NAVY DEPARTMENT

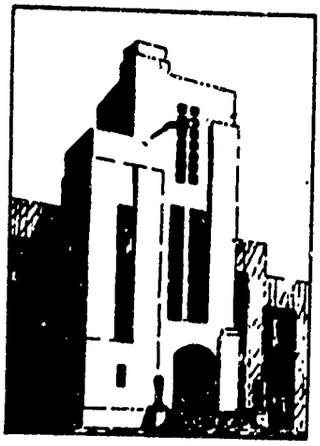
THE DAVID W. TAYLOR MODEL BASIN

WASHINGTON 7, D.C.

RECENT DEVELOPMENTS IN THE THEORY OF SHIP VIBRATION

by

R.T. McGoldrick, A.N. Gleyzal, R.L. Hess, and G.K. Hess, Jr.



October 1953

Report 739

NS 712-064

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First Printing
February 1951

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ABSTRACT

It is shown in this report that by considering the ship hull as a floating beam having shearing and bending flexibility with a distributed viscous damping proportional to mass it is possible to derive equations of motion under external forces by the general Rayleigh method which yields a solution in terms of normal modes of motion. Practical methods of determining natural frequencies and normal modes are discussed, and it is shown that solutions based on finite-difference equations can be obtained with the use either of a digital sequence calculator or an electrical analog. A discussion is given of coupled horizontal bending and torsional modes as well as vertical modes of vibration. It is pointed out that the finite-difference-equation method applied to vertical vibration of a vessel of merchant type has given results in good agreement with experimental values up to the sixth vertical mode.

1. INTRODUCTION

By the term ship vibration as used in this report is to be understood the vibration of the hull considered as a free-free beam. This does not include the vibration of items of machinery or "local" hull structures such as decks, bulkheads, and superstructures except in so far as these items participate in the motion of the entire system.

While the literature on ship vibration is by now quite extensive, the bulk of effort so far has been concentrated on methods of calculating the first vertical mode and of estimating ratios between the frequency of the first mode and that of the second and third modes by comparison with uniform-bar flexure theory.

What led to a recent revival of the study of ship vibration theory at the David Taylor Model Basin and at the University of Michigan under contract with the Office of Naval Research was the realization, during the recent war, that present knowledge of the dynamics of the ship structure was entirely inadequate for the needs of modern design. The same point is emphasized in a

paper presented to the Institution of Naval Architects in 1946 by J. Lockwood Taylor entitled "Dynamic Longitudinal Strength of Ships."

While an estimate of the frequency of the fundamental mode of flexural vibration is very helpful in avoiding the possibility of first-order vibration in this mode due to unbalance of propellers or shafting, it falls far short of enabling one to predict how the hull may respond to impacts from waves or how it will vibrate at blade frequencies. The latter are usually well above the frequency of the fundamental mode. If the ship could respond only in its fundamental mode there would be no wave traveling through it as a result of an impact at either end, for in this mode both ends move simultaneously in phase. On the other hand it is hard to imagine a structure of such size as a ship being struck at one end and responding instantly at the other end.

Although the impetus to this study came from the consideration of shock or transient phenomena, the steady-state vibration problem was considered equally important. It was felt that if the basic theory of the dynamics of ship structure could be worked out the steady-state vibration problem would automatically be included and therefore the problem was formulated on the general basis of the dynamics of ship structure within the elastic limit.

The problem was visualized as being similar to that considered by J. Lockwood Taylor but with more emphasis on steady-state vibration and with damping considered. It requires the mathematical formulation of the equation or equations defining the behavior of the ship girder under dynamic loading and the development of methods of solving such equations. Since there is abundant literature on vibration theory, dating back to Rayleigh's "Theory of Sound" first published in 1877, the problem is chiefly one of finding out how much of previous theory can be applied. This report represents only a start in this direction.

It is also clear that mathematical rigor would be out of the question in a problem in which so many unknown factors exist and experimental verification is necessary at every stage. It is also evident that limits must be set to the scope of the problem. The range of frequencies of vibration must be limited to those in which the hull preserves its beam-like characteristics. This does not mean that vibrations in the range of audible frequencies or even in ultrasonic frequencies are not important but that if such frequencies are set up by some particular dynamic loading applied to the ship they do not greatly affect its behavior in the range of frequencies in which mechanical vibration is ordinarily considered.

Another limitation must also be set, namely that for the permissible stresses. The ship will behave as an elastic system only when stresses remain below the elastic limit. Furthermore under sufficiently high loading, such as due to underwater explosion, buckling and rupture of plating may take place—giving the elastic system different characteristics.

The dynamic loading of a ship may assume many different types. There are the simple harmonic forces resulting from rotation of unbalanced propellers and shafting or from reciprocating engines. There are varying hydrodynamic forces due to the rotation of the propellers in a nonuniform wake. There are varying wave forces in heavy seas and finally, in the case of naval vessels, there have to be considered forces due to gunfire, projectile impact, and explosion phenomena. Again it is necessary to restrict the problem to forces of such magnitude that stresses beyond the elastic limit are not developed. It is further necessary to restrict the loading to certain types that are expressible either graphically or analytically for purposes of analysis.

The lateral load having the dimensions of force per unit length along the hull is assumed to be expressible in the form $P(x, t)$ which means that the load is given at each point x along the length of the ship at any time t.

The problem may therefore be stated in general terms as follows: Assuming the hull of a ship, together with all loading carried by the ship and the surrounding water which moves with the ship, to comprise a mass-elastic system of calculable elastic and inertia constants, what are the principal motions which it executes under external forces acting in planes perpendicular to its longitudinal axis and expressed as a function of time and distance along the length of the ship?

2. THE BASIC EQUATIONS FOUND APPLICABLE TO MOTION IN THE VERTICAL PLANE—NATURAL FREQUENCIES AND NORMAL MODES

Because of the general symmetry of most ships with respect to a vertical plane containing the fore-and-aft centerline, the problem of motion in the vertical plane will be dealt with first and used to illustrate the general approach that has been used in this investigation.

The beam-like nature of a ship's hull is self-evident and has formed the basis for the ordinary strength calculations universally used in design[1] wherein the ship is assumed supported on trochoidal waves which exert a buoyant force per unit length which varies with distance from the end but is

[1]References are listed on page 46.

considered constant in time, that is, the analysis is carried out as a problem in statics.

The simple bending theory of beams has been widely used in deriving a differential equation for the free transverse or flexural vibrations of uniform slender bars, the differential equation being

$$EI \frac{\partial^4 y}{\partial x^4} + \mu \frac{\partial^2 y}{\partial t^2} = 0$$

where E is Young's modulus,

I is the moment of inertia of the area of the section with respect to its neutral axis,

y is a displacement in the vertical plane,

μ is the mass of the bar per unit length, and

t is the time.

The solution of this equation for a uniform bar with free ends yields the formula for the natural frequencies:

$$\omega_n = \alpha_n^2 \sqrt{\frac{EI}{\mu L^4}}$$

where ω_n is circular frequency,

L is the length, and the

α_n's are the "characteristic numbers" arising in the solution of this differential equation with these specific boundary conditions.

As given by Rayleigh[2] in his "Theory of Sound" the characteristic numbers fall closely in the ratios of the odd numbers starting with 3, that is, 3, 5, 7, 9, etc. The first three characteristic numbers are 4.73; 7.853; and 10.996.

It was understood from the beginning of this study that the ship departed materially from the uniform bar whose vibrations have been treated theoretically by numerous authors. The principal elements of departure are the variation in mass along the longitudinal axis, the variation in bending and shear rigidity, and the addition of the inertia effect of the surrounding water. Nevertheless an empirical formula very similar to the above uniform-bar formula was found by Schlick[3] to give a fair estimate of the fundamental vertical mode of vibration provided an appropriate empirical "constant" is used. The well-known Schlick formula is

$$N = \phi \sqrt{\frac{I}{DL^3}}$$

and in Reference 4 are given values of ϕ found applicable to certain classes of ships for a particular system of units used in expressing the other quantities.

Since the work of Schlick, demonstrating that the uniform-bar formula could be utilized in devising a formula for estimating the fundamental frequency of a ship, numerous investigators have considered the possibility of solving the differential equation for the nonuniform bar in bending only, with the thought in mind that thereby the effects of the variation of mass and bending rigidity along the length of the ship would be taken into account. The literature on this subject is now extensive as may be seen by reference to the Bibliography.

In connection with the planning of the research program under disdussion it was concluded that the Stodola method of solving the differential equation of the nonuniform bar as applied by Schadlofsky[5] would give a fairly good estimate of the fundamental frequency and the two-noded vertical normal-mode shape. This is a method of iterated integration for finding normal modes and natural frequencies. This method applied by Stodola to rotors in bearings was adapted to the ship problem by J. Lockwood Taylor,[6] Schadlofsky, and others. Other methods also were considered, such as the energy method applied by Lewis[7] and the Rayleigh-Ritz method recommended by Timoshenko,[8] but what appeared to be lacking was a flexible method of dealing with the general dynamical problem.

The group working at the University of Michigan solved by means of operational calculus the partial differential equation for the uniform bar subject to bending deflection only, having a uniformly distributed viscous damping and acted upon by a transverse load which was an arbitrary function of time and position along the bar. This required the solution of the differential equation

$$EI \frac{\partial^4 y}{\partial x^4} + \mu \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t} = P(x, t)$$

where E is Young's modulus,

I is the moment of inertia of the section area,

μ is the mass per unit length,

c is the damping force per unit length of bar per unit velocity,

y is a displacement normal to the axis of the bar,

x is a coordinate along the axis of the bar,

t is the time, and

P(x, t) is the external force per unit length varying both with x and t.

A solution of this equation was found by the operational method employing the Laplace transformation. The derivation is given in the second progress report of the University of Michigan on its contract with the Office of Naval Research.[9]

The solution shows that, whatever form the function P(x, t) takes, the response of the bar is expressible in a series of normal modes; in other words, the system behaves in general like the systems whose small oscillations were studied by Rayleigh. He showed that, for systems in which the damping was proportional to the mass if the displacements in the normal modes were taken as generalized coordinates, the kinetic and potential energies could each be expressed in terms of sums of squares of these coordinates and the rate of energy dissipation could be expressed as a sum of terms involving the squares of the generalized velocities. Moreover such a system does not partake of wave motion in the ordinary sense in that there is no fixed rate of propagation of a flexural wave. If the bar is struck at one end, a finite time will be required before a finite motion takes place at the other end, but the process is the result of compounding motions in normal modes in each of which the system deflects simultaneously at all points rather than the result of a flexural wave traveling back and forth.

The normal modes of such a system also possess the property of orthogonality with respect to the mass per unit length; that is

$$\int_0^L \mu X_n(x)\, X_m(x)dx = 0$$

where $X_n(x)$ and $X_m(x)$ are any two distinct normal modes.

Another important result is that the influence function is the same as the normal-mode function. This means that the effect of a concentrated force in exciting any given mode of vibration is proportional to the amplitude of the mode shape at the point at which it acts and is therefore zero at the nodes. There also follows from this result a reciprocity theorem similar to Maxwell's theorem for statically loaded beams: If a simple harmonic force applied at x_1 produces a certain amplitude at x_2 the same force applied at x_2 would produce this same amplitude at x_1.

It also follows from the solution of the uniform-bar problem that in each normal mode the system behaves as a system of one degree of freedom would behave and as though this mode only were present. The amplitude produced in each mode by a given simple harmonic driving force depends on the magnitude of the force, the influence function, the effective mass, stiffness, and damping constant of the system in that mode and on the ratio of the frequency of the force to the natural frequency of the mode. The ordinary resonance curve for

a system of one degree of freedom is applicable to each normal mode individually. From this it follows that if the frequency of the force coincides with one of the normal-mode frequencies, unless the system is very heavily damped, the amplitude in that mode will so far outweigh the amplitude in any other mode that that mode will predominate.

The calculation of the response of such a system in each of its normal modes is greatly facilitated by the application of the concept of "effective" values. While each normal mode responds as a system of one degree of freedom, the values of the effective mass, spring, and damper of the equivalent one-degree system are different for each mode. The response in each normal mode is obtained by treating the familiar differential equation for one degree of freedom by any one of a number of standard methods. The equation for the nth mode is

$$M_n\ddot{q}_n + C_n\dot{q}_n + K_nq_n = Q_n(t)$$

The effective values to be used in this equation are defined as follows:

$$M_n = \int_0^L \mu X_n^2(x)dx$$

$$C_n = \int_0^L cX_n^2(x)dx$$

$$K_n = \omega_n^2 M_n$$

$$Q_n(t) = \int_0^L P(x, t)\ X_n(x)dx$$

where $X_n(x)$ is the nth normal-mode function in arbitrary dimensionless units and ω_n is the undamped frequency. The q's have the dimension of length so that the displacement at any point x is

$$\sum_{n=1}^{n=\infty} q_nX_n(x)$$

Further discussion of the process of using effective values for the various normal modes as applied to the nonuniform bar with both shear and bending is given in Section 4.

Thus the dynamic problem of the uniform bar as envisaged above can be readily handled once the normal modes and natural frequencies have been determined. Rayleigh[2] showed the great generality of the concept of normal modes and natural frequencies for small motions of mass-elastic systems. The ship, although having a nonuniform distribution of mass and stiffness, may be approximated by a system of lumped masses connected by inertialess elastic members. Thus it may be expected to enjoy the properties of the systems treated by Rayleigh. For elastic systems with damping the "dissipation function" must be of a certain type, namely, proportional to velocity and proportional to mass. Hence if a linear partial differential equation in x and t describes the dynamical behavior of the ship which is a nonuniform beam and if Rayleigh's theory applies, then although EI, μ, and c all vary with x, the ratio c/μ must be constant for all values of x.

An appreciation of the extraordinary properties of such a system is of the greatest assistance in understanding its response to both steady-state and transient forces and one of the major phases of the program undertaken by the Taylor Model Basin was to determine the extent to which the ship, in spite of its complexities, possesses the properties of the systems investigated by Rayleigh.

Considerable attention was given to the extension of the calculation of vertical flexural modes of ships into the higher modes by the Stodola and Rayleigh-Ritz methods (see Reference 9), but it was soon found that the method of finite differences as applied to the problem of critical speeds of shafts by Prohl[10] and Myklestad[11] offered the most promise. Study of the latter method showed that it could be readily adapted to the free-free beam by suitable allowance for the particular end conditions involved. A Taylor Model Basin report[12] was published in July 1947 showing how the method based on bending theory only could be carried out for a ship by means of a digital computer. While the necessary data for carrying out such calculations on a naval vessel were being worked up from ship plans, a test was made on the ship in question by means of a vibration generator. Details of this test will be found in another Taylor Model Basin report.[13] Ten modes of vibration of the vessel were found and investigated, the vessel being anchored in deep water. The vessel, the USS NIAGARA (AFA87)—an attack transport, was a converted merchant ship and for this purpose had the advantage over other classes that it carried no armor and that the guns had been removed so that the discontinuities existing on many classes of naval vessels were not a problem. Further details of this vessel are also given in Reference 14.

When calculations were made for the NIAGARA by the Prohl-Myklestad method, it was found that the frequency of the fundamental vertical mode checked the experimental value fairly well but that the frequencies corresponding to the second and third modes were much too high and the discrepancy increased with the order of the mode. It was to be expected that the frequencies would be too high since the method made no allowance for shear deflection, but the magnitude of the discrepancy in the second and third modes was so great as to make it apparent that unless a shear flexibility term could be incorporated into the differential equations of the nonuniform beam attempts to calculate the higher modes would be fruitless.

The task of incorporating a term for shear flexibility into the finite difference equations was not found too difficult and in fact it was found that a term for the effect of rotary inertia could also be readily included. The problem was arranged for digital computation with both shear and rotary inertia terms included and was carried out for the USS NIAGARA for a number of different assumptions: 1) That the hull could deflect in bending only, 2) that the hull could deflect in shear only, 3) that the hull could deflect in combined shear and bending with rotary inertia omitted, and 4) that the hull could deflect in shear and bending with the rotary inertia term included. These variations were readily tested without changing the coding of the problem for the digital sequence-controlled calculator simply by making the appropriate terms in the table of initial data equal to zero. The results of the calculation are summarized in Reference 15. The principal facts disclosed were the following: The calculation based on bending only is in fair agreement for the first vertical mode but becomes progressively too high beyond the first mode; the calculation based on shear deflection only is quite high for the first mode but becomes progressively nearer the true value as the order of the mode increases; in the case of USS NIAGARA the inclusion of rotary inertia had a negligible effect on the results.

As will be seen from the tabulation in Reference 15, the calculations based on shear and bending with rotary inertia neglected check the experimental values up to the sixth mode within 5 percent with the exception of the fundamental mode.

From the profile of this vessel it can be seen that its island or superstructure comprising three decks extends for about 30 percent of the length of the hull. When the moment of inertia of this island was added to

the moment of inertia previously computed up to the weather deck and the calculation repeated, it was found that the first mode checked within 1 percent but that the remaining frequencies were all too high. It thus appeared that the stiffening effect of a superstructure of such proportions cannot be neglected in the first mode but that it has little effect beyond the first mode. This does not seem at all unreasonable as the first mode is the only one in which bending predominates and the superstructure probably adds very little to the shear stiffness.

The method of finite differences, as described in Taylor Model Basin reports 632, 706, and 715, thus appears to offer the most promise at the present time. This may be carried out by a digital sequence-controlled calculator or, as is shown later in this report, by an electrical analog. At this writing, however, the question as to how much the superstructure contributes to the bending and shear rigidities remains undetermined. Empirical formulas may be given which will give effective values of EI for computing the fundamental vertical mode, but the value may not be applicable to the high modes. Until more experimental data on this subject are accumulated it appears advisable to make calculations with and without the superstructure. Judgment will be required in interpolating between the values obtained from the two cases. When improved methods of evaluating the coefficients to be used in the differential or difference equations are devised, such interpolations should no longer be necessary.

The difference equations used in this calculation (neglecting the term for rotary inertia) are as follows:

$$V_n = V_{n-1} + \mu_{n-1} \Delta x y_{n-1} \omega^2$$

$$M_n = M_{n-1} + V_n \Delta x$$

$$\gamma_n = \gamma_{n-1} + \frac{M_{n-1} \Delta x}{EI_{n-1}}$$

$$y_n = y_{n-1} + \gamma_n \Delta x - \frac{V_n \Delta x}{KAG_n}$$

where V is vertical shearing force,

M is bending moment,

γ is slope due to bending only,

y is vertical deflection

μ is mass per unit length including virtual mass of the surrounding water,

E is Young's modulus,

I is the moment of inertia of the section area up to the weather deck,

K is the ratio of the average shear stress to the shear stress at the neutral axis under vertical loading,

A is the section area,

G is the shear modulus of elasticity, and

n is the number identifying the station along the length of the ship.

In the calculations made for the NIAGARA the virtual mass was derived from Lewis' data[16] and checked by the approximation given by Prohaska.[17] The added mass was kept the same throughout the calculation which is not in accordance with Lewis' theory. If there were no other unknowns in the problem, the good results obtained could be used to support the view that the virtual mass for vertical vibration does not vary appreciably from one mode to another. However, the data on one ship cannot be used at the same time to verify a theory of virtual mass and a theory of elastic behavior. Hence the ultimate verdict on the soundness of the method presented here must await the outcome of further comparisons between calculated and experimental data.

It is now in order to inquire what answer the comparison between experimental and theoretical results obtained so far give to the question of the similarity between the dynamical behavior of the ship and the ideal systems treated by Rayleigh. In the case of the NIAGARA vertical normal modes up to the sixth have been both calculated and observed experimentally. However, beyond this point normal modes could not be found and no appreciable amplitudes could be produced although the vibration generator was operated well above this range.

While as yet the situation is obscure in the neighborhood of the sixth mode and beyond because of the limited amount of experimental data, this does not impair the validity of the method in obtaining components of response

up to modes of this order. It is believed at present that modes beyond the sixth or thereabouts are difficult to excite because of the increased impedance as shown by Equation [111] in Reference 9 and because of the tendency for cancellation with components in adjacent modes.

The naval architect is concerned chiefly with how the ship will respond to a first-order exciting force (one cycle per revolution of shaft) due either to an unbalanced propeller, unbalanced shafting, or a bent propeller blade and to blade frequency exciting forces whose order is equal to the rpm times the number of blades per propeller.

Except on small high-speed craft the first-order vibrations will fall chiefly in the range 50 to 500 cpm and the blade frequencies in the range 150 to 1500 cpm. When first-order vibration occurs, it is usually due to resonance with one of the lower modes of the hull and in that case the vibration will be felt throughout the ship except in the vicinity of the nodes. However, it is rather common to observe that from about half full power on up to full power a ship will vibrate at blade frequency at the fantail rather steadily over this speed range with no indication of a resonance and with the vibration extending only for about a quarter of the length of the vessel. According to the theory presented in this report an explanation of this phenomenon appears possible. The plane of the propellers is usually at a distance of only about 10 percent of the length of the ship from the after perpendicular, and the experiments on the NIAGARA showed the plane of the propellers to be aft of the node for the first five modes of vertical vibration. In steady-state vibration the displacement at the driving point is in phase with the driving force well below resonance, 90 degrees out of phase (lagging) at resonance, and 180 degrees out of phase well above resonance. It would be a common condition for the blade frequency at the normal running speed to fall above the natural frequency of the fifth vertical mode in which case all the first five modes would be excited at the driving frequency and in phase at the fantail. While the fifth mode would have a much greater resonance factor than the first, its influence function would be much less as its aftermost node is much nearer the stern. Thus the product of resonance factor and influence function for an excitation at the propeller could well be of the same order of magnitude for the first five vertical mode components. If this were the case, the components of amplitude in each mode would be the same at the fantail and they would all be in phase. They would not all reinforce one another, however, throughout the ship. In Figure 15 of Reference 18 is shown the result of combining the first five vertical modes of the NIAGARA when the amplitudes are made equal and in

phase at the after perpendicular. It can readily be seen from this figure that a forced vibration concentrated in the stern is consistent with the theory.

3. THE PROBLEM OF EVALUATION OF COEFFICIENTS APPLICABLE TO VERTICAL VIBRATION

It is obvious that the accuracy of any theoretical calculation is limited by the accuracy of the system constants used. The calculation of vertical modes as outlined in the preceding section requires the evaluation (at equidistant vertical cross sections of the ship) of the effective mass per unit length, the bending stiffness, and the shear stiffness. The evaluation of each of these quantities must obviously be based on certain simplifying assumptions.

In determining the mass per unit length the weights of hull, machinery, fuel, water, and stores are generally known with sufficient accuracy for such a calculation in the early stages of design. The mass that must be added for the inertia effect of the surrounding water is considerable and varies with the change of form of the hull in passing from stern to bow. Theoretical values were worked out for various ship-type forms from potential flow theory by F.M. Lewis[16] and, wherever values estimated in accordance with his paper on the subject have been used by the Taylor Model Basin in the calculation of the first mode of vertical vibration of full-scale ships, the results have been good. If there were no other uncertainties in the problem, this could be considered a sufficient verification of Lewis' method. This, however, is not the case and all that can be said at present is that the indication is that Lewis' method gives a good estimate of the virtual mass for the first vertical mode. For the calculation of higher modes it is of the utmost importance to know whether the virtual mass remains the same or whether it varies. It is also important to know whether it varies with frequency and amplitude. Lewis considered it to be independent of frequency and amplitude for any one mode but to vary with the order of the mode due to increasing departure from two-dimensional flow.

Model experiments made at the U.S. Experimental Model Basin[19] had not been too helpful in checking the virtual mass theory and, because of the high frequencies that had to be used, were not considered reliable. In view of the uncertainty of the whole question it was decided to use Lewis' values for the two-noded mode throughout the calculation and as has been pointed out

the results on the NIAGARA were fairly good up to the sixth vertical mode. Moreover, it was found that the data given in Prohaska's paper[17] yielded values very close to those given by Lewis' method without the necessity of graphical work. This procedure is as follows:

Divide the ship into twenty equal sections starting with the after perpendicular.

Let b = half breadth of a section at the waterline in feet,
B = whole beam of the section in feet,
L = length between perpendiculars in feet,
d = draft in feet,
β = sectional area coefficient, that is the ratio of the area of the underwater section to 2bd,
C = section coefficient,
J = longitudinal factor,
ρ = weight density of sea water in tons/ft^3, and
g = acceleration of gravity in ft/sec^2.

Estimate β for each section by inspection. Then compute B/d for each section. Next find the value of C corresponding to these values of β and B/d from Figure 16 of Prohaska's paper reproduced as Figure 1. Compute L/B where B is taken for the midship section, and evaluate J from Figure 2. This figure is based on Figure 17 of Prohaska's paper.

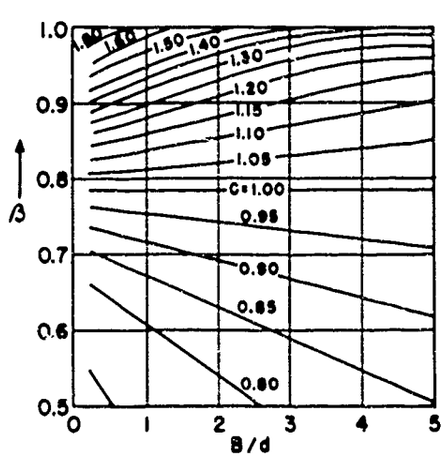

Figure 1 - Curves for Estimating the Coefficient C*

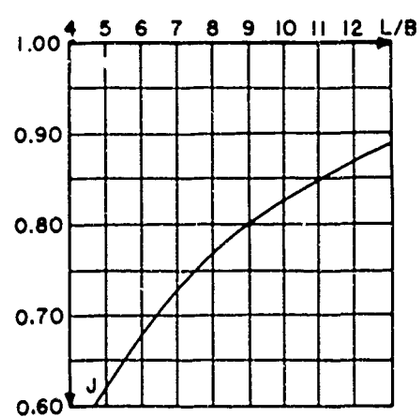

Figure 2 - Curve for Estimating the Coefficient J*

*The curves of Figures 1 and 2 were taken from Figures 16 and 17, respectively, of: Prohaska, C.W., "The Vertical Vibration of Ships, The Shipbuilder and Marine Engine Builder, October and November 1947.

The virtual mass to be added to the mass of each section of the ship is then obtained from the formula

$$VM = \frac{JC\pi b^2 \rho \Delta x}{2g}$$

The term representing bending stiffness in the calculations, namely $\frac{\Delta x}{EI}$, is evaluated on the basis of simple-beam theory but with allowance for judgment in selecting the items of the section area to be included in the moment of inertia. Although the subject is a very broad one, what experimental evidence is available[20] seems to indicate that in most usual designs simple-beam theory applies quite well when static bending loads are applied to a hull. The beam is considered to be the shell with its longitudinal framing up to the weather deck, and in general the superstructure is not included in the calculation. Also as in simple-beam theory stresses due to bending loads have been found to be proportional to the distance from the neutral axis. The following rules appear to be consistent with the experimental evidence. Shell plating and all longitudinal girders are included. Longitudinal bulkheads are included in the evaluation of the moment of inertia of a section only when they extend at least one bulkhead space beyond the section. Decks with expansion joints are not included. Side armor is not included but experimental justification for this is lacking; this same remark applies to armored decks. Machinery members are not considered to contribute to longitudinal strength of hulls. As pointed out elsewhere in this paper it appears that account need be taken of the increased bending strength due to a superstructure by making calculations with and without the superstructure included.

The shear flexibility term $\frac{\Delta x}{KAG}$ has been derived so far on the basis of the theory of rectangular beams. In this an expression for the shearing stress at any distance from the neutral axis is obtained by considering the equilibrium of the parallelepiped formed by two transverse planes a distance Δx apart and two horizontal planes one of which is the top surface of the beam and the other of which is the plane at the reference distance from the neutral axis. A is the total area of the section carrying bending stress, G is the shear modulus of elasticity, and K is given by the formula

$$K = \frac{Ib}{A(\bar{v}A)}$$

where I is the moment of inertia of the beam with respect to the neutral axis,

b is the width of the beam at the neutral axis,

A is the area of the section, and

$(\bar{v}A)$ is half the moment of area of the section with respect to the neutral axis.

If V is the vertical shearing force, the deflection between two sections distance Δx apart is given on this theory by the formula

$$\Delta y_s = \frac{-V}{KAG} \Delta x$$

Hovgaard[21] gives the following formula for the static deflection of a ship due to shear:

$$y_s = \frac{1}{G} \int_0^L \left[\frac{V}{I^2} \int \frac{m^2}{t} ds \right] dx$$

In this formula V is the vertical shearing force at the section,

I is the moment of inertia of the section, and

$\int \frac{m^2}{t} ds$ is obtained by integration around the shell, s being a distance measured along the perimeter of the plating, t being the plating thickness at the point in question, and m being the moment with respect to the neutral axis of the area between the point in question and the centerline of the top deck.

For the two formulas to yield the same deflection due to shear $\frac{1}{I} \int \frac{m^2}{t} ds$ must equal $(\bar{v}A)/b$. For a rectangle of width t and depth 2h the first formula gives h/2t while the second gives $h^2/2$. Hence in the case of a rectangle the two formulas agree only when ht = 1.

An exact stress analysis of a structure as complicated as the hull of a ship is a problem for the future. Another possibility is to make the simplifying assumption that the vertical shear load is carried only by the side shell plating and is uniformly distributed over the area of this plating. The shear flexibility then becomes simply $\Delta x/GA$ where A is the area term in question. An attempt has been made at the Taylor Model Basin to derive equations relating shear and bending flexibilities from energy considerations with certain simplifying assumptions but the results are not presented here because they have not been checked experimentally as yet. The problem may be described as that of reducing the three-dimensional elastic properties of a ship to those of an equivalent one-dimensional beam.

Although little has been said about it up to this point, it is essential to include in the discussion of the evaluation of coefficients the damping coefficient. While the damping is known to be quite low at low frequencies and small amplitudes so that it has a negligible effect on the natural frequencies and normal-mode shapes, an estimate of its magnitude is a prerequisite to a calculation of amplitudes of forced vibration at resonance since in this case it is the damping alone that limits the amplitude attained.

It might seem natural that since the ship is vibrating in a fluid medium most of the damping would be due to the absorption of energy by the medium. This, however, does not appear to be the case. Not only is the total damping quite low but the fraction of it due to the surrounding medium appears to be quite small. From measurements of the decay of the first mode of vertical vibration set up by dropping and then snubbing the anchor, it has been found that the logarithmic decrement is approximately 0.03, which for a system with viscous damping corresponds to a ratio of damping to critical damping of only about one-half of one percent.

The uniform-bar problem treated at the University of Michigan included a term for uniformly distributed damping. This coefficient represented the damping force per unit velocity per unit length along the axis of the bar. The assumption of uniformly distributed damping was made for simplicity, but it was not intended to convey the idea that in the case of the ship the damping action is due chiefly to the surrounding water. The solution of the Michigan problem shows that when a simple harmonic force acts on a uniform beam with this type of damping the steady-state vibration resulting is the sum of a series of terms each of which represents a deflection in a normal mode of the undamped beam multiplied by a resonance factor depending on the ratio of the frequency of the force to the natural frequency of the mode and to the magnitude of the damping. This is true also for the nonuniform bar provided c/μ is constant as is shown in the following section.

In the case of the USS NIAGARA an equivalent uniformly distributed viscous-damping constant was derived from the test data as follows: The energy per cycle fed into the system at resonance was computed from the formula

$$W = \pi P_0 y_0$$

where P_0 is the force amplitude of the vibration generator at the particular resonance in question and y_0 is the single amplitude of forced vibration at the driving point, i.e., the amplitude produced at the point of the hull where the vibration generator was installed. If this energy is all dissipated by a uniformly distributed viscous damping, the equation for energy dissipation per cycle is

$$W = \pi c \omega \int_0^L y^2 dx$$

where c is the damping force per unit velocity per unit length,

ω is the circular frequency at resonance, and

y is the single amplitude at any point along the hull at resonance.

Therefore

$$c = \frac{P_0 y_0}{\omega \int_0^L y^2 dx}$$

As shown in TMB Report 699[13] consistent values of c were thus obtained for the first two vertical modes of the NIAGARA but for the third mode the value doubled and for the fourth mode it was about eight times as great. Moreover, when the attempt was made to reconcile the values thus obtained with values known from data on towing of friction planes or from flow-in pipes—by expressing the damping constant in terms of damping resistance per unit velocity per unit projected area—it was found that the values for the NIAGARA were over 100 times as great. This suggested that in the case of the ship much more of the energy must be absorbed in the structure than is absorbed by the surrounding water.

From a practical point of view, if an effective damping constant per unit mass per unit length could be found which would hold for different classes of ships and for the different modes it would not make any difference as far as estimating forced vibration amplitudes is concerned whether the energy was dissipated internally or externally. It seems likely that a better value would be obtained by assuming c to be proportional to μ (the mass per unit length including virtual mass), especially since the logarithmic decrements are observed to be about the same for different ships. This would also have the advantage of conforming to the dissipative systems treated by Rayleigh.

4. THE GENERAL DYNAMICAL PROBLEM OF VERTICAL MOTION WITHIN THE ELASTIC RANGE—TRANSIENT AND FORCED VIBRATION

So far the subject of chief interest in ship-vibration theory has been the calculation of the vertical normal modes and natural frequencies, but the methods and equations discussed so far may be employed to obtain the deflection of a ship as a function y(x, t) under any external load which is given in the form of a function P(x, t). This problem, one of increasing importance in naval architecture, may be called the dynamics of a ship's structure. The standard procedure in strength calculations for ships as outlined by Rossell and Chapman[1] has served well in the past and will continue to serve in the future, but it has become increasingly evident in recent years that many weaknesses in design show up only under dynamic conditions. In the future it may be possible to set up a standard procedure for dynamic-strength calculations but already the answers to a number of vital questions are furnished by the basic theory involved in the vibration calculations.

Before any dynamical theory is adopted it is well to have clearly in mind its limitations. Firstly, the theory presented here is limited to deflections within the elastic range so that any formulas given cannot be considered valid once the calculated deflections exceed this range. This makes possible the treatment by means of linear equations. Moreover, aside from the problem of the design of protective structures to withstand underwater explosions, a strictly naval problem, a ship should be so designed that under the most severe sea conditions the stresses developed would at all points be well below the elastic limit.

Secondly, the theory must for the present be confined to the ship girder as a whole and cannot take into account various local structures such as masts, deckhouses, and skegs, important as they may be. It is clear that the validity of such a treatment becomes more and more questionable as the frequency increases. It is an important part of the present problem to define the frequency limits within which the theory may be expected to apply.

With computational methods now available the treatment is facilitated by the assumption of viscous damping. As previously stated this does not mean that the energy must be absorbed by the surrounding water. On the contrary the scheme can be used where all the energy is absorbed internally in the form of heat. But the use of a viscous-damping constant is based on the assumption that an equivalent viscous-damping constant can be found and that its value is independent of frequency and amplitude. What little data are available on the subject suggest, however, that the damping "constant" increases with both amplitude and frequency and thus the theory is less valid in the higher modes.

Lastly, the methods described are based on the assumption that the inertia terms remain constant throughout the calculation. Lewis' theory[16] indicates that for a ship of the proportions of the NIAGARA the virtual mass for the three-noded mode should be about 10 percent less than the virtual mass for the two-noded mode. As previously pointed out, this variation, which is due to the increased departure from two-dimensional flow, was not taken into account in the calculation of the vertical modes for the NIAGARA. It may well be possible in the future to take into account this variation in virtual mass, but at the present time it is felt that neither the state of knowledge of the virtual mass itself nor the methods of computation available make it practical to take this into account in the general dynamical problem of the ship hull.

The above restrictions apply only to the estimate of the "constants" defining the properties of the mass-elastic system. It is well to observe that in the case of the ship very little is known about the dynamic loads to which it will be subjected in service. There are some data as to the

magnitude of blade frequency forces due to the action of the propellers based on the model experiments made by F.M. Lewis[22] but this was carried out for only one vessel and to the authors' knowledge little progress has since been made in further developing the method. As far as transient forces due to wave impacts are concerned little data seem to be available at present. See, however, References 23 and 24.

In dynamical problems of mass-elastic systems a distinction is generally made between the steady-state and the transient problem. However, it may be noted that in the theory of linear systems the response to a transient load can be found once the impedance to a simple harmonic force is known. In this report the steady-state-vibration problem is treated first and the transient next.

The steady-state problem of the ship as applied to vertical vibration involves the calculation of the amplitude of vibration set up at any point of the hull by a simple harmonic vertical driving force acting at any point of the hull and having a frequency within the range in which the theory is found to apply. Tentatively this range may be taken to be 50 to 1500 cpm.

Two alternative methods of dealing with this problem are considered. The first is based on the assumption that the dynamical system can be represented by a set of linear equations and that its natural frequencies and normal modes have already been found by the method outlined previously. The data considered available for the ship in this case are the values of μ, EI, and KAG as functions of distance x from the after perpendicular where μ is mass per unit length including virtual mass of entrained water, EI is bending stiffness as previously defined on page 15, and KAG is shear stiffness as previously defined on page 17. There is also required the value of c, the effective damping force per unit velocity per unit length, given as a function of length and assumed proportional to μ.

If a driving function P(x, t) acts on such a system, the equations governing the ensuing motion comprise the following set:

$$\mu \frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} + \frac{\partial V}{\partial x} = P(x, t)$$

$$V = \frac{\partial M}{\partial x}$$

$$\frac{\partial y}{\partial x} = \beta + \gamma \qquad [1]$$

$$V = -KAG\beta$$

$$M = EI\frac{\partial \gamma}{\partial x}$$

where x is the distance along the longitudinal axis of the hull,

μ is the mass per unit length (including virtual mass),

c is damping force per unit length per unit velocity,

y is displacement in the vertical direction,

V is the total vertical shearing force exerted by one section of the hull on the section to the right,

$P(x, t)$ is the external driving force per unit length,

M is the bending moment exerted by a section of the hull on the section to the right,

β is the component of the slope of y due to shear deflection only,

γ is the component of the slope of y due to bending deflection only,

KAG is the vertical shearing rigidity of the hull, and

EI is the bending rigidity of the hull.

In Equations [1], μ, EI, KAG, and c all vary with x but c/μ is assumed to remain constant.

Let it be assumed that $P(x, t)$ can be represented by the series

$$P(x, t) = \sum_{i=1}^{i=\infty} \frac{\mu(x) Q_i(t)\, X_i(x)}{\int_0^L \mu(x)\, X_i^2(x)\, dx}$$

where X_i is the ith normal mode of the system in arbitrary dimensionless units and $Q_i(t)$ has the dimension of force. This implies that $Q_i(t) = \int_0^L P(x, t)\, X_i(x)dx$.

Let it then be assumed that solutions exist in the form

$$y(x, t) = \sum_{i=1}^{i=\infty} q_i(t)\, X_i(x)$$

$$M(x, t) = \sum_{i=1}^{i=\infty} q_i(t)\, M_i(x) \qquad [2]$$

$$V(x, t) = \sum_{i=1}^{i=\infty} q_i(t)\, V_i(x)$$

where q_i has the dimension of length and where $V_i = \int^x \mu\omega_i^2\, X_i(x)dx$ and $M_i = \int_0^x\int \mu(x)\omega_i^2\, X_i(x)\, dx\, dx$, and the X_i's satisfy Equation [7] on page 23.

After elimination of β and γ from Equations [1] there results the set

$$\mu \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t} + \frac{\partial V}{\partial x} - P(x, t) = 0$$

$$V - \frac{\partial M}{\partial x} = 0 \qquad [3]$$

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{V}{KAG}\right) - \frac{M}{EI} = 0$$

Substitution of Equations [2] in [3] yields

$$\sum_{i=1}^{i=\infty}\left[\mu \ddot{q}_i X_i + c\dot{q}_i X_i + q_i V_i' - \frac{\mu \ddot{q}_i(t) X_i}{\int_0^L \mu X_i^2(x)dx}\right] = 0$$

$$\sum_{i=1}^{i=\infty}\left[q_i V_i - q_i M_i'\right] = 0 \qquad [4]$$

$$\sum_{i=1}^{i=\infty}\left[q_i X_i'' + q_i\left(\frac{V_i}{KAG}\right) - \left(\frac{q_i M_i}{EI}\right)\right] = 0$$

where the dot notation signifies differentiation with respect to t and the prime notation signifies differentiation with respect to x.

Equations [4] are satisfied if each term in the summations is set equal to zero. Then

$$\mu \ddot{q}_i X_i + c\dot{q}_i X_i + q_i V_i' - \frac{\mu \ddot{q}_i X_i}{\int_0^L \mu X_i^2(x)dx} = 0$$

$$V_i' = M_i'' \qquad [5]$$

$$M_i'' = \left[EI\, X_i'' + EI\left(\frac{V_i}{KAG}\right)'\right]''$$

By substitution in [5]

$$\mu \ddot{q}_i X_i + c q_i X_i + q_i \left[EI\, X_i'' + EI \left(\frac{V_i}{KAG} \right)' \right]'' = \frac{\mu Q_i X_i}{\int_0^L \mu X_i^2(x)dx} \qquad [6]$$

As shown on page 17 of Reference 25, for a normal mode

$$\left[EI\, X_i'' + EI \left(\frac{V_i}{KAG} \right)' \right]'' = \mu \omega_i^2 X_i \qquad [7]$$

Hence

$$\mu \ddot{q}_i X_i + c q_i X_i + q_i \mu \omega_i^2 X_i = \frac{\mu(x) Q_i X_i}{\int_0^L \mu(x) X_i^2(x)dx} \qquad [8]$$

Whence

$$\ddot{q}_i + \frac{c}{\mu} \dot{q}_i + \omega_i^2 q_i = \frac{Q_i(t)}{\int_0^L \mu X_i^2(x)dx} \qquad [9]$$

If both sides of [9] are multiplied by $\int_0^L \mu X_i^2(x)dx$ there results the equation

$$M_i \ddot{q}_i + C_i \dot{q}_i + K_i q = Q_i(t) \qquad [9a]$$

where the effective values M_i, C_i, and K_i are as defined on page 7.* This is the ordinary differential equation for a system of one degree of freedom acted on by a driving force $Q_i(t)$.

If in Equation [9] $q_i(0) = \dot{q}_i(0) = 0$ the solution is

$$q_i(\tau) = \int_0^t \frac{Q_i(\tau)}{\lambda_i M_i} e^{-\frac{C_i}{2M_i}(t-\tau)} \sin \lambda_i (t - \tau) d\tau \qquad [10]$$

where $\lambda_i = \sqrt{\omega_i^2 - \frac{1}{4}\left(\frac{C_i}{M_i}\right)^2} = \sqrt{\omega_i^2 - \frac{1}{4}\left(\frac{c}{\mu}\right)^2}$

and

$$y(x, t) = \sum_{i=1}^{i=\infty} q_i(t)\, X_i(x) \qquad [11]$$

*M as used here for effective mass is not to be confused with M as used for bending moment on page 21.

If a concentrated sinusoidal driving force F sin ωt is acting at x_0 then

$$Q_i(t) = F\, X_i(x_0) \sin \omega t$$

If this value is substituted for $Q_i(t)$ in Equation [9a] the steady-state solution becomes in complex form

$$q_i(t) = \frac{F\, X_i(x_0)\, e^{j\omega t}}{\left(\omega_i^2 - \omega^2 + j\,\frac{c}{\mu}\,\omega\right)\int_0^L \mu X_i^2(x)\,dx}$$

where $j^2 = -1$, or in real form

$$q_i(t) = \frac{F\, X_i(x_0) \sin(\omega t - \phi_i)}{\omega_i^2 \sqrt{\left[1-\left(\frac{\omega}{\omega_i}\right)^2\right]^2 + \left(\frac{c\omega}{\mu\omega_i^2}\right)^2} \int_0^L \mu X_i^2(x)\,dx}$$

where $\phi_i = \arctan \dfrac{\frac{\omega c}{\omega_i^2 \mu}}{1-\left(\frac{\omega}{\omega_i}\right)^2}$

Therefore the steady-state response of a damped bar with shear and bending flexibility (representing the ship hull) to a sinusoidal force of amplitude F acting at x_0 is

$$y(x,\, t) = \sum_{i=1}^{i=\infty} \frac{F\, X_i(x_0) \sin(\omega t - \phi_i) X_i(x)}{\omega_i^2 \sqrt{\left[1-\left(\frac{\omega}{\omega_i}\right)^2\right]^2 + \left(\frac{c}{\mu}\,\frac{\omega}{\omega_i^2}\right)^2} \int_0^L \mu X_i^2(x)\,dx}$$

The number of components to be used in practice appears to be not greater than six. If the two rigid-body motions of heaving and pitching are not included in the normal modes, they must be computed separately and added to the series. The above relatively simple equations theoretically tell the complete story of the steady-state vibration of a ship in the vertical plane under the action of a known sinusoidal driving force acting at one point.

It is particularly to be observed that, while in the steady-state condition a driving force excites to varying degrees amplitudes in all the normal modes, all these components have the frequency of the driving force and not the natural frequency of the mode as they would have in a free vibration. Moreover, while the different components will have different phases relative

to the driving force, there will be no shift in phase of the various components relative to one another during the steady-state vibration. Practically the damping is usually so small that all components whose natural frequency is above the driving frequency are in phase with the driving force at the driving point and all components whose frequency is below the driving frequency are 180 degrees out of phase with the driving force at the driving point.

It is next in order to consider a method of computing the steady-state vibration of a ship in the vertical plane when nothing is known initially about its normal modes but when only the EI, KAG, μ, and c functions are given. The method is based on the same set of finite-difference equations as used in the method for determining normal modes previously described. This method, though considerably more complex when applied to forced vibration, will also give the normal modes.

Whether the motion is free or forced, the same difference equations apply to the system. The method is more complicated because external forces now have to be taken into account whereas in the normal-mode problem external forces and moments can be considered absent. The further complication is due to the fact that the damping forces under the assumed condition of viscous damping are 90 degrees out of phase with the displacement since they are in phase with the velocity.

The process consists in setting up the same finite-difference equations as used for the normal-mode calculations, but introducing the damping forces and the external driving force as additional terms to be added to the vertical shearing force at the appropriate stations. Details of the method together with an example are given in Reference 18.

The same two general approaches to the forced vibration problem are also applicable to the transient problem and will be considered in the same order. When the normal modes have been previously determined, the response of the ship to an arbitrary transient load can be found in a manner similar to that previously given for the steady-state response. An expression is derived for the transient response in each normal mode and the resultant response is obtained by summation of the series of terms representing the component in each mode. Again, although a convergent infinite series describes the exact solution, usually only about six terms or less of the series need be considered to obtain an accuracy consistent with the accuracy of the assumptions underlying the theory.

Before considering the formula for the response in any normal mode it is helpful to recall the transient response of a system of one degree of freedom to a force which is an arbitrary function of time. Let the mass be at rest at time $t = 0$. Then it is well known that the displacement at time $t > 0$

caused by a force P(t) is

$$x(t) = \int_0^t P(t_0)h(t - t_0)dt_0$$

where $h(t - t_0)$ is the response to a unit impulse applied at time t_0.

To extend this idea to the ship it is necessary only to remember that as in the case of forced vibration the system behaves in each normal mode as though it were a system of one degree of freedom. Instead of the mass m for the system of one degree of freedom there must be substituted the effective mass for the nth normal mode of the ship

$$M_n = \int_0^l \mu X_n^2(x)dx$$

and the effective spring constant

$$K_n = \omega_n^2 M_n$$

In the case of the ship a distributed damping assumed proportional to μ must also be considered and the effective damping constant is:

$$C_n = \int_0^L cX_n^2(x)dx$$

If a vertical impulse H acts at distance x_0 from the after perpendicular, the response in the nth normal mode is obtained by finding the effective impulse and then applying the formula for a one-degree system to the normalized mass and spring values and in this case also introducing an exponential damping term. Thus

$$q_n = \frac{H \cdot X_n(x_0)}{\lambda_n M_n} e^{-\frac{C_n}{2M_n}} \sin \lambda_n t$$

where $\lambda_n = \sqrt{\omega_n^2 - \frac{1}{4}\left(\frac{C_n}{M_n}\right)^2}$, q_n has the dimension of length and, to obtain the displacement at any point x, q_n must be multiplied by the value of the normal-mode function at x, namely $X_n(x)$.

The total response of the ship to an impulse H at x_0 is obtained by summation of the series of responses in the various normal modes, and the response to an impulse H at x_0 and at time t_0 becomes

$$y_H(x,t,x_0,t_0) = \sum_{n=1}^{n=\infty} q_n X_n(x) = H\sum_{n=1}^{n=\infty} X_n(x)\frac{X_n(x_0)}{\lambda_n M_n} e^{-\frac{C_n}{2M_n}(t - t_0)} \sin \lambda_n(t-t_0)$$

As in the case of the system of one degree of freedom the response of the ship to an arbitrary force function $Q(x, t)$ (where Q has the dimensions of force per unit length) can be obtained by summation of the responses to successive impulses of the form $Qdtdx$, and the total response to a transient force function $Q(x, t)$ takes the form of a double integral

$$y(x, t) = \int_0^L dx_0 \int_0^t \frac{Q(x_0, t_0)}{H} y_H(x, t, x_0, t_0) dt_0$$

While the complete response can thus be written in a fairly compact expression it is clear that, given the arbitrary function $Q(x_0, t_0)$, the working out of the formula to obtain $y(x,t)$ will in general be quite laborious.

The degree to which the solution must be worked out depends on the particular information required. If for instance the excitation consists of an impulse applied at the bow and it is desired to know only the response in the first mode, the formula for q_n on page 26 is sufficient.

The application of the method of finite differences to the transient problem involves more computation than the steady-state problem but the basic principle is the same. The same difference equations apply as in the steady-state problem but in this case it is not permissible to assume that a solution exists in the form $Y(x) \sin \omega t$. Instead steps in time as well as the distance x are employed. Given an initial condition for the ship, namely, the deflection and velocity at every point from stern to bow at time $t = 0$ and an initial set of forces acting, the condition at a short interval of time later $t + \Delta t$ can be found by computing the acceleration at every point at $t = 0$. These accelerations depend not only on the external and damping forces but also on the shearing forces applied by adjoining sections.

It is necessary in this case to represent each variable by both a subscript and a superscript, the former to designate that position along the length of the ship at which the variable is being considered and the superscript to designate the instant of time at which it is being considered. In addition to the four variables considered in the steady-state problem, namely, the shearing force V, the bending moment M, the slope γ due to bending only, and the displacement y, there must now be added the velocity $\dot{y}$ and the acceleration $\ddot{y}$, the dot notation indicating differentiation with respect to time.

The difference equations in space (t fixed) are similar to those used in the steady-state problem, but since y is no longer of the form $f(x) \sin \omega t$ the equations have the following more general form

$$V_{n+1} = V_n - m_n\ddot{y}_n - C\dot{y}_n + F_n$$

$$M_{n+1} = M_n + V_{n+1}\Delta x$$

$$\gamma_{n+1} = \gamma_n + \frac{M_n\Delta x}{EI}$$

$$y_{n+1} = y_n + \gamma_n\Delta x - \frac{V_n\Delta x}{KAG_n}$$

The difference equations in time are as follows:

$$\dot{y}_n^{n+1} = \dot{y}_n^n + \ddot{y}_n^n\,\Delta t$$

$$\ddot{y}_n^n = -\frac{\Delta V_n^n}{m_n} - \frac{C\dot{y}_n^n}{m_n} + \frac{F_n^n}{m_n} = \frac{V_n^n - V_{n+1}^n - C\dot{y}_n^n + F_n^n}{m_n}$$

$$y_n^{n+1} = y_n^n + \dot{y}_n^n\,\Delta t$$

If the ship is at rest when the transient disturbance starts to act, the initial conditions are:

$$\gamma_n^0 = y_n^0 = \dot{y}_n^0 = 0$$

Since the shearing force and bending moment remain zero at both ends throughout the motion, the boundary conditions become

$$V_0^n = M_0^n = V_{20_A}^n = M_{20_A}^n = 0$$

Whereas in the steady-state problem y_0 is taken as unity and γ_0 is carried as an unknown, in this problem $\ddot{y}_0^n$ and γ_0^n are both carried through as unknowns and found from the boundary conditions.

Although the problem involves a considerable amount of computation, if the external forces are tabulated for intervals of time from $t = 0$ at each of the twenty stations, the calculation of the deflection and velocity of all points of the ship at any future time is given by the routine use of the above formulas. For each interval of time there has to be carried out a calculation

of the same type as is made for each assumed frequency in the normal-mode calculation. So far opportunity has not permitted testing out this scheme.

5. HORIZONTAL BENDING, TORSIONAL, AND COUPLED TORSION—HORIZONTAL BENDING MODES

While offhand it appears that horizontal bending modes can be dealt with in the same manner as vertical bending modes with the exception that the area moments of inertia are to be taken with respect to the vertical centroidal axis, consideration of the lack of symmetry leads one to expect the former problem to be more complicated.

In the case of the USS NIAGARA, data were lined up for digital computation of horizontal modes by means of the same set of difference equations as used for the vertical modes. The moments of inertia of areas included the same members as were included in the vertical calculation, no members above the weather deck being considered.

The virtual-mass values for horizontal vibration were arrived at by taking as the "entrained area" the area of the inscribed circle, that is a circle having a diameter equal to the draft. Thus at each section a mass of water equal to that of the water displaced by a circular cylinder, having the length of the section and a diameter equal to the draft, was added to the mass curve of the ship in making the horizontal-vibration calculations. As in the case of the vertical calculations no allowance was made for a possible variation in the horizontal virtual mass with frequency.

The calculations of horizontal modes were carried out by the digital process for the same set of conditions as tried for the vertical calculations, namely, bending only, shear only, combined bending and shear, and combined shear, bending, and rotary inertia. The calculated frequencies followed the same general pattern as for the vertical case. For the case with bending only the first-mode frequency checked fairly well but beyond this the frequencies became progressively too high. Likewise the calculations based on shear only were too high for the first mode and better for the second. Rotary inertia again had little effect but the calculations based on bending and shear, while giving the best over-all results, were too low for all three modes found experimentally. This could have been due either to using too large a value of virtual mass or too small a value of rigidity.

Whereas experimental evidence is available[20] to support the simple-beam theory for static bending in a vertical plane, similar evidence for static bending in the horizontal plane has not yet come to the authors' attention. The term KAG used to represent shear stiffness in the calculations was based on the assumption that inner decks carried shear load. In the case of

the calculations for vertical modes the one based on shear only and that based on bending only gave very nearly the same frequency for the second mode (which of course was considerably higher than the experimental value). In the calculation of horizontal modes of vibration the frequency of the second mode calculated on the basis of shear only was considerably higher than that based on bending only. Because of the possibility of coupling between horizontal bending and torsion in the transverse plane, it has not seemed profitable to pursue the question of horizontal bending and shear rigidity further during this investigation.

The number of cases reported in the literature in which torsional modes have definitely been identified is relatively small. This may not be a true indication of their importance as it seems not only possible but probable that many cases of horizontal vibration in which the mode has not been definitely established were torsional modes in which the motion was predominantly horizontal at the measuring station (as for example, on an upper deck along the centerline plane).

Mathematically the calculation of uncoupled torsional modes presents no great difficulty provided rational values of torsional rigidity and mass moment of inertia can be determined for a sufficient number of stations between stern and bow. This proviso, however, is not easily disposed of. If it be assumed for the moment that these quantities can be determined, two rather well-known methods are available for the calculation of torsional modes.

The Holzer method widely used in the calculation of torsional vibration of engine-shafting systems[26] can be applied to the ship problem by first breaking down the system into a dynamically equivalent system consisting of a series of disks (inertia members) connected by massless rods (torsion members). This is really the method of finite differences which has been discussed under vertical flexural modes. In the torsional case the calculation is much simpler since the problem involves only a second-order differential equation. In the case of a uniform cylindrical shaft, for example, the differential equation is

$$GJ_e \frac{\partial^2 \theta}{\partial x^2} = I_\mu \frac{\partial^2 \theta}{\partial t^2}$$

where G is the shear modulus of elasticity,

J_e is the effective polar moment of inertia of the section area,

I_μ is the effective mass moment of inertia per unit length with respect to the longitudinal (x) axis, and

θ is the instantaneous angular displacement at any point along the shaft.

Besides the Holzer method the Stodola method can also be applied to the torsional calculation. To do this the effective mass moment of inertia per unit length and the area polar moment of inertia are first plotted against axial position and a normal-mode shape for torsion is assumed just as in the case of flexural vibration.[5] Again as for flexural vibration the normal-mode shape assumed is that for a uniform cylindrical shaft. The solution of the above differential equation gives for the fundamental normal mode of a uniform shaft the function

$$\theta = \cos \frac{\pi x}{\ell}$$

If this assumed amplitude function is multiplied by the mass moment of inertia function, a dynamic torque function is obtained and a base correction to the assumed amplitude curve can be made so as to satisfy the end conditions that in a free oscillation the torque at either end must be zero. A double integration of the T/B curve (torque over torsional stiffness $B = GJ_e$) will again yield an amplitude function from the absolute value of which the frequency may be computed.

In order to use either the Holzer or Stodola method for calculating the torsional modes of a ship it is necessary to evaluate for a sufficient number of sections of the ship (say 20) the effective polar moment of inertia of the section area and the mass moment of inertia. At the present stage of development of torsion theory of ship-type structures simplifying assumptions are made.

A theory of the torsion of ships was proposed by G. Vedeler.[27] According to this theory the total torque developed by any hollow section of thickness t is given by the formula

$$T = 2A\ t\tau$$

where T is the torque,
A is the enclosed area,
t is the thickness, and
τ is the shearing stress.

To allow for stress concentrations at deck edges an empirical coefficient k is introduced and the formula for torque becomes

$$T = k\ A\ t\tau$$

where k is less than 2. For rectangles of width a and height b $k = 2 - \frac{b}{a}$. For a thin rectangular solid

$$T = \frac{1}{3}\ t^2 h\tau$$

where h is the width and t the thickness. For a hollow section of variable wall thickness Vedeler gives for the torsional constant per unit length (restoring torque per unit angle of twist per unit length)

$$\frac{2kA^2G}{\Sigma \frac{\ell}{t}}$$

where G is the shear modulus and ℓ is the peripheral length of each wall of thickness t. Since for a circular section the torsional constant per unit length is equal to GJ where J is the polar moment of inertia of the section area ($\pi d^4/32$), it is convenient to introduce the term J_e for the effective polar moment of inertia of a noncircular section. This does not mean that J_e can be evaluated in the way polar moments of inertia are ordinarily evaluated. It simply means that the product GJ_e gives the torsional stiffness or restoring torque per unit angle of twist per unit length just as GJ does for a circular section. Vedeler's theory has not been extended to multiple-deck ships.

Horn[28] proposed for the effective polar moment of inertia

$$J_e = \frac{4F_0^2}{\Sigma \frac{\Delta s}{\delta}}$$

where Δs is a small distance along the wall enclosing the section,

δ is the plating thickness, and

F_0 is the enclosed area.

Horn's formula for the effective polar moment of inertia of section area agrees with the formula

$$J_e = \frac{2kA^2}{\Sigma \frac{\ell}{t}}$$

given by Vedeler if k is assumed equal to 2.

For a free-free uniform circular cylinder the natural frequencies in torsion are given by the formula

$$\omega_n = n\pi\sqrt{\frac{GJ}{\ell^2 I_\mu}}$$

where G is the shear modulus,

J is the polar moment of inertia of section area,

I_μ is the mass polar moment of inertia per unit length.

ℓ is the length,

ω_n is the natural circular frequency of the nth mode in radians per second, and

n has the successive values 1, 2, 3, ...

On the basis of the formula for the uniform cylinder Horn developed an empirical formula for the torsional frequencies of ships somewhat similar to Schlick's formula for vertical flexural vibration. Horn's formula is

$$N_e = 60k \sqrt{\frac{gG\ J_{eo}}{D(B^2+H^2)L}}$$

Although Horn used metric units in this formula, since the part of the formula under the radical must have the dimensions t^{-2}, the following English units may be used:

N_e = natural frequency (cpm)

g = acceleration of gravity (ft/sec^2)

J_{eo} = effective polar moment of inertia of midship section area according to formula on page 32 (ft^4)

D = displacement in tons

B = beam in feet

H = depth in feet

L = length in feet

G = shear modulus of elasticity (tons/ft^2)

k is an empirical coefficient which Horn found for the freighter WASGENWALD to be 1.58 for the fundamental mode, 3.00 for the second mode, and 4.07 for the third mode.

A computation of the fundamental torsional mode of the USS NIAGARA by Horn's formula based on the value of k found for the SS WASGENWALD gave a frequency of 239 cpm against the experimental value of 322 cpm. A calculation by Holzer's method based on polar moments of inertia of area taken equal to the actual polar moment of inertia of the section areas up to the weather deck and mass polar moments of inertia obtained by estimating radii of gyration with respect to the centroidal axis gave for the fundamental torsional frequency 629 cpm, a rather wide discrepancy.

In addition to the obstacle of correctly evaluating the mass and area polar moments of inertia, there appears to be another fundamental difficulty in the way of calculating torsional modes of ships, namely the coupling that may exist with bending modes. Apparently the extent of such coupling has not been investigated up to the present and is unknown. Because of the symmetry of most vessels with respect to a vertical plane through the longitudinal axis there appears to be no need of considering coupling between

vertical flexure and torsion. The center of mass of all sections may in general be assumed to lie on the vertical centerline of the section, and hence there is no tendency to produce a rotation when the ship vibrates vertically. On the other hand the vertical position of the center of mass of a section may vary appreciably in going from stern to bow.

In Figure 3 is shown an element of the ship of length Δx and a set of rectangular coordinates with X-axis parallel to the longitudinal axis of the ship. This axis is arbitrarily located to pass through the centroid of area of the midship section. This choice of the X-axis implies neither the assumption that the various sections of the ship are oscillating about this axis nor that this is the "center of twist" for torsion of the hull. If the equations are set up correctly according to the laws of dynamics and if the mass moments of inertia are correctly computed with reference to this axis as well as the vertical distance of the center of mass and center of shear of each section from this axis, the solution will show how much torsional and how much bending vibration takes place in any normal mode.

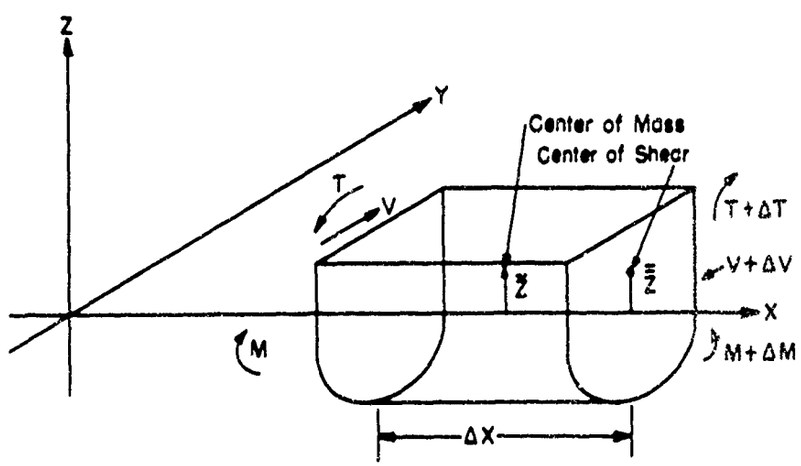

Figure 3 - Free Body Diagram of Section of Ship of Length Δx Subject to Forces and Moments Due to Coupled Horizontal Bending-Torsion Vibration

In the element represented in Figure 3 the Y-axis is taken in a horizontal plane since the coupling is assumed to exist between torsion and horizontal flexure. A right-handed coordinate system is used and the following definitions apply:

- m is the mass or linear inertia ($\mu\Delta x$) including the virtual mass of water moving with the hull in the horizontal direction
- I_{mx} is the mass polar moment of inertia of element about X-axis ($=I\mu\Delta x$)

$\bar{z}$ is the coordinate giving location of the center of mass of each section of the ship (including virtual mass of the surrounding water for horizontal vibration) with respect to the X-axis

$\bar{\bar{z}}$ is the coordinate giving the location of the center of horizontal shear.

I_{mz} is the mass polar moment of inertia of the element about a vertical (Z) axis

γ is the slope in the horizontal (XY) plane due to component of deflection due to bending only.

ϕ is the angular displacement about X-axis considered positive when clockwise as viewed looking in positive X direction

y is the linear displacement in Y direction of points of ship sections initially on X-axis.

y' is the linear displacement of the center of mass of the section in the Y direction

y" is the linear displacement of the center of shear in the y direction

T is the summation of the moments with respect to the X-axis of the shearing forces acting on all elements of area of the section considered positive if, when looking in the positive X direction from the origin, the part of the system nearest the origin tends to rotate the part farthest from the origin in a clockwise di-direction

V is the horizontal shearing force at any section considered positive when the section to the left tends to move the section to the right in the positive Y direction

M is the bending moment at any section acting in the horizontal plane considered positive when it tends to make the system concave when viewed from a point in the positive Y half-plane

Δx is the length of each element

EI is the bending rigidity of the section about the Z-axis

GJ_e is the effective torsional rigidity of the section

KAG is the horizontal shearing rigidity of the section

The finite difference equations are based on the laws of mechanics applicable to the linear and angular motions of the elements. Thus if Newton's law is applied to forces in the Y direction

$$m\ddot{y}' = -\Delta V$$

$$y' = y - \bar{z}\phi$$

$$\ddot{y}' = \ddot{y} - \bar{z}\ddot{\phi}$$

hence

$$-\Delta V = m\ddot{y} - m\bar{z}\ddot{\phi}$$

A change in bending moment exists between the ends of the element due to a change in the lever arm of the shearing force, and there is also a change in moment due to angular acceleration about a vertical (Z) axis. Hence

$$\Delta M = V\Delta x + I_{mz}\ddot{\gamma}$$

From the definition of γ as slope due to bending only, it follows from simple-beam theory that

$$\Delta\gamma = \frac{M\Delta x}{EI}$$

From the definition of shear rigidity it follows that the deflection due to shear in passing from one element to the next is

$$\Delta y_s'' = -\frac{V\Delta x}{KAG}$$

The change in deflection due to bending is

$$\Delta y_b'' = \gamma\Delta x$$

Since

$$y'' = y - \bar{\bar{z}}\phi,\ \Delta y = \Delta y'' + \bar{\bar{z}}\Delta\phi$$

whence

$$\Delta y = \Delta y_s'' + \Delta y_b'' + \bar{\bar{z}}\Delta\phi$$

Since as shown on page 37

$$\Delta\phi = \frac{-(T+V\bar{\bar{z}})\Delta x}{GJ_e}$$

$$\Delta y = \gamma\Delta x - \frac{V\Delta x}{KAG} - \frac{\bar{\bar{z}}T\Delta x}{GJ_e} - \frac{V\bar{\bar{z}}^2\Delta x}{GJ_e}$$

The change in moment about the X-axis between the ends of the element is determined by the mass polar moment of inertia about the X-axis and the angular acceleration as well as the vertical distance of the center of mass from the X-axis. The expression for the change in moment can be derived by considering the rate of change of moment of momentum with respect to the X-axis. The moment of momentum of a rigid body with respect to an axis fixed in space is equal to the moment of momentum of a particle having the mass of the rigid body and the velocity of its center of mass plus the moment of momentum of the rigid body with respect to a parallel axis through the center of mass. The moment of momentum with respect to an axis through the center of mass and parallel to the X-axis is

$$I_0\dot{\phi} = I_{mx}\dot{\phi} - m\bar{z}^2\dot{\phi}$$

Therefore the moment of momentum with respect to the X-axis is

$$-m\dot{y}'\bar{z} + I_0\dot{\phi} = -m\dot{y}'\bar{z} + I_{mx}\dot{\phi} - m\bar{z}^2\dot{\phi}$$

Since the change in moment must be equal to the time rate of change of moment of momentum

$$-\Delta T = -m\ddot{y}'\bar{z} + I_{mx}\ddot{\phi} - m\bar{z}^2\ddot{\phi} = -m\bar{z}\ddot{y} + I_{mx}\ddot{\phi}$$

Therefore during a vibration

$$\Delta T = -m\bar{z}y\omega^2 + I_{mx}\phi\omega^2$$

The moment T is the resultant of the moment of the shear force V acting through the center of shear and a torque. The latter is equal to $(T + V\bar{\bar{z}})$, a quantity which is independent of the choice of the X-axis.

The only remaining difference equation to be derived is that giving the relation between the change in angle of twist and the torque. This follows from the definition of effective area polar moment of inertia J_e and the equation is

$$\Delta\phi = \frac{-(T+V\bar{\bar{z}})\Delta x}{GJ_e}$$

If the system is assumed to be vibrating at circular frequency ω, all quantities are simple harmonic functions of time and reach their peak values simultaneously. It is therefore permissible in writing the difference equations to substitute, for the second derivatives with respect to time, $-\omega^2$ times the amplitude. Thus $-\phi\omega^2$ is substituted for $\ddot{\phi}$ and $-y\omega^2$ for $\ddot{y}$, where ϕ and y are now understood to be functions of x only.

Also in the coupled torsion-bending problem the natural frequencies are determined by the end conditions that the shear force, bending moment, and torque are zero at both ends of the ship. The relation between y, ϕ, and γ at station 0 cannot be assumed known, but a linear combination of solutions in which each of these terms in turn is taken equal to unity, while the other two are zero, can be found which will satisfy the end conditions. Thus three cases can be taken as follows:

Case	y	γ_0	ϕ_0
I	1	0	0
II	0	1	0
III	0	0	1

The end conditions require that there exist three quantities a, b, and c such that

$$aV^{I}_{20\,A} + bV^{II}_{20\,A} + cV^{III}_{20\,A} = 0$$

$$aM^{I}_{20\,A} + bM^{II}_{20\,A} + cM^{III}_{20\,A} = 0$$

$$aT^{I}_{20\,A} + bT^{II}_{20\,A} + cT^{III}_{20\,A} = 0$$

For a solution to exist other than a = b = c = 0 the determinant

$$\begin{vmatrix} V^{I}_{20\,A} & V^{II}_{20\,A} & V^{III}_{20\,A} \\ M^{I}_{20\,A} & M^{II}_{20\,A} & M^{III}_{20\,A} \\ T^{I}_{20\,A} & T^{II}_{20\,A} & T^{III}_{20\,A} \end{vmatrix}$$

must equal zero. Any frequency at which this condition is found to hold is a natural frequency. A convenient way of finding the critical frequencies is to plot the value of the determinant against ω. This should give a smooth curve which crosses the axis at the natural frequencies.

It may be noted that in the case of the uniform symmetrical bar, if the X-axis is taken through the center of mass, the quantities $\bar{z}$ and $\bar{\bar{z}}$ vanish and the torsional and bending modes are no longer coupled.

When the natural frequencies have been found, the normal modes are obtained simply by plotting the values of y and ϕ appearing in the calculations made at those frequencies. Where a natural frequency does not occur at one of the computed values, it is necessary to interpolate between the values of y or the values of ϕ obtained in the two calculations on either side of the critical value and to use proportionality factors according to the proximity of the critical frequency to the frequencies for which the calculations were carried out.

6. EXCITING FORCES

If an estimate can be made in the early stages of design of the vibratory forces to be expected to act on the vessel under service conditions, then there is some hope of anticipating the amplitudes that may be encountered in service by making use of the forced-vibration computations previously outlined.

Pioneering work on the problem of determining the vibratory forces was carried out by means of model tests some years ago by F.M. Lewis.[22] The forces were determined on self-propelled models by neutralizing the vibration with a rotating weight of adjustable eccentricity. The vibratory force thus neutralized could be computed from the known weight, eccentricity, and speed of rotation.

In the self-propelled model test the relation between model shaft speed and full-scale shaft speed is given by the equation

$$(\text{model rpm}) = (\text{ship rpm}) \times \sqrt{\frac{\text{length of ship}}{\text{length of model}}}$$

and the model forces are stepped up to full-scale values by the equation

$$\left(\begin{array}{c}\text{full-scale vibratory}\\ \text{force}\end{array}\right) = \left(\begin{array}{c}\text{model vibratory}\\ \text{force}\end{array}\right) \times \frac{(\text{displacement of ship})}{(\text{displacement of model})}$$

Although Lewis' work on this project was carried out at the U.S. Experimental Model Basin, the Model Basin itself did not experiment with the method until recently, when an attempt was made to determine the vibratory

forces on a model of a proposed passenger vessel for the U.S. Maritime Commission. Many experimental difficulties were encountered in the attempt to extend Lewis' method. In particular the wooden model had a natural frequency in the operating range of model blade frequencies. Moreover the sensitivity of the pickup system was such that signals could be detected only in the vicinity of resonance of the model which was not related to the natural frequency of the prototype. No attempt has been made at all to build complete dynamical models for this investigation and aside from the difficulties in the construction of steel models it would be impossible to satisfy the condition that the frequency vary inversely as the scale inasmuch as the self-propelled model shaft speed must vary inversely as the square root of the scale.

In view of these difficulties the development of the model technique for determining vibratory forces is now in an inactive status at the Model Basin.

It appears that whatever data are to be obtained in the near future as to the magnitude of exciting forces must be obtained from full-scale measurements. It would not be a simple matter to apply Lewis' method to a full-scale ship, but it does appear feasible to obtain a rough estimate of the exciting force by installing a vibration generator on the weather deck over the propellers and running the machine through the speed corresponding to propeller-blade frequency while the ship speed is maintained constant and a continuous record of vertical vibration is made by means of a pallograph. It is to be expected that, as the frequency of the vibration generator approaches the propeller-blade frequency or the shaft frequency in the case of first-order vibration, the vibration of the hull will assume a beat characteristic and that from a measurement of the maximum and minimum amplitudes, occurring during beating, the ratio of the exciting force caused by propeller action to the exciting force produced by the vibration generator can be obtained.

From what scant information is now available it appears that force amplitudes of the order of 10,000 pounds are to be expected, and the forces in general will vary as the square of the shaft speed. It can also be stated qualitatively that vibratory forces will diminish with increasing tip clearance and with increasing uniformity of the wake over the area of the propeller disk. In general the vertical and lateral forces may be expected to be large if the thrust and torque variations are large.

It is to be hoped that as full-scale information on propeller exciting forces is accumulated in the future, it will be possible to correlate the data with wake-pattern data so that some idea of the magnitude of exciting forces can be obtained from the model wake surveys.

7. ELECTRICAL ANALOGY METHODS

The calculations previously described are readily carried out by means of sequence-controlled digital calculators. It is natural to inquire whether true electrical analogies are possible in this case. Do there exist circuits whose elements represent definite physical properties of the ship and in which voltages and currents represent definite forces and motions in the mechanical system?

Electrical analogies of mechanical systems which vibrate longitudinally or torsionally are well known and of considerable practical utility. Such analogs are usually of the conventional (mass-inductance) type or the mobility (mass-capacitance) type. They involve second-order differential equations, and each type of circuit element used in the circuit represents the same type of mechanical quantity wherever it occurs.

In attempting to set up electrical analogs of vibrating beams or ships fourth-order differential equations must be dealt with and the difficulties in establishing the analog are considerably greater.

Kron[29] has worked out a scheme of great generality by means of which circuits may be derived not only for flexural, but for combined flexural, torsional, and longitudinal modes and where these different types of motion may be coupled or uncoupled. Kron's general tensor method for accomplishing this derivation yields circuits that are analogous to the mechanical system in the mathematical sense, but in the majority of cases the capacitance, resistance, or inductance values of the elements of the analogous circuit must be varied with frequency so as to maintain the required impedances or admittances. Furthermore, even where large network analyzers are available so that the circuit elements can be changed readily by merely turning dials, the range of parameters required to cover the necessary frequency range for the problem is often prohibitive. However, the circuit may serve as a basis for computation in the field of alternating-current theory which is generally more familiar than the field of mechanical vibration.

A problem considered by the authors has been to find a circuit analogous to the ship vibrating in flexure, the circuit itself being independent of frequency It has also been an objective, if possible, to find a circuit without transformers because of the difficulty in realizing the ideal properties required of transformers in electrical-analog computations. So far a circuit with passive elements only and without transformers has not been found. Flexural vibration involves both forces and bending moments as well as displacements and slopes. Since voltage and current are the only measurable electrical quantities, the circuit will require some components in which a current

has a different significance than it has for certain other components. Likewise voltages in general will represent more than one mechanical quantity according to their location in the circuit.

Before considering an analog employing transformers (which has already given very encouraging results) it is important to consider the laws of similitude which should be observed in transforming the equations, so as to permit the use of physically realizable values of circuit elements which make up the analogy. In other words, before the analogous circuit is derived from the mechanical system it is in general necessary to transform the latter into a similar system, that is, one derived according to the laws of similitude and which has a similar response over a frequency range commonly encountered in electrical circuits. The rules governing such transformations, as applied to the specific problem of a vibrating beam having both shear and bending flexibility, may be obtained by dimensional analysis. The Pi Theorem of Buckingham[30] facilitates the process of establishing the equivalent system.

The three fundamental mechanical quantities are taken to be mass, length, and time, and the dimensional relations for the quantities entering into the ship vibration problem, see Reference 31, are:

Quantity	Dimensions
$\frac{EI}{\Delta x}$	$m\ell^2 t^{-2}$
$\frac{KAG}{\Delta x}$	mt^{-2}
$m = \mu\Delta x$	m
L	ℓ
ω	t^{-1}
C	mt^{-1}

Since there are six quantities involved and three fundamental units, the number of π's is three. These are as follows:

$$\pi_{\frac{EI}{\Delta x}} = \frac{EI}{\Delta x}(\mu\Delta x)^{-1}(L)^{-2}(\omega)^{-2}$$

$$\pi_{\frac{KAG}{\Delta x}} = \frac{KAG}{\Delta x}(\mu\Delta x)^{-1}(\omega)^{-2}$$

$$\pi_C = (C)(\mu\Delta x)^{-1}(\omega)^{-1}$$

where E is Young's modulus,

I is the area moment of inertia of the ship section,

μ is mass per unit length including virtual mass of the surrounding water,

A is the section area,

G is the shear modulus of elasticity,

K is the ratio of average shearing stress to shearing stress at the neutral axis, and

C is the damping force per unit velocity acting over a section of length Δx and is equal to $c\Delta x$ where c is damping constant per unit length.

If the flexibility in shear and damping are neglected only $\pi_{\frac{EI}{\Delta x}}$ has to be considered.

A transformer analog of the nonuniform beam with shear and bending flexibility has been proposed by H.M. Trent[32] which represents the same set of finite difference equations (page 11) as has been used in the digital-computation process. This circuit, which is described in detail in Reference 33, will not be discussed further here but a typical section of the circuit with a damping element included but the rotary inertia element omitted is illustrated in Figure 4. It is to be noted that it conforms to the general scheme

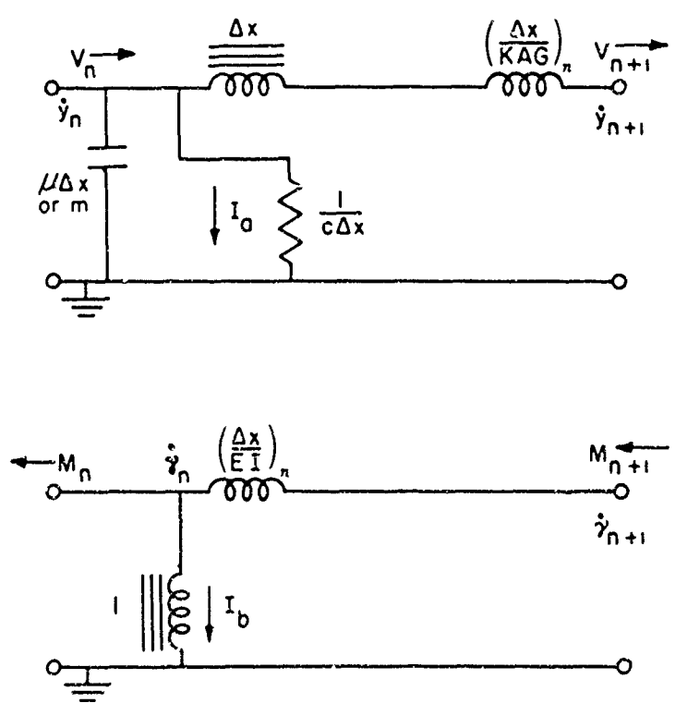

Figure 4 - Typical Section of Analogy of Vibrating Ship Using Ideal Transformer

The transformer [illegible] that when the instantaneous polarity across the moment w[illegible] is positive from the moment line to ground the instantaneous polarity acr[illegible] the shear wi[illegible] positive from right to left.

of the mobility analogy and employs an "ideal transformer." It allows for the effects of both shear and bending flexibility. In the figure the mechanical symbols for shearing force, bending moment, slope due to bending only, and deflection are used directly to indicate the magnitudes of the analogous currents or voltages as the case may be, in the electrical circuit. V and M represent currents and $\dot{y}$ and $\dot{\gamma}$ represent voltages. As usual the dot notation represents differentiation with respect to time.

The transformer shown in Figure 4 has one winding in series with the coil representing $\Delta x/KAG$ while the other winding shunts the moment line to ground. As indicated in this figure the transformer ratio is $1:\Delta x$, which means that if a unit difference of potential exists across the shunt winding, then a potential difference Δx is induced across the series winding. In general, the required transformation will involve an increase in the frequency, that is, the oscillator will drive the circuit over a higher frequency range than the range of frequencies encountered in hull vibration. Symmetry may be obtained here by dividing the condenser m and the coil $\Delta x/EI$ in two and placing half at each end of the circuit. When this is done, these halves will combine with the halves from the adjoining section so that, except at the ends, the circuit will have the same general appearance as shown in Reference 33.

A 20-section transformer circuit representing the USS NIAGARA was set up on the network analyzer at the Taylor Model Basin, and both normal modes and natural frequencies were determined. From a comparison between the results obtained with this analog and the results obtained by digital computation, it appears that the analog permits as much accuracy in solving this problem as is warranted by the accuracy of the given data, that is, the quantities EI, μ, and KAG.

The time required for obtaining normal modes and natural frequencies by means of this analog is scarcely more than the time required to set up the circuit.

One of the great advantages of the electrical analog, if it is a true dynamic analogy, over other methods of calculation in problems of this sort is the ease with which forced steady-state vibration and transient vibrations can be obtained. Certain of the forced-vibration calculations for the USS NIAGARA (with a sinusoidal driving force of one-ton amplitude acting at various positions along the hull and for a uniformly distributed viscous damping) which were made on the digital sequence calculator were also set up for solution on the transformer analog, and the results are discussed in Reference 33.

8. SUMMARY AND CONCLUSIONS

The methods described in this report show promise of yielding at least the first six vertical modes of vibration of a ship. The limits of accuracy are close enough to offer a basis for designing so as to avoid having to operate close to a critical frequency.

Where in a design it is not possible to reduce the exciting forces or to alter the operating speed it may still be possible to alter the longitudinal position of the propellers or unbalanced machinery in such a way as to reduce greatly the combined amplitudes in the various normal-mode components at the operating speed. The methods given in this report should indicate the improvement to be expected from such alterations.

Improvement in the accuracy of calculating the vertical modes can be expected only when improved methods of estimating the parameters that enter into the calculation have been found. These parameters are chiefly the mass (including the virtual mass of surrounding water), the bending-rigidity factors, and the shear-rigidity factors. In special cases the rotary inertia may also be of importance.

The methods described are not limited to the calculation of natural frequencies and normal modes but are applicable to forced vibration also. For this purpose, however, the additional factors of driving force and damping must be taken into account. At present, reliance must be placed on the scant experimental data available in estimating these values. As experimental data are collected in this field in the future forced-vibration estimates will become more reliable.

The calculation of horizontal and torsional modes of vibration for the present must involve more assumptions than the calculation of the vertical modes as less is known about all the parameters entering into the calculation in these cases. The possibility of coupling between torsional and horizontal vibration suggests the desirability of setting up calculations for such coupled modes by the use of the equations given here. The extent of such coupling is at present unknown, but the calculation if correctly set up will yield the uncoupled modes also in case the coupling should prove to be of negligible extent.

9. ACKNOWLEDGMENTS

In carrying out the studies on which this report is based the authors had the cooperation of numerous individuals. In particular, it was due to the encouragement of Captain H.E. Saunders, former Director of the David Taylor Model Basin and to Professor J. Ormondroyd, of the University of Michigan, that a renewed effort was recently made by the Model Basin and the University of Michigan in the extension of hull-vibration theory. The investigations described would not have been possible without the enthusiastic support furnished by Captain J.A McNally and Mr. E.F. Noonan of the Bureau of Ships, as well as Captain R.A. Hinners, former Officer-in-Charge of the Structural Mechanics Laboratory and Admiral C.O. Kell, former Director of the Model Basin. The assistance of Mr. N.H. Jasper, Miss E.J. Adams, Mrs. A.W. Mathewson. and Mr. E. Kapiloff of the Taylor Model Basin staff and Mr. J.L. Edman of the University of Michigan has been most helpful. Dr. E.H. Kennard, Chief Scientist of the Structural Mechanics Laboratory at the Model Basin offered many constructive suggestions in reviewing this report. In the development of methods of computation the authors are indebted to the assistance of Dr. H.M Trent of the Naval Research Laboratory and to Mr. R. Hopkins and Mr. R.R. Seeber, Jr., of the International Business Machines Corporation.

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