material, say, a boron-epoxy composite, bounds exist on the range of variability of the elastic properties. As shown in Table II, each of the six independent components of the A, B, D matrices is governed by a constant term, which is not affected by lamina orientation, and variable terms expressed by $V_i[A, B, D]$ in Equations 23 and 24.

SECTION IV

SPECIAL PROPERTIES OF LAMINATED COMPOSITES

We will examine a number of special laminated composites and hope to shed light on the nature of $V_i[A, B, D]$, in this section. Since the limits of integration are $\pm h/2$, integration of an odd function (antisymmetric function with respect to z = 0) will be zero; that of an even function, not zero. Let us examine the following cases:

(a) If θ is an odd function of z, which may be represented by a 2-layer angle-ply with $\pm\theta$ orientation shown in Figure 1a, the following integrands are odd:

$$\cos p\theta\left[z\right] \quad \sin p\theta\left[1, z^2\right].$$

The following integrands are even:

$$\cos p\theta\left[1, z^2\right] \quad \sin p\theta\left[z\right]$$

where p = 2 or 4.

Thus, the following integrals among those in Equation 23 vanish:

$$V_{1B} = V_{3B} = V_{2A} = V_{2D} = V_{4A} = V_{4D} = 0. \tag{29}$$

From Table II:

$$\begin{aligned} A_{16} &= A_{26} = 0 \\ B_{11} &= B_{22} = B_{12} = B_{66} = 0 \\ D_{16} &= D_{26} = 0 \end{aligned} \tag{30}$$

Hence, A_{ij} and D_{ij} are orthotropic.

(b) If θ is an even function of z, which is known as a symmetric laminate and may be represented by Figure 1b, the following integrands are odd:

$$\cos p\theta\left[z\right] \qquad \sin p\theta\left[z\right]$$

The following integrands are even:

$$\cos p\theta\left[1, z^2\right] \quad \sin p\theta\left[1, z^2\right]$$

Thus, the following integrals among those in Equation 23 vanish:

$$V_{1B} = V_{2B} = V_{3B} = V_{4B} = 0 \qquad (31)$$

From Table II:

$$B_{ij} = 0 \qquad (32)$$

which means that there is no coupling between bending and extension in the laminated composite. Furthermore, A_{ij} and D_{ij} are, in general, anisotropic.

(c) Let θ be a random function of z, i.e., layers are randomly oriented, as shown in Figure 1c, and define $\overline{V_i}$ as the space average of V_i (Reference 5):

$$\begin{aligned} \overline{V}_i &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} V_i \, d\theta \\ &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \int_{-h/2}^{h/2} \begin{pmatrix} \cos p\theta \\ \sin p\theta \end{pmatrix} \left[1, z, z^2\right] dz \, d\theta \end{aligned} \qquad (33)$$

where p is even. We have dropped the second subscript in $V_{i\,[A,B,D]}$ since it is immaterial here. Interchanging the order of integration, we get

$$\overline{V}_i = \frac{1}{\pi} \int_{-h/2}^{h/2} \int_{-\pi/2}^{\pi/2} \begin{pmatrix} \cos p\theta \\ \sin p\theta \end{pmatrix} d\theta \left[1, z, z^2\right] dz$$

$$= 0 \qquad (34)$$

Thus, for random orientation of the constituent layers, all the $\overline{V_i}$ with the exception of the constant terms in Table II will vanish. The laminated composite becomes isotropic, since

$$\begin{aligned} A_{11} &= A_{22} = U_1 h \\ A_{12} &= U_4 h \\ A_{66} &= U_5 h \\ A_{16} &= A_{26} = 0 \end{aligned} \qquad (35)$$

and from the above and Equation 15,

$$A_{11} - A_{12} = 2A_{66} \qquad (36)$$

which satisfies the condition of isotropy of A_{ij}. The isotropy of A_{ij} only implies that the moduli of a laminated composite are isotropic. The stress distribution, however, is not the same as that in an isotropic body. Similarly it can be shown that

$$B_{ij} = 0 \qquad D_{ij} = A_{ij} h^2/12 \tag{37}$$

Thus D_{ij} is also isotropic. The laminated composite satisfies the condition of homogeneity as well, although the stress distribution is different from that in a homogeneous material.

(d) If a laminated composite has n equal layers (n > 2) and the orientation angles of the layers are at increments of π/n, the integral V_{1A} may be expressed as

$$V_{1A} = (\cos 2\pi/n + \cos 4\pi/n + \cdots \cos 2\pi) h/n \tag{38}$$

From Pierce's table (4th Edition), Formula (639):

$$\cos x + \cos 2x \cdots + \cos nx = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x} - \frac{1}{2} \tag{39}$$

For $x = 2\pi/n$

$$V_{1A} = 1/2 - 1/2 = 0$$

Similarly, from Pierce, Formula (637),

$$\sin x + \sin 2x \cdots + \sin nx = \frac{\sin\frac{1+n}{2}x \cdot \sin\frac{n}{2}x}{\sin\frac{x}{2}} \tag{40}$$

For $x = 2\pi/n$

$$V_{3A} = 0$$

Using Equations 39 and 40, we can show for $x = 4\pi/n$:

$$V_{2A} = V_{4A} = 0$$

Since V_{iA} $(i \neq 0)$ vanish for this type of laminated composite, A_{ij} is isotropic. The same relations as those in Equations 35 and 36 are obtained. This, of course, is the well-known result for in-plane quasi-isotropic composites, where the lamina orientations are (-60) - 0 - 60, (-90) - (-45) - 0 - 45, etc., shown in Figures 1d and 1e. B_{ij} and D_{ij} can be made quasi-isotropic by more complex stacking sequences than that for A_{ij}.

Finally, the area under the A'_{ij} versus ϕ curve from $\phi = 0$ to $\phi = 2\pi$ can be obtained by the integration of the transformation equations listed in Table III. Since,

$$\int_0^{2\pi} \left(\begin{matrix} \cos p\phi \\ \sin p\phi \end{matrix} \right) d\phi = 0 \qquad (41)$$

where p is an integer, only the constant terms remain. Thus the areas under the A'_{ij} are constant and the average numerical values are the isotropic constants in Equation 35 for the randomly oriented lamina composites and those quasi-isotropic laminates described in the previous subsection. This leads to the conclusion that the invariant properties of constants U_1 and U_5, U_4 being dependent on U_1 and U_5, may constitute a measure of the performance of orthotropic materials and laminates. Lamina orientation variations only change the shape of the A'_{ij} curve as ϕ varies but the area under the curve remains constant. We can also conclude that the area under the B'_{ij} curve is zero and that under D'_{ij}, constant.

SECTION V

ISOTROPIC CONSTANTS

We have shown that the elastic properties of a unidirectional composite are strongly influenced by two independent invariants,

$$U_1 = (3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})/8 \quad (42)$$

$$U_5 = (Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66})/8 \quad (43)$$

For laminated composites, the same invariants exist, except that corrections for thickness of h and $h^3/12$ must be applied for the A and D matrices, respectively. The invariants for B_{ij} are identically zero, as shown in Equation 28.

If the material is isotropic, the resulting relations shown in Equation 17 are

$$U_1 = Q_{11}, \quad U_5 = Q_{66} \quad (44)$$

Because U_1 and U_5 reduce to the stiffness and shear rigidity of an isotropic material, we shall designate U_1 and U_5 defined in Equations 42 and 43 as the isotropic stiffness and isotropic shear rigidity, respectively. These isotropic properties, which are specific combinations of orthotropic properties, represent a realistic measure of the minimum stiffness capability of composite materials, which can be compared directly with isotropic materials as well as other orthotropic materials. This measure of stiffness is different from the common practice of comparing the longitudinal stiffness Q_{11} with isotropic materials. Although Q_{11} for many modern composites can be several times higher than lightweight metals on the weight basis, this is not a fair comparison because the weakness of most composites in transverse stiffness and shear rigidity is ignored.

In addition to affording a basis of comparison with isotropic materials, the proposed use of invariant or isotropic properties may lead to a better understanding of the variability of lamina optimization of composite materials. If we start initially with a unidirectional composite, for which

$$A_{ij} = Q_{ij}h$$

any change in fiber orientation of some layers within the same composite will change A_{ij} according to Table II. These changes are governed by the integrals V_1, V_2, V_3 and V_4, while V_0 remains invariant. The V's dictate the magnitude of the variability in the elastic properties of a laminated composite and the variation oscillates above or below the isotropic constants. Since the absolute value of sine and cosine functions are bounded between 0 and 1, the variability of the V's are also bounded.

The concept of invariant properties may simplify the lamina optimization process. Structural optimization can begin with the isotropic constants. They should represent the minimum stiffness of composite materials. Any lamina design that falls below the performance of that based on isotropic constants should be automatically rejected.

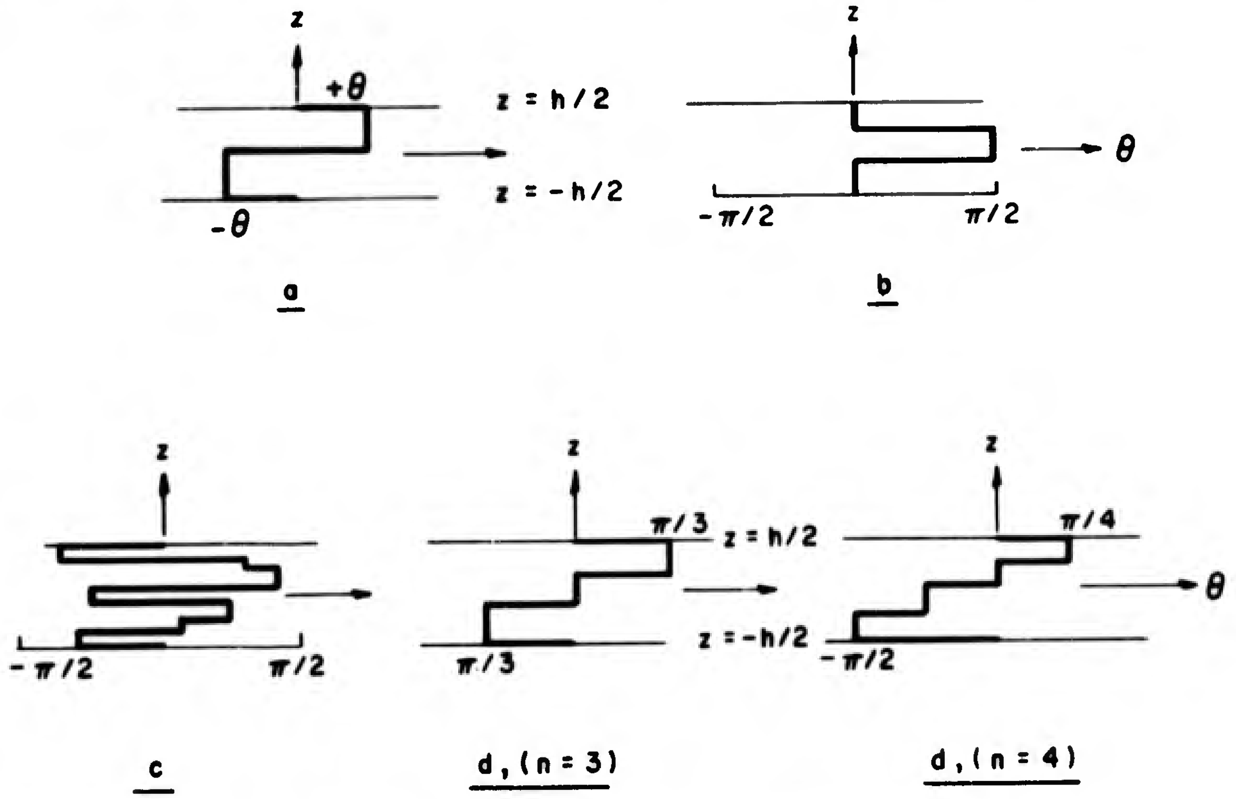
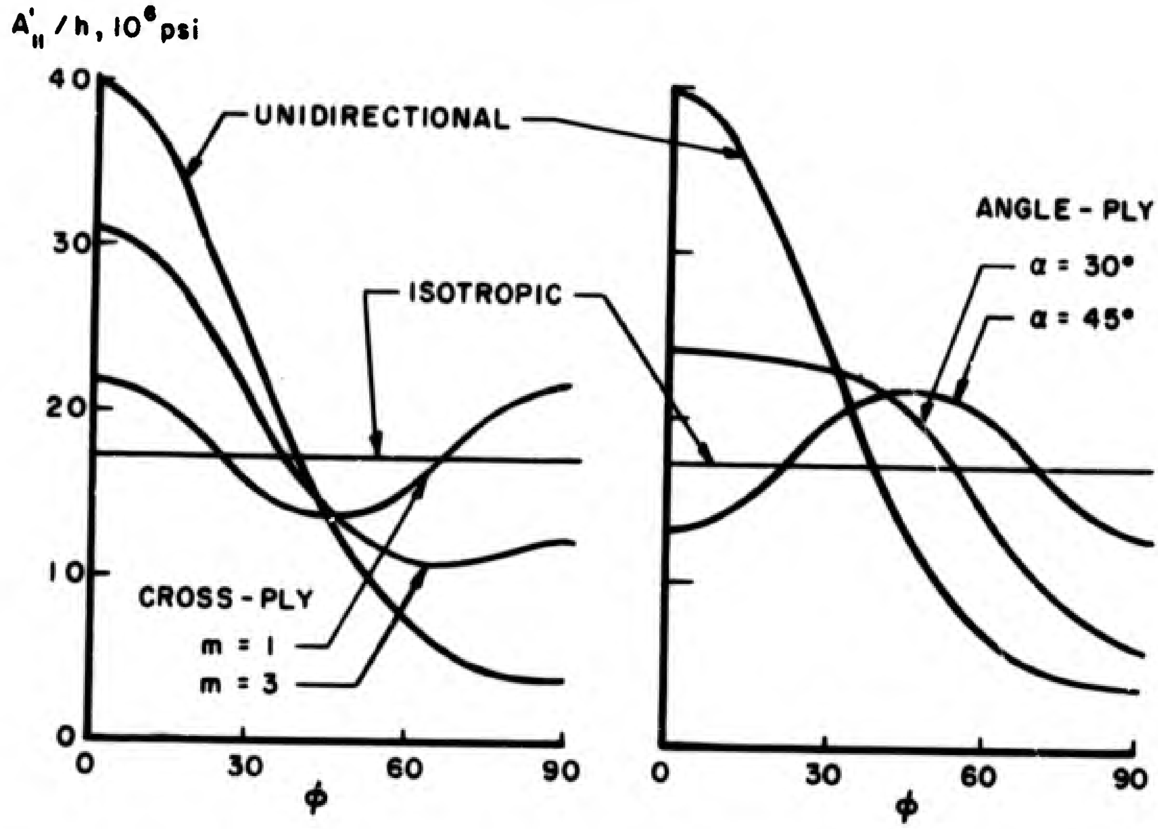


Figure 1. Examples of Lamina Orientations

Figure 2 shows the variation of A'_{11} for various boron-epoxy composites, using the following data:

$$
\begin{aligned}
Q_{11} &= 40 \times 10^6 \text{ psi} \\
Q_{22} &= 4 \times 10^6 \text{ psi} \\
Q_{12} &= 1.0 \times 10^6 \text{ psi} \\
Q_{66} &= 1.5 \times 10^6 \text{ psi}
\end{aligned}
\qquad (45)
$$



The unidirectional and isotropic composites are shown in both diagrams. On the left diagram, two cross-ply composites with cross-ply ratio m equal to 1 and 3 are shown. Cross-ply ratio is the ratio of the thickness of the 0-degree to 90-degree layers. On the right diagram, two angle-ply composites with helical angle α equal to 30 degrees and 45 degrees are also shown. Angle-ply composites consist of equal numbers of layers oriented at $+\alpha$ and $-\alpha$. These diagrams illustrate that the areas under all the A'_{11} curves are the same. If a cross-ply with m = 1 is combined with an angle-ply with α = 45 degrees, the resulting composite is isotropic. This agrees with the conclusion of the previous section and is shown in Figure 1e. All the cross-ply composites have the same value at ϕ = 45 degrees. This can be shown from the transformation equation. Finally, when the number of lamina orientations increases, the resulting laminated composite will approach the isotropic state. Thus, depending on the nature of the design data, a more effective lamina optimization program may be achieved by beginning with the isotropic laminate, rather than the unidirectional composite.

It may be useful to determine approximately the numerical values for the invariant properties represented by Equations 42 and 43. We will define

$$U_1 = \bar{E}, \qquad U_5 = \bar{G} \tag{46}$$

For the highly orthotropic composites like glass-epoxy and boron-epoxy composites

$$Q_{11} \cong E_{11}, \quad Q_{22} \cong E_{22} \tag{47}$$

because the minor Poisson ratio ν_{21} is usually less than 0.1. If ν_{12} is 0.3, $1 - \nu_{12}\nu_{21} > 0.97$. The approximation of Equation 47 introduces an error less than 3%. From elasticity solutions by Adams and Doner on longitudinal shear (Reference 7) and transverse loading (Reference 8) of a unidirectional composite, we have

$$Q_{66}/G_m = G_{12}/G_m = F_1(G_f/G_m, v_f) \tag{48}$$

$$Q_{22}/E_m = E_{22}/E_m = F_2(E_f/E_m, v_f) \tag{49}$$

Although E_{22} also depends upon the Poisson ratios of the constituents, this dependence is not considered in Equations 48 and 49 since representative values of the Poisson ratio for typical plastic matrix composites were assumed in Reference 8. We also assume that small changes in constituent Poisson ratios do not greatly affect the transverse modulus E_{22}. Let

$$F_1 = \beta F_2$$

where β is a function of constituent stiffness ratio and fiber volume fraction. By comparing Figure 5 of Reference 7 and Figure 4 of Reference 8, it can be seen that

$$\frac{3}{4} < \beta < 1 \tag{50}$$

for a fiber volume of 70% or less. Since

$$G_m = E_m/2(1+\nu_m) \tag{51}$$

we obtain from Equations 48 and 49, for $\nu_m = 0.33$,

$$Q_{66} = \frac{3}{8}\beta E_{22} \tag{52}$$

Hence, if we substitute

$$\begin{aligned} Q_{11} &= E_{11}, \quad Q_{22} = E_{22} \\ Q_{12} &= \nu_{12} Q_{22} = 0.25 E_{22} \\ Q_{66} &= G_{12} = \frac{3}{8}\beta E_{22} \end{aligned} \tag{53}$$

into Equations 42 and 43, we obtain approximately

$$\begin{aligned} \bar{E} &= \frac{3}{8}E_{11} + \frac{1}{16}(7+3\beta)E_{22} \\ \bar{G} &= \frac{1}{8}E_{11} + \frac{1}{16}(1+3\beta)E_{22} \end{aligned} \tag{54}$$

Since $E_{22} < E_{11}$ for fiber-reinforced composites, the error introduced by putting $\beta = 1$ in Equation 54 is quite small. For example, if $\frac{E_{11}}{E_{22}} = 10$, the maximum possible errors are 3.3 and 0.9% in $\overline{G}$ and $\overline{E}$, respectively, owing to variations in β. Setting $\beta = 1$ then, yields

$$\text{Isotropic stiffness} = \overline{E} = \frac{3}{8} E_{11} + \frac{5}{8} E_{22} \qquad (55)$$

$$\text{Isotropic shear rigidity} = \overline{G} = \frac{1}{8} E_{11} + \frac{1}{4} E_{22} \qquad (56)$$

These approximate equations are simple to use and give reasonable values to represent the invariant properties.

Cox (Reference 6) derived isotropic constants for randomly oriented fiberous composites as

$$\overline{E} = E_{11}/3, \quad \overline{G} = E_{11}/8 \qquad (57)$$

These values are lower than those of Equations 55 and 56. Loewenstein (Reference 9) also showed the 3/8 factor for in-plane random orientation (the transverse stiffness is taken to be zero). Bishop (Reference 10) also derived a theory which has results similar to that reported by Loewenstein (Reference 9). Both References 9 and 10 may be considered as having

$$Q_{22} = Q_{12} = Q_{66} = 0 \qquad (58)$$

The conditions implied by this equation, however, are not reasonable for modern fiber-reinforced composites. The transverse and shear moduli are significant quantities in determining the elastic behavior of composite materials.

An estimate of the performance of fiber-reinforced composites is shown in terms of invariant properties in Figure 3. The normalized $\overline{E}$ is derived from

$$\overline{E}/E_m = \frac{3}{8}(E_{11}/E_m) + \frac{5}{8}(E_{22}/E_m)$$

$$= \frac{3}{8}\left[(1-v_f) + v_f E_f/E_m\right] + \frac{5}{8} F_2 \qquad (59)$$

where the rule of mixtures equation is used:

$$E_{11} = (1-v_f)E_m + v_f E_f \qquad (60)$$

and F_2 is expressed in Equation 49, the numerical values of which are obtained from Reference 8. From Equation 56, using $\beta = 1$ as discussed earlier,

$$\overline{G}/G_m = \frac{2.66}{8}\left[(1-v_f) + v_f E_f/E_m\right] + \frac{5.32}{8} F_2 \qquad (61)$$

where Equation 51 is used with ν_m = 0.33. Comparing Equations 59 and 61, we notice that

$$\bar{E}/E_m \cong \bar{G}/G_m \qquad (62)$$

Figure 3 shows the normalized $\bar{E}$ and $\bar{G}$ for fiber-reinforced composites with v_f = 70 and 40%. For convenience, absolute units for $\bar{E}$ are also shown for boron-aluminum, glass-epoxy, and boron-epoxy composites. Figure 3 represents the minimum capabilities of the composite materials; the advantage of designed anisotropy to meet a specific loading condition has not been claimed.

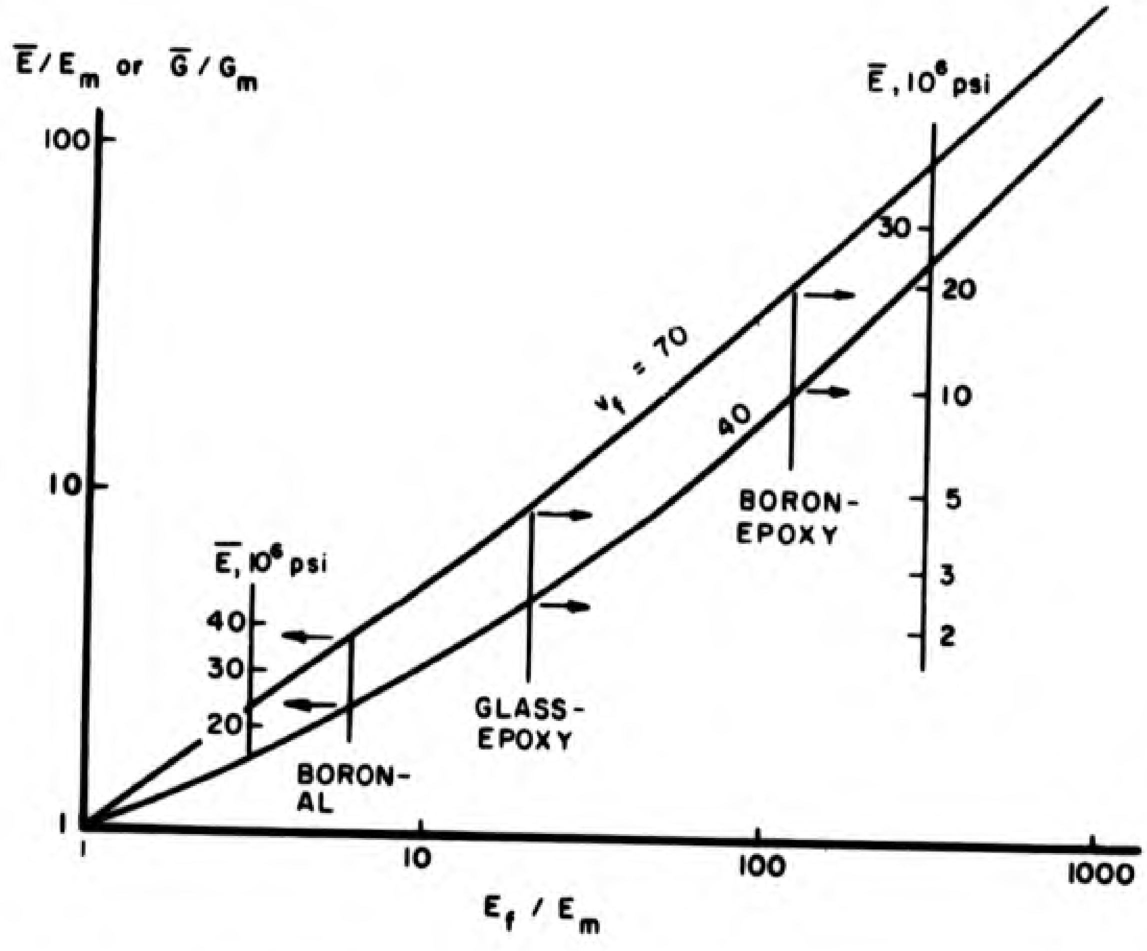


Figure 3. Isotropic Constants of Fiber-Reinforced Composites

SECTION VI

SUMMARY

We have shown that the transformation equations of tensors can be expressed in multiple angles instead of the usual powers of sines and cosines. In the multiple angle representation, the transformation properties consist of invariant terms, which correspond to the isotropic constants, and cyclic terms, which control the variation and directionality of properties due to anisotropy. The transformation equations for the moduli of two-dimensional layers (Q_{ij}) and laminated composites (A_{ij}, B_{ij}, and D_{ij}) can be readily derived.

The elastic properties of laminated composites as functions of lamina orientation are shown in Table II. The components of A_{ij}, B_{ij}, and D_{ij} are governed by invariant terms, plus variable terms in terms of integrals V_i. It is proposed that isotropic properties for anisotropic materials be used as a measure of the minimum stiffness capability. They may be considered intrinsic properties of the material because they are independent of the lamina orientations. Direct comparisons of the stiffness represented by $\bar{E}$ and $\bar{G}$ with isotropic materials appear to be more realistic than the use of the longitudinal stiffness of unidirectional composites. Approximate expressions for these isotropic constants are shown in Equations 55 and 56 and their numerical results in Figure 3. The results may be helpful in systems application of composite materials. The relative merits of controllable variables like E_f/E_m and v_f can be determined directly from Figure 3 which should be of value to materials engineers.

Finally, the basis of lamina optimization may be more easily carried out and better understood by the multiple-angle relations than the conventional treatment. The degree of variability can be determined from the values of the integrals V_i. If anisotropy is to be beneficial for a given loading condition, the performance of the composite should in all cases exceed that of the isotropic laminate. Thus optimization can begin with the isotropic constants. The isotropic constants, the integrals V_i and the invariants P_i should be considered as additional constraints to lamina optimization procedures. For practical design, the number of lamina orientations in a laminated composite may be kept to, say, no more than four orientations. The variation of the properties may be more effectively controlled through the lamina thickness than the orientation. The reduction in lamina orientations may introduce immediate simplification in structural analysis, design procedures, and automated fabrication techniques of laminated composites. The present concept may lead to an optimum design based on strain energy from which the advantage of anisotropy in a composite material may be readily established for specific load conditions. A similar approach to the problem of strength seems possible.

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Invariant properties of the elastic coefficient matrices of laminated composite plates are presented. The use of these invariants in materials evaluation and design optimization is discussed. Simple formulas, based upon micromechanics results, are derived for the invariants in terms of constituent material properties.

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