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SUBJECT: A Note on the Conical Pendulum Analogy to Rotary Sloshing

FROM: R. R. Berlot

REFERENCES:

1. R-W Memo GM45.3-397, Revision 1, "Representation of Sloshing Modes by the Motions of a Set of Conical Pendulums," to File from R. R. Berlot, dated 19 September 1957.
2. R-W Memo "Motion of Fluid in a Cylindrical Tank," Rev. 2, to File from G. J. Gleghorn, dated 7 February 1956.
3. R-W Memo GM45.3-138, "Propellant Tank Fluid Motion Including Rotation," to File from R. R. Berlot, dated 21 November 1956.
4. R-W Memo GM61.4-9, "On the Damping of Liquid Sloshing by Rings," by J. W. Miles, dated 6 May 1957.

In Reference 1, a model for the sloshing modes was proposed. The purpose of the present note is to call attention to certain features which lend credence to this model and to indicate the necessary modification in baffle damping theory.

As in Reference 1, consider a simple pendulum of mass, m_i, and length l_i, which is rotating about the gravity vector, $\bar{g}$, with an angular velocity, ω, at an angle, θ, with respect to the gravity vector. Then the kinetic energy of rotation is given by:

$$E = (1/2)\, m_i \omega^2 l_i^2 \sin^2 \theta \qquad (1)$$

To determine θ, we equate the moment due to centrifugal force to the moment due to gravity and obtain:

$$\cos \theta = g/\omega^2 l_i \qquad (2)$$

Using equation (2) to replace $\sin^2 \theta$ in equation (1), we obtain:

$$E = \frac{m_i}{2} \omega^2 l_i^2 (1 - g^2/\omega^4 l_i^2) \qquad (3)$$

Now the kinetic energy cannot be negative, so that for rotational motion to be possible it is necessary that:

$$\omega > \sqrt{g/l_i} \tag{4}$$

But g/l_i is the natural frequency ω_i, of a simple pendulum, so we conclude that rotational motion cannot exist below this frequency. This fact has been observed in sloshing experiments.

More generally, the motion is represented by a set of pendulums so that equation (3) becomes:

$$E = \sum_{i=1}^{\infty} \frac{m_i}{2} \, i^2 \omega^2 l_i^2 (1 - g^2/\omega^4 i^4 l_i^2) \tag{5}$$

Now for $\omega < g/l_i$ all of the terms in the series will be negative so that the conclusion deduced from the inequality (4) still holds. The threshold for rotational motion is given by the root of the equation:

$$\sum_{i=1}^{\infty} \frac{m_i l_i^2 i^2}{2} (1 - g^2/\omega^4 i^4 l_i^2) = 0; \text{ namely } \omega = \left\{ \left| \frac{\sum (m_i g^2/ i^2)}{\sum (m_i l_i^2 i^2)} \right| \right\}^{1/4} \tag{6}$$

We turn now to some fluid dynamical considerations. In References 2 and 3, a potential function was derived for linear, uncoupled sloshing of an ideal fluid. We shall now show how the theory of these references may be extended to include circular motion, which, however, still maintains the curl of the velocity as zero, except along one line. This is an important point, since, by a theorem of Helmholtz, if the curl of the velocity is zero at any time in an ideal fluid, it will be zero forever. Now it has been observed that it is possible to set up rotary sloshing in water without any initial circulation, and since water should behave essentially like perfect fluid under the test conditions, it should be possible to explain the rotary sloshing phenomenon essentially in terms of perfect fluid theory.

We consider the case of a cylindrical tank and assume the motion to be made up of two sloshing waves oriented at right angles and a two-dimensional vortex centered around the axis of the tank. The potential function for this motion is given by:

$$\phi = K\theta + \sum_{i=1}^{\infty} J_1(k_i r)\left[(A_i \sinh k_i z + B_i \cosh k_i z)\cos\theta + (C_i \sinh k_i z + D_i \cosh k_i z)\sin\theta\right] + (\gamma + z)\dot{\beta}_y r\cos\theta - (\gamma + z)\dot{\beta}_x r\sin\theta \tag{7}$$

Where r, θ, z are circular cylindrical coordinates, K is constant in both space and time, and β and γ have the same significance as in Reference 3. Now the satisfaction of the boundary conditions at the radial wall, $r = R$, at the free surface, $z = 0$, and at the tank bottom $z = -h$ does not involve the vortex term, $K\theta$, since this contributes only a tangential velocity component and, for an ideal fluid, the boundary conditions are expressed only in terms of normal velocity components. Therefore, the relations between the sloshing amplitude constants and the corresponding driving acceleration constants are the same as derived in Reference 3, namely:

$$\begin{pmatrix} A_i \\ C_i \end{pmatrix} = \frac{4R\left[s^2\begin{pmatrix}\dot{\beta}_y \\ \dot{\beta}_x\end{pmatrix}/k_i + s\left\{s\gamma\begin{pmatrix}\dot{\beta}_y \\ \dot{\beta}_x\end{pmatrix} - \begin{pmatrix}A_x \\ A_y\end{pmatrix} + 2A_z\begin{pmatrix}\beta_y \\ \beta_x\end{pmatrix}\right\}(\sinh k_i h)/2\right]}{(k_i^2 R^2 - 1) J_1(k_i R)\cosh k_i h\left[s^2 + k_i A_z \tanh k_i h\right]} \tag{8}$$

$$\begin{pmatrix} B_i \\ D_i \end{pmatrix} = \frac{4R\left[s\left\{s\gamma\begin{pmatrix}\dot{\beta}_y \\ \dot{\beta}_x\end{pmatrix} - \begin{pmatrix}A_x \\ A_y\end{pmatrix} + sA_z\begin{pmatrix}\beta_y \\ \beta_x\end{pmatrix}\right\}(\cosh k_i h)/2 - \begin{pmatrix}\dot{\beta}_y \\ \beta_x\end{pmatrix} A_z\right]}{(k_i^2 R^2 - 1) J_1(k_i R)\cosh k_i h\left[s^2 + k_i A_z \tanh k_i h\right]} \tag{9}$$

(We have changed the notation slightly from Reference 3 to conform with that of Reference 2.) It should be noted that A_x, A_y, $\dot{\beta}_x$, and β_y contain contributions from the nonlinear cross-coupling terms.

Now, although equation (7) contains sufficient features to explain the phenomenon of rotary sloshing, the velocity associated with the vortex becomes infinite for $r = 0$. When integrated over the volume of the fluid, this would lead to infinite energy. To overcome this defect, we must recognize that the actual fluid cannot be entirely curl-free, and introduce a rankine-combined vortex through the following stream function:

$$\psi = \frac{\zeta}{4}(a^2 - r^2) \qquad \text{for } r < a \tag{10}$$

$$\psi = \frac{\zeta a^2}{2}\ln\frac{r}{a} \qquad \text{for } r > a \tag{11}$$

This function yields the same velocity in the θ-direction as the vortex potential, $k\theta$, (with a proper choice of K), for $r > a$, namely:

$$v_\theta = \zeta a^2/2r \tag{12}$$

For $r < a$, the velocity in the θ-direction is given by:

$$v_\theta = \zeta r/2 \tag{13}$$

No velocity terms are contributed by ψ to either the r or z-directions.

The equations (10) and (11) contain two constants, the vortex strength, ζ, and the vortex filament radius, a. These constants may be chosen so that the rotational energy and angular momentum associated with the vortex motion equal the corresponding quantities for the rotating conical pendulums. The energy of the pendulums has already been given in equation (5).

The angular momentum to be associated with the vortex motion is:

$$H_p = \omega \sum_{i=1}^{\infty} m_i l_i^2 \left(1 - g^2/l_i^2\, i^4 \omega^4\right) \tag{14}$$

The corresponding quantities, as obtained from the fluid stream function ψ, are:

$$E_f = \frac{\pi \rho h}{4} \zeta^2 a^4 \left(1/4 + \ln \frac{R}{a}\right) \tag{15}$$

$$H_f = 2\pi \rho h\, \zeta a^2 \left(\frac{R^2}{4} - \frac{a^2}{8}\right) \tag{16}$$

Equating E_p to E_f, and H_p to H_f, and eliminating ζ leads to:

$$\ln \frac{R}{a} = \frac{\pi \rho h \left(2R^2 - a^2\right)}{8 \sum_{i=1}^{\infty} i \left(m_i l_i^2 - m_i g^2/\omega^4 i^4\right)} - 1/4 \tag{17}$$

From Reference 1, the summations are obtained as:

$$\sum_{i=1}^{\infty} i\, m_i l_i^2 = 2\pi \rho R^5 \sum_{i=1}^{\infty} \frac{i}{k_i^3 R^3 \left(k_i^2 R^2 - 1\right) \tanh k_i h} \tag{18}$$

$$\sum_{i=1}^{\infty} \frac{m_i}{i^3} = 2\pi\rho R^3 \sum_{i=1}^{\infty} \frac{\tanh k_i h}{i^3 k_i R (k_i^2 R^2 - 1)} \tag{19}$$

If $k_i h$ is greater than 1.84, these can be approximated to within 5% by:

$$\sum_{i=1}^{\infty} m_i l_i^2 = .426\ \rho R^5 \tag{20}$$

$$\sum_{i=1}^{\infty} \frac{m_i}{i^3} = 1.44\ \rho R^3 \tag{21}$$

This corresponds to tanks in which the liquid level exceeds the tank radius. If we substitute equations (20) and (21) into equation (17) and neglect a^2 with respect to $2R^2$, we obtain:

$$a/R = 1.28\ e^{-3.7\ h/R}\ (1 - 3.4\ g^2/\omega^4 R^2) \tag{22}$$

The smallness of a/R, as given by equation (21) for $h > R$, justifies the neglect of a^2. Using equations (5), (15), and (22), we obtain for ζ:

$$\zeta = 35.6\ \omega\ (R/h)\ (1 - 3.4\ g^2/\omega^4 R^2)\ e^{7.4h/R}\ (1 - 3.4\ g^2/\omega^4 R^2) \tag{23}$$

The solution for a and ζ for those cases in which the approximation given by equation (20) does not hold, while more complicated, presents no theoretical difficulty, provided the liquid level, h, is not too low. For, in that event, the quantity

$$\sum_{i=1}^{\infty} i\, m_i l_i^2$$

goes to infinity from equation (18), and the energy, E_p, given by equation (5) would also be infinite for any finite ω. From equation (2), we see that θ approaches $\pi/2$ under the circumstances, and we may conclude that the reason for the breakdown of the theory is that the assumption of small oscillations, which was made in order to satisfy the free surface boundary condition, has been violated.

We may note that the form of the pressure equation, as derived in Reference 3, is unchanged by the inclusion of the vortex term. This is because in the linearized pressure equation it is only the partial derivative of the velocity with respect to time which enters and the vortex term is assumed to be independent of time. However, the vortex will undoubtedly make a significant contribution to the non-linear pressure equation which we include here for completeness.

$$p/\rho = \partial\phi/\partial t - [illegible] - q^2/2 - [illegible] \cdot (\bar{r} \times \nabla\phi) \tag{24}$$

$\bar{r} = x\,\bar{i} + y\,\bar{j} + ([illegible] + z)\,\bar{k}$; [illegible] = instantaneous angular velocity vector

Here, as in Reference 3, Ω is the potential of external accelerations, and q is the total velocity amplitude.

Finally, we turn to the question of damping by horizontal ring baffles. In Reference 4, an expression was derived for the mean rate of dissipation of energy. In the notation of that reference, the expression is:

$$\frac{dE}{dt} = -(1/2)\rho\alpha \; S \, C_D \left[\omega \, \zeta_1 \; f(-d) \right]^3 \left(\frac{4}{3\pi}\right)^2 \tag{25}$$

The damping ratio was taken as:

$$\gamma = -(2\,\omega\,E)^{-1} \frac{dE}{dt} \tag{26}$$

The total energy was given by

$$E = (1/4)\,\rho\, g\, S \left[1 - (kR)^{-2} \right] \zeta_1^2 \tag{27}$$

For the case in which vortex motion is present, we may still use equation (26) for the damping of a ring baffle provided we replace E by the total energy E_T, including that in the vortex as given by equation (15), namely

$$E_T = E + E_f \tag{28}$$

Since E_T is always greater than E, it is apparent that the damping due to a horizontal ring baffle will be less in the presence of rotary sloshing than in its absence. However, if we were to include vertical baffles so that the damping in equation (26) was related to

$$\frac{dE_T}{dt}$$

instead of

$$\frac{dE}{dt}$$

the damping could be increased.

In closing, we emphasize that the theory presented here is an extrapolation from linear theory which must be checked by experiment before it can be accepted finally. It is a complete theory in that no constants are left to be determined empirically and it affords many opportunities for experimental verification. These include the calculation of wave heights, fluid pressures, forces, moments, and effectivity of various baffle configurations, all of which can also be obtained experimentally. The inclusion of the vortex term, while not necessary for explaining a rotating line of nodes, provides a means of carrying angular momentum which is not present in pure wave theory, since in the latter case the fluid particles do not actually move in circular trajectories. As a consequence, centrifugal force cannot be present in pure wave theory and this should provide a further criterion for the validity of the vortex term. The need for some sort of term to carry angular momentum arises through the equivalence of tank and inertial coordinates for describing the motion of the fluid. If we accelerate the tank in the x - direction, for example, when the wave happens to be maximum in the y - direction, then we may equally well assume the tank stationary and the fluid moving. The direction of motion of the off-center fluid mass will be deflected by the curved wall of the tank and converted to angular momentum. By the momentum principle, this will result in a force on the tank wall.

Further theoretical studies are continuing.

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