PARAMETER ESTIMATION FOR WAVEFORMS IN ADDITIVE GAUSSIAN NOISE

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SUMMARY

A method is developed for computing the greatest lower bound for the variance of unbiased estimates of waveform parameters, when the waveform is observed in additive Gaussian noise.

The greatest lower bound is approximately evaluated in several illustrative cases. The waveform parameters occurring in these examples are amplitude, time delay, and doppler shift.

I. INTRODUCTION

The subject considered is the inherent accuracy with which the parameters of a waveform observed in additive Gaussian noise can be estimated. Woodward[1] considers a special case of this problem, using an approach based on a posteriori probability. The approach to be followed here is based on results in statistical estimation theory concerning the greatest lower bound for the variance of estimates of statistical parameters. Before proceeding to a more precise formulation of the problem, it is convenient to summarize these results of estimation theory. The following is a summary, in a notation and in a form convenient for the proposed application, of results contained in Refs. 2, 3, and 4, or of results which can be obtained by straightforward generalization of these references.

Let Ω be a sample space with points ω, and let μ be a measure defined on Ω. Let Π be any set of points (finite or infinite) called the parameter set, with individual points denoted by ξ. Let $\{p(\omega,\xi)\}$ for $\xi \in \Pi$ be a family of probability densities in Ω with respect to the measure μ. Let $f(\xi)$ denote a real valued function of ξ. We call a real random variable $\phi(\omega)$ an unbiased estimate of $f(\xi)$ if

$$\int_{\Omega} \phi(\omega)\, p(\omega,\xi)\, d\mu = f(\xi), \quad \text{all } \xi \in \Pi. \qquad (1)$$

Now pick some parameter value ξ_o (which we may interpret as the true value of ξ) and consider

$$\sigma^2_{glb}\{f,\xi_o\} = \text{g.l.b.} \int_{\Omega} \left[\phi(\omega) - f(\xi_o)\right]^2 p(\omega,\xi_o)\, d\mu \qquad (2)$$

where g.l.b. means greatest lower bound for all ϕ satisfying (1). Any ϕ satisfying (1) which, when $\xi = \xi_o$, has variance equal to $\sigma^2_{glb}\{f, \xi_o\}$, is called an unbiased estimate of $f(\xi)$ which is locally best for $\xi = \xi_o$.

Now let us suppose that

$$G(\xi, \xi' \mid \xi_o) = \int_{\Omega} \frac{p(\omega, \xi)\ p(\omega, \xi')}{p(\omega, \xi_o)}\ d\mu < \infty, \quad \text{for all } \xi_o, \xi, \xi' \in \Pi. \qquad (3)$$

(We suppose the integrand to be defined almost everywhere in Ω.) It is easy to show that $G(\xi, \xi' \mid \xi_o) - 1$ has the property

$$\sum_{i,j=1}^{n} \left\{ G(\xi_i, \xi_j \mid \xi_o) - 1 \right\} a_i a_j \geq 0 \qquad (4)$$

for any choice of real numbers a_i and points $\xi_i \in \Pi$.

Let us also denote by λ the difference between any two measures over Π, each of which assigns weight to only a finite (but otherwise arbitrary) set of points of Π. In other words, if f is any function of ξ,

$$\int_{\Pi} f(\xi)\ d\lambda(\xi) = \sum_{i=1}^{n} a_i f(\xi_i) \qquad (5)$$

where a_i are real (positive or negative) numbers.

Then

$$\sigma^2_{glb}\{f, \xi_o\} = \text{l.u.b.} \left[\frac{\left\{ \int_{\Pi} [f(\xi) - f(\xi_o)]\ d\lambda(\xi) \right\}^2}{\int_{\Pi}\int_{\Pi} G(\xi, \xi' \mid \xi_o)\ d\lambda(\xi)\ d\lambda(\xi')} \right] \qquad (6)$$

where the l.u.b. means lowest upper bound over all possible λ which assign non-zero weight to at least one point of Π .

Also, it can be shown that there exists a sequence $\{\lambda^{(n)}\}$ such that the quantity in brackets on the right side of (6) approaches $\sigma^2_{glb}\{f, \xi_o\}$ as $n \to \infty$, and such that

$$\lim_{n \to \infty} \int_{\Pi} G(\xi, \xi' \mid \xi_o)\, d\lambda^{(n)}(\xi') = f(\xi) - f(\xi_o) \tag{7}$$

In addition,

$$\sigma^2_{glb}\{f, \xi_o\} = \lim_{n \to \infty} \int_{\Pi}\int_{\Pi} G(\xi, \xi' \mid \xi_o)\, d\lambda^{(n)}(\xi)\, d\lambda^{(n)}(\xi') \tag{8}$$

$$= \lim_{n \to \infty} \int_{\Pi} \left[f(\xi) - f(\xi_o)\right] d\lambda^{(n)}(\xi)$$

It is apparent from (7) that this sequence $\{\lambda^{(n)}\}$ has the property that $\lim_{n \to \infty} \int d\lambda^{(n)}(\xi) = 0$. (Just let $\xi = \xi_o$ in (7).) Moreover, if we can find a function (or generalized function) λ over Π (not necessarily assigning weight to only a finite number of points) such that

$$\int_{\Pi} G(\xi, \xi' \mid \xi_o)\, d\lambda(\xi') = f(\xi) - f(\xi_o) \tag{9}$$

then, under certain conditions,

$$\sigma^2_{glb}\{f, \xi_o\} = \int_{\Pi}\int_{\Pi} G(\xi, \xi' \mid \xi_o)\, d\lambda(\xi)\, d\lambda(\xi') \tag{10}$$

$$= \int_{\Pi} \left[f(\xi) - f(\xi_o)\right] d\lambda(\xi)$$

(The condition under which this holds is that $\int_{\Pi} \frac{p(\omega, \xi)}{p(\omega, \xi_o)} d\lambda(\xi)$ can be defined as an element of the closed linear manifold determined by the family of random variables $\left\{ \frac{p(\omega, \xi)}{p(\omega, \xi_o)} \right\}$, having mean $f(\xi) - f(\xi_o)$ with respect to $dP(\omega, \xi)$, and having variance, with respect to $dP(\omega, \xi_o)$, equal to the right side of (10).)

It must be expected in general that one would only be able to find a sequence of functions satisfying (7) in the limiting sense, although there are some cases in which (9) can be solved in closed form. Inability to solve (9) exactly should not trouble one too much in practical cases, however, since a lower bound for the quantity

$$\int_{\Omega} \left[\phi(\omega) - f(\xi_o) \right]^2 p(\omega, \xi_o)\, d\mu \quad , \quad \phi \text{ satisfying (1)} \qquad (11)$$

is obtained by inserting any λ, satisfying (5), into the quantity in brackets on the right side of (6).

Another fact important for applications is that if λ is any generalized function satisfying the above mentioned conditions (and therefore, in particular, if λ satisfies (5) and is such that $\int_{\Pi} d\lambda(\xi) = 0$), then the expression

$$\int_{\Pi} \int_{\Pi} G(\xi, \xi' \mid \xi_o)\, d\lambda(\xi)\, d\lambda(\xi')$$

is equal to $\sigma^2_{glb}\{f, \xi_o\}$ for any f of the form

$$f(\xi) = \int_{\Pi} G(\xi, \xi' \mid \xi_o)\, d\lambda(\xi') + \text{constant.}$$

The solution λ or $\{\lambda^{(n)}\}$ to Eq. (9) or (7) will in general depend on ξ_0.

Now suppose that a family of stochastic processes is defined corresponding to the family $\{dP(\omega,\xi)\}$ of measures over the sample space Ω. Also suppose that if a subset S of Ω has zero measure for any one of the measures $dP(\omega,\xi)$, then it has zero measure for all $dP(\omega,\xi)$.

Select any particular point ξ_0 in Π. Then[5] there exist non-negative functions $p(\omega,\xi \mid \xi_0)$ such that for any measurable subset S of Ω,

$$\int_S dP(\omega,\xi) = \int_S p(\omega,\xi \mid \xi_0)\, dP(\omega,\xi_0) \qquad (12)$$

If we set $d\mu = dP(\omega,\xi_0)$ in the above equations, then

$$G(\xi,\xi' \mid \xi_0) = \int_\Omega p(\omega,\xi \mid \xi_0)\, p(\omega,\xi' \mid \xi_0)\, dP(\omega,\xi_0) \qquad (13)$$

One may also ask what estimator actually attains the minimum variance, at ξ_0, of unbiased estimates of $f(\xi)$. The answer is that it is that element of the linear manifold determined by the family of random variables $\{p(\omega,\xi \mid \xi_0)\}$ whose mean with respect to $dP(\omega,\xi)$ is $f(\xi)$. Since $p(\omega,\xi \mid \xi_0)$ can often be evaluated (see Eq. (25) below), it is possible in many cases to give the minimum variance estimator.

II. ESTIMATION OF WAVEFORM PARAMETERS

Let $\{F(t, \xi)\}$ be a family of real-valued functions of time t, where ξ ranges over some set Π. Suppose that $\{x(t)\}$ is a (real) Gaussian random process defined over the time interval $T_1 \leq t \leq T_2$, having mean zero and covariance function $\psi_x(s,t)$. (Stationarity is not required.) Suppose that we observe functions $z(t)$ over the interval $T_1 \leq t \leq T_2$, where

$$z(t) = x(t) + F(t,\xi) \tag{14}$$

We will evaluate $G(\xi,\xi'|\xi_0)$ for some situations of this kind.

First, suppose T_1 and T_2 are finite and that $\{x(t)\}$ is continuous in the mean over $[T_1,T_2]$;then[5],

$$x(t) = \sum_{\nu=1}^{\infty} x_\nu \frac{\psi_\nu(t)}{\sqrt{\lambda_\nu}} \tag{15}$$

where λ_ν and $\psi_\nu(t)$ are the eigenvalues and (orthonormal) eigenfunctions of the integral equation

$$\psi(s) = \lambda \int_{T_1}^{T_2} \psi_x(s,t)\, \psi(t)dt \;. \tag{16}$$

In (15), convergence is in the mean for every t in $[T_1,T_2]$.

Also,

$$x_\nu = \sqrt{\lambda_\nu} \int_{T_1}^{T_2} x(t)\, \psi_\nu(t)dt \tag{17}$$

The random variables x_ν are independent and normally distributed with means zero and standard deviations unity.

We will also suppose that for each ξ in Π,

$$F(t,\xi) = \sum_{\nu=1}^{\infty} \beta_\nu(\xi) \; \psi_\nu(t) \tag{18}$$

where the convergence is pointwise as well as in mean square in $[T_1, T_2]$, and

$$\beta_\nu(\xi) = \int_{T_1}^{T_2} F(t,\xi) \; \psi_\nu(t)dt \; . \tag{19}$$

We also suppose that, for each ξ in Π,

$$\sum_{\nu=1}^{\infty} \lambda_\nu \, \beta_\nu^2(\xi) < \infty \; . \tag{20}$$

Then

$$z(t) = \sum_{\nu=1}^{\infty} z_\nu \, \frac{\psi_\nu(t)}{\sqrt{\lambda_\nu}} \tag{21}$$

where

$$z_\nu = x_\nu + \beta_\nu(\xi)\sqrt{\lambda_\nu} \tag{22}$$

$$= \sqrt{\lambda_\nu}\int_{T_1}^{T_2} z(t) \; \psi_\nu(t)dt$$

The joint density function (with respect to Lebesgue measure) of $z_1, \ldots, z_N$ is

$$g_N(z_1, \ldots, z_N) \tag{23}$$

$$= (2\pi)^{-N/2} \exp\left\{-\frac{1}{2} \sum_{\nu=1}^{N} \left[z_\nu - \beta_\nu(\xi)\sqrt{\lambda_\nu}\right]^2\right\}$$

We may regard $z_1, z_2, \ldots$ as observable coordinates of the stochastic process $\{z(t)\}$.

Also, with probability one(5),

$$p(\omega, \xi \mid \xi_o) = \lim_{N \to \infty} \frac{g_N(z_1, \ldots, z_N \mid \xi)}{g_N(z_1, \ldots, z_N \mid \xi_o)} \tag{24}$$

Thus, by (23),

$$p(\omega, \xi \mid \xi_o) = \exp\left\{-\frac{1}{2}\sum_{\nu=1}^{\infty} \lambda_\nu \left[\beta_\nu^2(\xi) - \beta_\nu^2(\xi_o)\right]\right\} \tag{25}$$

$$\times \exp\left\{-\sum_{\nu=1}^{\infty} z_\nu \sqrt{\lambda_\nu}\left[\beta_\nu(\xi_o) - \beta_\nu(\xi)\right]\right\}$$

Consequently,

$$p(\omega, \xi \mid \xi_o)\, p(\omega, \xi' \mid \xi_o)$$

$$= \exp\left\{-\frac{1}{2}\sum_{\nu=1}^{\infty} \lambda_\nu \left[\beta_\nu^2(\xi) + \beta_\nu^2(\xi') - 2\,\beta_\nu^2(\xi_o)\right]\right\} \tag{26}$$

$$\times \exp\left\{-\sum_{\nu=1}^{\infty} z_\nu \sqrt{\lambda_\nu}\left[2\,\beta_\nu(\xi_o) - \beta_\nu(\xi) - \beta_\nu(\xi')\right]\right\}$$

Thus, by (13),

$$G(\xi, \xi' \mid \xi_o) = \exp\left\{\sum_{\nu=1}^{\infty} \lambda_\nu\, \beta_\nu^2(\xi_o)\right\} \tag{27}$$

$$\times \exp\left\{\sum_{\nu=1}^{\infty} \lambda_\nu \left[\beta_\nu(\xi)\,\beta_\nu(\xi') - \beta_\nu(\xi)\,\beta_\nu(\xi_o) - \beta_\nu(\xi')\,\beta_\nu(\xi_o)\right]\right\}$$

It is possible in many cases to express $G(\xi, \xi' \mid \xi_o)$ in a more convenient form: suppose

$\tilde{k}(s,\xi)$ satisfies the integral equation

$$\int_{T_1}^{T_2} \psi_x(s,t)\, \tilde{k}(s,\xi)\,ds = F(t,\xi) \tag{28}$$

Here $\tilde{k}(s,\xi)$ is permitted to contain a finite number of delta functions of various orders.

Then, under conditions which will be stated,

$$\sum_{\nu=1}^{\infty} \lambda_\nu \beta_\nu(\xi)\, \beta_\nu(\xi') = \int_{T_1}^{T_2} F(t,\xi')\, \tilde{k}(t,\xi)\, dt \tag{29}$$

To show this, consider the following linear prediction problem: let $q(\xi)$ be a random variable defined on Ω such that $q(\xi)$ has zero mean, finite variance, and such that

$$E\left[q(\xi)\, x(t)\right] = F(t,\xi) \tag{30}$$

Such a random variable exists by (20). $E[\;]$ denotes the expected value of the quantity in brackets.

Let $\tilde{q}(\xi)$ be a random variable such that

(i) $\tilde{q}(\xi)$ is a linear operation over $[T_1, T_2]$ on the random process $\{x(t)\}$;

(ii) $\tilde{q}(\xi)$ minimizes $E\left[\hat{q} - q(\xi)\right]^2$ for all linear operations $\hat{q}$ over $[T_1, T_2]$ on $\{x(t)\}$.

By 'linear operation on $\{x(t)\}$ over $[T_1, T_2]$' is meant any random variable of the form

$\hat{q} = \sum_{\nu=1}^{\infty} c_\nu x_\nu$ where $\sum_{\nu=1}^{\infty} c_\nu^2 < \infty$.

Now,[6]

$$\tilde{q}(\xi) = \sum_{\nu=1}^{\infty} x_\nu \, \beta_\nu(\xi)\sqrt{\lambda_\nu} \tag{31}$$

Also, $\tilde{q}(\xi)$ is given by

$$\tilde{q}(\xi) = \int_{T_1}^{T_2} x(t)\, \tilde{k}(t,\xi)dt \tag{32}$$

provided the integration in (32) is legitimate and provided the satisfaction of (28) is a sufficient condition that $\tilde{q}(\xi)$ given by (32) minimize $E\left[\hat{q} - q(\xi)\right]^2$. Equation (29) is then established by calculating $E\left[\tilde{q}(\xi)\ \tilde{q}(\xi')\right]$.

There is a large literature[7] on the solution of the integral equation (28), and in many cases the conditions of validity of (29) can be established.

One special case is the following:

Suppose

$$\Psi_x(s,t) = d\ e^{-c|s-t|} \tag{33}$$

Then, denoting $\frac{d}{dt} F(t,\xi)$ by $F'(t,\xi)$, etc.,

$$\begin{aligned}\tilde{k}(t,\xi) = {} & \frac{c}{2d}\left[F(t,\xi) - \frac{1}{c^2}\ F''(t,\xi)\right] \\ & + \frac{1}{2d}\left[F(T_1,\xi) - \frac{1}{c}\ F'(T_1,\xi)\right]\ \delta(t-T_1) \\ & + \frac{1}{2d}\left[F(T_2,\xi) + \frac{1}{c}\ F'(T_2,\xi)\right]\ \delta(t-T_2)\end{aligned} \tag{34}$$

Thus,

$$\sum_{\nu=1}^{\infty} \lambda_\nu \beta_\nu(\xi)\ \beta_\nu(\xi') \tag{35}$$

$$= \frac{c}{2d}\left\{ \int_{T_1}^{T_2} \left[F(t,\xi)F(t,\xi') + \frac{1}{c^2} F'(t,\xi)F'(t,\xi')\right] dt \right.$$

$$\left. + \frac{1}{c} F(T_1,\xi)F(T_1,\xi') + \frac{1}{c} F(T_2,\xi)F(T_2,\xi')\right\}$$

One reasonable representation for 'white noise' of spectral density N_o is noise having a covariance function given by (33) with

$$N_o = \frac{4d}{c} \tag{36}$$

and c and d very large.

From (35) and (27), for $\{x(t)\}$ with covariance function given by (33) with $\frac{4d}{c} = N_o$ and $c, d \to \infty$,

$$\lim G(\xi,\xi' \mid \xi_o) = \exp\left\{\frac{2}{N_o}\left[\int_{T_1}^{T_2} F^2(t,\xi_o)dt \right.\right. \tag{37}$$

$$+ \int_{T_1}^{T_2} F(t,\xi)F(t,\xi')dt - \int_{T_1}^{T_2} F(t,\xi)F(t,\xi_o)dt$$

$$\left.\left. - \int_{T_1}^{T_2} F(t,\xi')F(T,\xi_o)dt\right]\right\}$$

It is reasonable to suppose that a close approximation to $\sigma^2_{glb}\{f,\xi_o\}$ can be obtained for this case with very large c and d by inserting G as given by (37) into (6), though this seems difficult to prove rigorously.

Another case in which G may be evaluated is as follows: let the interval of observation be the whole real axis $-\infty \leq t \leq \infty$, and let

$\{x(t)\}$ be stationary with constant spectral density N_o over $0 \leq f \leq W$ and zero spectral density for $f > W$. Also suppose that for each ξ, $F(t, \xi)$ has a Fourier transform which vanishes for $|f| > W$.

Then, the process $\{z(t)\}$ may be regarded[8] as being determined by the random variables $z(\frac{i}{2W})$, $i = 0, \pm 1, \pm 2, \ldots$ These are independent, normally distributed random variables with means $F(t_i, \xi)$, where $t_i = \frac{i}{2W}$, and standard deviations $N_o W$. We obtain the result that $G(\xi, \xi' \mid \xi_o)$ is given exactly by the right side of (37), with $T_1 = -\infty$ and $T_2 = \infty$.

To recapitulate what can be accomplished by using the above expressions for G: by inserting any function λ satisfying (5) into the quantity in brackets on the right side of (6), one obtains a lower bound for the variance, at ξ_o, of unbiased estimates of $f(\xi)$; also, if λ is any generalized function,* then

$$\int_{\Pi} \int_{\Pi} G(\xi, \xi' \mid \xi_o)\, d\lambda(\xi) d\lambda(\xi') \quad \text{is equal to} \quad \sigma^2_{glb} \{f, \xi_o\}$$

for any function f of the form

$$f(\xi) = \int_{\Pi} G(\xi, \xi' \mid \xi_o) d\lambda(\xi') + \text{constant}.$$

A possible computational procedure for approximation of $\sigma^2_{glb} \{f, \xi_o\}$ is to select n points ξ_i from Π, and solve the equations

$$\sum_{j=1}^{n} G(\xi_i, \xi_j \mid \xi_o)\, a_j = f(\xi_i) - f(\xi_o) \qquad (38)$$

Then the function $\lambda = \sum_{i=1}^{n} a_i \delta(\xi - \xi_i)$ could be inserted into the brackets on the right side of (6).

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*satisfying certain conditions mentioned in Section I, including the condition $\int_{\Pi} d\lambda(\xi) = 0$

III. EVALUATION OF $\sigma^2_{glb}\{f, \xi_0\}$ IN SPECIAL CASES

We will suppose from now on that the interval of observation and the noise process $\{x(t)\}$ are such that $G(\xi,\xi'|\xi_0)$ is given, or very closely approximated, by (37) with $T_1 = -\infty$, $T_2 = \infty$.

Suppose first that

$$F(t,\xi) = \alpha F(t-\tau) \tag{39}$$

where α belongs to some real non-negative interval A, and τ belongs to some finite real interval $[a, b]$. Here $\xi = (\alpha,\tau)$; Π = direct product of A and $[a, b]$.

Then

$$G(\xi,\xi'|\xi_0) = \frac{e^R\,[H(\tau-\tau')]^{\alpha\alpha'/\alpha_0^2}}{[H(\tau-\tau_0)]^{\alpha/\alpha_0}\,[H(\tau'-\tau_0)]^{\alpha'/\alpha_0}} \tag{40}$$

where

$$R = \frac{2\alpha_0^2}{N_0}\int_{-\infty}^{\infty} F^2(t)\,dt \tag{41}$$

$$H(\tau) = \exp\{R\,\rho(\tau)\} \tag{42}$$

$$\rho(\tau) = \frac{\int_{-\infty}^{\infty} F(t)F(t+\tau)\,dt}{\int_{-\infty}^{\infty} F^2(t)\,dt} \tag{43}$$

Equations (9) and (10) become

$$\int_{\Pi} \left[H(\tau-\tau')\right]^{\alpha\alpha'/\alpha_o^2} d\lambda^*(\xi') = \left[H(\tau-\tau_o)\right]^{\alpha/\alpha_o} \left[f(\xi) - f(\xi_o)\right] \qquad (44)$$

where

$$d\lambda^*(\xi') = e^R \left[H(\tau'-\tau_o)\right]^{-\alpha'/\alpha_o} d\lambda(\xi')$$

and

$$\sigma^2_{glb}\left\{f, \xi_o\right\} = e^{-R} \int_{\Pi}\int_{\Pi} \left[H(\tau-\tau')\right]^{\alpha\alpha'/\alpha_o^2} d\lambda^*(\xi) d\lambda^*(\xi') \qquad (45)$$

A. PURE AMPLITUDE ESTIMATION

Suppose in (39) that τ is known, and hence may without loss of generality be taken equal to zero. Also suppose A is non-degenerate and that α_o is interior to A.

Thus, G is given by (40) with $\tau = \tau' = \tau_o = 0$.

Let $f(\xi) = \alpha$. Then (44) becomes

$$\int_{A} \left[H(0)\right]^{\alpha\alpha'/\alpha_o^2} d\lambda^*(\alpha') = \left[H(0)\right]^{\alpha/\alpha_o} (\alpha-\alpha_o) \qquad (46)$$

This is solved (since $H(0) = e^R$) by

$$d\lambda^*(\alpha) = \left[\frac{\alpha_o^2}{R} \delta'(\alpha-\alpha_o) - \alpha_o \delta(\alpha-\alpha_o)\right] d\alpha \qquad (47)$$

where δ and δ' are, respectively, the delta function and its derivative; that is, for any function $g(\alpha)$ differentiable at α_o,

$$\int_A g(\alpha)\, \delta'(\alpha-\alpha_o)\, d\alpha = g'(\alpha_o); \quad \int_A g(\alpha)\, \delta(\alpha-\alpha_o)\, d\alpha = g(\alpha_o).$$

Thus, from (45) with $f(\xi) = \alpha$,

$$\sigma^2_{glb}\{f,\alpha_o\} = \frac{\alpha_o^2}{R} = \frac{N_o}{2}\left[\int_{-\infty}^{\infty} F^2(t)dt\right]^{-1} \tag{48}$$

or

$$\frac{1}{\alpha_o^2}\;\sigma^2_{glb}\{f,\alpha_o\} = \frac{1}{R} \tag{49}$$

One interesting feature of this result is that the answer is independent of A, the a-priori range of variation of α, provided A is non-degenerate. This means that decreasing A does not decrease the minimum error variance of unbiased estimates of α -- in other words, if one has an unbiased estimate attaining the variance σ^2_{glb} in (49), and then if A is decreased, one cannot use this increase in a-priori information to provide an unbiased estimate of decreased variance.

B. PURE TIME-DELAY ESTIMATION

In this case we will suppose that $F(t,\xi)$ is given by (39) but that α_o is known, so that $\Pi = [a,b]$; $\xi = \tau$. Thus G is given by (40) with $\alpha = \alpha' = \alpha_o$. We will derive, under certain conditions, an asymptotic expression for $\sigma^2_{glb}\{f,\tau_o\}$ as $\tau_o - a$ and $b-\tau_o$ approach infinity, for certain functions f for which $f(\tau) \approx \tau$.

We will first assume that $\tau_o = 0$; the answer for general τ_o will be obtained by a minor modification of the result for $\tau_o = 0$. Under the assumed conditions, we must solve for $d\lambda^*(\tau)$ the equation

$$\int_a^b H(\tau-\tau')\, d\lambda^*(\tau') = [f(\tau) - f(0)]\, H(\tau) \tag{50}$$

Also

$$\sigma^2_{glb} = e^{-R} \int_a^b \left[f(\tau) - f(0)\right] H(\tau)\, d\lambda^*(\tau) \tag{51}$$

We first make the following definitions:

let

$$L(\tau) = H(\tau) - 1 \tag{52}$$

We assume that $L(\tau)$, $\tau L(\tau)$, and $\tau^2 L(\tau)$ are integrable over $(-\infty, \infty)$.

Also let

$$\mathcal{L}(u) = \int_{-\infty}^{\infty} e^{-iu\tau} L(\tau)\, d\tau \tag{53}$$

It is not hard to show that $\mathcal{L}(u)$ is a real, strictly positive function of u. We will suppose that $\dfrac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}$ is integrable over $(-\infty, \infty)$.

Now, let

$$d\lambda^*(\tau) = \mu(\tau)\, d\tau + \frac{\tau}{\mathcal{L}(0)}\, d\tau - I\, d\tau \tag{54}$$

where

$$I = \frac{b^2 - a^2}{2\,\mathcal{L}(0)\left[\mathcal{L}(0) + b - a\right]} \tag{55}$$

and

$$M(u) = \int_{-\infty}^{\infty} e^{-iu\tau} \mu(\tau)\, d\tau = \frac{i\,\mathcal{L}'(u)}{\mathcal{L}(u)}\, K(u) \tag{55}$$

where $K(u)$ is a function -- such as e^{-ku^2} -- which insures that $M(u)$ has the necessary integrability properties to justify the following steps, but for which the Fourier transform is close to a delta function. We obtain

$$\int_a^b H(\tau-\tau')\,d\lambda^*(\tau') = \tau H(\tau) + \text{remainder.} \tag{57}$$

The remainder term can, as $a \to -\infty$, $b \to +\infty$, be made very close to zero except for values of τ near the end points a and b.

Also, using (51), we obtain with the aid of Parseval's formula

$$\sigma^2_{glb}\{f,0\} \approx e^{-R}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\,du + \frac{2}{\mathcal{L}(0)}\int_{-\infty}^{\infty}\tau^2\,L(\tau)\,d\tau\right\}$$

$$+ e^{-R}\left\{\frac{b^3-a^3}{3\mathcal{L}(0)} - \frac{(b^2-a^2)^2}{4\mathcal{L}(0)\left[\mathcal{L}(0)+b-a\right]}\right\} \tag{58}$$

(as $a \to -\infty$, $b \to \infty$).

It is clear that if $\tau_0 = 0$, $\sigma^2_{glb}\{f,\tau_0\}$ can be obtained from (58) simply by replacing a and b respectively by $a-\tau_0, b-\tau_0$ in (58).

It will be noticed that the term depending on $[a, b]$ increases as $(b-a)^3$. On the other hand, the estimate $\frac{a+b}{2}$ for τ would have bias $(\tau - \frac{a+b}{2})$, but would have mean square error not greater than $\frac{1}{4}(b-a)^2$. Thus, when R is so small that the a-priori range of variation of τ is the main factor determining mean square error, the requirement of unbiasedness is clearly disadvantageous. On the other hand, if R is large enough that mean square error is much smaller than $(b-a)^2$, as will be true in most cases of interest, one would expect on intuitive grounds that any optimum estimate based on a reasonable criterion would be approximately unbiased (except when the a-priori distribution of τ over $[a, b]$ is known and non-uniform). The term in (58) dependent on $[a, b]$ is useful chiefly as a criterion for how large R should be in order that the error variance be effectively independent of $(b - a)$.

In many cases of interest the parameter values will be such that

$$\sigma^2_{glb}\{f, \tau_o\} \approx \frac{e^{-R}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\, du \tag{59}$$

It is therefore of interest to evaluate, approximately, the expression on the right side of (59). This has been done in a number of cases in Ref. 9. Here we will simply recapitulate the results:

Case 1

Here it will be assumed that $\rho(\tau)$, defined by Eq. (43), is given by

$$\rho(\tau) = \bar{\rho}(\tau) \cos \omega_o \tau \tag{60}$$

with, for small τ,

$$\bar{\rho}(\tau) \approx 1 - \frac{1}{2} \beta^2 \tau^2 \tag{61}$$

then, for sufficiently large R,

$$\frac{e^{-R}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\, du \approx \frac{1}{R\beta^2} \qquad \text{for} \quad \omega_o = 0 \tag{62}$$

$$\approx \frac{1}{R\beta^2} \qquad \text{for} \quad \omega_o >> \beta \quad \text{and} \quad \frac{\omega_o^2}{2R\beta^2} >> 1$$

$$\approx \frac{1}{R\omega_o^2} \qquad \text{for} \quad \omega_o >> \beta \quad \text{and} \quad \frac{\omega_o^2}{2R\beta^2} << 1.$$

These results can be interpreted as follows: for $\frac{\omega_o^2}{2R\beta^2} >> 1$, the result is the same as for $\omega_o = 0$, that is, it is the result determined by the envelope $\bar{\rho}(\tau)$; for $\frac{\omega_o^2}{2R\beta^2} << 1$, the result is that associated with a sinusoidal fine structure of frequency ω_o. The transition occurs at $\frac{\omega_o^2}{2R\beta^2} \approx 1$; that is, when the minimum error standard deviation associated with the envelope becomes roughly equal to the wavelength of the fine structure of $\rho(\tau)$.

Case 2

Here we assume $\rho(\tau)$ to be given by (60), but with

$$\bar{\rho}(\tau) \approx 1 - \gamma|\tau| \qquad (63)$$

for τ near the origin.

Then, for sufficiently large R,

$$\frac{e^{-R}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\,du \approx \frac{1}{2R^2\gamma^2} \quad \text{for} \quad \omega_o = 0 \qquad (64)$$

$$\approx \frac{1}{2R^2\gamma^2} \quad \text{for} \quad \omega_o >> \gamma \text{ and } \frac{\omega_o}{R\gamma} >> 1$$

$$\approx \frac{1}{R\omega_o^2} \quad \text{for} \quad \omega_o >> \gamma \text{ and } \frac{1}{\sqrt{R}} << \frac{\omega_o}{R\gamma} << 1$$

$$\frac{1}{2R^2\gamma^2} \quad \text{for} \quad \omega_o >> \gamma \text{ and } \frac{\omega_o}{R\gamma} << \frac{1}{\sqrt{R}}$$

It would be possible to carry out much the same sort of analysis as in Ref. 9 for $\rho(\tau)$ of the form

$$\rho(\tau) = \rho_1(\tau)\cos\omega_0\tau + \rho_2(\tau)\sin\omega_0\tau \tag{65}$$

where $\rho_1(\tau)$ and $\rho_2(\tau)$ can be expanded in a suitable manner at the origin.

C. PURE DOPPLER SHIFT ESTIMATION

Here it is assumed that

$$F(t;\xi) = e^{\xi/2}\, F(e^{\xi} t) \tag{66}$$

where $\xi = \zeta$.

The parameter ζ is a measure of the 'doppler shift.' For example, if we are dealing with a reflected radar signal from a target moving with radial velocity v, and if we assume $\frac{v}{c}$ to be small (c = velocity of light) then approximately $\zeta = \frac{2v}{c}$.

It will henceforth be assumed that the a-priori range of variation of ζ is a real interval $[a, b]$, and that $|a| << 1$, $|b| << 1$.

The factor $e^{\zeta/2}$ multiplying F makes the total received energy independent of the value of ζ . This would not actually be the case, since the received energy actually would be greater for approaching targets and less for receding targets. However, the factor $e^{\zeta/2}$ makes for a great simplification in the mathematics, so it will be assumed that the received energy is in fact independent of ζ . In cases of practical interest, this assumption probably has little effect on the calculated value of the minimum variance of unbiased estimates of $f(\zeta)$.

We retain the assumption that G can be obtained from (37) with $T_1 = -\infty, T_2 = \infty$; the result is as follows:

Let

$$R = \frac{2}{N_0} \int_{-\infty}^{\infty} F^2(t)\,dt \tag{67}$$

$$\phi(\zeta) = \frac{e^{\zeta/2} \int_{-\infty}^{\infty} F(t)F(e^{\zeta} t)\,dt}{\int_{-\infty}^{\infty} F^2(t)\,dt} \tag{68}$$

$$H(\zeta) = \exp\left[R\,\phi(\zeta)\right] \tag{69}$$

Then

$$G(\zeta, \zeta' \mid \zeta_0) = \frac{e^R H(\zeta - \zeta')}{H(\zeta - \zeta_0)\; H(\zeta' - \zeta_0)} \tag{70}$$

It is seen that this is of precisely the same form as Eq. (40) with $\alpha = \alpha' = \alpha_0$ except that ζ, ζ' are substituted for τ, τ' and the definition of H is given by Eqs. (67), (68), and (69). Therefore, the developments of part B can be followed, finally resulting in:

Let

$$L(\zeta) = H(\zeta) - 1 \tag{71}$$

$$\mathcal{L}(u) = \int_{-\infty}^{\infty} e^{-iu\zeta}\, L(\zeta)\,d\zeta \tag{72}$$

then $\sigma^2_{glb}\{\zeta\}$, the greatest lower bound for the variance of ~~nearly~~ unbiased estimates of ζ when the true value is ζ_0, is given approximately by Eq. (58) with $a - \zeta_0$, $b - \zeta_0$ substituted for a, b, and of course with the new definitions of $\mathcal{L}(u)$, $L(\zeta)$, R, etc.

In particular for sufficiently large R,

$$\sigma^2_{glb}\{f, \varsigma_0\} \approx \frac{e^{-R}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\,du \tag{73}$$

We will now turn to the evaluation of $\phi(\varsigma)$ in a large class of cases. Namely, we assume F of the form

$$F(t) = F_1(t)\cos\omega_0 t + F_2(t)\sin\omega_0 t \tag{74}$$

It can then be established with some elementary trigonometric identities that

$$\phi(\varsigma) = \frac{e^{\varsigma/2}}{\int_{-\infty}^{\infty}F^2(t)dt}\int_{-\infty}^{\infty}\mathcal{F}(t)\,dt \tag{75}$$

where

$$\begin{aligned}\mathcal{F}(t) = {} & \tfrac{1}{2}F_1(t)F_1(e^{\varsigma}t)\left\{\cos[\omega_0(e^{\varsigma}-1)t] + \cos[\omega_0(e^{\varsigma}+1)t]\right\} \\ & + \tfrac{1}{2}F_2(t)F_2(e^{\varsigma}t)\left\{\cos[\omega_0(e^{\varsigma}-1)t] - \cos[\omega_0(e^{\varsigma}+1)t]\right\} \\ & + \tfrac{1}{2}F_1(t)F_2(e^{\varsigma}t)\left\{\sin[\omega_0(e^{\varsigma}-1)t] + \sin[\omega_0(e^{\varsigma}+1)t]\right\} \\ & - \tfrac{1}{2}F_2(t)F_1(e^{\varsigma}t)\left\{\sin[\omega_0(e^{\varsigma}-1)t] - \sin[\omega_0(e^{\varsigma}+1)t]\right\}\end{aligned} \tag{76}$$

Thus, $\phi(\varsigma)$ can be readily evaluated for cases in which the Fourier transforms of $F_1(t)\,F_1(e^{\varsigma}t)$ and $F_2(t)\,F_2(e^{\varsigma}t)$ and of $F_1(t)\,F_2(e^{\varsigma}t)$ and $F_2(t)\,F_1(e^{\varsigma}t)$ are known.

Also, in most cases, if ω_0 is sufficiently large, the terms involving $\omega_0(e^{\varsigma}+1)t$ can all be neglected.

As an illustration, suppose

$$F_1(t) = \exp\left[-\tfrac{1}{2}\beta^2t^2\right] \quad (-\infty \leq t \leq \infty) \tag{77}$$

$$F_2(t) = 0.$$

Then, since $|\zeta|$ is always assumed $<< 1$, and assuming $\omega_o >> \beta$,

$$\phi(\zeta) \approx \exp \frac{-\omega_o^2 \zeta^2}{4\beta^2} \tag{78}$$

$$\frac{e^{-R}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)} du \approx \frac{2\beta^2}{R\omega_o^2} \tag{79}$$

It is also not difficult to obtain the result for

$F_1(t) = \exp\left[-\frac{1}{2}\beta^2 t^2\right]$, $F_2(t) = kt \exp\left[-\frac{1}{2}\gamma^2 t^2\right]$,

for example.

Appendix I

A proposition which may be of use in evaluation of G or of σ^2_{glb} is as follows:

Suppose T_1 and T_2 are finite and $\{x(t)\}$ is continuous in the mean over $[T_1, T_2]$.

Let $\{S_n\}$ be a sequence of sets of points, S_n consisting of the points $t_1^{(n)}, \ldots, t_n^{(n)}$ belonging to the interval $[T_1, T_2]$. Let S_n^+ be the union of S_n and the set consisting of the points T_1 and T_2.

Let Δ_n be the length of the maximum interval between neighboring points of S_n^+, and suppose $\Delta_n \to 0$ as $n \to \infty$.

Let the matrices $\left(\psi_x\left[t_i^{(n)}, t_j^{(n)}\right]\right)$ be nonsingular with matrix inverses $(\eta_{ij}^{(n)})$.

Then

$$\sum_{\nu=1}^{\infty} \lambda_\nu \beta_\nu(\xi)\, \beta_\nu(\xi') = \lim_{n\to\infty} \sum_{i,j=1}^{n} \eta_{ij}^{(n)}\, F(t_i^{(n)}, \xi)\, F(t_j^{(n)}, \xi') \tag{80}$$

To show this, consider the linear prediction problem introduced in the proof of Eq. (29). It can be shown[6] that

$$\tilde{q}(\xi) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{i,j=1}^{n} \eta_{ij}^{(n)}\, F(t_i^{(n)}, \xi)\, x(t_j^{(n)}) \tag{81}$$

Equation (80) then follows from computing $E\left[\tilde{q}(\xi)\, \tilde{q}(\xi')\right]$.

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