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SHOCK SPECTRA

by

Paul G. Hershall

January 1968

U.S. ARMY MATERIEL COMMAND
HARRY DIAMOND LABORATORIES
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ABSTRACT

Numerical solutions in graphical form are presented for the shock spectra of a one-degree-of-freedom system subjected to some idealized forcing functions. Elementary procedures are outlined for developing the solutions. Analog and digital computer programs are submitted where applicable.

1. INTRODUCTION

In some applications of engineering, knowledge of the maximum displacement of a driven member of a system is desired as a function of a characteristic frequency of the system. This spectrum is called a shock spectrum (ref 1, 2) when transient-type forces, such as in an explosion, are applied to the system.

The system of interest in this report is a one-degree-of-freedom (ODOF) undamped system with a linear restoring force, i.e., a simple mass-spring system. The interest lies in the effect of different driving functions. The study is further limited to a system initially at rest. Ground accelerations and velocities are not considered.

It is not irrelevant whether the maximum displacement occurs in the positive or the negative direction. A structural member generally has different values of Young's modulus in tension and compression. In the situation chosen, the positive maximum is usually larger, but the exceptions will be noted.

Two chief types of forcing functions are considered:

(1) A half-sine pulse with superimposed Dirac impulses and

(2) A $t^2 \exp(-\beta t)$ function.

Pulse (1) will have impulses of two types:

(1a) five equispaced, equistrength, positive Dirac impulses with the first at time zero and

(1b) the preceding with five additional negative impulses symmetrically interspersed (alternating impulses).

A Dirac impulse of strength F_1 will be one for which

$$\int_{-\infty}^{+\infty} F_1\, \delta(t)\, dt = F_1$$

The scope of this report is limited to presenting the mathematical details used in evaluating particular shock spectra. Their

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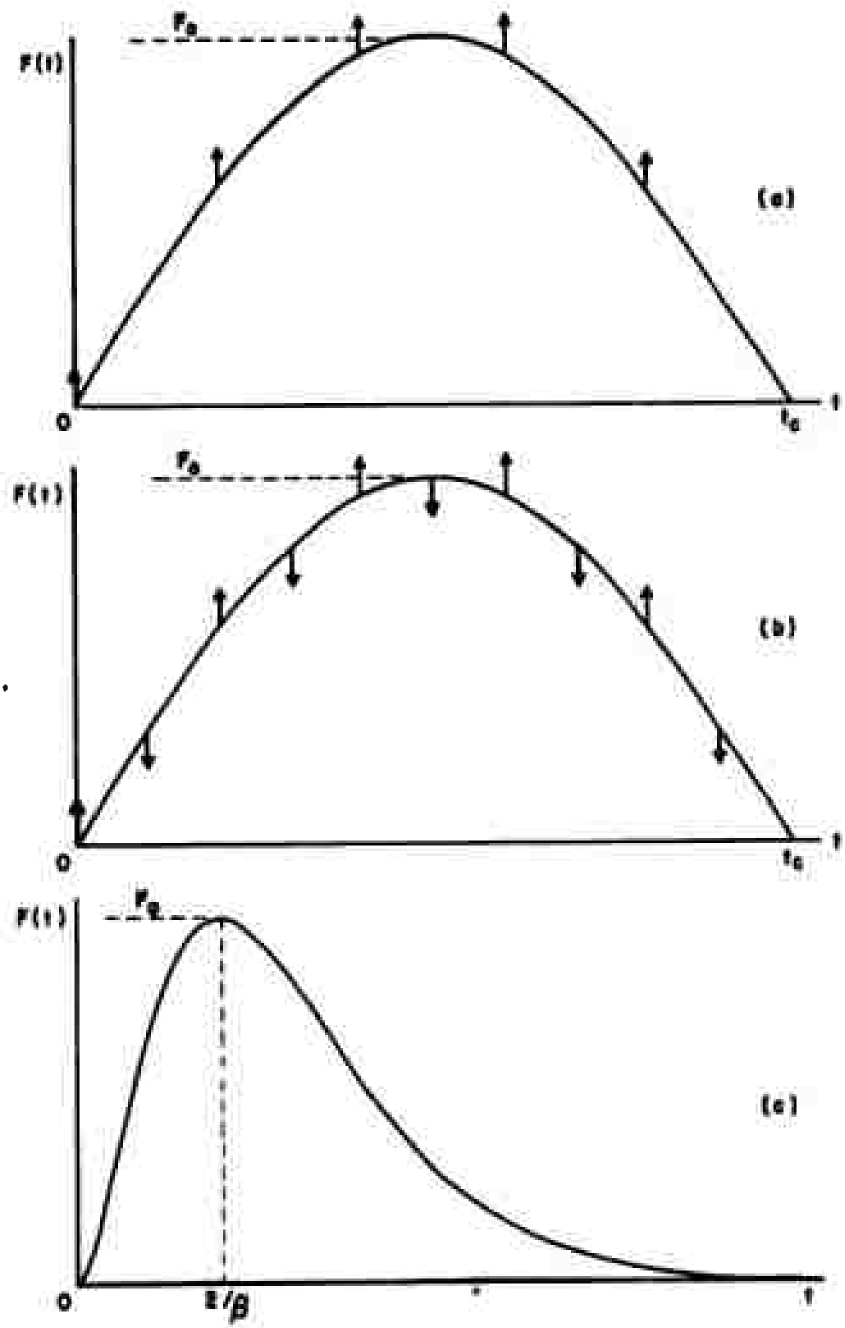

Figure 1. Forcing functions.

relevance will not be pursued here but will be treated in a later report (ref 3). However, one may posit the forcing function having Dirac impulses superimposed on a half-sine pulse as a possible idealization of the high-frequency oscillations often observed on shock pulses (ref 4). Moreover, the penetration resistance of certain projectiles striking steel plates (ref 5) is nearly of the form $t^2 \exp(-\beta t)$.

In section 2, the procedures used to evaluate the shock spectra are described, and general equations are developed. In sections 3 through 6 more detailed information is presented; in section 7 the results for several cases are given.

2. GENERAL DEVELOPMENT OF SOLUTION

We are concerned with ODOF systems, i.e., systems described by the equation

$$mx'' + kx = F(t) \tag{1}$$

The rest conditions assumed are

$$x(0) = 0, \quad x'(0) = 0 \tag{2}$$

The forcing functions F(t) considered are illustrated in figure 1.

The deletion of ground accelerations in our ODOF system is not a serious specialization. To clarify this, consider the generalization given in figure 2. Let x be defined by y-z, where y and z are changes in position experienced by the mass m and the ground, respectively, in relation to their rest or reference positions. Then the equation of motion of the system may be written

$$mx'' + kx = F(t) - mz'' \tag{3}$$

A ground force mz'' is thus equivalent to a force F(t) of opposite sign applied to mass m.

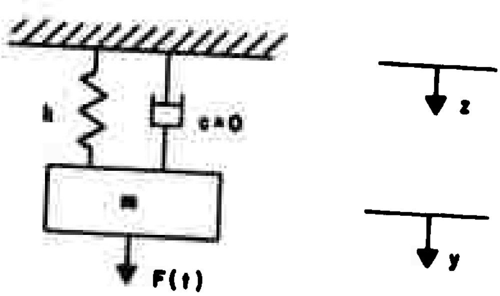

Figure 2. Physical model of generalized ODOF system.

Normalization of the variables in (1) will have two advantages: (1) the equations to be solved are easier to manipulate, having a minimum number of essential variables and parameters; (2) the solutions represent many cases resulting in economy of computation and representation. The first step is to normalize these variables.

One may define

$$F(t) = F_0 f(t) \tag{4}$$

where F_0 is the peak value of the forcing function excluding impulse components (fig. 1). If we further define

$$\omega_N^2 = k/m, \quad x_s = F_0/k, \quad u = x/x_s \tag{5}$$

equation (1) reduces to

$$u''/\omega_N^2 + u = f(t) \tag{6}$$

Derivatives so far have been with respect to t. In (6), ω_N is the natural frequency of the system, x_s the static displacement, and u the response factor. We have yet to normalize time; one may, in general, define a unit of time t_c and let

$$\tau = t/t_c \tag{7}$$

Examining figures 1a and 1b, we see that it is most natural to define

$$\omega_F t_c = \pi \tag{8}$$

where ω_F is a frequency characteristic of the driving function. To allow (8) to represent the situation in figure 1c, we may allow

$$\beta = \omega_F/\pi = 1/t_c \tag{9}$$

The characteristic forcing frequency is thus $\pi\beta$ and the peak value occurs at $2t_c$. Furthermore, we normalize the natural frequency as

$$\gamma = \omega_N/\omega_F \tag{10}$$

With these relations, (6) becomes

$$u''(\tau)/\gamma^2\pi^2 + u(\tau) = f(\tau) \tag{11}$$

Without ambiguity, derivatives of normalized variables are with respect to τ. In addition, some liberty has been taken with the symbols u and f; as functions, they are not the same functions of τ that they are of t. The symbols in (11) will, with no real ambiguity, represent the new functions. See appendix D for a table of symbols.

The next step in the analysis is the solution of (11) for u. For the functions presented in figure 1, the solution may be carried out explicitly. There are several known ways of achieving this. The author has chosen the Laplace transform method, but any of the other standard methods are equally quick.

Finally, the maximum displacement is determined for each value of the normalized natural frequency γ. For some very simple functions f, the explicit solution u may be differentiated and u_M found as

$$u_M = u(\tau_M)$$

where τ_M is the solution of $u'(\tau) = 0$. This is not feasible in our cases because of the presence of multiple peaks in u. There appears to be no better recourse than to scan u digitally for a suitable number of values of τ, and find the maximax (the global maximum or supremum) or minimax (the supremum in the algebraically negative sense) by comparison. In reference 6, a general program is described to deal with experimental forcing functions. The choice to write an ad hoc program was not easy to make but the ease of finding the response functions analytically, the general clumsiness of large canned programs, and the virtue of added insight with the ad hoc program ruled in its favor.

Except for large values of γ, the number of values of τ required to scan the peaks are not unreasonable (say ~ 1000) to achieve four-to five-digit precision. The reason for this is that near any peak of u, $du/d\tau$ is very small, and any error in τ is compressed into a much smaller error in u. In view of the use made of the results, it does not seem worthwhile to improve the accuracy by use of local convergence algorithms.

The preceding remarks apply, in general, to finding u_M in the forced-vibration era. It is necessary, also, to find the u_M in the free-vibration era (residual spectrum) as for certain values of γ it may be larger than the u_M in the forced era. In the case of some transient-like functions (cf fig. 1c) no free-vibration era exists, strictly speaking. In those cases where it does exist, the spectrum may be simply derived from the forced era response factor u, when damping is absent. Thus, from consideration of constant energy, we must have

$$\frac{1}{2} kx^2(t_c) + \frac{1}{2} mx'^2(t_c) = \frac{1}{2} kx_M^2 \qquad (12)$$

where t is chosen to be t_c to be sure that all the energy has been added to the system by the external force. The total energy is then equated to the maximum potential energy in the free-vibration era. Normalization of (12) results in the expression

$$u_M = \sqrt{u^2(1) + \{u'(1)/\gamma\pi\}^2} \qquad (13)$$

which is used in obtaining results of this study. Sometimes the u_M for the forced era is not desired. In that case, the alternative method of obtaining the free era spectrum by means of the Fourier spectrum (ref 7) — related by a constant factor to the free spectrum — may be more direct.

The remaining sections give further mathematical details on the methods and results obtained for the forcing functions considered.

3. HALF-SINE-PULSE-TYPE FORCING FUNCTIONS

As a first specialization of equation (11), let us consider driving functions of the type given in figures 1a and 1b. Since we are dealing with a linear system, we may find the responses to the pure half-sine pulse and the delta function train separately, then add the two proportionately to obtain the total response. (We may not, however, add their separate spectra to find the total spectrum.)

Since we use (13) to obtain u_M directly in the free era, the response is needed only for $\tau \le 1$. The normalized half-sine pulse may therefore be written as

$$f_{HS}(\tau) = \sin \tau\pi \tag{14}$$

and the equation of motion reads

$$u''/\gamma^2\pi^2 + u = \sin \tau\pi \tag{15}$$

By standard methods we obtain from (15) the transfer function

$$U_{HS}(s) = \frac{\gamma^2\pi^3}{(s^2 + \pi^2)(s^2 + \gamma^2\pi^2)} \tag{16}$$

With the aid of a partial-fraction expansion, one obtains the desired inverse transform

$$u_{HS}(\tau) = \frac{\gamma}{1 - \gamma^2} (\sin \gamma\tau\pi - \gamma \sin \tau\pi) \tag{17}$$

Next, for the finite train of positive delta functions, we have

$$f_{D+}(\tau) = \delta(\tau) + \delta(\tau - 1/5) + \cdot \cdot \cdot + \delta(\tau - (m-1)/5) \tag{18}$$

whose transform is

$$F_{D+}(s) = 1 + e^{-s/5} + \cdot \cdot \cdot + e^{-(m-1)\, s/5} \tag{19}$$

The function f_{D+} represents the situation after the m^{th} impulse and before the $m+1^{th}$, where m = 1, 2, 3, 4, 5. The train of negative impulses is represented by

$$f_{D-}(\tau) = -f_{D+}(\tau - 1/10) \tag{20}$$

and has a transform

$$F_{D-}(s) = -e^{-s/10} F_{D+}(s) \tag{21}$$

It follows that the transfer functions for these cases — the left side of (15) being still valid — are

$$U_{D+}(s) = \{\gamma^2\pi^2/(s^2 + \gamma^2\pi^2)\} F_{D+}(s) \tag{22}$$

and

$$U_{D-}(s) = -e^{-s/10} U_{D+}(s) \tag{23}$$

whose inverse transforms become

$$u_{D+}(\tau) = -\gamma\pi\{\sin \gamma\tau\pi + \sin \gamma(\tau - 1/5)\pi + \cdot\cdot\cdot + \sin \gamma(\tau - (m_+ - 1/5)\pi\} \tag{24}$$

and

$$u_{D-}(\tau) = -\gamma\pi\{\sin\gamma(\tau - \tfrac{1}{10})\pi + \sin\gamma(\tau - \tfrac{3}{10})\pi + \cdots + \sin\gamma(\tau - (2m_- - \tfrac{1}{10})\pi)\} \quad (25)$$

$$\tau \geq \tfrac{1}{10}$$

where m_+ and m_- have the significance of m in (19) and also either $m_+ = m_-$ or else $m_+ = m_- + 1$ holds.

The summations over the sine terms in (24) or (25) may be combined by trigonometric identities such as

$$\sin\theta + \sin(\theta - \alpha) + \cdots \text{ m terms} = \sin\{\theta - (m-1)\alpha/2\}\sin(m\alpha/2)\csc(\alpha/2)$$

However, this will not be done; the number of terms in the sum is never greater than five, and the lumped term artificially introduces poles for certain values of γ which require separate handling in a digital program.

Thus the final expression for the forced response to the half-sine pulse reads

$$u = u_{HS} + R(u_{D+} + Lu_{D-}) \quad (26)$$

Here $R = F_1/F_0$, the ratio of the strength of a delta function to the amplitude of the pure half-sine pulse; L = 0 for the pulse of figure 1a, and L = 1 for that of 1b. The programming is discussed in section 5.

The specialization of equation (13) to the response discussed in this section represented by (26) yields for the free-vibration maximax

$$u_M = \sqrt{\{u_{HS}(1) + R[u_{D+}(1) + Lu_{D-}(1)]\}^2 + \{u'_{HS}(1) + R[u'_{D+}(1) + Lu'_{D-}(1)]\}^2/\gamma^2\pi^2} \quad (13a)$$

The substitution of (17), (24), (25), and their derivatives evaluated at $\tau = 1$ into (13a) was carried out in order to program the residual spectrum.

4. FORCING FUNCTION OF THE TYPE $t^2\exp(-\beta t)$

The t-dependent function may be written, C_0 being an arbitrary constant, as

$$F(t) = C_0 t^2 e^{-\beta t} \quad (27)$$

The peak value of F(t) is found to be

$$F_0 = 4C_0/\beta^2 e^2 \quad (28)$$

Consequently

$$f(t) = \frac{\beta^2 e^2}{4} t^2 e^{-\beta t} \quad (29)$$

and

$$f(\tau) = \frac{e^2}{4}\tau^2 e^{-\tau} \quad (29a)$$

when we use (7) and (9), which together imply that $\tau = \beta t$.

The usual liberties have been taken with the symbol f. It is convenient to rescale u and γ as

$$u = e^2 w/4, \quad \xi = \gamma\pi \tag{30}$$

Then equation (11) becomes

$$w''/\xi^2 + w = \tau^2 e^{-\tau} \tag{31}$$

Using the Laplace transform, one readily obtains

$$W(s) = \frac{2\xi^2}{(s^2+\xi^2)(s+1)^3} \tag{32}$$

One may proceed now by two different paths. The partial fraction expansion may be effected and the inverse transform obtained from the fragments; or the convolution theorem may be applied followed by an integration by parts.

4.1 Partial Fraction Method

Writing

$$\frac{1}{(s^2+\xi^2)(s+1)^3} \equiv \frac{As+B}{s^2+\xi^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)^3} \tag{33}$$

one obtains by standard methods (apx A)

$$A = \frac{\xi^2 - 3}{(1+\xi^2)^3} \tag{34a}$$

$$B = \frac{1 - 3\xi^2}{(1+\xi^2)^3} \tag{34b}$$

$$C = -A \tag{34c}$$

$$D = 2/(1+\xi^2)^2 \tag{34d}$$

and

$$E = 1/(1+\xi^2) \tag{34e}$$

The solution follows by substitution of these coefficients into the inverse transform obtained from (32) and (33):

$$w(\tau) = 2\xi^2\{A\cos\xi\tau + (B/\xi)\sin\xi\tau + (C + D\tau + E^2\tau^2/2)e^{-\tau}\} \tag{35}$$

4.2 Convolution Method

Applying the convolution theorem to equation (32), one obtains the solution to (31) in the form

$$w(\tau) = \xi\int_0^\tau v^2 e^{-v}\sin\xi(\tau - v)\,dv \tag{36}$$

as may be readily verified by direct substitution. This elegant solution is, however, decidedly inconvenient for computation. As a check on the partial fraction method, one may convert (36) into the form of (35). More details on this are furnished in appendix B. In any case, equation (B-13) developed there is seen to be equivalent to (35).

For reference, the explicit solution is given below as

$$w(\tau) = \frac{2\xi}{(1+\xi^2)^3}\{[(1-3\xi^2)\sin\xi\tau - \xi(3-\xi^2)\cos\xi\tau] + \xi[(3-\xi^2) + 2(1+\xi^2)\tau + \tfrac{1}{2}(1+\xi^2)^2\tau^2]e^{-\tau}\} \quad (37)$$

5. PROGRAM FOR HALF-SINE-PULSE FUNCTIONS

The FORTRAN IV program does the following:

(1) The response factor u is calculated for $0 < \tau \leq 1$ and tabled.

(2) The values of u are simultaneously compared to determine the maximaxes u_M in both positive and negative senses.

(3) These u_M's are tabled as a function of γ, the normalized natural frequency.

(4) The positive u_M's are graphed against γ.

(5) The residual u_M's are tabled and graphed.

For ease of program checkout, options have been incorporated which, among other possibilities, allow the preceding items to be selected individually. These are noted briefly in the comments to the program MAXU in figure 3. These comments show, moreover, that the positive impulse case can be selected by letting M1 be not-unity; that special test values of γ may be read in when M5 = 1 (used with M2 = 1); and that tables for a reduced number of R values may be chosen when KTEST < KK. The particular values of R desired are represented by the KT array.

A selected number of curves may be plotted by using the feature contained in the linking subroutine PLTOPT: the number of plots is read in; for each plot (indexed by K), M values are read into the ICURVE array; these values represent the M values of R chosen to make up a family of curves.

SCALE6 — a subroutine modified from A. Hausner's SCALE2* — scales the family of curves for each graph. Thus one may plot five curves on one graph, then re-plot three of these, which are scaled independent of the five.

* One of several subroutines documented at HDL for internal use.

```
      DIMENSION TAU(1C00),TAUPI(1000),STAUPI(1000),SGTPI(1000),U(1000,5)
     1,CTAUPI(1000),UMAX(5),UMIN(5),R(5),KT(5),UR(5),CGP(10),RA(5),
     2 UMINA(5),UMAXA(5)
      DIMENSION G(500),UM(500,5),URESM(5C0,5)
    1 FORMAT(16I5)
   11 FORMAT(8F10.5)
    2 FORMAT(    20X4HR = F7.4,10X8HGAMMA = F6.2,10X7HUMAX = 1PE15.7,
     1 10X7HUMIN = 1PE15.7//10X4(7X1HT12X1HU9X))
   12 FORMAT((10X4(4X0PF6.3,1PE20.7)))
   22 FORMAT(    12X5HGAMMA5(7X7HUMAX R=F6.4))
   32 FORMAT(/11X0PF6.2,1P5E20.7)
   42 FORMAT(17X1P5E2C.7)
   52 FORMAT(    30X39HPOSITIVE IMPULSES     RESIDUAL SPECTRUM10X4HR = ,
     1 F7.4///)
   62 FORMAT((15X3(2X5HGAMMA8X5HURMAX4X)))
   72 FORMAT((15X3(0PF08.1,1PE16.7)))
   82 FORMAT(20X25HFORCED VIBRATION SPECTRUM//)
   92 FORMAT(2CX45HTEN ALTERNATING IMPULSES ON A HALF-SINE PULSE//)
  102 FORMAT(20X43HFIVE POSITIVE IMPULSES CN A HALF-SINE PULSE//)
  112 FORMAT(30X42HALTERNATING IMPULSES     RESICUAL SPECTRUM10X4HR = ,
     1 F7.4///)
  122 FORMAT(17X5(7X7HUMIN R=F6.4))
  132 FORMAT(1H1)
C
C     II      NUMBER OF TAU VALUES
C     JJ      NUMBER OF GAMMA VALUES
C     KK      NUMBER OF R VALUES
C     M1 = 1       ALTERNATING IMPULSES
C     M2 = 1       WRITE TABLE OF U VS TAU (FORCED ERA)
C     M3 = 1       WRITE TABLE OF UMAX,UMIN VS GAMMA (FORCED ERA)
C     M4 = 1       PLOT UMAX VS GAMMA (FORCED ERA)
C     M5 = 1       READ GAMMA IN (READ CARC)
C     M6 = 1       WRITE TABLE OF URESM VS GAMMA (RESIDUAL ERA)
C     M7 = 1       PLOT URESM VS GAMMA (RESICUAL ERA)
C     KTEST = NUMBER OF VALUES OF R FOR WHICH TABLES ARE WANTED
C
      READ(5,1) II,JJ,KK
      READ(5,1) M1,M2,M3,M4,M5,M6,M7,KTEST
      READ(5,11) DTAU,DG,(R(K),K=1,KK)
      READ(5,1) (KT(K),K=1,KTEST)
      PI = 3.1415927
      EL = M1
      IF(M1.NE.1) EL=C.
      DO 119 KA=1,KTEST
      K = KT(KA)
  119 RA(KA) = R(K)
      DO 9  I=1,II
      FI = I
      TAU(I) = FI*DTAU
      TAUPI(I) = PI*TAU(I)
      STAUPI(I) = SIN(TAUPI(I))
      CTAUPI(I) = COS(TAUPI(I))
    9 CONTINUE
      IF(M5.EQ.1) GO TO 60
      DO 19  J=1,JJ
```

Figure 3. Program MAXU and subroutine PLTOPT.

```
      FJ = J
      G(J) = FJ*DG
   19 CONTINUE
      GO TO 70
C
C     WARNING - READ STATEMENT CONTROLLEC BY IF STATEMENT
C
   60 READ(5,11) (G(J),J=1,JJ)
   70 DO 39  J=1,JJ
      DO 29  K=1,KK
      UMAX(K) = 0.0
      UMIN(K) = 0.0
   29 CONTINUE
      C1 = G(J)/(1.-G(J)*G(J))
      C2 = PI*G(J)
      DO 89  M=1,10
      FM = M
      FGP = FM*C2/10.
      CGP(M) = COS(FGP)
   89 CONTINUE
      DO 49  I=1,II
      SGTPI(I) = SIN(G(J)*TAUPI(I))
      PAREN = SGTPI(I) - G(J)*STAUPI(I)
      IF(I.GT.100) GO TO 15
      BRICK = 0.0
      GO TO 10
   15 IF(I.GT.200) GO TO 25
      IF(M1.EQ.1) GO TO 110
      GO TO 10
  110 BRICK = SGTPI(I-100)
      GO TO 10
   25 IF(I.GT.300) GO TO 35
      IF(M1.EQ.1) GO TO 120
      GO TO 20
  120 BRICK = SGTPI(I-100)
      GO TO 20
   35 IF(I.GT.400) GO TO 45
      IF(M1.EQ.1) GO TO 130
      GO TO 20
  130 BRICK = SGTPI(I-100) + SGTPI(I-300)
      GO TO 20
   45 IF(I.GT.500) GO TO 55
      IF(M1.EQ.1) GO TO 140
      GO TO 30
  140 BRICK = SGTPI(I-100) + SGTPI(I-300)
      GO TO 30
   55 IF(I.GT.600) GO TO 65
      IF(M1.EQ.1) GO TO 150
      GO TO 30
  150 BRICK = SGTPI(I-100) + SGTPI(I-300) + SGTPI(I-500)
      GO TO 30
   65 IF(I.GT.700) GO TO 75
      IF(M1.EQ.1) GO TO 160
      GO TO 40
  160 BRICK = SGTPI(I-100) + SGTPI(I-300) + SGTPI(I-500)
      GO TO 40
```

Figure 3. Program MAXU and subroutine PLTOPT.

```
 75 IF(I.GT.800) GO TO 85
    IF(M1.EQ.1) GO TO 170
    GO TO 40
170 BRICK = SGTPI(I-100) + SGTPI(I-300) + SGTPI(I-500) + SGTPI(I-700)
    GO TO 40
 85 IF(I.GT.900) GO TO 95
    IF(M1.EQ.1) GO TO 180
    GO TO 50
180 BRICK = SGTPI(I-100) + SGTPI(I-300) + SGTPI(I-500) + SGTPI(I-700)
    GO TO 50
 95 IF(M1.EQ.1) GO TO 190
    GO TO 50
190 BRICK = SGTPI(I-100) + SGTPI(I-300) + SGTPI(I-500) + SGTPI(I-700)
   1 + SGTPI(I-900)
    GO TO 50
 10 BRACK = SGTPI(I)
    GO TO 100
 20 BRACK = SGTPI(I) + SGTPI(I-200)
    GO TO 100
 30 BRACK = SGTPI(I) + SGTPI(I-200) + SGTPI(I-400)
    GO TO 100
 40 BRACK = SGTPI(I) + SGTPI(I-200) + SGTPI(I-400) + SGTPI(I-600)
    GO TO 100
 50 BRACK = SGTPI(I) + SGTPI(I-200) + SGTPI(I-400) + SGTPI(I-600) +
   1 SGTPI(I-800)
100 HS = C1*PAREN
    IF(ABS(G(J)-1.0).LT.1.E-5) HS=0.5*(STAUPI(I)-TAUPI(I)*CTAUPI(I))
    DELTP = C2*BRACK
    DELTM = C2*BRICK
    DO 59 K=1,KK
    U(I,K) = HS + R(K)*(DELTP - EL*DELTM)
    IF(U(I,K).GE.UMAX(K)) UMAX(K)=U(I,K)
    IF(U(I,K).LE.UMIN(K)) UMIN(K)=U(I,K)
    LM(J,K) = UMAX(K)
 59 CONTINUE
 49 CONTINUE
    DO 99  K=1,KK
    UR(K) = C1*(CGP(10)+1.) + C2*R(K)*(CGP(10)+CGP(8)+CGP(6)+CGP(4)+
   1 CGP(2)-EL*(CGP(9)+CGP(7)+CGP(5)+CGP(3)+CGP(1)))
    URESM(J,K) = SQRT(U(II,K)*U(II,K) + UR(K)*UR(K))
 99 CONTINUE
    DO 69  KA=1,KTEST
    K = KT(KA)
    UMAXA(KA) = UMAX(K)
    UMINA(KA) = UMIN(K)
    IF(M2.NE.1) GO TO 69
    IC = 0
  7 WRITE(6,132)
    IF(M1.EQ.1) WRITE(6,92)
    IF(M1.NE.1) WRITE(6,102)
    WRITE(6,82)
    WRITE(6,2) R(K),G(J),UMAX(K),UMIN(K)
    I1 = 1 + IC*200
    I2 = I1 + 49
    WRITE(6,12) (TAU(I),U(I,K),TAU(I+50),U(I+50,K),TAU(I+100),
   1 U(I+100,K),TAU(I+150),U(I+150,K),I=I1,I2)
```

Figure 3. Program MAXU and subroutine PLTOPT.

```
      IF(IC.GE.4) GO TO 69
      IC = IC + 1
      GO TO 7
   69 CONTINUE
      IF(M3.NE.1) GO TO 39
      IF(MOD(J,17).NE.1) GO TO 17
      WRITE(6,132)
      WRITE(6,82)
      IF(M1.EQ.1) WRITE(6,92)
      IF(M1.NE.1) WRITE(6,102)
      WRITE(6,22) (RA(K),K=1,KTEST)
      WRITE(6,122) (RA(K),K=1,KTEST)
   17 WRITE(6,32) G(J),(UMAXA(K),K=1,KTEST)
      WRITE(6,42) (UMINA(K),K=1,KTEST)
   39 CONTINUE
      IF(M6.NE.1) GO TO 125
      DO 109  KA=1,KTEST
      K = KT(KA)
      ICE = 0
  200 L1 = 1 + 150*ICE
      L2 = L1 + 49
      WRITE(6,132)
      IF(M1.EQ.1) WRITE(6,112) R(K)
      IF(M1.NE.1) WRITE(6,52) R(K)
      WRITE(6,62)
      WRITE(6,72) (G(L),URESM(L,K),G(L+50),URESM(L+50,K),G(L+100),
     1 URESM(L+100,K),L=L1,L2)
      ICE = ICE + 1
      IF(ICE.LT.3) GO TO 200
  109 CONTINUE
C
C     WARNING*****PLTOPT HAS READ CARD
C
  125 IF(M4.EQ.1) CALL PLTOPT(1,G,UM,JJ,KK)
      IF(M7.EQ.1) CALL PLTOPT(2,G,URESM,JJ,KK)
      REWIND 9
      STOP
      END
```

```
      SUBROUTINE PLTOPT(L,X,Y,JJ,KK)
      DIMENSION X(500),Y(500,5)
      COMMON/DIMENS/N,M,K,ICURVE(5,5),XAMIN,XAMAX
    1 FORMAT(16I5)
      N = JJ
      READ(5,1) NOPLTS
      DO 9  K=1,NOPLTS
      READ(5,1) M
      READ(5,1) (ICURVE(K,J),J=1,M)
      IF(L.EQ.1) CALL SCALE6(X,Y,7,10,1,2,0)
      IF(L.EQ.2) CALL SCALE6(X,Y,7,10,2,2,0)
    9 CONTINUE
      RETURN
      END
```

Figure 3. Program MAXU and subroutine PLTOPT.

The calculations in the bulk of the program are done so as to avoid repetitious calculations of the sine and cosine; arrays such as TAUPI, STAUPI, SGTPI are precalculated, keeping the running time to a reasonable 5 to 8 min on the 7094. A better method, though, uses recurrence formulas (ref 8) to generate all but the first members of the circular functions with multiplications.

When $\gamma = 1$, equation (17) has a pole; thus C_1, after statement 29, becomes zero on the 7094; HS in 100 is thus zero, but the IF statement following it — allowing for binary roundoff — calculates the limiting value obtained by L'Hopital's rule.

Many of the variables have names suggestive of their counterparts in the previous sections; hence most of the arithmetic can be followed easily.

6. PROGRAMMING THE $t^2 \exp(-\beta t)$ CASE

A simple analysis of the limiting behavior of equation (31) shows that as $\xi \to \infty$, $w \to \tau^2 e^{-\tau}$, $\tau_{MAX} \to 2$, and $w_{MAX} \to 4/e^2 = 0.54134\cdots$. When $\xi \to 0$, $w \to 0$, and moreover, from (37), when $\xi \ll 1$,

$$w \sim 2\xi(\sin \xi\tau - 3\xi \cos \xi\tau) + \xi^2(6 + 4\tau + \tau^2)e^{-\tau} \tag{38}$$

which is a sinusoid with a small contribution from the transient term.

It is clear then that for large ξ we may be certain to find the maximax upon scanning the domain of ξ between 1 and 3; likewise, for small ξ, we may expect the first peak to be the maximax. The situation for intermediate values of ξ is not clarified by a brief additional analysis: e.g., the transient term has no relative maximum at the level of approximation given in equation (38). Practically, this question is readily solved by the analog computer. The results of this study (details are furnished in section 6.1) show that for $\xi < 3$, the first-peak criterion is valid. Also the criterion for $\xi > 3$ is that $1 < \tau_{MAX} < 3$.

Based on these two criteria, a straightforward digital program was written in FORTRAN to calculate and plot the shock spectrum. A short explanation of the program is given in section 6.2.

6.1 Analog Computer Analysis

An elementary circuit is sufficient for generating $w(\tau)$. All the intermediate functions required can be generated by linear equipment (ref 9). The unscaled equations are:

$$G_1 = e^{-\tau}$$

$$G_1' = -kG_1$$

$$G_2' = k(G_1 - G_2)$$

$$G_3' = k(2G_2 - G_3)$$

$$H_1' = k\xi^2(G_3 - w)$$

$$w' = kH_1$$

$$\tau' = k = d\tau/ds$$

where all derivatives are with respect to s, the machine time. The scaled circuit is given in figure 4, in which S_1 and C_w are unknown scaling factors that depend on ξ. A table giving suitable values of these factors is given in appendix C. The constant c determines the arm speed of the recorder and thus the paper abscissa scaling.

Representative response curves for low, intermediate, and high values of ξ are given in figure 5.

The ease of programming the differential equation leading to these response curves suggests the use of the repetitive operation feature available on the EAI 231R analog computer. (The 131R was used here.) In this REPOP technique, the differential equation would be solved in fast time, and the parameter ξ swept in slow time. The dynamic range of ξ and a fortiori ξ^2 is unfortunately too great for the analog computer; scaling problems arise. It appears one can do no better than divide the ξ-axis into sections and scale each section separately. This naturally vitiates much of the attractiveness of the method, which was therefore not pursued.

6.2 Digital Computer Program

The FORTRAN IV program to calculate the shock spectrum is presented in figure 6. The main branch statement after DO 29 allows calculation and search for a maximum for $1 < \tau < 3$ when $\xi > 3$, storing the maximaxes in WMAX(J). If $\xi < 3$, a semiglobal algorithm is chosen, which, starting from $\tau = 0$, sweeps past the first maximum in fairly large steps, reverses two steps, cuts the interval by five, resweeps and repeats until the intervals are roughly equal to those used in the other path of the program, again storing the results in WMAX. The SCALIT subroutine, written by A. Hausner* for simple x-versus-y plotting, scales TAU and WMAX, and writes them on tape (in low density).

The shock spectra for all the forcing functions are presented and discussed in section 7.

* One of several subroutines documented by HDL for internal use.

7. RESULTS

The chief results of this study are the shock spectra shown in figures 7 through 12. As can be seen, figure 7 illustrates a family of forced-vibration spectra for different values of R corresponding to the forcing function of figure 1a. Figure 8 is the analogous family corresponding to figure 1b. Figures 9 and 10 are the free-vibration spectra for the same two cases. Figure 11 shows a forced- and a free-vibration spectrum on the same graph for R = 0.01. These are the upper curves in figures 7 and 9. Finally, figure 12 is the spectrum for the forcing function of figure 1c. These results have not been observed in the literature.

The important features of figures 7 through 11 are the resonance peaks near $\gamma = 1.5$, 10, 20, and 30. If we were to "continue" the half-sine pulse to a continuous monotonic frequency, then γ would measure the natural frequency of the system with respect to the former frequency. When there are five positive impulses per half-sine pulse, the system responds to this repetition rate of ten per "cycle" in giving the resonances at 10, 20, and 30 with negligible augmentation of the major peak of the pure half-sine pulse spectrum (R = 0). Additional energy is fed into the system when interspersed, negative impulses are added; the amplitude of the spectrum is thus larger except for suppression at values of γ at even multiples of 10. The suppression is due to the cancellation of the kinetic energy of the system oscillations by the forcing function. This happens when the system has just completed an oscillation cycle and is then "struck" by a negative impulse. The velocity at this moment is maximum in the positive direction, but the force is in the negative direction.

In figure 11, we see how the residual spectrum may predominate for certain values of γ, in this case for $\gamma < 1$; one must in general then consider both the forced and the free era.

The tabular data, not given here, show that in no case is the negative maximax larger than the positive for the forced vibrations. (They are equal in the residual era.) The trend of the data, however, strongly suggests that for resonances at larger γ, the negative maximaxes may become larger than the positive. Moreover, some results generated for R = 0.2 with the earlier program showed that for the alternating impulse case, negative maximaxes were larger near and at the resonances for $\gamma = 10$ and 30. Thus, although not investigated here, negative maximaxes may be very significant for large γ and large R.

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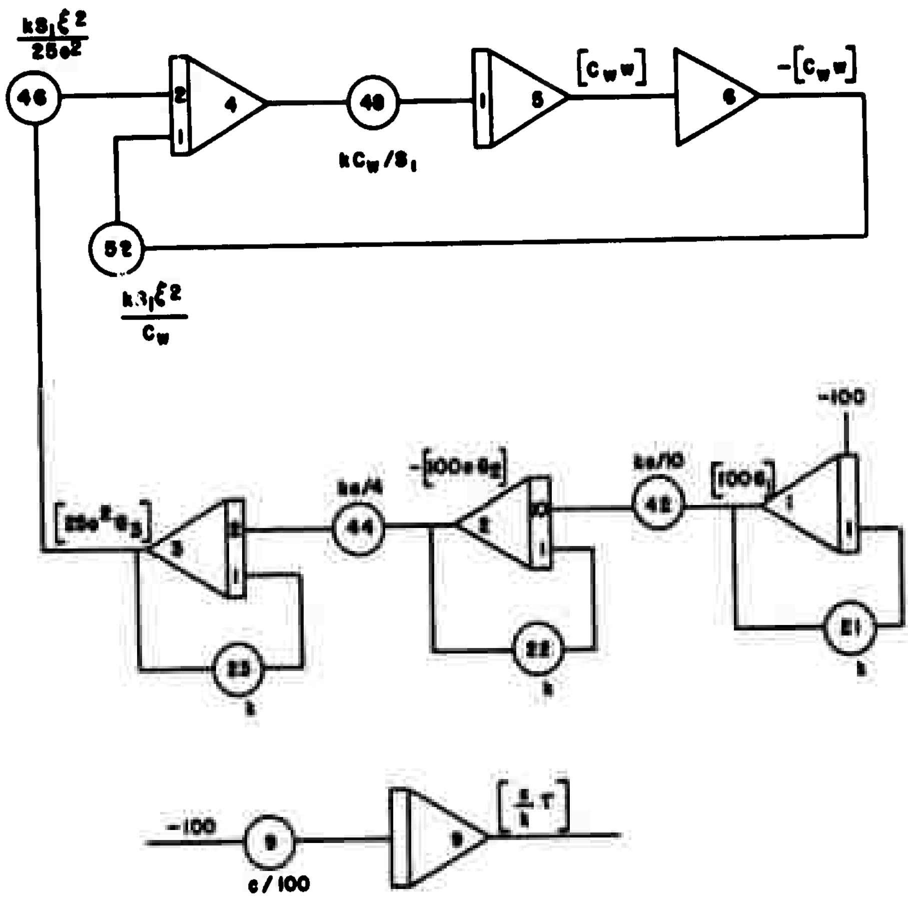

Figure 4. Analog computer circuit for undamped ODOF system forced by $-\xi e^{-\tau}$.

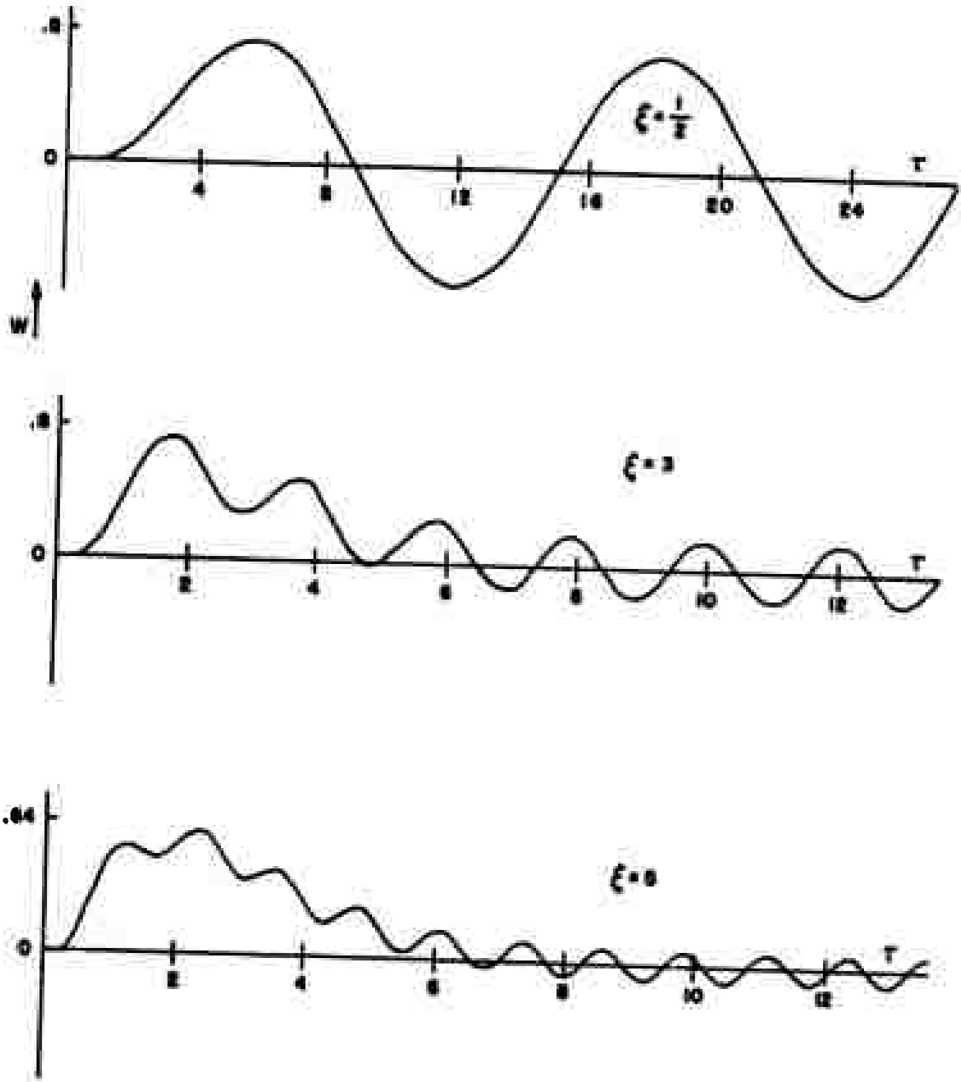

Figure 5. Representative response curves for low, medium and high values.

```
      DIMENSION TAU(1000),TAUS(1000),EXMTAU(1000),XI(350),WMAX(350),
     1 A(350),A1(350),A2(350),B1(350),B2(350),B3(350)
    1 FCRMAT(16I5)
   11 FCRMAT(8F10.5)
    2 FORMAT(1H1)
   12 FCRMAT((20X2(7X2HXI13X4HWMAX4X))//)
   22 FCRMAT((20X2(0PF10.2,1PE20.7)))
   32 FORMAT(37X5HII = I4,5X5HJJ = I3)
   42 FCRMAT(37X6HDXI = F5.2,5X7HDTAU = F7.4/37X9HDELTAM = F5.2,5X4HZ =
     1 F7.4///)
      READ(5,1) II,JJ,I1,I2,NX,NY,IDENT,ICODE
      READ(5,11) DXI,DTAU,DELTAM,Z
C
C     II = NUMBER CF VALUES COMPUTED OF EACH RESPONSE FUNCTION
C     JJ = NUMBER CF POINTS IN THE SPECTRUM
C     I1 = PLOTTING OPTICN (PLOT WHEN I1=1)
C     I2 = WRITING CPTION (WRITE TABLE OF RESULTS WHEN I2=1)
C     DXI = XI INCREMENT
C     DTAU = TAU INCREMENT
C     DELTAM = INITIAL INCREMENT IN TAM REPRESENTING TAU
C     DELTAN = UPDATED INCREMENT
C     Z = ERROR CRITERICN IN TAU
C
C     DC LOOP 9 GENERATES ARRAYS BASED ON 1.LT.TAU.LT.3
C
      DC 9  I=1,II
      FI = I
      TAU(I) = FI*DTAU + 1.
      TAUS(I) = TAU(I)*TAU(I)
      EXMTAU(I) = EXP(-TAU(I))
    9 CCNTINUE
      DC 19  J=1,JJ
      WMAX(J) = 0.0
      FJ = J
      XI(J) = FJ*DXI
      XIS = XI(J)*XI(J)
      ONEPXS = 1. + XIS
      ONPXSS = CNEPXS*CNEPXS
      ONPXSC = CNPXSS*CNEPXS
      A(J) = 2.*XI(J)/ONPXSC
      A1(J) = 1. - 3.*XIS
      A2(J) = -XI(J)*(3.-XIS)
      B1(J) = -A(J)*A2(J)
      B2(J) = 4.*XIS/ONPXSS
      B3(J) = XIS/CNEPXS
   19 CCNTINUE
      DC 29  J=1,JJ
      IF(XI(J).LT.3.0) GO TO 100
      DC 39  I=1,II
      SINXIT = SIN(XI(J)*TAU(I))
      CCSXIT = CCS(XI(J)*TAU(I))
      W = A(J)*(A1(J)*SINXIT + A2(J)*COSXIT) + (B1(J) + B2(J)*TAU(I) +
     1 B3(J)*TAUS(I))*EXMTAU(I)
      IF( W.GE.WMAX(J)) WMAX(J)=W
   39 CONTINUE
      GO TO 29
  100 TAM = 0.0
      DELTAN = DELTAM
      WHOLD1 = C.0
   80 WHOLD2 = C.0
```

Figure 6. Program TSEXP.

```
60 TAM = TAM + DELTAN
   XITAM = XI(J)*TAM
   W = A(J)*(A1(J)*SIN(XITAM) + A2(J)*COS(XITAM))+ (B1(J) + B2(J)*TAM
  1 + B3(J)*TAM*TAM)*EXP(-TAM)
   IF(W.LT.WHOLD1) GO TO 50
   WHOLD2 = WHOLD1
   WHOLD1 = W
   GO TO 60
50 IF(DELTAN.LT.Z) GO TO 70
   TAM = TAM - 2.*DELTAN
   WHOLD1 = WHOLD2
   DELTAN = DELTAN/5.
   GO TO 60
70 WMAX(J) = WHOLD1
29 CONTINUE
   IF(I1.NE.1) GO TO 10
   CALL SCALIT(XI,WMAX,JJ,NX,NY,IDENT,ICODE)
   REWIND 9
10 IF(I2.NE.1) STOP
   JCOUNT = 0
   WRITE(6,2)
   WRITE(6,32) II,JJ
   WRITE(6,42) DXI,DTAU,DELTAM,Z
30 J1 = 1 + 100*JCOUNT
   J2 = J1 + 49
   WRITE(6,12)
   WRITE(6,22) (XI(J),WMAX(J),XI(J+50),WMAX(J+50),J=J1,J2)
   JCOUNT = JCOUNT + 1
   IF(JJ.LE.100*JCOUNT) GO TO 20
   WRITE(6,2)
   GO TO 30
20 STOP
   END
```

Figure 6. Program TSEXP.

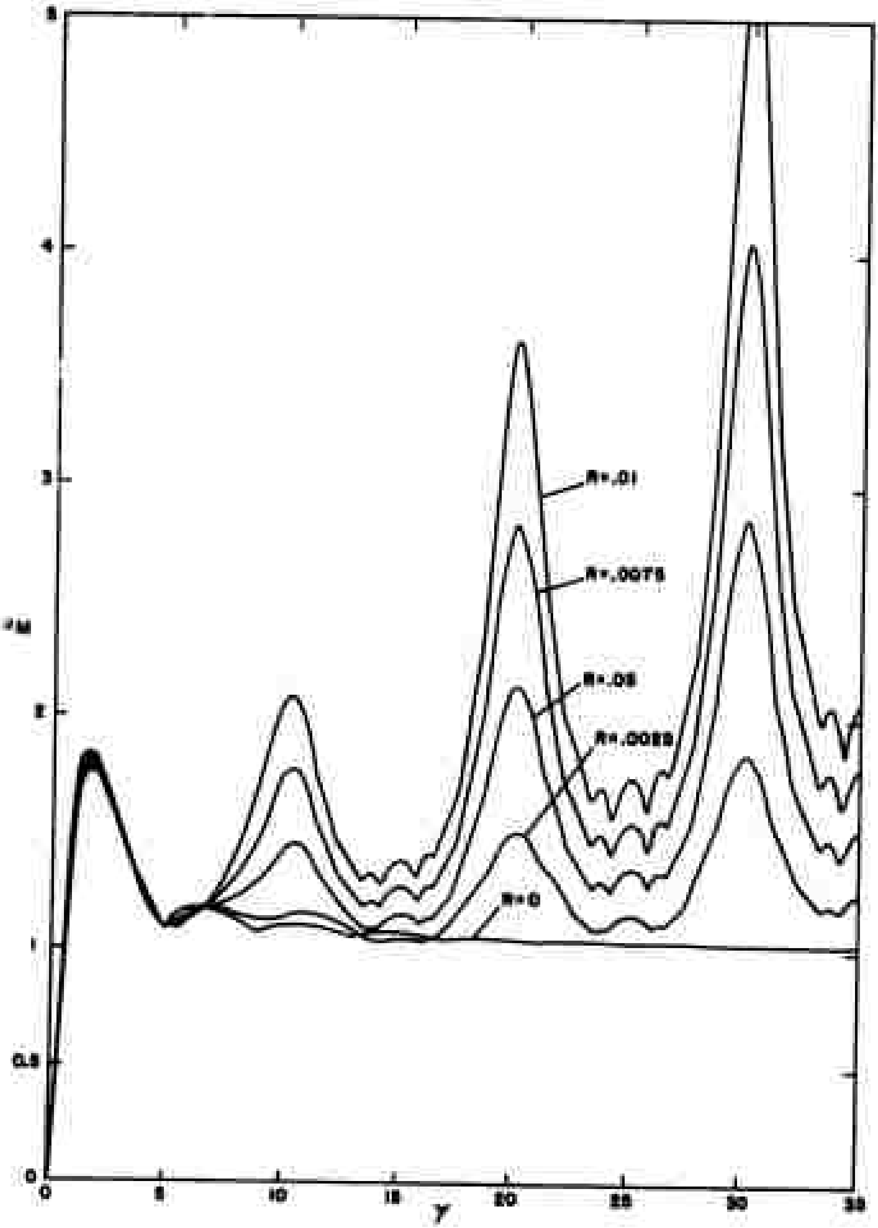

Figure 7. Forced vibration spectra of five positive impulses on a half-sine pulse driving an ODOF system.

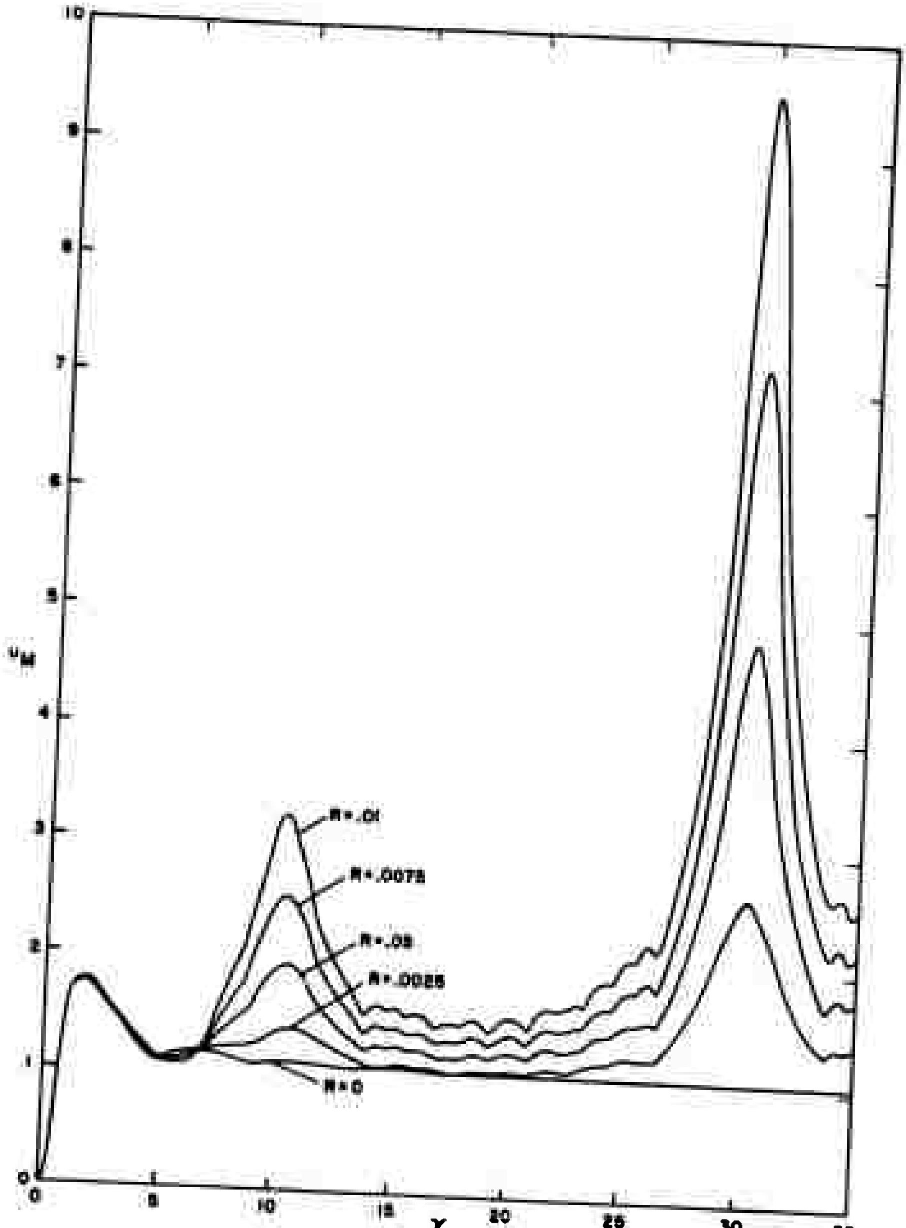

Figure 8. Forced vibration spectra of ten alternating impulses on a half-sine pulse driving an ODOF system.

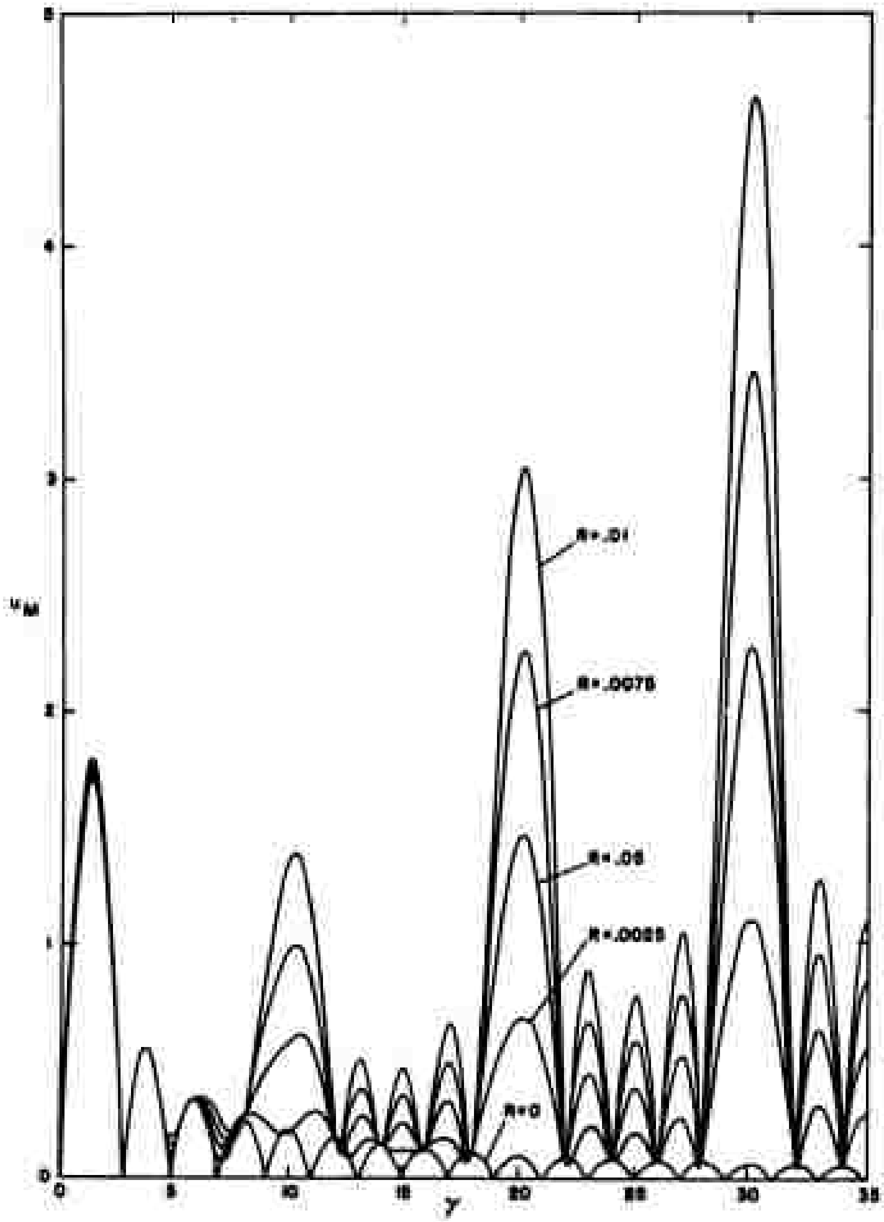

Figure 9. Free vibration spectra of five positive impulses on a half-sine pulse driving an ODOF system.

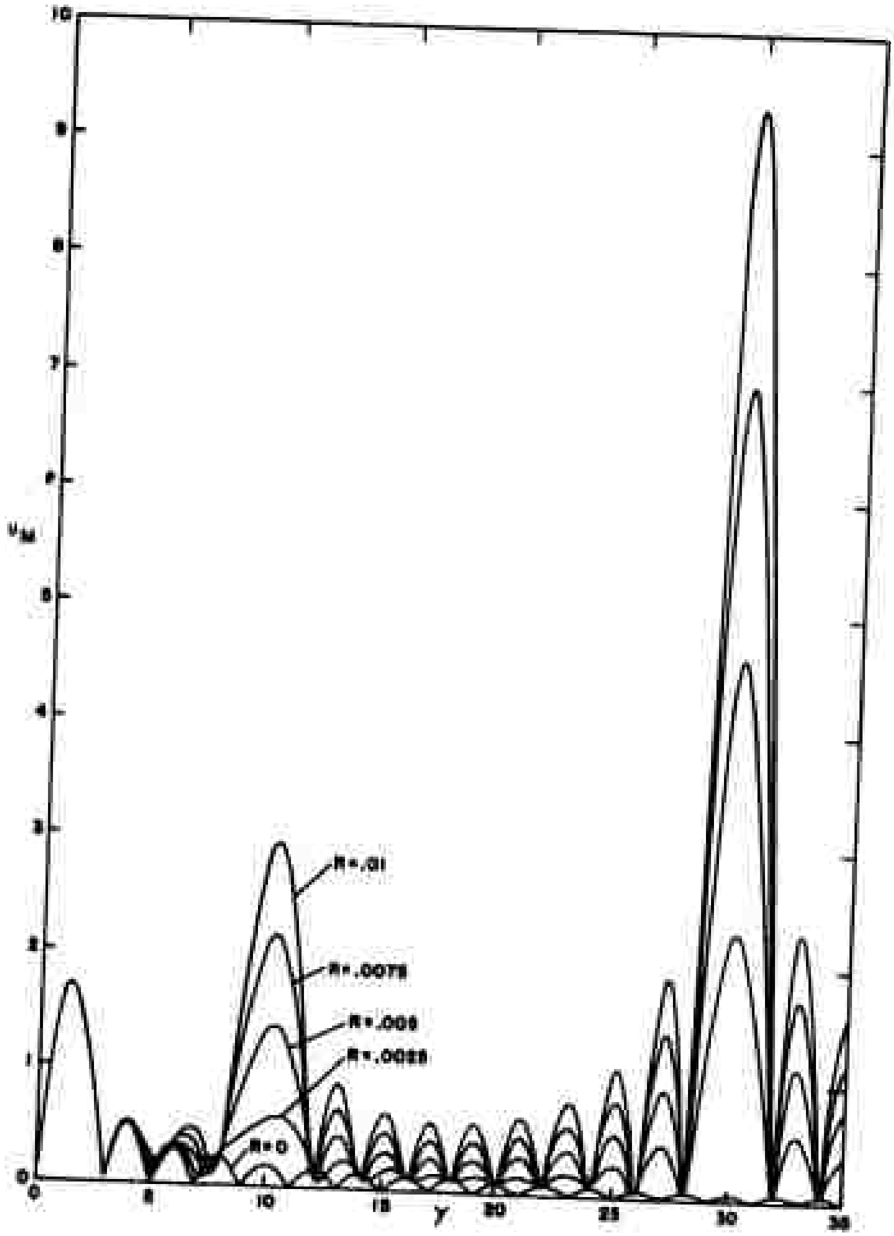

Figure 10. Free vibration spectra of ten alternating impulses on a half-sine pulse driving an ODOF system.

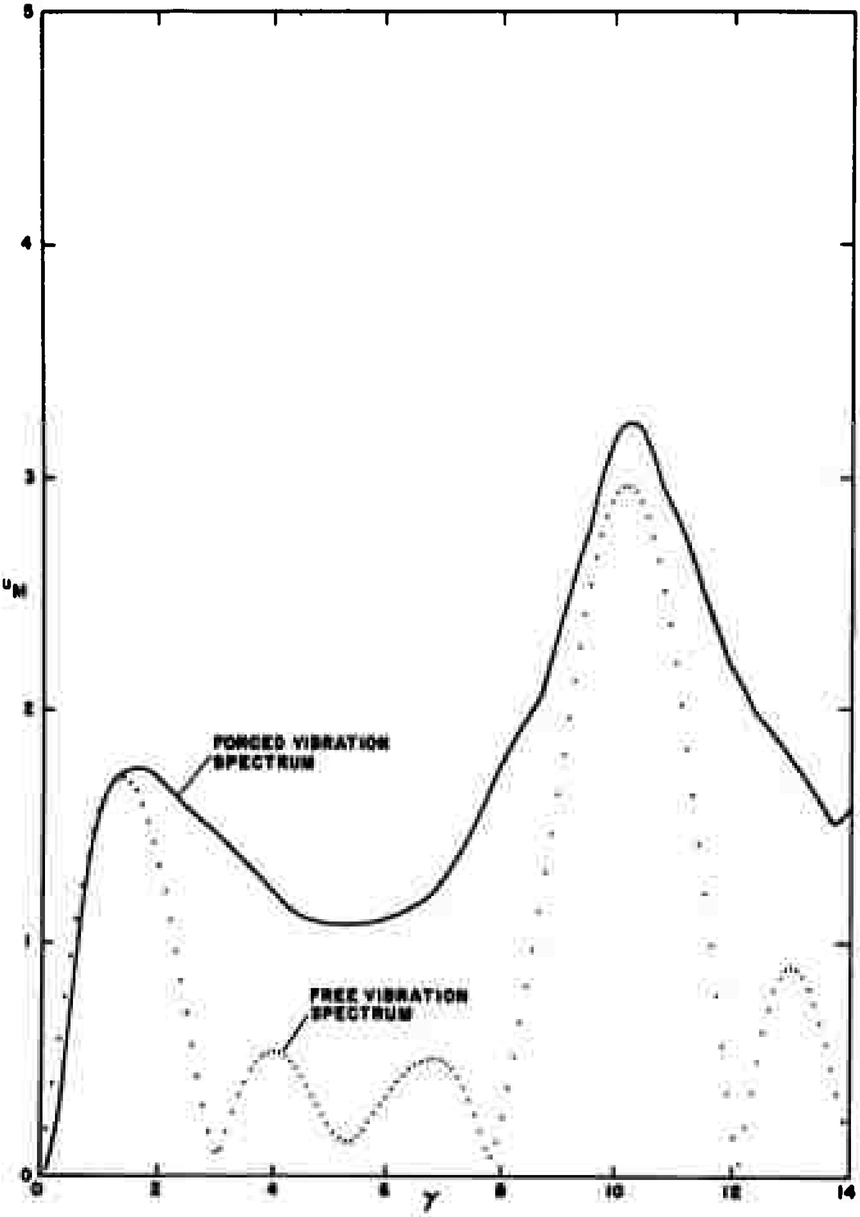

Figure 11. Comparison of forced and free vibration spectra for ten alternating impulses; R = 0.01.

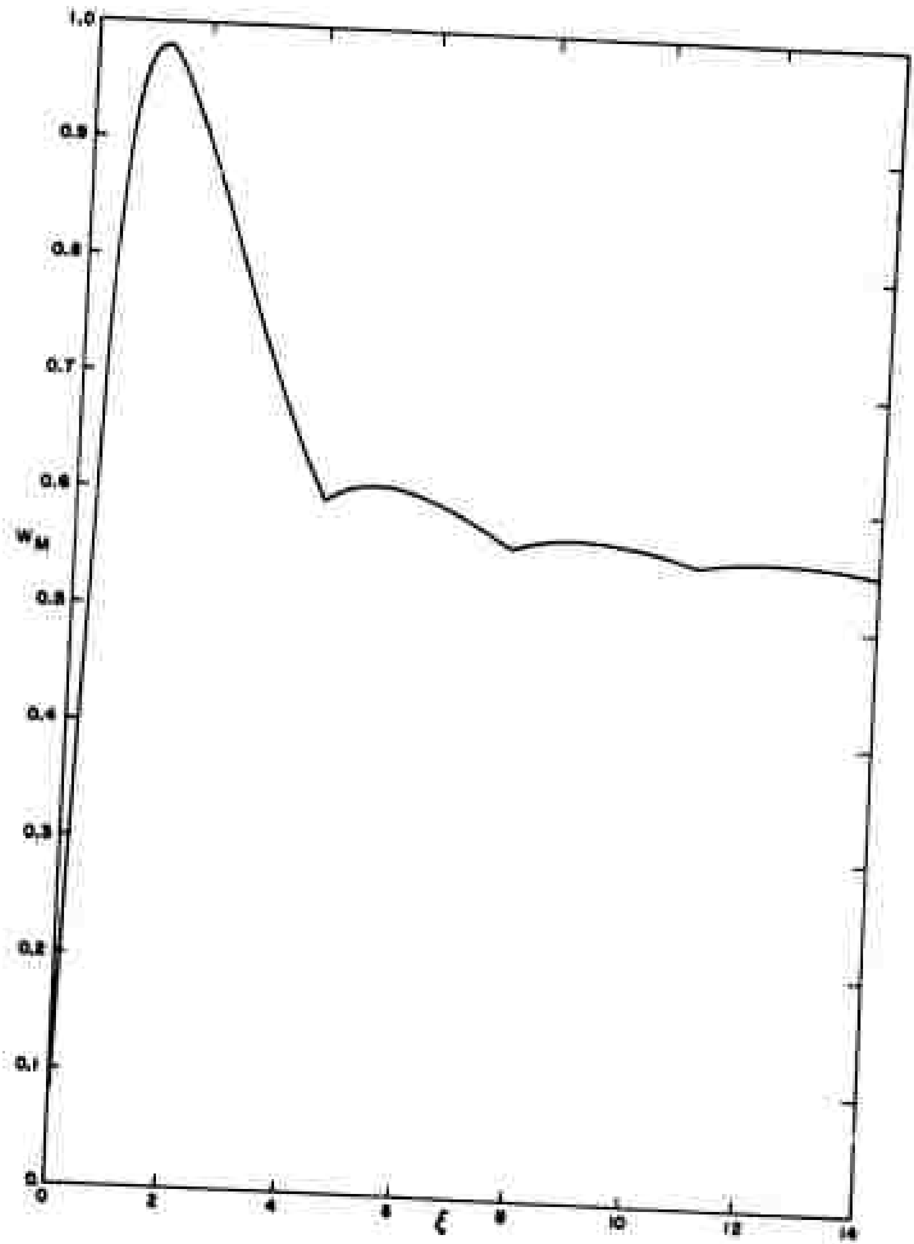

Figure 12. Spectrum of t exp(-αt)-type function driving an ODOF system.

APPENDIX A. — EVALUATION OF COEFFICIENTS OF PARTIAL FRACTION EXPANSION

If in (33) we multiply by s and let $s \to \infty$,

$$A + C = 0 \tag{A-1}$$

If we let $s \to 0$,

$$B/\xi^2 + C + D + E = 1/\xi^2 \tag{A-2}$$

If we multiply (33) by $(s+1)^3$ and let $s \to -1$,

$$E = 1/(1+\xi^2) \tag{A-3}$$

If we multiply by $s^2 + \xi^2$ and let $s \to i\xi$,

$$Ai\xi + B = 1/(1 + i\xi)^3 \tag{A-4}$$

Equation (A-4) yields

$$(Ai\xi + B)(1 + 3i\xi - 3\xi^2 - e\xi^3) \equiv 1 \tag{A-5}$$

which upon expansion and equating real and imaginary parts yields the pair of equations

$$(1 - 3\xi^2)B - \xi^2(3 - \xi^2)A = 1 \tag{A-6}$$

and

$$(3 - \xi^2)B + (1 - 3\xi^2)A = 0 \tag{A-7}$$

The determinant of the coefficient matrix reduces to $(1 + \xi^2)^3$. The solutions are readily found to be those given in (34a) and (34b). Finally D is obtained from (A-2)

APPENDIX B — INTEGRATION BY PARTS OF THE CONVOLUTION INTEGRAL

One defines

$$I = \int_0^{\tau} v^2 e^{-v} \sin \xi(\tau - v)\, dv \qquad \text{(B-1)}$$

$$J = \int_0^{\tau} v^2 e^{-v} \cos \xi(\tau - v)\, dv \qquad \text{(B-2)}$$

$$K = \int_0^{\tau} v e^{-v} \sin \xi(\tau - v)\, dv \qquad \text{(B-3)}$$

$$L = \int_0^{\tau} v e^{-v} \cos \xi(\tau - v)\, dv \qquad \text{(B-4)}$$

$$M = \int_0^{\tau} e^{-v} \sin \xi(\tau - v)\, dv \qquad \text{(B-5)}$$

$$N = \int_0^{\tau} e^{-v} \cos \xi(\tau - v)\, dv \qquad \text{(B-6)}$$

where $w = \xi I$. Integration by parts, retaining e^{-v} under the differential each time, yields the relations

$$I = 2K - \xi J \qquad \text{(B-7)}$$

$$J = -\tau^2 e^{-\tau} + 2L + \xi I \qquad \text{(B-8)}$$

$$K = M - \xi L \qquad \text{(B-9)}$$

$$L = -\tau e^{-\tau} + N + \xi K \qquad \text{(B-10)}$$

$$M = \sin \xi\tau - \xi N \qquad \text{(B-11)}$$

$$N = -e^{-\tau} + \cos \xi\tau + \xi M \qquad \text{(B-12)}$$

whose solution for I may be given as

$$I(1+\xi^2)^3 = \xi(1+\xi^2)^2 \tau^2 e^{-\tau} + 4\xi(1+\xi^2)\tau e^{-\tau} + 2\xi(3-\xi^2)e^{-\tau} + 2(1-3\xi^2)\sin \xi\tau - 2\xi(3-\xi^2)\cos \xi\tau \qquad \text{(B-13)}$$

APPENDIX C. — SCALE FACTORS OF ANALOG CIRCUIT

Table CI summarizes the essential, approximate scaling data obtained by trial using the circuit in figure 4. The potentiometer numbers in the five columns at the right are, of course, arbitrary. The circled numbers denote gains on the EAI 131-R analog computer. The values and gains are easily translated into corresponding values for computers having different gains available.

Table CI. Scale Factors and Potentiometer Settings for Analog Circuit

ζ	k	C_W	S_1	42	44	46	48	52
0.2	2	500	2500	.5437 (10)	.1259 (10)	.5413 (2)	.4	.4
0.5	1	125	250	.2718 (10)	.6796 (2)	.3383	.5	.5
1.0	1	100	100	.2718 (10)	.6796 (2)	.5413	(1)	1
$\sqrt{3}$	1	100	100	.2718 (10)	.6796 (2)	.8120 (2)	(1)	.6 (5)
2.0	1	100	100	.2718 (10)	.6796 (2)	.4331 (5)	(1)	.8 (5)
3.0	1	100	100	.2718 (10)	.6796 (2)	.9744 (5)	(1)	.9 (10)
4.0	1	100	100	.2718 (10)	.6796 (2)	.8661 (10)	(1)	.8 (20)
5.0	0.2	125	100	.5437	.2718	.5413 (5)	.25	.8 (5)
10.0	0.1	125	100	.2718	.1359	.5413 (10)	.125	.8 (10)

APPENDIX D.—TABLE OF SYMBOLS

m	mass
k	spring constant (on analog diagram, used as time scale factor)
x	displacement
t	time
$F(t)$	forcing function
F_0	maximum value of half-sine pulse forcing function (excluding impulses)
β	given decay constant of one forcing function
$\omega_N = \sqrt{\frac{k}{m}}$	natural frequency of system
$x_s = \frac{F_0}{k}$	static displacement
$u = \frac{x}{x_s}$	response factor
t_c	duration of half-sine pulse
$\tau = \frac{t}{t_c}$	normalized time
$\omega_F = \frac{\pi}{t_c}$	angular frequency characteristic of forcing function
$\gamma = \frac{\omega_N}{\omega_F}$	normalized natural frequency
τ_M	time at which u reaches a global maximum
u_M	value of u at its global maximum
F_I	strength of Delta impulse function
$R = \frac{F_I}{F_0}$	ratio of impulse strength to half-sine pulse peak
$\psi = \frac{4}{e} u$	scaled response factor
$\Gamma = \gamma\pi$	scaled normalized natural frequency

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13. ABSTRACT

Numerical solutions in graphical form are presented for the shock spectra of a one-degree-of-freedom system subjected to some idealized forcing functions. Elementary procedures are outlined for developing the solutions. Analog and digital computer programs are submitted where applicable.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Dirac impulses						
Half-sine pulse						
One-degree-of-freedom system						
Idealized forcing functions						
Digital programs						
Analog computer as auxiliary						