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ON THE DETECTION OF A SINE WAVE IN GAUSSIAN NOISE

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I Introduction

1. Historical Background

The first systematic statistical approach to the problem of detection in the presence of noise seems to be that of Wiener (Ref. 10), who studied the problem of designing a linear filter for separating two stationary random processes optimally, using the least-squares error criterion. Wiener's work was extended by Zadeh (Ref. 11) to a case where the signal which it is desired to filter is non-stationary. Van Vleck and Middleton (Ref. 8) also treated the problem of designing a "best" linear filter, but used a criterion different from Wiener's. Their work was generalized by Dwork (ref. 3). These analyses made use of the assumption that an infinite sample of observed signal is available, and adopt a more or less artificial criterion for the optimum detection process. There has recently been some work on analysis of finite samples of observed signal (Ref. 2), but the problem of obtaining a more fundamentally acceptable detection criterion has received only scant attention in the engineering literature.

2. Definition of the Optimum Detector

Before defining optimum detector it seems advisable to define detector. Let

$$S(t) = A \sin (\omega t + \theta) \qquad (I.2.1)$$

be the signal which it is desired to detect. A and ω are known in advance, and θ is purely random. (By this we mean that θ varies randomly from sample to sample, with a uniform probability density in the interval $0 \leq \theta \leq 2\pi$.) Let $N(t)$ denote the noise voltage. If $z(t)$ denotes the observed sample of length T, we have

$$z(t) = S(t) + N(t) \qquad (0 \leq t \leq T) \qquad (I.2.2)$$

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Summary

This paper deals with the problem of finding the "optimum" method of detecting a sine wave of known frequency and amplitude in the presence of noise. The type of noise considered is the so-called stationary Gaussian process, which is obtained when thermal noise is passed through an arbitrary linear passive device.

The analysis takes into account the fact that in practice only a finite sample of observed signal is available.

The optimum detection method is defined as that which maximizes the probability of recognizing the presence of a sine wave if one has actually appeared; while the probability of falsely announcing the presence of a sine wave, if none has actually appeared, does not exceed some pre-chosen value.

It is shown that when the noise has a flat spectrum, all the relevant information is contained in the amplitude and phase of the Fourier transform of the received sample at the frequency of the sine wave. Almost the same result holds in the case where the noise has an exponentially decaying autocorrelation, except that in this case the values of the observed sample at the endpoints of the sample also play a role.

when S(t) is present; and

$$z(t) = N(t) \qquad (0 \leq t \leq T) \qquad (I.2.3)$$

when S(t) is absent.

By a <u>detector</u> is meant a black box whose input is z(t), and whose output is a yes-no indication (Fig. 1), indicating the detector's guess as to whether or not S(t) was present.

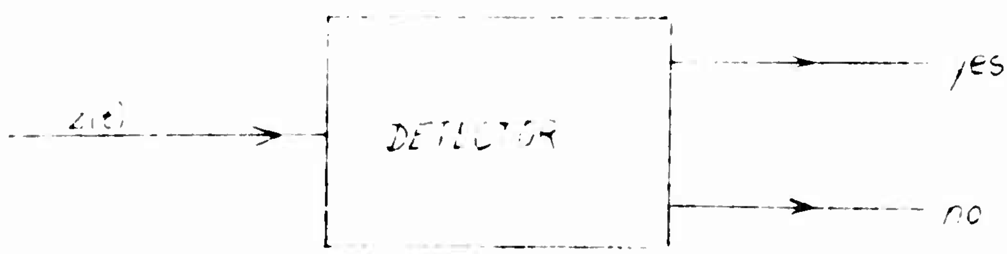

Fig. 1 Block Diagram of Basic Detector

If an infinite sample were available ($T = \infty$) it would be possible, in principle, to design an arbitrarily good detector by using a sufficiently narrow band-pass filter centered at ω, or by performing some such process as looking for a periodic component in the autocorrelation of z(t). Since, however, T is finite, it will, except for degenerate cases (such as a missing band of noise frequencies around ω) be impossible to design a perfect detector. The errors committed by the detector can be of the following two exhaustive and mutually exclusive types:

(a) Detector says "yes" even though S(t) is absent;

(b) Detector says "no" even though S(t) is present.

Let

P_F = probability that the detector falsely announces the presence of S(t) when S(t) is not actually present.

P_R = probability that the detector recognizes S(t) when S(t) actually is present.

We have

$$P_F = \text{probability of error of type (a)} \tag{I.2.4}$$

$$P_R = 1\text{-probability of error of type (b)} \tag{I.2.5}$$

We define the optimum detector as the one which maximizes P_R subject to $P_F \leq$ pre-chosen value $P_F{}^*$. Generally speaking, the higher the upper limit on P_F is set, the higher it will be possible to make P_R.

3. The Likelihood Ratio

The information upon which the detector must base its decision is all contained in the observed sample $z(t)$, $(0 \leq t \leq T)$.

Put

$$t_k = k\,\frac{T}{n} \qquad (k = 1, 2, \ldots, n) \tag{I.3.1}$$

We can assume first that the detector knows only $z_k = z(t_k)$, and later let $n \to \infty$. It is physically obvious that the information obtained by the detector as $n \to \infty$ is all the information in $z(t)$. This has been shown rigorously in Ref. 4.

In the general case, $(z_1, z_2, \ldots, z_n)$ is a set of random variables with a certain joint probability density which depends on whether or not $S(t)$ is present. Let $L_o(u_1, u_2, \ldots, u_n)$ = probability density of $(z_1, z_2, \ldots, z_n)$ when $S(t)$ is absent, and $L_1(u_1, u_2, \ldots, u_n)$ = probability density of $(z_1, z_2, \ldots, z_n)$ when $S(t)$ is present; i.e.,

$$L_o(u_1, u_2, \ldots, u_n)\, du_1\, du_2 \ldots du_n = \text{joint probability that } u_i \leq z_i \leq u_i + du_i \quad (i = 1, 2, \ldots, n) \text{ when } S(t) \text{ is absent.} \tag{I.3.2}$$

$L_1(u_1, u_2, \ldots, u_n)\, du_1\, du_2 \ldots du_n$ = joint probability that

$u_i \leq z_i \leq u_i + du_i \quad (i=1,2,\ldots,n)$

when $S(t)$ is present. (I.3.3)

(L_o and L_1 will be given explicitly in section II, 1.)

The detection problem is equivalent to testing the hypothesis that $(z_1, \ldots, z_n)$ has density function L_1 against the hypothesis that it has density function L_o. It is a known theorem in the theory of testing statistical hypotheses (Ref's. 1, 4, 5, 6, 9) that in order to maximize P_R subject to $P_F \leq P_F^*$ one proceeds as follows:

(a) Define the "likelihood ratio"

$$\Lambda(z_1, z_2, \ldots, z_n) = \frac{L_1(z_1, z_2, \ldots, z_n)}{L_o(z_1, z_2, \ldots, z_n)} \qquad \text{(I.3.4)}$$

Λ is a function of the random variables $z_1, z_2, \ldots, z_n$ and thus is itself a random variable.

(b) Let

$\varphi_o(\lambda)$ = probability density of Λ under the hypothesis that $S(t)$ is absent. (I.3.5)

$\varphi_1(\lambda)$ = probability density of Λ under the hypothesis that $S(t)$ is present. (I.3.6)

Knowing the functions $L_o(u_1, u_2, \ldots, u_n)$ and $L_1(u_1, u_2, \ldots, u_n)$, and using (I.3.4), φ_o and φ_1 can be found.

(c) Find $\mu = \mu(P_F^*)$ such that

$$\int_\mu^\infty \varphi_o(\lambda)\, d\lambda = 1 - \int_0^\mu \varphi(\lambda) d\lambda = P_F^* \qquad \text{(I.3.7)}$$

(d) For a given observed sample $(z_1, z_2, \ldots, z_n)$, let the detector say "yes" when

$$\Lambda(z_1, z_2, \ldots, z_n) > \mu \qquad (I.3.8)$$

and "no" when

$$\Lambda(z_1, z_2, \ldots, z_n) \leq \mu \qquad (I.3.9)$$

Thus, by (I.3.7), the probability of falsely announcing a signal to be present will be exactly P_F^*.

The resultant value of P_R will be

$$P_R = \int_{\mu}^{\infty} \varphi_1(\lambda)\, d\lambda \qquad (I.3.10)$$

II Derivation of Optimum Detector

1. Noise with Arbitrary Autocorrelation

The noise $N(t)$ is assumed to be a stationary Gaussian random process with mean 0; variance σ^2; and autocorrelation function

$$\varphi(\tau) = E\left[N(t)\ N(t+\tau)\right] = E\left[N(t)\ N(t-\tau)\right]^{(1)} \tag{II.1.1}$$

If signal is present, it has the form

$$S(t) = A \sin(\omega t+\theta) \tag{II.1.2}$$

where A and ω are known constants while θ is equally likely to be anywhere in the interval $0 \le \theta \le 2\pi$.

The total voltage is

$$z(t) = N(t) \qquad \text{(signal absent)} \tag{II.1.3}$$

$$z(t) = N(t) + S(t) \qquad \text{(signal present)} \tag{II.1.4}$$

Let this voltage be observed at times $t_i = \frac{iT}{n}$ $\quad (i=1,2,\ldots,n)$

and denote $z(t_i)$ by z_i.

We will assume that the matrix $\left[\varphi_{ij}\right] = \left[\varphi(t_i - t_j)\right]$ $\quad (i,j = 1,2,\ldots,n)$

is non-singular (i.e., has an inverse) for each n.

If signal is present, the conditional probability density for $(z_1, \ldots, z_n)$ when the phase is θ is given by[(2)]

$$L_1'(u_1,u_2,\ldots,u_n|\theta) = \frac{1}{(2\pi)^{\frac{n}{2}}|\varphi|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}\sum_1^n \varphi_{ij}\left[u_i - A\sin(\omega t_i+\theta)\right]\left[u_j - A\sin(\omega t_j+\theta)\right]\right\} \tag{II.1.5}$$

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(1) $E[X]$ means: expected value of X. Sometimes this is denoted by $\bar{X}$.

(2) Ref. 1.

where $\left[\varphi_{ij}\right] = \left[\varphi_{ji}\right] = \left[\varphi(t_i - t_j)\right]$

$|\varphi| = \text{determinant} \left[\varphi_{ij}\right]$

$\left[\xi_{ij}\right] = \left[\xi_{ji}\right] = \left[\varphi_{ij}\right]^{-1}$

Thus the joint density function for $(z_1, z_2, \ldots, z_n)$ when signal is present is

$$L_1(u_1, u_2, \ldots, u_n) = \frac{1}{2\pi} \int_0^{2\pi} L_1'(u_1, u_2, \ldots, u_n | \theta)\, d\theta \qquad \text{(II.1.6)}$$

If signal is absent then the joint density function for $(z_1, z_2, \ldots, z_n)$ is

$$L_o(u_1, u_2, \ldots, u_n) = \frac{1}{(2\pi)^{\frac{n}{2}} |\varphi|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} \sum_1^n \xi_{ij} u_i u_j \right\} \qquad \text{(II.1.7)}$$

Hence the likelihood ratio Λ_n is

$$\Lambda_n = \frac{L_1(z_1, z_2, \ldots, z_n)}{L_o(z_1, z_2, \ldots, z_n)} \qquad \text{(II.1.8)}$$

or

$$\Lambda_n = \frac{1}{2\pi} \int_0^{2\pi} \exp\left\{ -\frac{1}{2} \sum_1^n \xi_{ij} \left[z_i - A \sin(\omega t_i + \theta)\right] \left[z_j - A \sin(\omega t_j + \theta)\right] - \sum_1^n \xi_{ij} z_i z_j \right\} d\theta$$

or

$$\Lambda_n = \frac{e^{-b_n}}{2\pi} \int_0^{2\pi} \exp\left\{ c_n \sin\theta + d_n \cos\theta + g_n \cos 2\theta - h_n \sin 2\theta \right\} d\theta \qquad \text{(II.1.9)}$$

where

$$b_n = \frac{A^2}{4} \sum_1^n \zeta_{ij} \cos \omega(t_i - t_j)$$

$$c_n = A \sum_1^n \zeta_{ij}\, z_i \cos \omega t_j$$

$$d_n = A \sum_1^n \zeta_{ij}\, z_i \sin \omega t_j \qquad \text{(II.1.10)}$$

$$g_n = \frac{A^2}{4} \sum_1^n \zeta_{ij} \cos \omega (t_i + t_j)$$

$$h_n = \frac{A^2}{4} \sum_1^n \zeta_{ij} \sin \omega (t_i + t_j)$$

The integral appearing in (II.1.9) is clearly a continuous function of b_n, c_n, d_n, f_n, g_n.

Assume that, as $n \to \infty$, we have[3]

$$b_n \longrightarrow b$$

$$c_n \longrightarrow c$$

$$d_n \longrightarrow d \qquad \text{(II.1.11)}$$

$$g_n \longrightarrow g$$

$$h_n \longrightarrow h$$

---

(3) Finite limits b, c, d, g, and h do not exist for all possible autocorrelation functions. In most interesting cases, however, the limits will exist. (Since c_n and d_n are functions of the random variables z_i, we can only say that c and d exist with probability one.) The existence of these (finite) limits is closely connected with the possibility of perfect detection; if these limits do exist, perfect detection is impossible.

We will then have

$$\Lambda_n(z_1,\ldots,z_n) \longrightarrow \Lambda\left\{z(t)\right\} = M(c,d;\, b,g,h) \qquad \text{(II.1.12)}$$

where

$$M(c,d;\, b,g,h) = \frac{e^{-b}}{2\pi}\int_0^{2\pi} \exp\left\{c \sin\theta + d\cos\theta + g\cos 2\theta - h\sin 2\theta\right\} d\theta \qquad \text{(II.1.13)}$$

Here c and d are functionals of the observed $z(t)$ and are thus random variables.

If signal is actually present, the vector (c,d) has a density function which we shall denote by $F_1(u,v)$; if signal is actually absent (c,d) has a density function denoted by $F_0(u,v)$.

According to what was said in Part I, the optimum method of detection is as follows:

(a) Find $F_0(u,v)$. Thus find (via II.1.12 and II.1.13) the probability density for Λ when signal is absent.

(b) Let $P_F^* =$ pre-chosen upper bound on P_F.

Choose $\mu(P_F^*)$ such that $\text{Prob}\left[\mu < \Lambda \mid \text{signal absent}\right] = P_F^*$.

(c) For a given observed sample $z(t)$, compute Λ and

(i) if $\Lambda \le \mu(P_F^*)$ say signal is <u>not</u> present.

(ii) if $\Lambda > \mu(P_F^*)$ say signal <u>is</u> present.

The intervals $0 \le \Lambda \le \mu$ correspond to regions R_μ in the (c,d)-plane which can be determined from II.1.13. Thus, the above procedure is entirely equivalent to the following one:

(a) Find (numerically or otherwise) the regions R_μ in the (c,d)-plane corresponding to $0 \le \Lambda \le \mu$.

(b) Find $F_0(u,v)$ - the density function for (c,d) when signal is absent.

(c) Choose R_μ so that

$$\int_{R-R_\mu} F_0(u,v)\,du\,dv = P_F^*$$

[R denotes the whole (c,d)-plane; $R-R_\mu$ = complement of R_μ.]

(d) For a given observed sample z(t), compute c and d and

(i) if $(c,d) \in R_\mu$ say signal is not present.

(ii) if $(c,d) \notin R_\mu$ say signal is present.

The first three of these steps do not require any observations to be made; they can be carried out once and for all, once the autocorrelation function of noise is known.

A fifth step which would be of interest is the calculation of the probability of detection P_R (which depends on P_F^*).

(e) Find $F_1(u,v)$ - the density function for (c,d) if signal is present.

Then

$$P_R(P_F^*) = \int_{R-R_{\mu(P_F^*)}} F_1(u,v)\,du\,dv.$$

The calculation of $F_0(u,v)$ and $F_1(u,v)$:

We have $c_n = A \sum_1^n \gamma_{ij} z_i \cos\omega t_j$

$$d_n = A \sum_1^n \gamma_{ij} z_i \sin\omega t_j$$

(a) Signal not present (calculation of $F_o(u,v)$)

the z_i are normal; mean 0; variance $\sigma^2 = \varphi(0)$

c_n, d_n are normal with mean 0. Also

$$\overline{c_n^{\,2}} = \overline{A^2 \sum_{i,j,k,\ell} \varphi_{ij} \varphi_{k\ell} z_i z_k \cos\omega t_j \cos\omega t_\ell}$$

$$= A^2 \sum_{i,j,k,\ell} \varphi_{ij} \varphi_{k\ell} \overline{z_i z_k} \cos\omega t_j \cos\omega t_\ell$$

$$= A^2 \sum_{i,j,k,\ell} \varphi_{ij} \varphi_{k\ell} \varphi_{ik} \cos\omega t_j \cos\omega t_\ell$$

$$= A^2 \sum_{j,\ell} \varphi_{j\ell} \cos\omega t_j \cos\omega t_\ell \qquad \left(\text{since} \left[\varphi_{ij}\right] = \left[\varphi_{ij}\right]^{-1}\right)$$

Thus

$$\overline{c_n^{\,2}} = A^2 \sum_1^n \varphi_{ij} \cos\omega t_i \cos\omega t_j = 2(b_n + g_n) \qquad \text{(II.1.14)}$$

similarly

$$\overline{d_n^{\,2}} = A^2 \sum_1^n \varphi_{ij} \sin\omega t_i \sin\omega t_j = 2(b_n - g_n) \qquad \text{(II.1.15)}$$

and

$$\overline{c_n d_n} = A^2 \sum_1^n \varphi_{ij} \cos\omega t_i \sin\omega t_j = 2h_n \qquad \text{(II.1.16)}$$

Hence, in the limit, (c,d) has the density function

$$F_o(u,v) = \frac{1}{2\pi D^{1/2}} \exp\left\{-\frac{1}{2}\left[\alpha_{11}u^2 + 2\alpha_{12}uv + \alpha_{22}v^2\right]\right\} \qquad \text{(II.1.17)}$$

where

$$D = \text{determinant} \begin{bmatrix} 2(b+g) & 2h \\ 2h & 2(b-g) \end{bmatrix}$$

and

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} 2(b+g) & 2h \\ 2h & 2(b-g) \end{bmatrix}^{-1}$$

(b) Signal is present (calculation of $F_1(u,v)$)

The z_i are normal with means $A \sin(\omega t_i + \theta)$; variance $\sigma^2 = \psi(0)$.

For a fixed θ we now have

$$\overline{c_n} = \overline{c_n}(\theta) = A^2 \sum_1^n \xi_{ij} \sin(\omega t_i + \theta) \cos \omega t_j \quad \text{(II.1.18)}$$

$$\overline{d_n} = \overline{d_n}(\theta) = A^2 \sum_1^n \xi_{ij} \sin(\omega t_i + \theta) \sin \omega t_j \quad \text{(II.1.19)}$$

so

$$c_n - \overline{c_n} = A \sum_1^n \xi_{ij} \left[z_i - A \sin(\omega t_i + \theta)\right] \cos \omega t_j \quad \text{(II.1.20)}$$

$$d_n - \overline{d_n} = A \sum_1^n \xi_{ij} \left[z_i - A \sin(\omega t_i + \theta)\right] \sin \omega t_j \quad \text{(II.1.21)}$$

But the variables $z_i - A \sin(\omega t_i + \theta)$ have the same statistical properties as have the z_i when only noise is present.

Hence $\overline{(c-\bar{c})^2}$, $\overline{(d-\bar{d})^2}$, and $\overline{(c-\bar{c})(d-\bar{d})}$ have the same values as when signal is absent. The only difference is in the means:

$$\bar{c}_n(\theta) = A^2 \sum_1^n \xi_{ij} \sin(\omega t_i + \theta) \cos \omega t_j$$

$$= A^2 \sum_1^n \xi_{ij} \cos \omega t_j \left[\sin \omega t_i \cos\theta + \cos \omega t_i \sin\theta\right]$$

$$\bar{c}_n(\theta) = 2h_n \cos\theta + 2(b_n + g_n) \sin\theta$$

Hence,

$$\bar{c}(\theta) = 2h\cos\theta + 2(b+g)\sin\theta \qquad (II.1.22)$$

similarly,

$$\bar{d}(\theta) = 2h\sin\theta + 2(b-g)\cos\theta \qquad (II.1.23)$$

The conditional density function for (c,d) when the phase is θ is then

$$F_1'(u,v|\theta) = \frac{1}{2\pi D^{1/2}} \exp\left(-\frac{1}{2}\right)\left\{\alpha_{11}\left[u-\bar{c}(\theta)\right]^2 + 2\alpha_{12}\left[u-\bar{c}(\theta)\right]\left[v-\bar{d}(\theta)\right] + \alpha_{22}\left[u-\bar{d}(\theta)\right]^2\right\} \qquad (II.1.24)$$

where D, α_{ij} have the same values as in II.1.17.

Methods for explicitly calculating functionals of the type c and d for general autocorrelations are outlined in Appendix II. Sometimes it is possible to calculate b, c, d, g, h by explicit calculation of b_n, c_n, d_n, g_n, h_n; an example of this appears in II.2. Once $\bar{c}(\theta)$ and $\bar{d}(\theta)$ have been obtained, the density function $F_1(u,v)$ for (c,d) in the presence of signal is given by

$$F_1(u,v) = \frac{1}{2\pi}\int_0^{2\pi} F_1'(u,v|\theta)\, d\theta \qquad (II.1.25)$$

II. 2. Noise with Exponentially Decaying Autocorrelation

In this section we consider the case

$$\varphi(\tau) = \beta\, e^{-\alpha|\tau|} \qquad (\alpha,\ \beta > 0) \qquad \text{(II.2.1)}$$

We will obtain expressions for b, c, d, g, h.

It is possible to calculate ξ_{ij} explicitly. We have

$$[\varphi_{ij}] = \beta \begin{bmatrix} 1 & \gamma & \gamma^2 & \cdots & \gamma^{n-1} \\ \gamma & 1 & \gamma & & \gamma^{n-2} \\ \gamma^{n-1} & \gamma^{n-2} & \gamma^{n-3} & & 1 \end{bmatrix} \qquad \text{(II.2.2)}$$

$$\text{where } \gamma = e^{-\alpha\delta}; \quad \delta = \frac{T}{n}$$

As can easily be verified, the inverse matrix is

$$[\xi_{ij}] = \frac{1}{\beta(1-\gamma^2)} \begin{bmatrix} 1 & -\gamma & 0 & 0 & \cdots & 0 & 0 \\ -\gamma & 1+\gamma^2 & -\gamma & 0 & \cdots & 0 & 0 \\ 0 & -\gamma & 1+\gamma^2 & -\gamma & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1+\gamma^2 & -\gamma \\ 0 & 0 & 0 & 0 & \cdots & -\gamma & 1 \end{bmatrix} \qquad \text{(II.2.3)}$$

where the main diagonal is 1, $1+\gamma^2$, $1+\gamma^2$, ..., $1+\gamma^2$, 1 ; the diagonals immediately above and below the main diagonal consist entirely of $-\gamma$'s; and all other elements are 0.

The functionals b_n, c_n, d_n, g_n, h_n (II.1.10), the limiting values of which must be calculated, all contain a factor of the form

$$B_n = \sum_1^n \xi_{ij}\, x_i\, y_j \qquad \text{(II.2.4)}$$

where $x_k = x(k\delta)$, $y_k = y(k\delta)$; and where at least one of the functions, say $y(t)$, is a bounded analytic function of t while the other function, say $x(t)$, is a bounded continuous function of t.

Putting (2.3) into (2.4) gives

$$B_n = \frac{1}{\rho(1-\gamma^2)}\left[x_1y_1+x_ny_n+(1+\gamma^2)\sum_2^{n-1}x_ky_k-\gamma\sum_1^{n-1}x_ky_{k+1}-\gamma\sum_1^{n-1}x_{k+1}y_k\right] \qquad \text{(II.2.5)}$$

Using the relations

$$y_{k+1} = y_k + \delta\dot{y}_k + \frac{\delta^2}{2}\ddot{y}_k + \frac{\delta^3}{6}\dddot{y}_k + O(\delta^4)$$

$$y_{k-1} = y_k - \delta\dot{y}_k + \frac{\delta^2}{2}\ddot{y}_k - \frac{\delta^3}{6}\dddot{y}_k + O(\delta^4)$$

where $\dot{y}_k = \left.\frac{dy(t)}{dt}\right]_{t = k\delta}$, etc.,

we can write

$$\sum_1^{n-1}x_ky_{k+1} = \sum_1^{n-1}x_ky_k + \delta\sum_1^{n-1}x_k\dot{y}_k + \frac{\delta^2}{2}\sum_1^{n-1}x_k\ddot{y}_k + \frac{\delta^3}{6}\sum_1^{n-1}x_k\dddot{y}_k + O(\delta^3) \qquad \text{(II.2.6)}$$

$$\sum_1^{n-1}x_{k+1}y_k = \sum_2^{n}x_ky_k - \delta\sum_2^{n}x_k\dot{y}_k + \frac{\delta^2}{2}\sum_2^{n}x_k\ddot{y}_k - \frac{\delta^3}{6}\sum_2^{n}x_k\dddot{y}_k + O(\delta^3) \qquad \text{(II.2.7)}$$

Putting (2.6), (2.7) into (2.5) and collecting terms gives

$$B_n = \frac{1}{\rho(1-\gamma^2)}\Bigg[(1-2\gamma+\gamma^2)\sum_1^n x_ky_k - \gamma\delta^2\sum_1^n x_k\ddot{y}_k + \gamma(1-\gamma)(x_1y_1+x_ny_n) \qquad \text{(II.2.8)}$$

$$+\gamma\delta(x_n\dot{y}_n - x_1\dot{y}_1) + \frac{\gamma\delta^2}{2}(x_n\ddot{y}_n + x_1\ddot{y}_1) + \frac{\gamma\delta^3}{6}(x_n\dddot{y}_n - x_1\dddot{y}_1) + O(\delta^3)\Bigg]$$

This can be rewritten as

$$B_n = \frac{\alpha T}{2\beta n}\sum_1^n x_ky_k - \frac{T}{2\alpha\beta n}\sum_1^n x_k\ddot{y}_k + \frac{1}{2\beta}(x_1y_1+x_ny_n) \qquad \text{(II.2.9)}$$

$$+\frac{1}{2\alpha\beta}(x_n\dot{y}_n - x_1\dot{y}_1) + O\left(\frac{1}{n}\right)$$

So if $B = \lim_{n \to \infty} B_n$,

$$B = \frac{\alpha}{2\beta} \int_0^T x(t) \left[y(t) - \frac{1}{\alpha^2} y''(t) \right] dt - \frac{1}{2\alpha\beta} x(0) \left[y'(0) - \alpha y(0) \right] \tag{II.2.10}$$

$$+ \frac{1}{2\alpha\beta} x(T) \left[y'(T) + \alpha y(T) \right]$$

We are now in a position to evaluate the limiting values of the functionals (II.1.11):

$$\left.\begin{aligned}
b &= \frac{A^2}{4} \left[\frac{\alpha}{2\beta}\left(1 + \frac{\omega^2}{\alpha^2}\right) T + \frac{1}{\beta} \right] \\
c &= \frac{\alpha A}{2\beta}\left(1 + \frac{\omega^2}{\alpha^2}\right) \int_0^T z(t) \cos \omega t \, dt + \frac{A}{2\beta} z(0) + \frac{A}{2\alpha\beta} z(T) \left[\alpha \cos \omega T - \omega \sin \omega T \right] \\
d &= \frac{\alpha A}{2\beta}\left(1 + \frac{\omega^2}{\alpha^2}\right) \int_0^T z(t) \sin \omega t \, dt - \frac{\omega A}{2\alpha\beta} z(0) + \frac{A}{2\alpha\beta} z(T) \left[\omega \cos \omega T + \alpha \sin \omega T \right] \\
g &= \frac{A^2}{4} \left[\frac{\alpha}{4\beta\omega}\left(1 - \frac{\omega^2}{\alpha^2}\right) \sin 2\omega T + \frac{1}{2\beta}(1 + \cos 2\omega T) \right] \\
h &= \frac{A^2}{4} \left[\frac{\alpha}{4\beta\omega}\left(1 - \frac{\omega^2}{\alpha^2}\right)(1 - \cos 2\omega T) + \frac{1}{2\beta} \sin 2\omega T \right]
\end{aligned}\right\} \tag{II.2.11}$$

II. 3. White Noise (with Flat Spectrum)

A definition is in order here since, strictly speaking, there is no stationary Gaussian random process with a spectrum that is flat for frequencies from 0 to + ∞. Still it is possible to "approach" white noise, in the sense that it is possible to define a sequence $\left\{N_m(t)\right\}$ (m = 1,2,...) of Gaussian random processes with the property that their power spectra approach a flat spectrum.

Thus, consider white noise with the power density spectrum

$$G(f) = K \text{ volt}^2/\text{cps} \qquad (0 \leq f < \infty) \tag{II.3.1}$$

This will be defined as the "limiting" case, as $m \to \infty$, of a sequence $\left\{N_m(t)\right\}$ of Gaussian random processes with autocorrelations $\psi_m(\tau) = \frac{Km}{4} e^{-m|\tau|}$. (See Ref. 5, pp 42, for relation between power spectrum and autocorrelation.) Putting $\alpha = m$, $\beta = \frac{Km}{4}$ in (II.2.11) and letting $m \to \infty$ gives the following limiting values:

$$b = \frac{A^2 T}{2K}$$

$$c = \frac{2A}{K} \int_0^T z(t) \cos \omega t \, dt$$

$$d = \frac{2A}{K} \int_0^T z(t) \sin \omega t \, dt \tag{II.3.2}$$

$$g = \frac{A^2}{4\omega K} \sin 2\omega T$$

$$h = \frac{A^2}{4\omega K} (1 - \cos 2\omega T)$$

Note that the endpoints of the observed sample no longer play a part in the optimum detection process.

A particularly simple result is obtained when T is an integral multiple of the half period, $\frac{\pi}{\omega}$, of S(t). The remainder of this section is devoted to this case. Note that g and h vanish. Thus the likelihood ratio becomes (II.1.12)

$$\Lambda\left\{z(t)\right\} = \frac{1}{2\pi} e^{-b} \int_0^{2\pi} \exp\left\{c \sin\theta + d \cos\theta\right\} d\theta \qquad (II.3.3)$$

$$= e^{-b} I_o\left(\sqrt{c^2+d^2}\right)$$

where I_o is the modified Bessel function of the first kind, of order zero.

From (II.3.3) we see that the only significant statistic is (c^2+d^2), which is proportional to the squared amplitude of the Fourier transform of the observed sample at radian frequency ω. An explicit formula for the bias level μ can be obtained for this case. Since I_o is a monotonic function of its argument, the critical value $\mu(P_F^*)$ for Λ corresponds to a critical value, say, $\nu(P_F^*)$ for $V = (c^2+d^2)$. When S(t) is absent, the probability that V is between v and v+dv is (Ref. 1, pp 236)

$\frac{1}{4b} e^{-\frac{v}{4b}}$. ν is defined as a function of P_F^* by the relation corresponding to (I.3.7):

$$\frac{1}{4b}\int_{\nu}^{\infty} e^{-\frac{v}{4b}} dv = P_F^* \qquad (II.3.4)$$

or

$$\nu(P_F^*) = 4b \ln\left(\frac{1}{P_F^*}\right) = \frac{2A^2T}{K} \ln\left(\frac{1}{P_F^*}\right) \qquad (II.3.5)$$

Summing up, the rule for this optimum detector in the case of white noise, when T an integral multiple of $\frac{\pi}{\omega}$, is:

$$\text{Announce presence of } S(t) \text{ if } \left| \int_0^T z(t)\, e^{-i\omega t}\, dt \right|^2 > \frac{KT}{2} \ln\left(\frac{1}{P_F^*}\right)$$

$$\text{Announce absence of } S(t) \text{ if } \left| \int_0^T z(t)\, e^{-i\omega t}\, dt \right|^2 \leq \frac{KT}{2} \ln\left(\frac{1}{P_F^*}\right) \qquad \text{(II.3.6)}$$

To calculate P_R it is necessary to obtain the probability density of c^2+d^2 when $S(t)$ is present. The conditional probability density for phase θ turns out to be independent of θ; it is equal to

$$\text{Prob}\left\{v \leq c^2+d^2 < v+dv\right\} = \frac{1}{4b}\, e^{-b}\, e^{-\frac{v}{4b}}\, I_o(\sqrt{v})\, dv \qquad \text{(II.3.7)}$$

where b is given by (II.3.2). Therefore,

$$P_R = e^{-b} \int_{-\ln P_F^*}^{\infty} e^{-v}\, I_o(2\sqrt{bv})\, dv \qquad \text{(II.3.8)}$$

For small ratios of average received signal energy to noise power/cps, we have $0<b\ll 1$; (II.3.8) then implies

$$P_R \doteq P_F \quad \text{for} \quad \frac{A^2T}{K} \ll 1 \qquad \text{(II.3.9)}$$

In general, it follows from (II.3.8) that if $A \neq 0$,

$$P_R > P_F \qquad \text{(II.3.10)}$$

Of course, (3.9) and (3.10) hold for any noise autocorrelation function and any value of T if the detector is designed on the basis of the theory of testing statistical hypotheses.

APPENDIX I

The foregoing can be formally generalized as follows: Suppose the signal to be detected is given by $S(t;\alpha_1,\ldots,\alpha_K) = S(t;\vec{\alpha})$ where $\alpha_1,\ldots,\alpha_K$ are K parameters having probability distribution function $G(\vec{\alpha})$. Let the noise be stationary, Gaussian with autocorrelation $\phi(\tau)$. Then

$$L_1'(u_1,u_2,\ldots,u_n|\vec{\alpha}) = \frac{1}{(2\pi)^{n/2}|\phi|^{1/2}} \exp\left(-\frac{1}{2}\right)\left\{\sum_1^r \xi_{ij}\left[u_i-S(t_i;\vec{\alpha})\right]\left[u_j-S(t_j;\vec{\alpha})\right]\right\} \qquad (AI.1)$$

Therefore,

$$L_1(u_1,u_2,\ldots,u_n) = \frac{1}{(2\pi)^{n/2}|\phi|^{1/2}}\int \exp\left(-\frac{1}{2}\right)\left\{\sum_1^n \xi_{ij}\left[u_i-S(t_i;\vec{\alpha})\right]\left[u_j-S(t_j;\vec{\alpha})\right]\right\}dG(\vec{\alpha}) \qquad (AI.2)$$

Also $$L_o(u_1,\ldots,u_n) = \frac{1}{(2\pi)^{n/2}|\phi|^{1/2}} \exp\left(-\frac{1}{2}\right)\left\{\sum_1^n \xi_{ij}u_i u_j\right\} \qquad (AI.3)$$

$\therefore$

$$\Lambda_n(z_1,\ldots,z_n) = \int \exp\left(-\frac{1}{2}\right)\left\{\sum_1^n \xi_{ij}\left[z_i-S(t_i;\vec{\alpha})\right]\left[z_j-S(t_j;\vec{\alpha})\right] - \sum_1^n \xi_{ij} z_i z_j\right\}dG(\vec{\alpha}) \qquad (AI.4)$$

This can be applied to the case where $\alpha_1,\ldots,\alpha_K$ are all fixed instead of being random, (i.e., signal shape and phase are known).

Then (denoting the signal simply by $S(t)$)

$$\Lambda_n = \exp\left(-\frac{1}{2}\right)\left\{\sum_1^n \xi_{ij}\left[z_i-S(t_i)\right]\left[z_j-S(t_j)\right] - \sum_1^n \xi_{ij} z_i z_j\right\}$$

$$\Lambda_n = \exp\frac{1}{2}\left\{\sum_1^n \xi_{ij} S(t_i)\,S(t_j)\right\}\exp\left\{-\sum_1^n \xi_{ij} S(t_i)\,z_j\right\} \qquad (AI.5)$$

Thus the statistic which it is necessary to calculate is $\sum \xi_{ij} S(t_i)\,z_j$.

If the noise is white noise, this quantity, as $n \rightarrow \infty$, approaches a quantity proportional to

$$\frac{1}{T} \int_0^T S(t)\, z(t)\, dt.$$

In other words, all the information as to whether or not the signal is present is contained in the cross-correlation between signal and observed sample.

APPENDIX II

The problem of evaluating the quantities b, c, d, g, and h can be formally reduced to the problem of solving a certain integral equation.

Let the correlation function of noise be given by

$$E\left\{N(t)\ N(s)\right\} = \varphi(s,t) = \varphi(t,s)^{(4)} \qquad \text{(AII.1)}$$

The quadratic forms to be evaluated are of the form

$$B_n = \sum_1^n \xi_{ij}\ x_j y_i \qquad \text{(AII.2)}$$

where $\left[\xi_{ij}\right] = \left[\varphi(t_i, t_j)\right]^{-1}$

$x_j = x(t_j)$, where $x(t)$ is a bounded continuous function in $0 \leq t \leq T$

$y_i = y(t_i)$, where $y(t)$ is a function possessing derivatives of all orders.

(In some cases we may have $y(t) = x(t)$.)

Now let

$$\sum_{i=1}^n \xi_{ij}\ y_i = w_j \qquad \text{(AII.3)}$$

then

$$\sum_j \varphi_{ij}\ w_j = y_i \qquad \text{(AII.3)}$$

Hence we must evaluate $B_n = \sum_{j=1}^n w_j x_j$ where $\left\{w_j\right\}$ is the solution of (AII.3).

As $n \to \infty$ (and as the maximum interval $t_i \leq t \leq t_{i+1} \to 0$) the problem formally reduces to the evaluation of

(4) In the stationary case $\varphi(s,t) = \varphi(s-t)$.

$$B = \int_{-0}^{T+0} x(t)\, dW(t) \qquad \text{(AII.4)}$$ [5]

where $W(t)$ is the solution of

$$y(t) = \int_{-0}^{T+0} \rho(s,t)\, dW(s) \qquad (0<t<T) \qquad \text{(AII.5)}$$

As an example consider the case $\rho(s,t) = \rho e^{-\alpha|s-t|}$ discussed in section II.2. Let us evaluate c, for example. In this case $y(t) = A \cos\omega t$; $x(t) = z(t)$. (AII.5) becomes

$$A \cos\omega t = \int_{-0}^{T+0} \rho e^{-\alpha|s-t|}\, dW(s) \qquad \text{(AII.6)}$$

The solution $W(t)$ is[6]

$$W(t) = \frac{\alpha A}{2\rho}\left[1 + \frac{\omega^2}{\alpha^2}\right] \int_0^t \cos\omega s\, ds + R(t)$$

where $R(t)$ is a function which is everywhere constant except for a jump of $\frac{A}{2\rho}$ at $t=0$ and a jump of $\frac{A}{2\rho}\left[\cos\omega T - \frac{\omega}{\alpha}\sin\omega T\right]$ at $t=T$.

Hence

$$\int_{-0}^{T+0} z(t)\, dW(t)$$

$$= \frac{A}{2\rho} z(0) + \frac{A}{2\alpha\rho} z(T)\left[\alpha \cos\omega T - \omega \sin\omega T\right]$$

$$+ \frac{\alpha A}{2\rho}\left(1 + \frac{\omega^2}{\alpha^2}\right) \int_0^T z(t) \cos\omega t\, dt$$

which agrees with the value previously obtained (See II.2.11).

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(5) $\int_{-0}^{T+}$ means $\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{T+\varepsilon}$.

(6) To within an arbitrary additive constant.

Our ability to solve the integral equation (XII.6) in closed form was somewhat fortuitous. The authors are investigating the corresponding equation for autocorrelation functions of type $\psi(\tau) = P(|\tau|)\, e^{-|\tau|}$, where P is a polynomial. The possibility that general autocorrelation functions might be approximated by Laguerre polynomials, and approximations to the values of c, d, etc., obtained in this way, is being investigated.

EK/PS:ime

LIST OF SYMBOLS

$S(t) = A \sin(\omega t + \theta)$ = sine wave signal

$N(t)$ = noise

$z(t)$ = observed signal

T = duration of observed sample

P_F = probability that detector falsely announces the presence of $S(t)$ when $S(t)$ is not actually present.

P_F^* = upper limit set on P_F

P_h = probability that detector recognizes $S(t)$ when $S(t)$ actually is present.

$t_k = \frac{kT}{n}$ $(k = 1, 2, \ldots, n)$

Λ_n = likelihood ratio with the values $z(t_i)$, $(i = 1, \ldots, n)$ as the observed variables.

$\Lambda = \lim_{n \to \infty} \Lambda_n$

μ = critical value of Λ = bias level

V = a monotonic function of Λ

ν = critical value of V = bias level

$\psi(\tau)$ = autocorrelation of $N(t)$

$\psi_{ij} = \psi(\tau_i - \tau_j)$

$\left[\xi_{ij}\right] = \left[\psi_{ij}\right]^{-1}$

b_n, c_n, d_n, g_n, h_n = functionals occurring in calculation of Λ_n

b, c, d, g, h = limits of above as $n \to \infty$

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