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ENGINEERING GRAPHICS SEMINAR

FOUR-DIMENSIONAL DESCRIPTIVE GEOMETRY
PROBLEMS ON 3-D SPACES

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ABSTRACT

The paper presents new solutions of five problems in four-dimensional descriptive geometry. These problems were briefly discussed in previous papers. The new solutions have already been used in papers preceding this one but were not fully examined.

FIRST PROBLEM

"Given a 3-D space Λ, determine the orthographic projections of a line of the 3-D space perpendicular to a plane α, also of the 3-D space, through a point (a) of the plane."

The first consideration is the question of the determination of a plane α belonging to the 3-D space Λ. Figure 1 shows a plane (mnp) satisfying the condition, where each point belongs to the traces of Λ in the planes π_1, π_2, and π_3 of the 4-D system of reference.

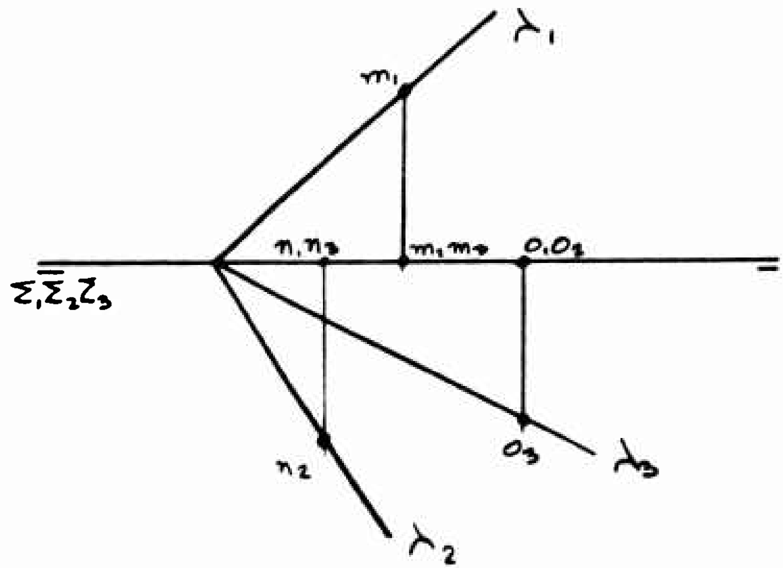

Figure 1

The procedure for the determination of the projection of a line perpendicular to this plane is outlined in our paper "Descriptive Geometry of Four Dimensions"[1] and consists in making changes of 3-D spaces in the 3-D system of reference, so that the space Λ is superimposed on one of those 3-D spaces of the system. Thus, the solution is obtained by methods of the three-dimensional descriptive geometry.

Evidently the above approach may be applied to all other cases, but it is more or less apparent that some "direct" solution may be obtained for some particular cases when the plane occupies a special position in relation to one of the 3-D spaces of the 4-D system of reference. Therefore, the use of a plane in one of these particular positions would not justify the application of the generalized solution.

With this in mind we searched for the particular positions of the plane, within a 3-D space Λ. Those positions may be obtained as results of the intersection of Λ with a 3-D space Ω which is a geometric locus. For example, the plane α, intersection of Λ and Ω being 3-D space Ω parallel to the 3-D space Σ_2 of the system of reference, is a plane parallel to Σ_2. (Figure 2).

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[1] Engineering Graphics Seminar, Technical Seminar Series, Report No. 9, pp. 29-30, December 5, 1963, Department of Graphics and Engineering Drawing, Princeton University.

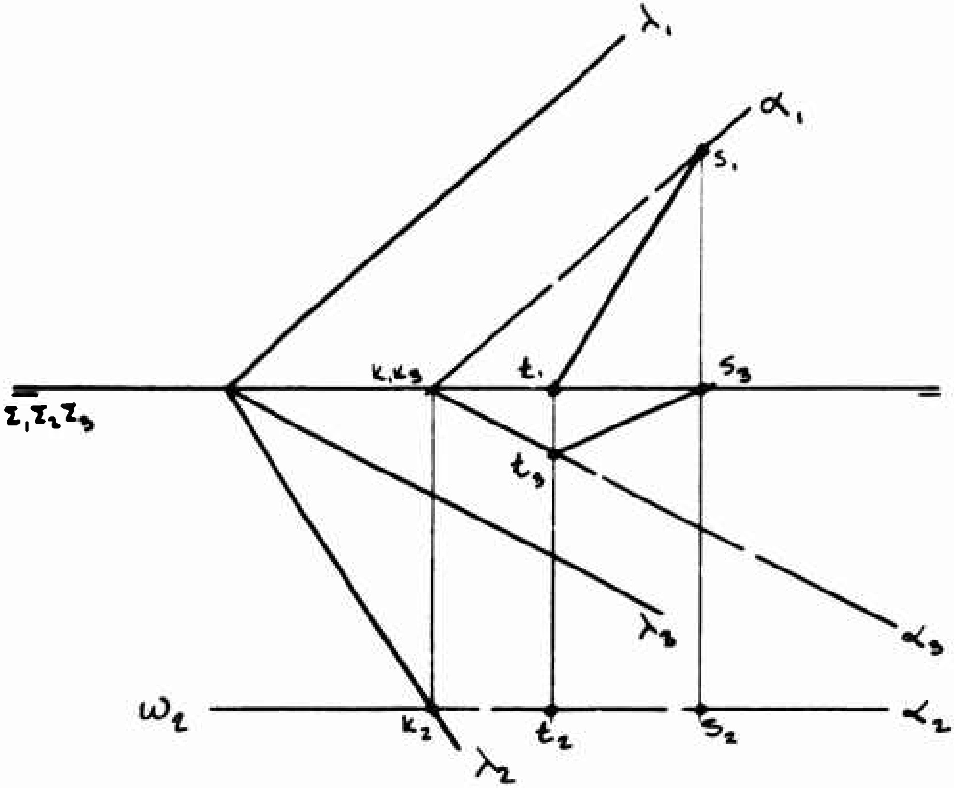


Figure 2

If we now compare the arrangement of ths lines (sk,tk) of the plane α , with the arrangement of the lines (sk,tk) of a plane β (Figure 3) where they belong to a 3-D space, we can contemplate the idea that since it is immediately possible to determine the projections of a line (ab) perpendicular to β , (Figure 4), it should be also possible to determine the projections of (ab) perpendicular to α in the 4-D space, without the use of the generalized solution indicated for the case of the plane in figure 1.

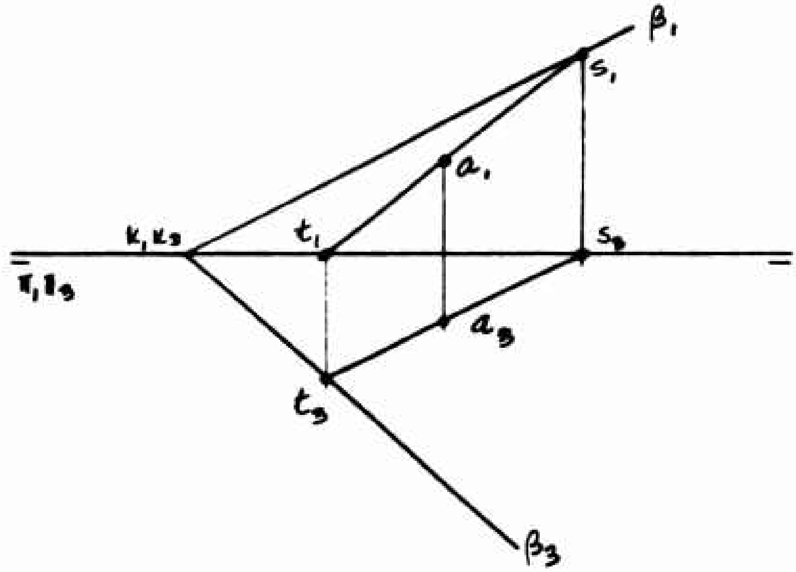


Figure 3

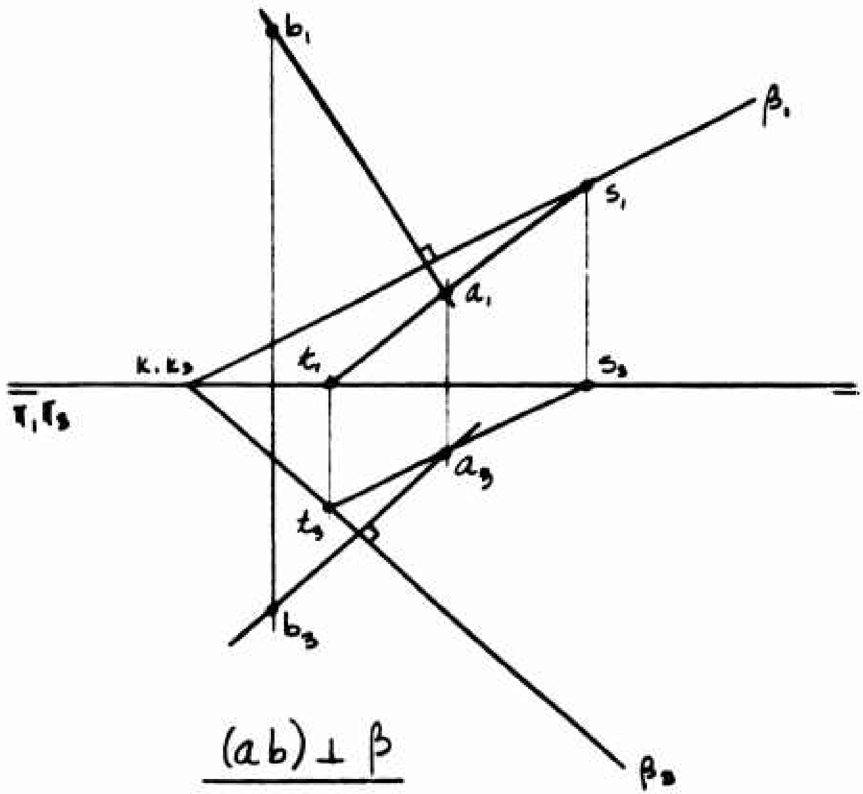
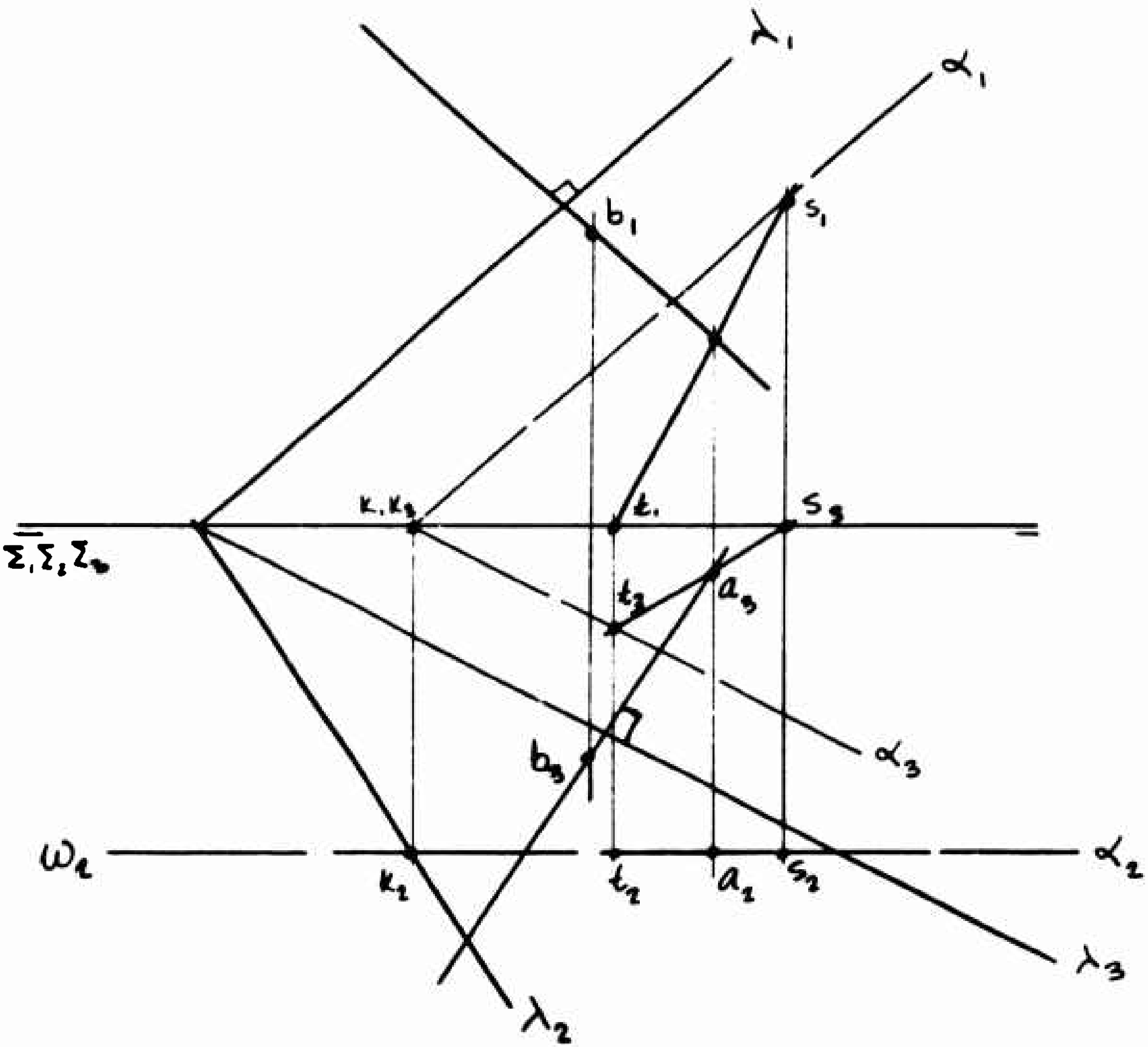


Figure 4

Further analysis leads then to the conclusion that there exists the same relationship, in 4-D descriptive geometry, between the projections of a line perpendicular to a plane and those of the lines (sk) and (st) shown in figure 2, as in the case of the 3-D descriptive geometry shown in figure 4: the projections of the line are perpendicular to the traces of same sub-index, of the plane.

Thus it may be demonstrated that the projections a_1b_1 and a_3b_3 of a line (ab) perpendicular to the plane α of a 3-D space Λ are perpendicular, respectively, to the projections s_1k_1 and s_3t_3 of two lines (sk) and (st) of that plane, being each one of these lines, parallel to two of the traces of the 3-D space Λ . (See figure 5). The only remaining problem is the determination of the projection a_2b_2 of the line (ab).



$\Lambda \times \Omega \Rightarrow$ PLANE α

(st) AND (sk) BELONG TO α

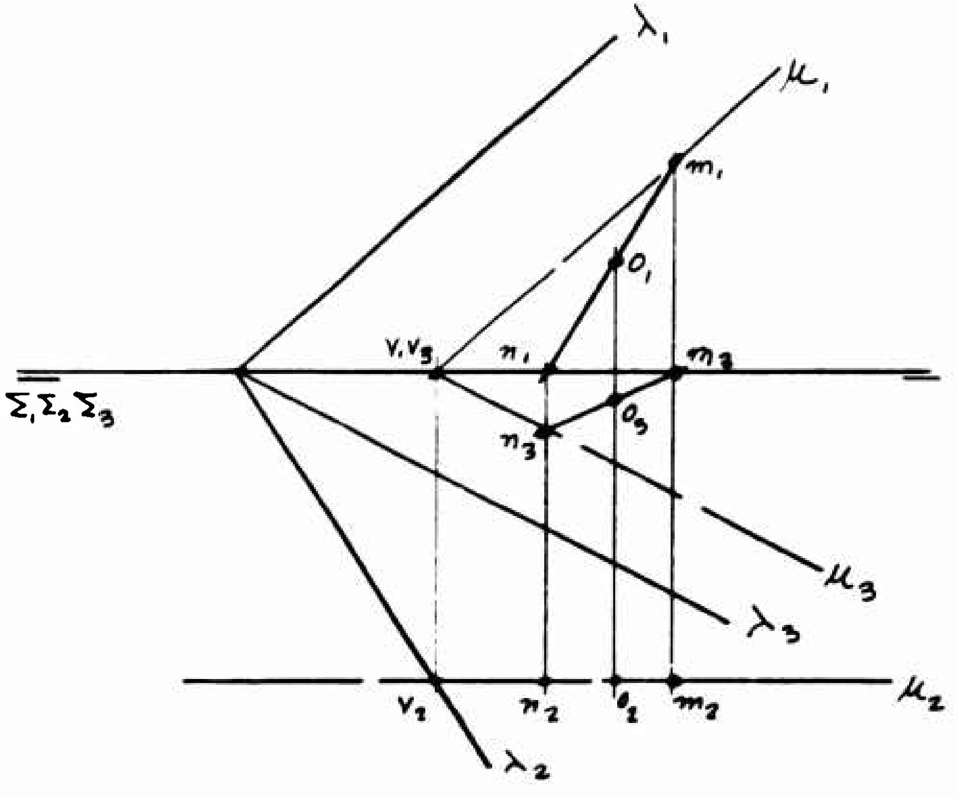
$\alpha_1 \equiv (sk) \parallel \lambda_1$; $\alpha_3 \equiv (tk) \parallel \lambda_3$

$a_1 b_1 \perp \alpha_1$; $a_3 b_3 \perp \alpha_3$

PROBLEM: DETERMINE $a_2 b_2$

Figure 5

To determine the projection b_2 of the point (b) we shall make use of the conditions of belonging between a point and a 3-D space: the point belongs to a line of a plane in the 3-D space. Figures 6, 7, 8, show a point (o) of the 3-D space Λ, which condition of belonging is satisfied by use of planes and lines belonging to the 3-D space.



$$\left.\begin{array}{l}(o) \rightarrow (mn) \\ (mn) \rightarrow \Lambda\end{array}\right\} (o) \rightarrow \Lambda$$

Figure 6

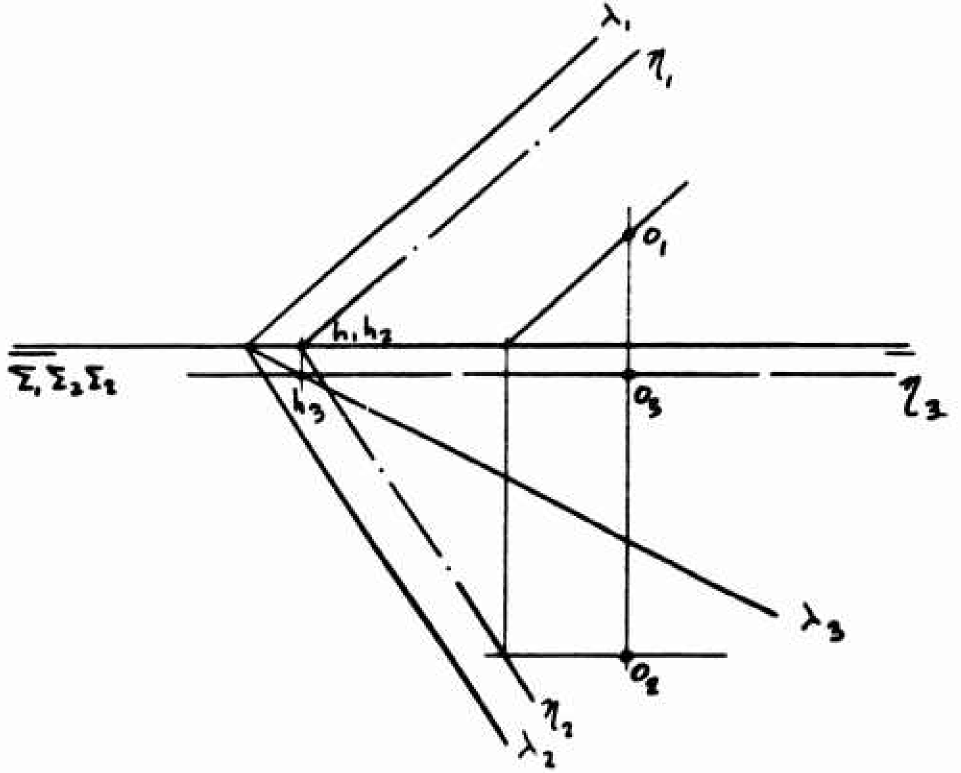


Figure 7

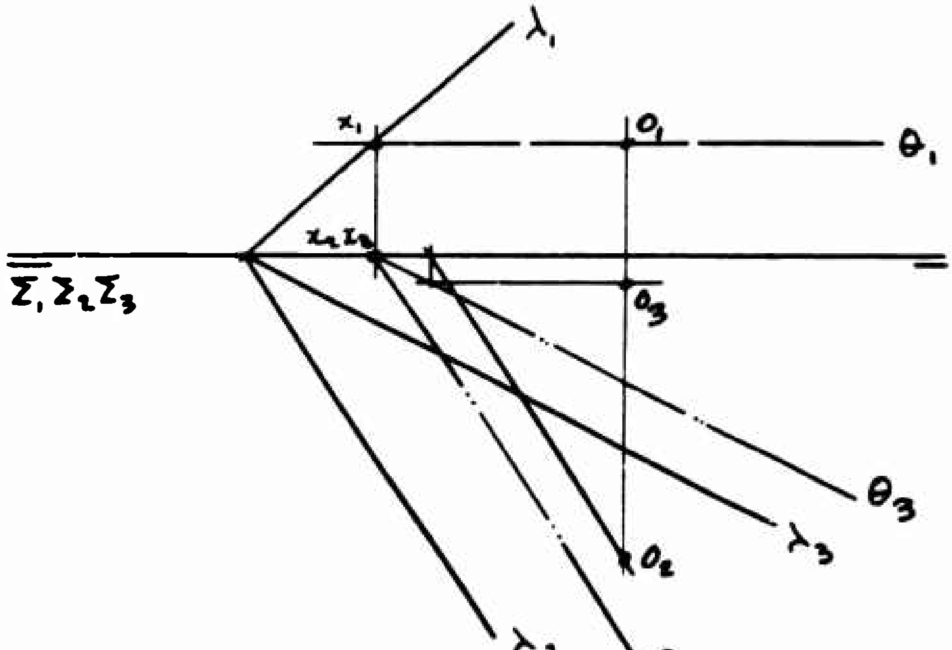


Figure 8

Thus it can be seen that if one or two of the projections of a point are given, it is possible to determine the other projections. This is the case of the point (b), being known b_1 and b_3. Figure 9 shows how the projection b_2 is determined. Therefore, a_1b_1, a_3b_3 are the projections of a line (ab) of Λ, perpendicular to the plane α, of Λ, through the point (a) of α.

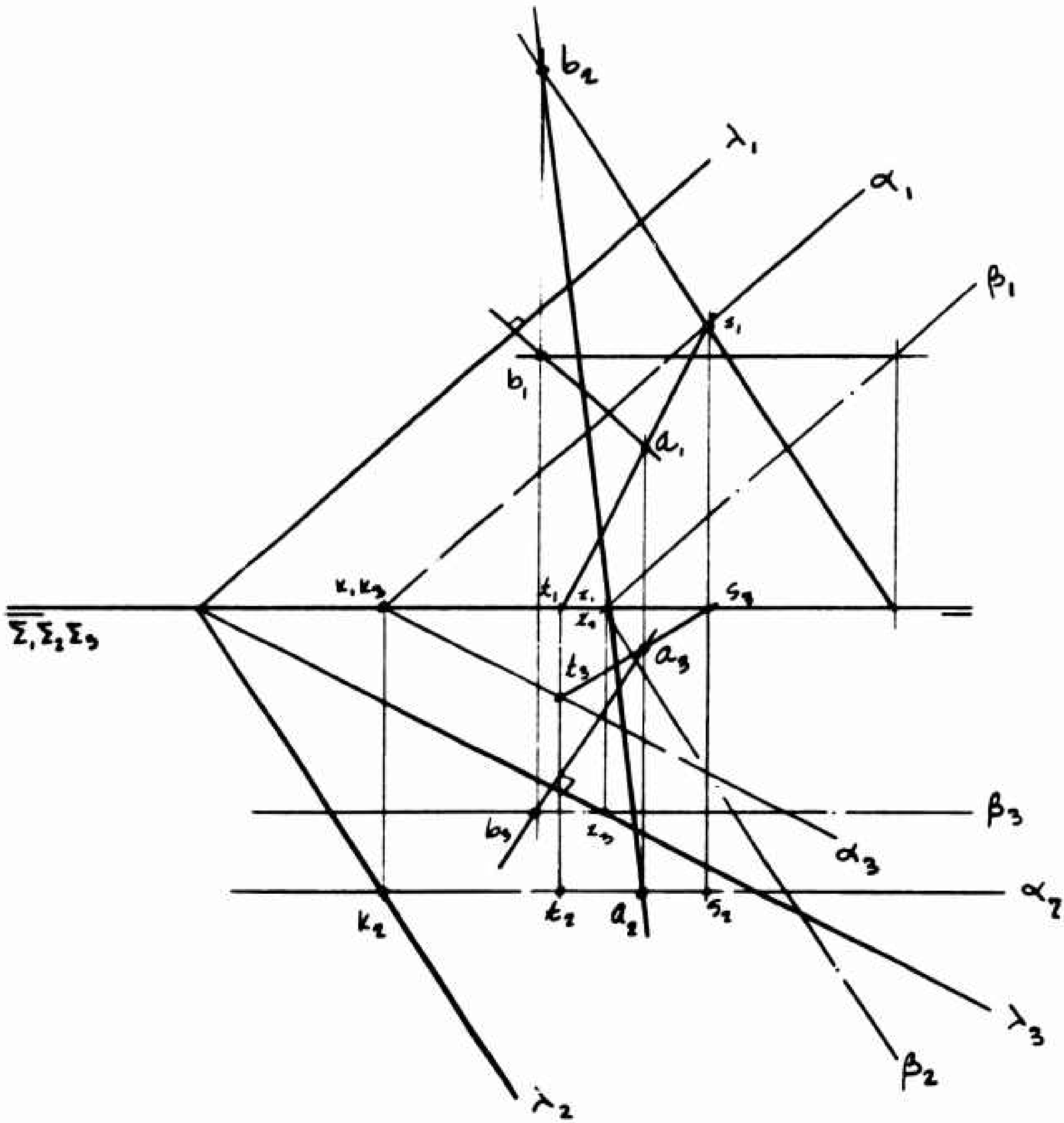

b2
λ1
α1
β1
s1
b1
a1
k1,k3
t1
z1
z2
s3
Σ1,Σ2,Σ3
t3
a3
β3
b3
z3
α3
α2
k2
t2
a2
s2
λ3
β2
λ2

Figure 9

SECOND PROBLEM

"Given a 3-D space T by its traces and a point (a) determine the traces of the 3-D space Λ parallel to T and belonging to (a)".[2)]

Shown in figure 10 are the given 3-D space T and the point (a).

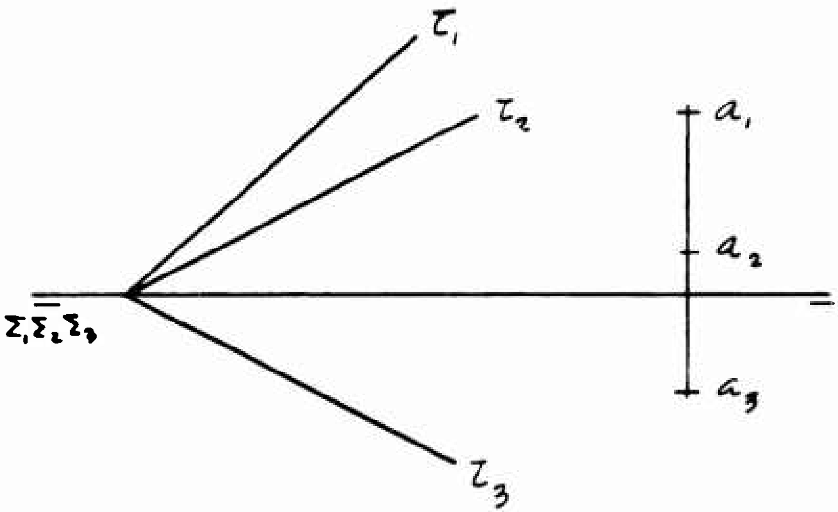


Figure 10

2) Report No. 9, Ibid, pp. 19-20.

SOLUTION

If through (a) we consider the 3-D space Ω parallel to Σ_1, the intersections with the 3-D spaces T and Λ are the planes α and β, respectively, shown in figure 11.

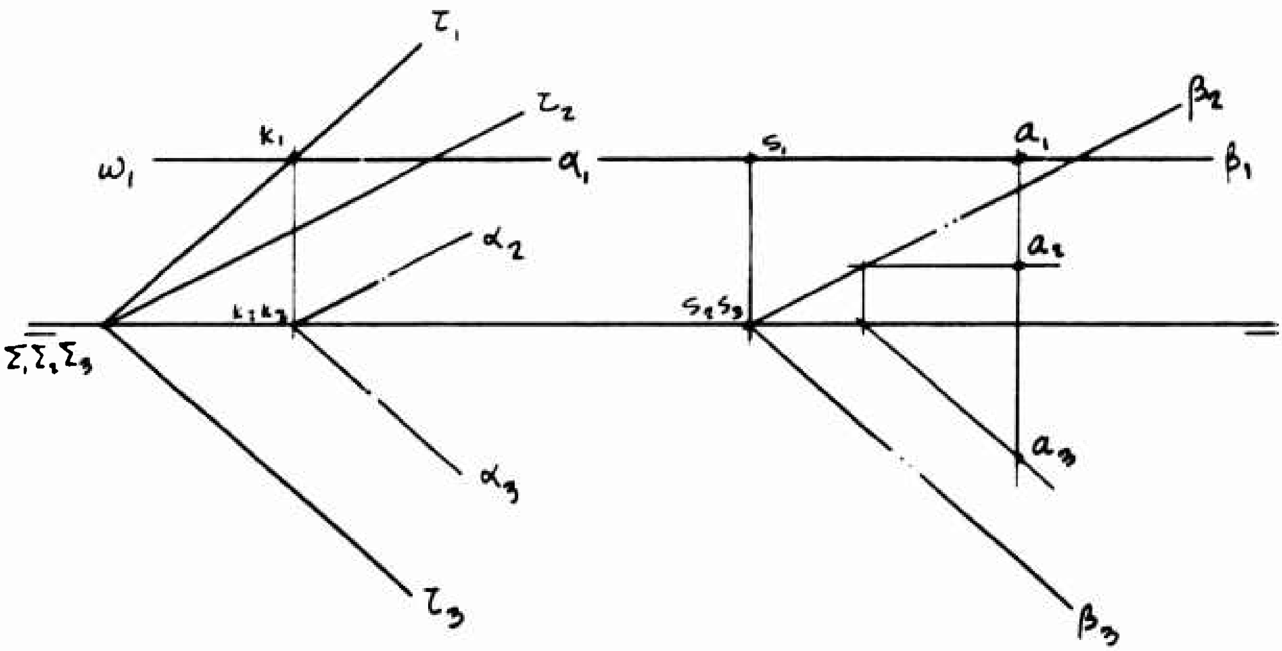


Figure 11

The 3-D space Λ belongs to β and to the point (s), this point belonging to the trace λ_1. Therefore, to determine the traces of Λ, through s_1 draw parallel to τ_1, to obtain λ_1 and the point (o) on the reference line. Through (o), draw parallels to τ_2 and τ_3, to obtain λ_2 and λ_3. See figure 12.

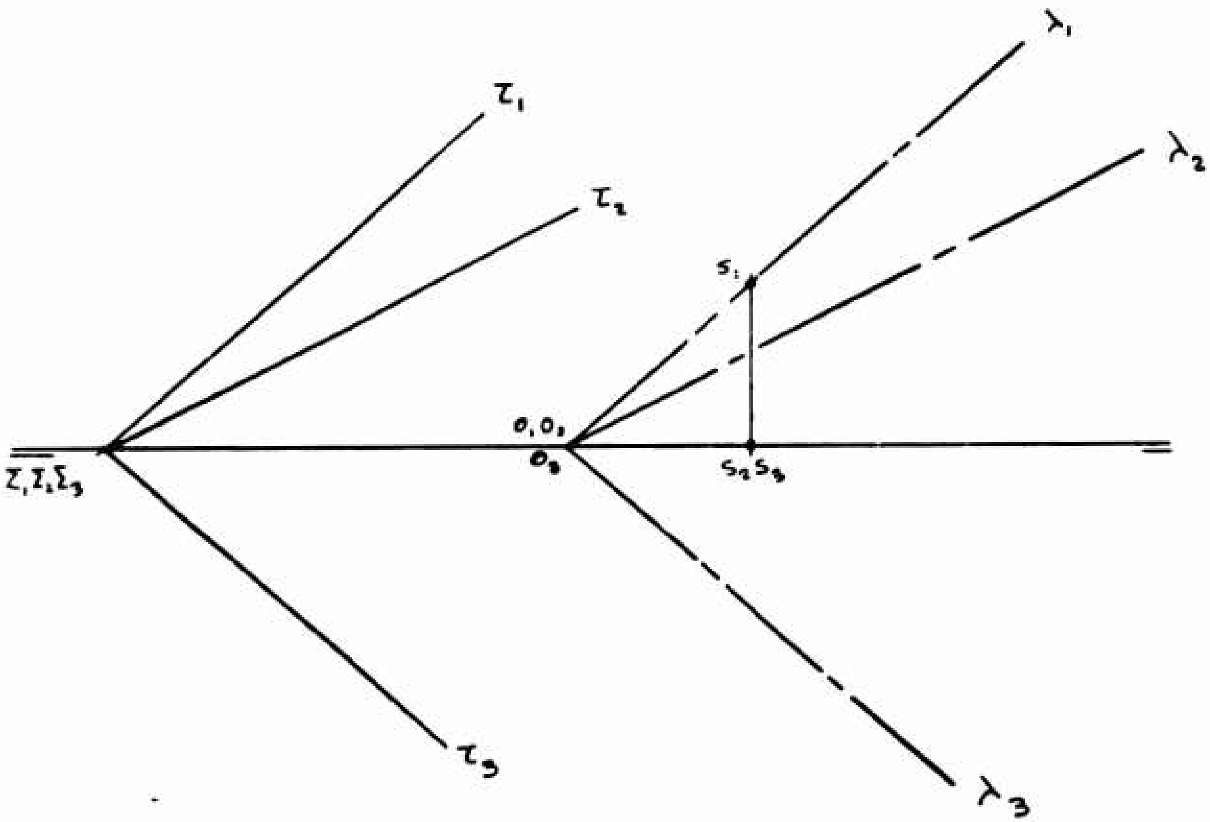


Figure 12

THIRD PROBLEM

"To rotate a 3-D space Ω until its superimposition on one of the 3-D spaces of the 4-D system of reference."

SOLUTION

The rotation is made about a plane of Ω , also belonging to the 3-D space of reference.

Let Ω be the given 3-D space (figure 13) which is to be rotated until its superimposition on Σ_2.

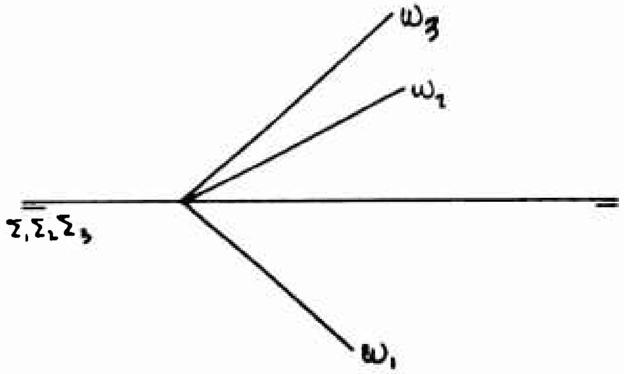
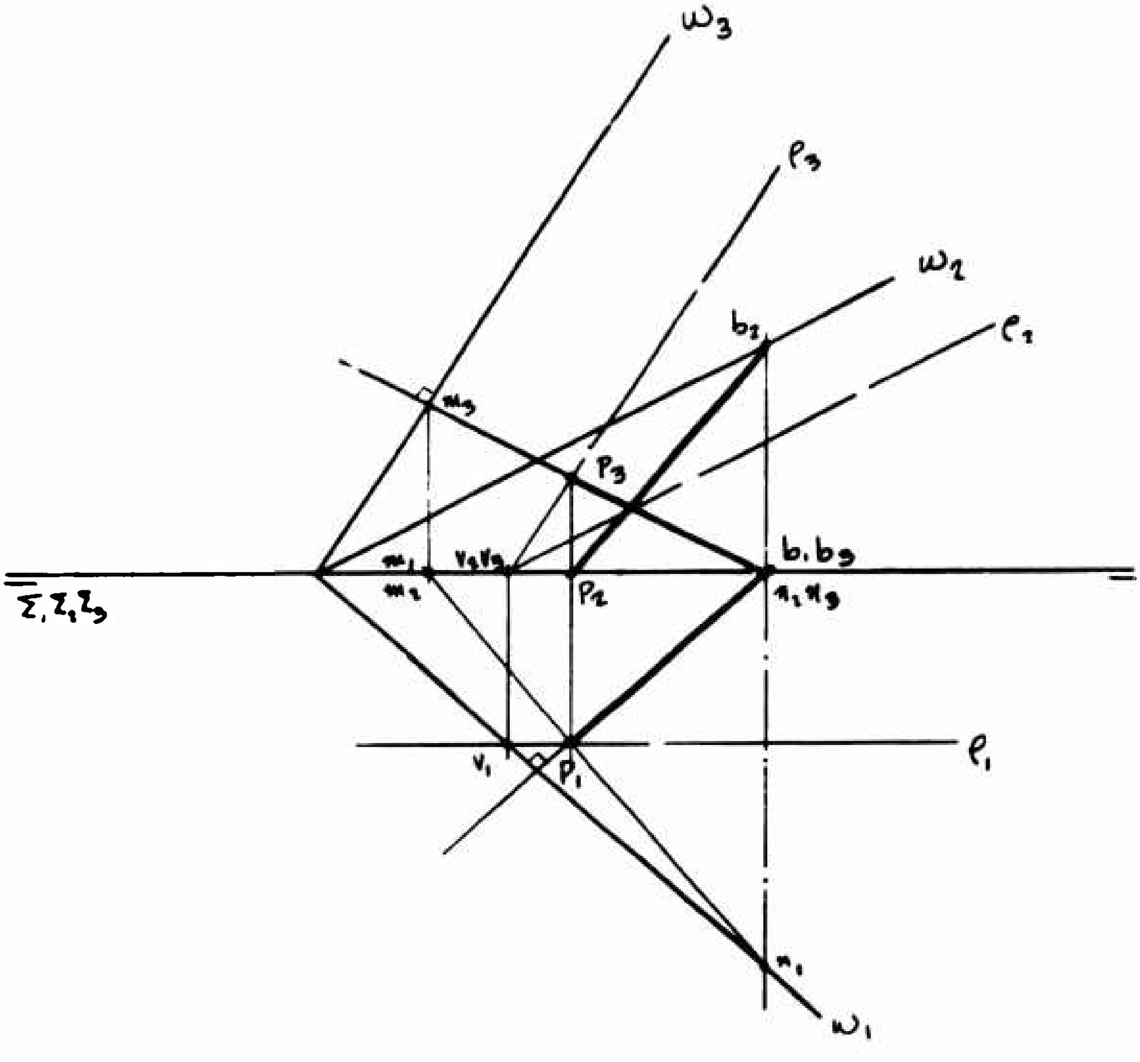


Figure 13

Consider that the three traces, ω_1, ω_2, and ω_3 are the edges of a trihedral angle whose faces are the planes $\omega_1\omega_2$, $\omega_2\omega_3$, $\omega_1\omega_3$, and whose vertice is (o). Therefore, the point (b) of the edge ω_2 can be referred to the opposite face $\omega_1\omega_3$ (in the 3-D space Σ_2) by its projection on that face and by its distance to it.

The determination of this projection and of that distance are indicated in figures 14 and 15. First we draw a perpendicular through (b) to the plane $\omega_1\omega_3$ and determine the foot of the perpendicular, point (p), which is then, the projection of (b) in the plane $\omega_1\omega_3$. The true length of (bp) is the distance from (b) to $\omega_1\omega_3$.

To determine the projection p_3 of the foot of the perpendicular we made use of a plane $\rho(\rho_1,\rho_2,\rho_3)$ belonging to (p) and to the 3-D space Ω. As expected, the projection p_2 is found on the reference line, for the plane $\omega_1\omega_3$ belongs to the 3-D space Σ_2.



<u>**Figure 14**</u>

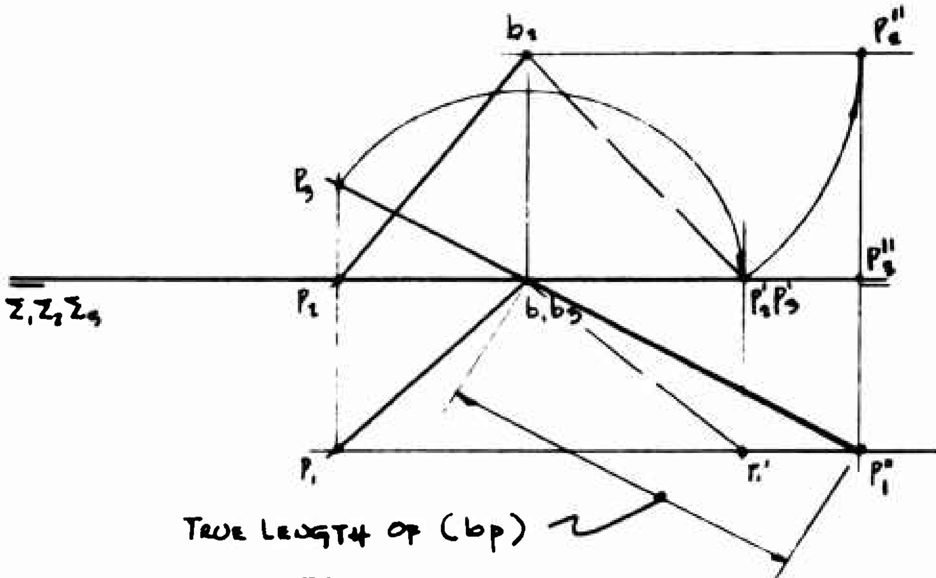


Figure 15

To obtain the location of the edge ω_2 in the 3-D space Σ_2, all we have to do now is to consider the point (p) and the plane $\omega_1\omega_3$ of that 3-D space, and operate the following construction by three-dimensional descriptive method: from (p) raise perpendicular to $\omega_1\omega_3$, in the 3-D space Σ_2 and mark a point (s) on this perpendicular so that (ps) = (bp). Figure 16.

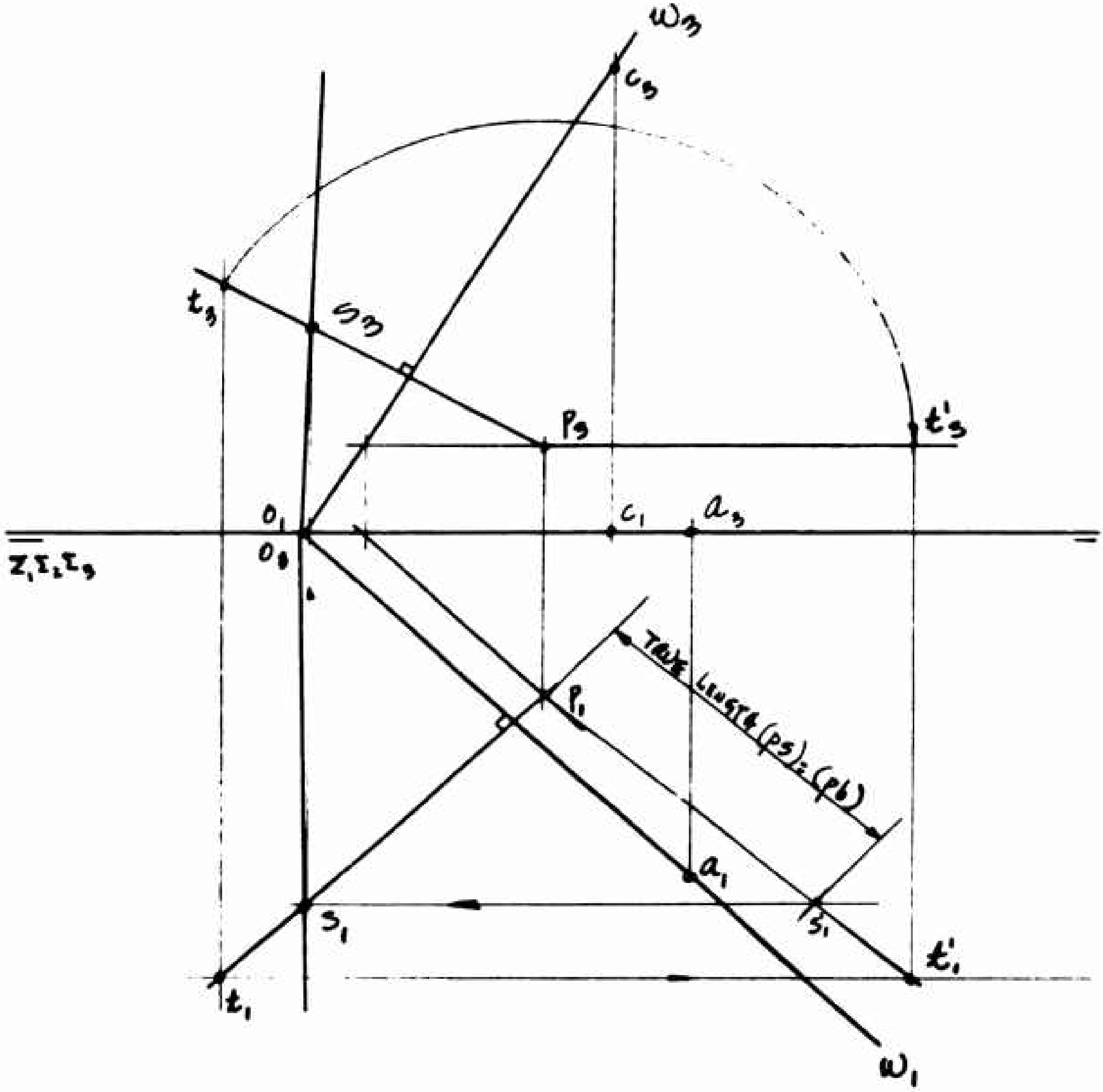

ω3
c3
t3
s3
p3
t'3
o1
c1
a3
o3
Z1Z2Z3
TRUE LENGTH (ps)=(pb)
p1
a1
s1
s1
t'1
t1
ω1

Figure 16

Thus, in figure 16 we can identify a trihedral angle of edges $\omega_1 \equiv$ (oa), $\omega_3 \equiv$ (oc), and (os) whose faces are equal to the trihedral angle $(\omega_1 \omega_2 \omega_3)$ of the 4-D space (figure 13). This trihedral angle (o-acs) characterizes the position of the trihedral $(\omega_1 \omega_2 \omega_3)$ until its super-imposition with the 3-D space Σ_2.

If we want we may check the true value of the faces of each trihedral and see that they are, in fact, equal. Naturally, we can see that the face $\omega_1 \omega_3$ is common to both trihedral angles.

The determination of the true value of the faces of the trihedral $(\omega_1 \omega_2 \omega_3)$ is shown in figure 17.

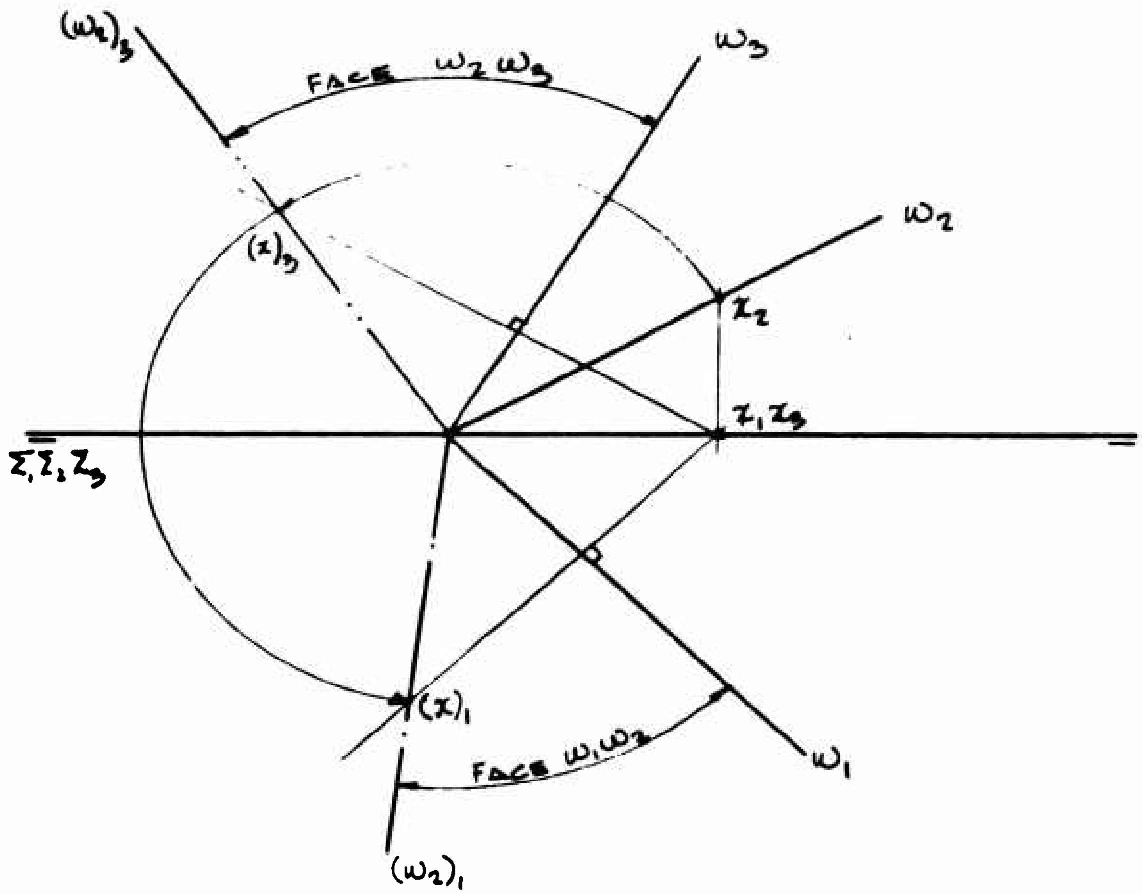


Figure 17

In figure 18 we show the determination of the true value of the faces (o-sc) - corresponding to face $(\omega_2\omega_3)$ - and (o-sa) - corresponding to face $(\omega_1\omega_2)$.

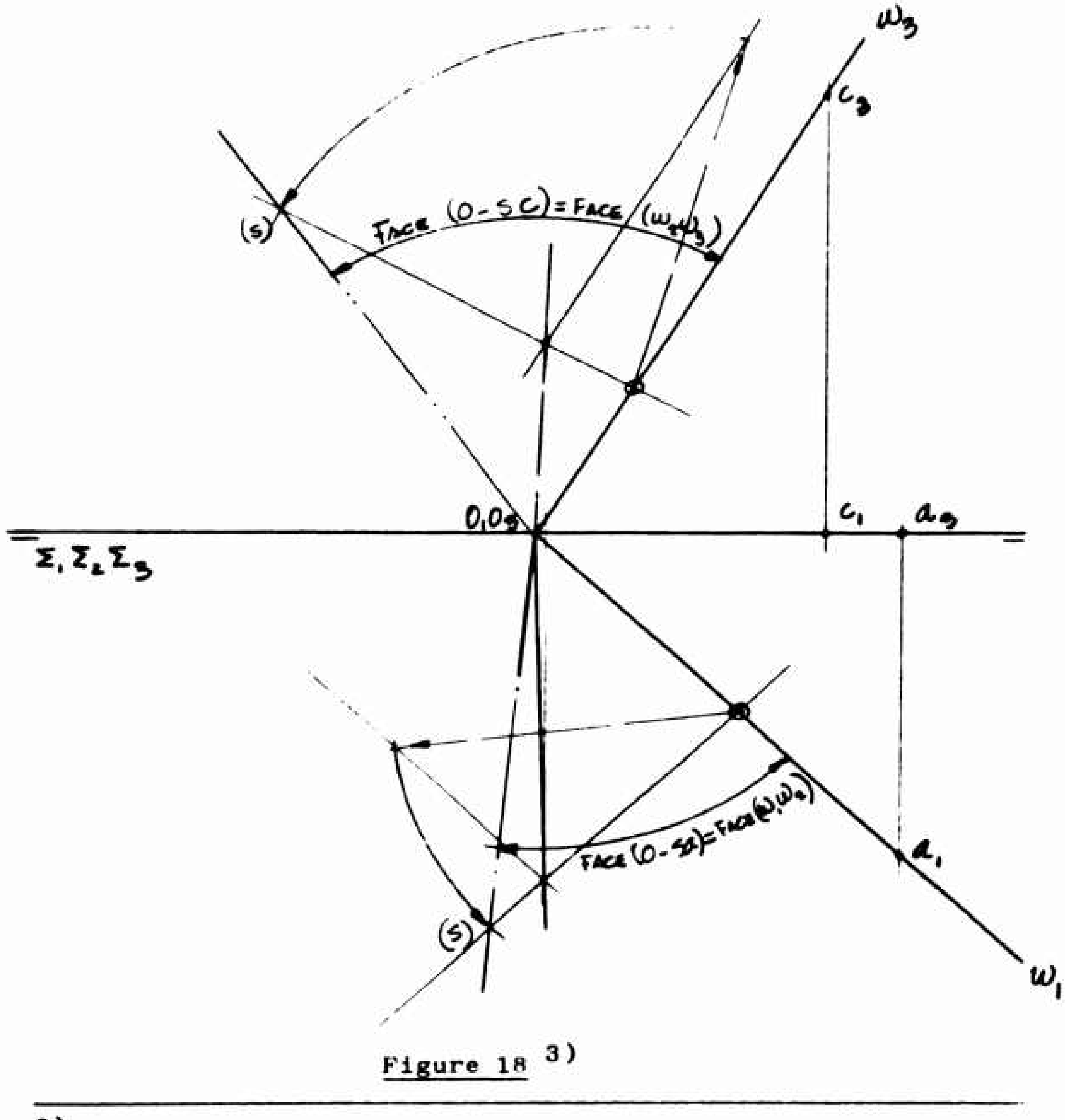


Figure 18 [3]

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[3] Consult a treatise on theoretical three-dimensional descriptive geometry for justification of the constructions.

Any point of the 3-D space Ω may be referred to the plane $\omega_1\omega_2$ as it was done with the point (b) and its position, after the rotation, determined in function of the foot of the perpendicular through the point to $\omega_1\omega_2$ and in function of its distance to that plane. In resume, it becomes a problem of trihedral angle, where are given a face, the projection of a point of the opposite edge on that face, and the distance from the point to the face.

FOURTH PROBLEM

"To rotate a point of a 3-D space about one of its traces until it belongs to one of the planes of the 4-D system of reference."

Let τ be a given 3-D space and (o) a point belonging to a plane γ of that 3-D space. Figure 19.

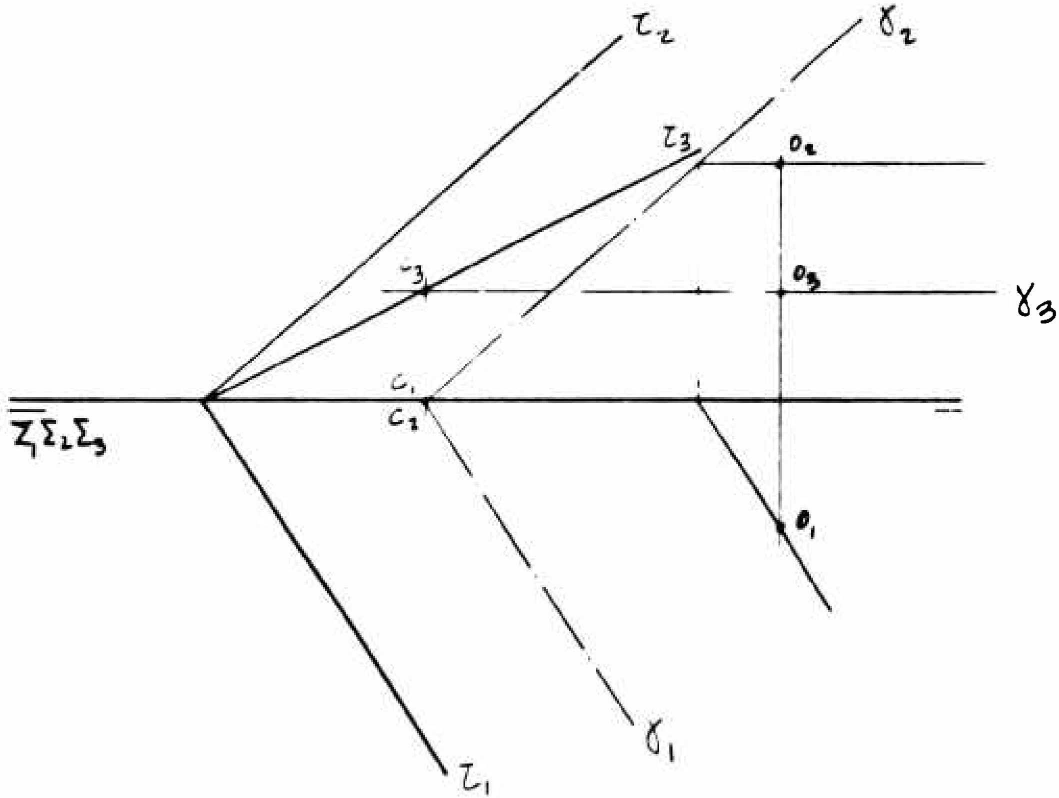

Figure 19

Because of the conditions of belonging between a point and a 3-D space, we may also consider two other planes, α and β, belonging to the point (o) and to the 3-D space T, as indicated in figure 20. Thus, the point (o) is the intersection of planes α, β, and γ.

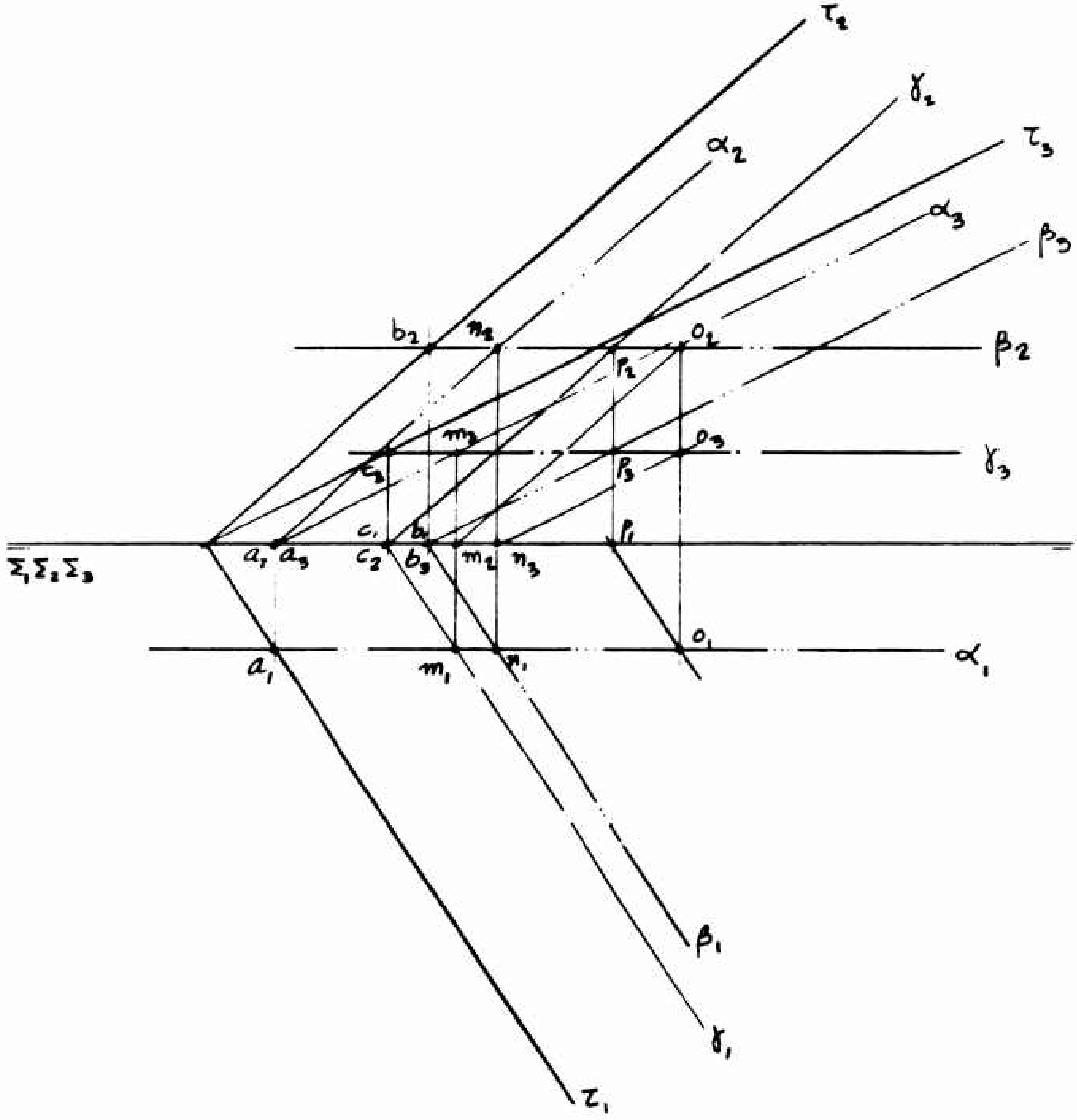

Figure 20

We can write:

Lines (mo) and (no) belong to plane α

$(mo)//\alpha_2 \; : \; (\alpha_2//\tau_2)$

$(no)//\alpha_3 \; : \; (\alpha_3//\tau_3)$

Lines (po) and (no) belong to plane β

$(po)//\beta_1 \; : \; (\beta_1//\tau_1)$

$(no)//\beta_3 \; : \; (\beta_3//(\tau_3)$

Lines (mo) and (po) belong to plane γ

$(mo)//\gamma_2 \; : \; (\gamma_2//\tau_2)$

$(po)//\gamma_1 \; : \; (\gamma_1//\tau_1)$

Therefore,

$$\alpha \times \beta \rightleftarrows (no)$$
$$\alpha \times \gamma \rightleftarrows (mo)$$
$$\beta \times \gamma \rightleftarrows (po)$$

Furthermore, we observe the following conditions of belonging:

Point (n) belongs to γ_1 and to α_3.

Point (n) belongs to β_1 and to α_2.

Point (p) belongs to β_3 and to γ_2.

Point (a) belongs to τ_1 and to plane α.

Point (b) belongs to τ_2 and to plane β.

Point (c) belongs to τ_3 and to plane γ.

If we proceed in obtaining the superimposition of planes $\tau_1 - \tau_2$, $\tau_1 - \tau_3$ on plane π_1, by rotating each of them about the line τ_1, we arrive to the results shown in figure 21.

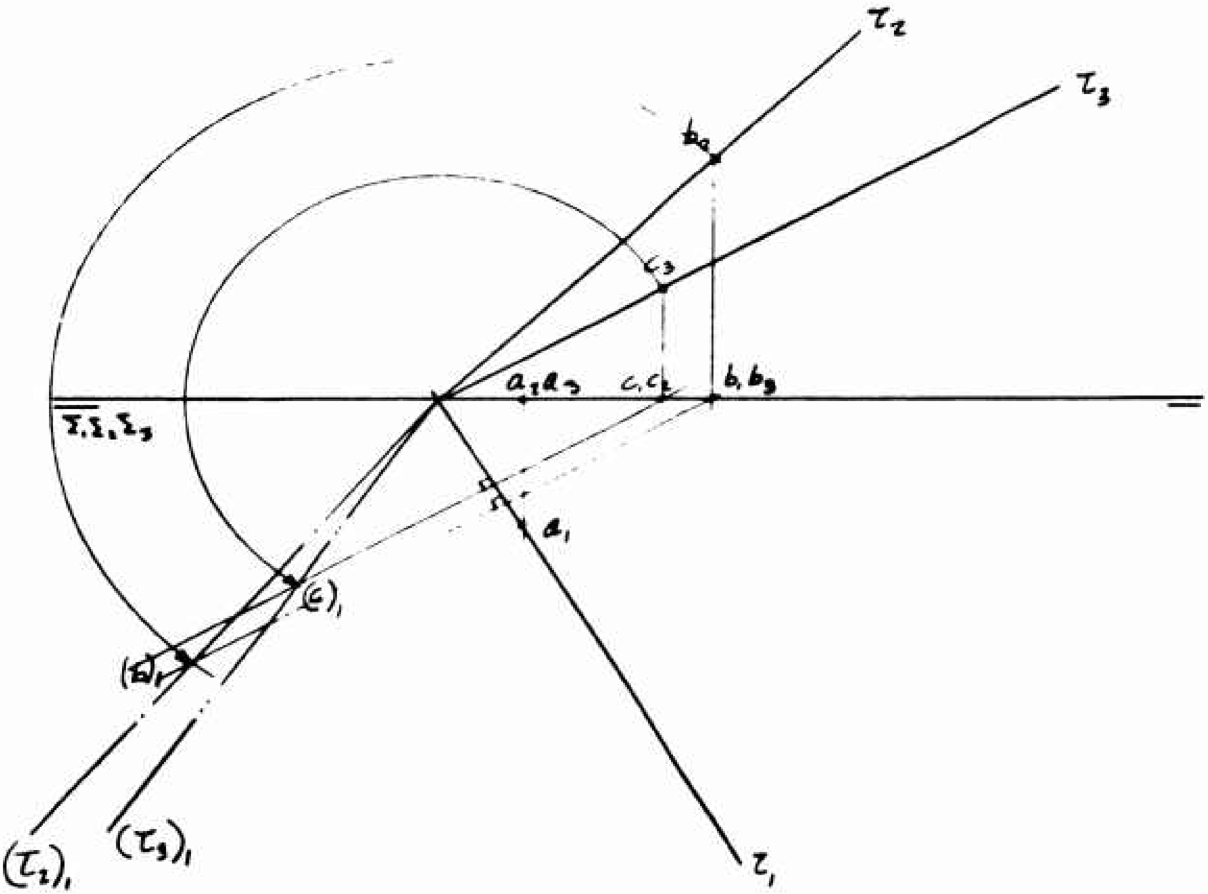

Figure 21

Therefore, due to the conditions of belonging among the points, lines and planes, we obtain the location of these points, lines and planes, when each is superimposed on π_1

by rotation about the line τ_1. See figure 22.

To obtain the position of $(o)_1$, draw perpendicular to τ_1 through o_1: from $(m)_1$, $(n)_1$, and $(p)_1$, draw parallels to $(\tau_2)_1$, $(\tau_3)_1$, and $(\tau_1)_1$, respectively. All these lines will meet in $(o)_1$.

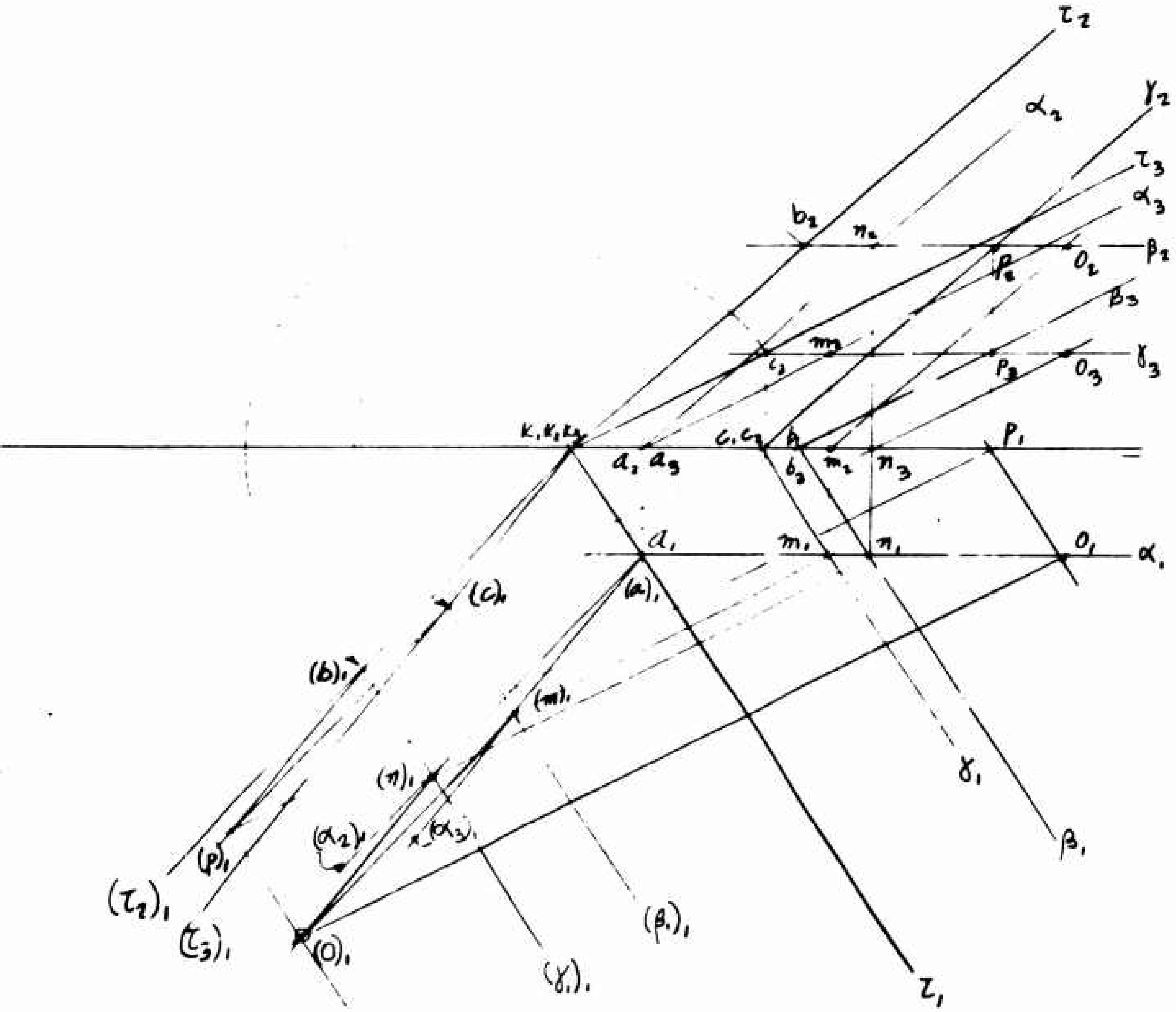

Figure 22

A verification of the construction just outlined is as follows and may be used as a method for determining the position of the points of a plane determined by a trace of the 3-D space and a point. However, if the position of the point is known, in the plane π_1, for example, we will have to make use of the constructions shown in figure 22, in order to determine its projection.

The second method is based on the determination of the true length of the segment (ko). This true length will also appear as $[(k)-(o)_1]$ in the plane π_1. Therefore, to determine the position of $(o)_1$, first determine the true length of (ko) and center in (k) and radius (ko), cut the perpendicular drawn through o_1 to τ_1, in the point $(o)_1$. Figure 23.

<u>Verification:</u>

$[(a)_1 - (o)_1]$ in figure 22

$[(k) - (o)_1]$ in figure 23

$$[(a)_1 - (o)_1] = [(k) - (o)_1]$$

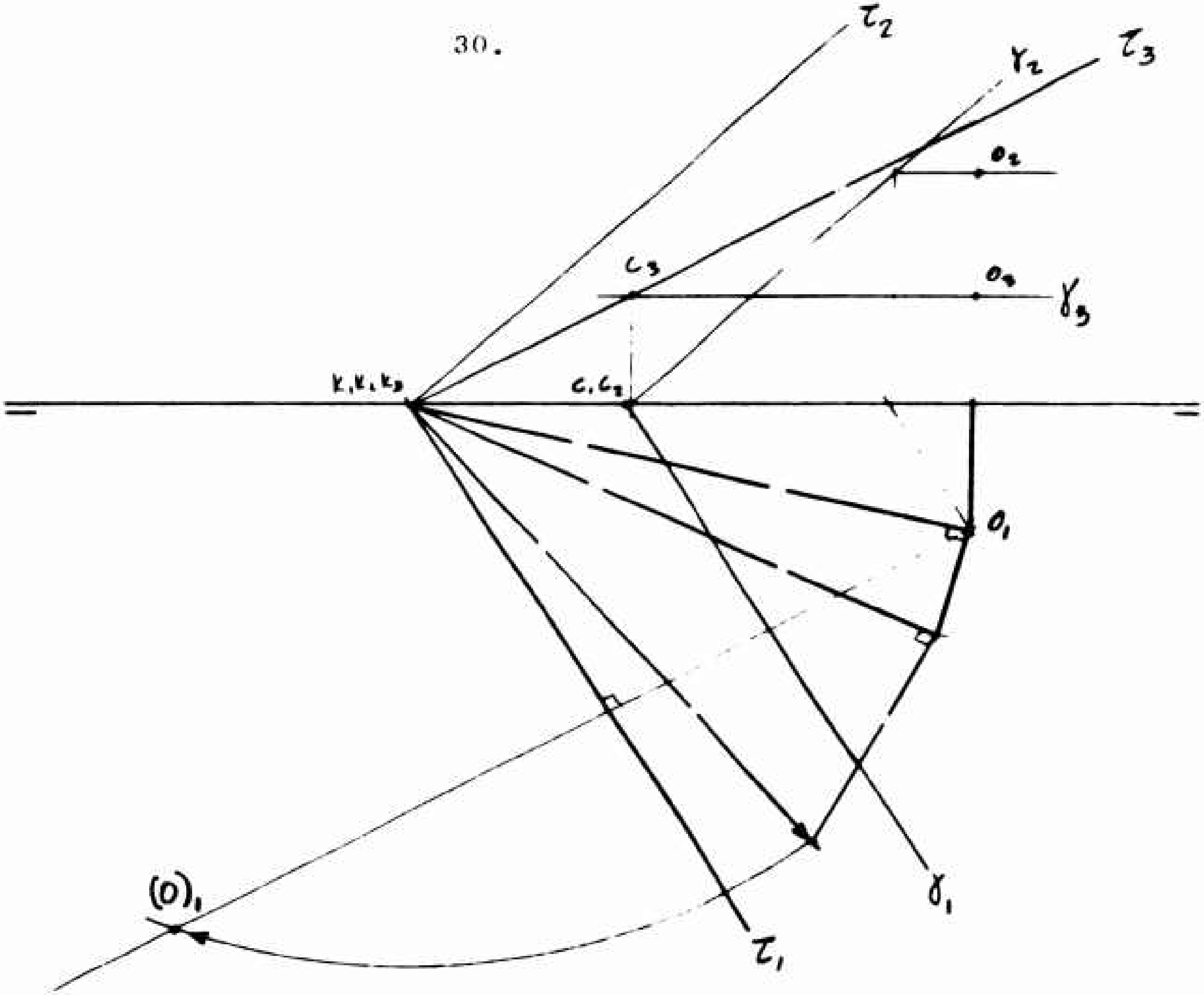

Figure 23

FIFTH PROBLEM

"In the 4-D system of reference there are six planes, two by two perpendicular, three by three belonging to the same line. Question: what is the section made by a plane belonging to a point of the line of three planes and perpendicular to it?"

SOLUTION

Any problem involving plane sections in the four-dimensional space, ought to be preceded by the section of a 3-D space that belongs to the plane.

Let then the planes π_1, π_2, and π_3, (figure 24), be the planes of the 4-D system of reference. They belong to the line L.

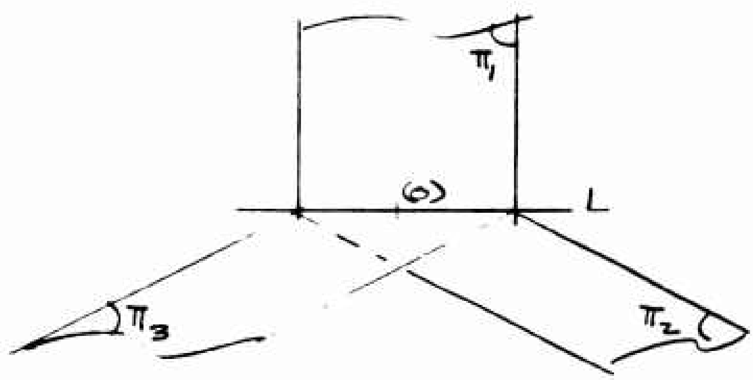

Figure 24

Through the point (o) we shall consider a plane α, perpendicular to L, and also, a 3-D space Ω, belonging to α. Therefore, line L is perpendicular to Ω.

To determine the section made by α, we shall, first determine the section made by Ω in each of the planes π_1, π_2, π_3, that is, in the 3-D spaces Σ_1, Σ_2, and Σ_3

that they determine. According to the study of the representation of a 3-D space, we will obtain three lines of Ω, in π_1, π_2, π_3, all perpendicular to L and belonging to (o). Let these lines be ω_1, ω_2, and ω_3.

We can write:

$$\Omega \times \Sigma_1 \rightleftarrows \text{plane } \omega_2 - \omega_3$$
$$\Omega \times \Sigma_2 \rightleftarrows \text{plane } \omega_1 - \omega_3$$
$$\Omega \times \Sigma_3 \rightleftarrows \text{plane } \omega_1 - \omega_2$$

These are three distinct planes, all belonging to (o), two by two perpendicular, and each one perpendicular to a plane π_i,

$$(\omega_1 - \omega_2) \perp \pi_3$$
$$(\omega_1 - \omega_3) \perp \pi_2$$
$$(\omega_2 - \omega_3) \perp \pi_1$$

and all perpendicular to L.

$$(\omega_1 - \omega_2) \perp L$$
$$(\omega_1 - \omega_3) \perp L$$
$$(\omega_2 - \omega_3) \perp L$$

Therefore, due to the above results, we conclude that the plane α is one of these planes. Thus, the section made by plane α in the 4-D system, are <u>two</u> of the <u>three</u> lines $\omega_1, \omega_2, \omega_3$, depending on the condition of belonging involving α and one of the <u>three</u> 3-D spaces $\Sigma_1, \Sigma_2, \Sigma_3$.

With this result, we want to observe that the observations made in our paper "On the Generation of a Spherical Surface in a Four-Dimensional Space"[3], are not correctly stated, since we assumed that the three lines ω_1, ω_2, and ω_3, (noted in that paper as OA, OD, OE), belong to <u>one</u> unique plane.

3) Technical Seminar Series, Report No. 14, December 10, 1964, Department of Graphics and Engineering Drawing, Princeton University.

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PRINCETON, NEW JERSEY 08540

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Four-Dimensional Descriptive Geometry Problems on 3-D Spaces, March 1965.

Graphical Plotting of Data. An Application of Three and Four Dimensional Theoretical Descriptive Geometries, March 1965.

Four-Dimensional Descriptive Geometry Metric Problems: Angles - Descriptive Solution, May 1965.

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