KOLMOGOROV'S HYPOTHESES AND EULERIAN TURBULENCE THEORY

Robert H. Kraichnan

Peterborough, New Hampshire

Research Report No. 6

Task NR 062-309 Contract Nonr 4307(00)

June 1964

This work was supported by the Fluid Dynamics Branch of the Office of Naval Research under Contract Nonr 4307(00). Reproduction in whole or in part is permitted for any purpose of the United States Government.

KOLMOGOROV'S HYPOTHESES AND EULERIAN TURBULENCE THEORY

Robert H. Kraichnan

Peterborough, New Hampshire

It is argued that Eulerian formulations are intrinsically unsuited for deriving the Kolmogorov theory because low-order Eulerian moments do not express sufficiently well a statistical dependence of nonsimultaneous amplitudes that accompanies the convection of small spatial scales by large spatial scales. Illustration is made by applying the direct-interaction approximation and a related, higher Eulerian approximation to an idealized convection problem and to a modified Navier-Stokes equation. Convection effects of low wavenumbers on high wavenumbers are removed in the modified equation, and as a consequence the direct-interaction approximation for it yields the Kolmogorov spectrum. Low-order Lagrangian moments provide a promisingly more complete description of the convection of small spatial scales by large, and a search for satisfactory Lagrangian closure approximations seems highly desirable.

BLANK PAGE

1. INTRODUCTION AND SUMMARY

The similarity theory proposed by Kolmogorov[1,2] for the small-scale structure of turbulence is intuitively attractive, and the theory has received strong experimental support. Yet Kolmogorov's ideas have resisted deduction from the Navier-Stokes equation. The present paper suggests that Eulerian analytical formulations are intrinsically unsuited to the task of deriving the Kolmogorov theory, and it ends with a plea for attacks by Lagrangian methods.

The argument can be summarized simply: An underlying assumption of the Kolmogorov theory is that very large spatial scales of motion convect very small scales without directly causing significant internal distortion of the small scales. The assumption usually is considered to be consistent with, and to imply, statistical independence of small and large scales. But this is true only for the simultaneous velocity distribution. It is pointed out below that simple convection of small scales by large scales implies statistical dependence in the many-time distribution and that this dependence cannot adequately be specified by low-order Eulerian moments.

If only a few Eulerian moments are specified, it is not possible to distinguish between two quite different physical situations. In one, the Eulerian time-dependence of the small-scale structures is due to their being swept by, undistorted, by the large-scale motion. In the other, the time-dependence is due to intrinsic, internal distortion of the small scales. The implication is that no closure approximation which retains only low-order Eulerian moments uses enough information about the flow to sort out convection effects from intrinsic distortion effects. In contrast, low-order Lagrangian moments do appear to provide this information.

These considerations arose out of an investigation of the inertial- and dissipation-range predictions of the direct-interaction approximation[3] and a related, higher approximation.[4] A number of results from this investigation are given in the present paper in order to build and support the argument.[5] The direct-interaction approximation gives a self-consistent closure of the Eulerian statistical equations at the lowest nontrivial level. It expresses triple moments in terms of covariances and yields closed equations for the covariances. It is shown below that this approximation is incapable of describing adequately the convection of small scales by large scales. The higher approximation gives closure at the next level. It expresses higher moments in terms of triple moments and yields closed equations for the triple moments. This approximation is found to give an improved but still seriously imperfect representation of convection effects. Because of the difficulty with convection effects, neither approximation is capable, without modification, of giving correct predictions of the inertial-range spectrum law.

These matters are clarified by considering a modified Navier-Stokes equation in which the convection of any given spatial scale by much larger scales is consistently removed. The modification may be interpreted as an effective transformation to the quasi-Lagrangian coordinate system invoked by Kolmogorov.[1] When applied to the modified equation, the direct-interaction approximation reproduces the Kolmogorov scaling laws and inertial-range power law. The higher approximation offers the hope of yielding a reasonably accurate numerical prediction of the Kolmogorov inertial-range constant and the universal Kolmogorov dissipation-range spectrum. However, this assumes in advance that the Kolmogorov hypotheses are valid. It does not justify them.

The paper concludes with a discussion of a Lagrangian framework which seems capable of discriminating between convection effects and internal distortion effects even at the covariance level of statistical description. The associated equations are more complicated than the Navier-Stokes equation, however, and it is not clear that simple but consistent closure approximations can be formulated.

2. STATEMENT OF THE KOLMOGOROV HYPOTHESES

The basic assumption of the Kolmogorov theory is that dynamical effects significant for energy transfer do not exist over indefinitely large ratios or spatial scales of motion. An equivalent statement is that such dynamical effects do not extend over indefinitely large wavenumber ratios when the velocity field is Fourier-analyzed. The physical picture is that large-scale motions should carry small eddies about without distorting them. It is not obvious that this need be true, but the idea is certainly intuitively plausible.

The basic assumption suggests the following more detailed hypotheses (but probably does not fully imply them in a rigorous way): 1) The transfer of energy from small to large wavenumbers at high Reynolds numbers proceeds by a cascade process which is effectively local in wavenumber. 2) Detailed information about the low-wavenumber structure is degraded and lost during the cascade. 3) In consequence, there is an asymptotic high-wavenumber range whose structure is universal and depends only on the kinematic viscosity ν and the rate ϵ at which energy-per-unit-mass is cascaded. 4) The turbulence is isotropic in this universal range, regardless of anisotropy at low wavenumbers.

It follows that the universal energy spectrum has the form[2]

$$E(k) = \epsilon^{2/3}k^{-5/3}C(k/k_s), \qquad k_s = (\epsilon/\nu^3)^{1/4}. \tag{2.1}$$

Here C is some universal function and k_s is a nominal dissipation wavenumber. A final assumption is that 5) there exists an inertial subrange $\kappa \ll k_s$ in which direct effects of viscous dissipation are negligible so that $E(k)$ depends only on ϵ. The inertial-range spectrum is then

$$E(k) = C(0)\epsilon^{2/3}k^{-5/3}. \tag{2.2}$$

An extensive discussion of the Kolmogorov theory and its implications is given in Ref. 2.

3. AN IDEALIZED CONVECTION PROBLEM

It is not obvious that Kolmogorov's hypothesis about convection is correct for actual flows. The small-scale (high wavenumber) structure of high-Reynolds number turbulence appears visually to include narrow but quite long vortex filaments and sheets. It is not wholly clear that structures of this kind are carried about by the large-scale motion without internal distortion. In the present Section, a very simple, in fact physically trivial, problem is posed in which the large-scale structures unquestionably convect the small scales without distorting them. In Secs. 4 and 5, it will be seen that this idealized problem presents severe difficulties for Eulerian closure approximations.

Consider a total velocity field $\underline{u} + \underline{v}$ with the following properties: $\underline{v}$ is constant in space and time and has a Gaussian and isotropic distribution over an ensemble of realizations; $\underline{u}$ varies in space and is very weak compared to $\underline{v}$; at time $t = 0$, $\underline{u}$ is statistically independent of $\underline{v}$ and has a Gaussian, homogeneous, isotropic ensemble distribution. Assume that viscous effects and terms bilinear in $\underline{u}$ both may be neglected in the Navier-Stokes equation. Then any Fourier component of $\underline{u}$ satisfies

$$\partial \underline{u}(\underline{k},t)/\partial t = -i(\underline{k}\cdot\underline{v})\underline{u}(\underline{k},t), \tag{3.1}$$

whence

$$\underline{u}(\underline{k},t) = e^{-i\underline{k}\cdot\underline{v}t}\underline{u}(\underline{k},0). \tag{3.2}$$

All moments of the $\underline{u}$ field can be obtained from (3.2) and the stated statistical properties of $\underline{v}$ and $\underline{u}(\underline{k},0)$. For example, the time-correlation

$$R(k;t,t') = \langle \underline{u}(\underline{k},t)\underline{u}^*(\underline{k},t')\rangle / [\langle |\underline{u}(\underline{k},t)|^2\rangle\langle |\underline{u}(\underline{k},t')|^2\rangle]^{1/2} \tag{3.3}$$

has the value

$$R(k;t,t') = \langle e^{-i\underline{v}\cdot\underline{k}(t-t')}\rangle = \exp[-\tfrac{1}{2} v_0{}^2k^2(t-t')^2], \tag{3.4}$$

where v_0 is the rms value of $\underline{v}$ along any axis. The last member of (3.4) is obtained by expanding the exponential in the average, using the rules for evaluating moments of a Gaussian distribution, and then resumming the series. Any other desired moment can be obtained similarly.

At $t = 0$, the $\underline{u}$ and $\underline{v}$ fields are statistically independent by assumption. Since $\underline{v}$ gives simple translation of $\underline{u}$, clearly the simultaneous values of the two fields are also statistically independent at any later time. However, the many-time distributions are not independent. The joint distribution has nonzero cumulants of all orders. For example, it follows from (3.2) that

$$\langle v_i u_j(\underline{k},t)u_m^*(\underline{k},t')\rangle = -iv_0^2k_i(t-t')\exp[-\tfrac{1}{2} v_0^2k^2(t-t')^2]\langle u_j(\underline{k},0)u_m^*(\underline{k},0)\rangle \tag{3.5}$$

Similarly, it may be verified that the more general cumulant of the form

$$\langle v_i v_n \cdots v_\ell u_j(k,t)u_m^*(k,t')\rangle - \langle v_i v_n \cdots v_\ell\rangle\langle u_j(k,t)u_m^*(k,t')\rangle \tag{3.6}$$

does not vanish unless $t = t'$. These results illustrate an important fact: Simple convection of small-scale flow components by large-scale components does not imply statistical independence of the two scales. On the contrary, it requires a statistical dependence which involves cumulants of all orders.

It has been assumed so far that the initial $\underline{u}$ field is purely Gaussian, which implies statistical independence of different Fourier amplitudes. Now, suppose that at $t = 0$ an increment

$$\Delta u_j(\underline{k},0) = ik_m(\delta_{jn}-k_jk_n/k^2)u_n^*(\underline{p},0)u_m^*(\underline{q},0) \tag{3.7}$$

is added to the amplitude for a particular $\underline{k}$, where $\underline{p}$ and $\underline{q}$ are two wave-vectors such that $\underline{k} + \underline{p} + \underline{q} = 0$. As before, let $\underline{u}(\underline{p},0)$ and $\underline{u}(\underline{q},0)$ be statistically independent of each other. The factor $ik_m(\delta_{jn}-k_jk_n/k^2)$ ensures that $\Delta\underline{u}(\underline{k},0)$ is incompressible and real.

Consider the triple correlation

$$S(\underline{k},\underline{p},\underline{q};t,t',t'') = \frac{\langle k_iu_i(\underline{p},t')u_j(\underline{q},t'')\Delta u_j(\underline{k},t)\rangle}{\langle k_iu_i(\underline{p},0)u_j(\underline{q},0)\Delta u_j(\underline{k},0)\rangle}. \tag{3.8}$$

The quantity $\Delta\underline{u}(\underline{k},t)$ evolves according to (3.2). It then follows readily that

$$S(\underline{k},\underline{p},\underline{q};t,t',t'') = \langle e^{-i(\underline{v}\cdot\underline{k}t+\underline{v}\cdot\underline{p}t'+\underline{v}\cdot\underline{q}t'')}\rangle = \exp[-\tfrac{1}{2}v_0^2|\underline{k}t+\underline{p}t'+\underline{q}t''|^2]. \tag{3.9}$$

This triple correlation is of a kind which, in actual turbulence, would be associated with energy transfer among the modes $\underline{k}$, $\underline{p}$, and $\underline{q}$. Note that on the diagonal $t = t' = t''$ the correlation is independent of t [since $\underline{k} + \underline{p} + \underline{q} = 0$]. This is a formal expression of the fact that translation by the uniform $\underline{v}$ field does not distort the $\underline{u}$ field. Away from the diagonal, the translation does induce a time-dependence, as it does for the double correlation $R(k;t,t')$.

The results for $R(k;t,t')$ and $S(\underset{\sim}{k},\underset{\sim}{p},\underset{\sim}{q};t,t',t'')$ have been obtained under the assumption that $\underset{\sim}{v}$ is strictly constant in space. However, these results also represent asymptotic limits for a more general situation, in which $\underset{\sim}{v}$ is confined to very small (but nonzero) wavenumbers, has a homogeneous, isotropic, Gaussian, statistical distribution, and is statistically independent of the initial $\underset{\sim}{u}$ field. This correspondence is fairly clear physically but not immediately obvious mathematically. If $\underset{\sim}{v}$ varies slowly in space, (3.1) must be replaced by

$$\partial \underset{\sim}{u}(\underset{\sim}{k},t)/\partial t = -i\sum_{\underset{\sim}{k}'} \underset{\sim}{k}\cdot\underset{\sim}{v}(k')\underset{\sim}{u}(\underset{\sim}{k}-\underset{\sim}{k}',t), \qquad (3.10)$$

which, in contrast to (3.1), couples different wavevectors of the $\underset{\sim}{u}$ field. However, it is not difficult to verify by direct calculation (expansion in power series, averaging using homogeneity properties,[2] and summing) that (3.9) and (3.4) still are true, provided that the following conditions are satisfied: 1) The quantity $\langle|\underset{\sim}{u}(\underset{\sim}{k},0)|^2\rangle$ is a smooth function of $\underset{\sim}{k}$. 2) Equation (3.7) is replaced by

$$\Delta u_j(\underset{\sim}{k},0) = ik_m \sum_{\underset{\sim}{p},\underset{\sim}{q}}^{\underset{\sim}{k}+\underset{\sim}{p}+\underset{\sim}{q}=0} (\delta_{jn}-k_jk_n/k^2)u_n^*(\underset{\sim}{p},0)u_m^*(\underset{\sim}{q},0), \qquad (3.11)$$

where $\underset{\sim}{k}$ is any wavevector within some finite volume of wavevector space and the summation is over all $\underset{\sim}{p}$ and $\underset{\sim}{q}$ which fall within some other finite volumes of wavevector space (the three volumes not overlapping). 3) The wavevectors excited in the $\underset{\sim}{v}$ field are very small compared to $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$ and also very small compared to the width of the volumes invoked in condition 2). 4) Times large compared to $(v_0k)^{-1}$ are not considered. Conditions 1) and 2) (dense distribution in wave-vector) ensure ergodic properties. Conditions 3) and 4) ensure that the shear associated with the $\underset{\sim}{v}$ field produces negligible distortion of the $\underset{\sim}{u}$ field in x space.

This more general example still differs in two ways from real turbulence. In an actual flow the clean split into low wavenumber $\underset{\sim}{v}$ field and high wavenumber $\underset{\sim}{u}$ field is not possible. Moreover, the assumption of statistical independence of low and high wavenumbers at some given time is not clearly justified in actual flows.

4. DIRECT-INTERACTION APPROXIMATION

The direct-interaction approximation has been described in a series of papers.[3,4,6-9] The basic idea may be most simply described as follows, for the case of homogeneous turbulence: The energy transfer among the Fourier modes is associated with triple correlations among triads of interacting modes. In the direct-interaction approximation, it is assumed that the correlation of any given triad of Fourier modes is induced by the continuous direct dynamical interaction of the triad, acting against a relaxation process which destroys the correlation. The relaxation process involves two contributions: viscous decay and dynamical relaxation due to interaction of each of the three modes with all the rest of the Fourier modes. A consistent analytical expression of these ideas yields formulas for triple correlations in terms of two functions, the velocity covariance and the average response function of a Fourier amplitude to infinitesimal disturbances. The approximation then yields closed integrodifferential equations which determine these functions.

Numerical integration of the direct-interaction equations has suggested that the approximation gives a fairly adequate quantitative description of the decay of isotropic turbulence at modest Reynolds numbers.[7] At high Reynolds numbers, the approximation yields a universal high-k spectrum

$$E(k) = (\epsilon v_0)^{1/2} k^{-3/2} C'(k/k_d), \tag{4.1}$$

where v_0 is the rms value of velocity in any direction and C' is a universal function. The dissipation wavenumber k_d is related to the Kolmogorov wavenumber k_s by

$$k_d = k_s (R_\lambda/\sqrt{15})^{-1/6}, \tag{4.2}$$

where R_λ is the Reynolds number based on v_0 and the Taylor microscale.[2] The inertial-range spectrum corresponding to (4.1) is

$$E(k) = (\epsilon v_0)^{1/2} k^{-3/2} C'(0). \tag{4.3}$$

It was deduced in Ref. 3 that the direct-interaction approximation yields a local cascade of energy, as called for by Kolmogorov's assumptions. By this it is meant that contributions to the energy-transfer into an inertial wavenumber k from modes p, q is negligible if the ratio of any pair of the wavenumbers k, p, q is large. The discrepancy between (4.1) and (2.1) arises from a rather subtle effect. According to Kolmogorov, the excitation in the energy-containing range does not affect the internal dynamics of the energy-transfer process at high wavenumbers. In the direct-interaction approximation, the rate at which local transfer takes place at high k depends on the magnitude v_0 of the excitation in the energy range. This is because the energy-range excitation turns out to contribute to and, in fact, dominate the relaxation process which destroys triple correlations among high wavenumbers.

Recent experimental evidence[10] gives strong support to (2.1) and rules out (4.1) as a correct asymptotic law. In the present Section, theoretical support for this conclusion is obtained by applying direct-interaction to the idealized problem of Sec. 3. The approximation is found to yield a relaxation effect of the $\underline{v}$ field on triple correlations of the $\underline{u}$ field which here is clearly spurious. This does not establish the Kolmogorov theory, but it does mean that the result (4.1) should not be considered evidence against the theory.

In order to formulate the approximation for the idealized convection problem, it is necessary to introduce the average response function $G(k;t,t')$ of the Fourier mode k to infinitesmal perturbations. This may be defined, in the present case, by

$$\delta_{ij}G(k;t,t') = \langle \zeta_{ij}(\underline{k};t,t') \rangle,$$

$$\zeta_{ij}(\underline{k};t,t') = \delta u_i(\underline{k},t)/\delta f_j(\underline{k},t'), \qquad \zeta_{ij}(\underline{k};t,t') = 0 \quad (t<t'), \qquad (4.4)$$

where f is a solenoidal forcing term introduced on the right-hand-side of (3.1) and δ/δ denotes functional differentiation. (See Refs. 3 and 7.) Once induced, $\delta\underline{u}$ is simply translated by $\underline{v}$. Hence the exact value of G is

$$G(k;t,t') = \langle e^{-i\underline{v}\cdot\underline{k}(t-t')} \rangle = R(k;t,t') \quad (t \geq t'). \qquad (4.5)$$

The direct-interaction approximations for G and R have been obtained previously for problems formally identical with the present one. The results are

$$G(k;t,t') = J_1[2v_0k(t-t')]/v_0k(t-t') \qquad (t \geq t'), \qquad (4.6)$$

$$R(k;t,t') = G(k;t,t') \qquad (t \geq t'), \qquad (4.7)$$

where J_1 is the first-order Bessel function. This result for G can be represented as a particular partial summation, to all orders, of the series obtained by expanding the middle member of (4.5) in powers of t-t' and then averaging.[4] Equation (4.6) can be derived in either of two ways. If the v field is strictly constant in space, as assumed at the beginning of Sec. 3, the problem becomes formally identical with the random oscillator problem treated in Ref. 4, and the direct-interaction results are obtained by considering a collection of systems. If the $\underset{\sim}{v}$ field contains small nonzero wavenumbers, the same results are obtained by the more physical analysis of Ref. 3.

The direct-interaction approximation for $S(\underset{\sim}{k},\underset{\sim}{p},\underset{\sim}{q};t,t',t'')$ is

$$S(\underset{\sim}{k},\underset{\sim}{p},\underset{\sim}{q};t,t',t'') = G(k;t,0)R(p;t',0)R(q;t'',0). \qquad (4.8)$$

This result also is obtained for either choice of $\underset{\sim}{v}$ field. For present purposes assume that the $\underset{\sim}{v}$ field contains nonzero wavenumbers. The initial correlated increment $\Delta u_j(\underset{\sim}{k},0)$ given by (3.11) may be considered the result of an impulsive interaction among the $\underset{\sim}{u}$ field modes at $t = 0$. For $t > 0$, there is no further direct dynamical interaction among $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$. The procedure of Ref. 3, Sec. 2 yields

$$\begin{aligned}\langle u_i(\underset{\sim}{p},t')u_j(\underset{\sim}{q},t'')\Delta u_j(\underset{\sim}{k},t)\rangle &= ik_m(\delta_{rn}-k_rk_n/k^2) \\ &\times \langle u_i(\underset{\sim}{p},t)u_n^*(\underset{\sim}{p},0)u_j(\underset{\sim}{q},t'')u_m^*(\underset{\sim}{q},0)\zeta_{jr}(\underset{\sim}{k};t,0)\rangle = ik_m(\delta_{rn}-k_rk_n/k^2) \\ &\times \langle u_i(\underset{\sim}{p},t)u_n^*(\underset{\sim}{p},0)\rangle\langle u_j(\underset{\sim}{q},t'')u_m^*(\underset{\sim}{q},0)\rangle\langle\zeta_{jr}(\underset{\sim}{k};t,0)\rangle, \qquad (4.9)\end{aligned}$$

whence (4.8). The factoring of the average is not part of the approximation but is an exact consequence of homogeneity.[3] The direct-interaction approximation is invoked only in asserting the first equality in (4.9). The approximation excludes the "indirect" contributions to $S(\underset{\sim}{k},\underset{\sim}{p},\underset{\sim}{q};t,t',t'')$ which arise from initial increments $\Delta u_j(\underset{\sim}{k}-\underset{\sim}{k}',0)$, where k' is a wavenumber excited in the $\underset{\sim}{v}$ field. If no approximation is made, the interaction with $\underset{\sim}{v}$ induces a statistical dependence of $\Delta u_j(\underset{\sim}{k},t)$ on $\Delta u_j(\underset{\sim}{k}-\underset{\sim}{k}',0)$ and a dependence of $u_i(\underset{\sim}{p},t')u_j(\underset{\sim}{q},t'')$ on $u_i(\underset{\sim}{p}+\underset{\sim}{k}',0)u_j(\underset{\sim}{q},0)$ and $u_i(\underset{\sim}{p},0)u_j(\underset{\sim}{q}+\underset{\sim}{k}',0)$. In addition, more complicated dependencies are induced.

To what extent are the direct-interaction results faithful approximations to the exact G, R, and S functions? The results for G and R may be considered qualitatively acceptable. They give relaxation and correlation times $\sim(v_0 k)^{-1}$, in accord with the exact results. The principal flaw in the direct-interaction expressions is small damped oscillations which have no counterpart in the exact expressions. (See Fig. 2 of Ref. 3.) The approximation for S shows a much more serious fault. The exact function is constant on the diagonal $t = t' = t''$, while the direct-interaction result decays on the diagonal. Away from the diagonal, the qualitative behavior of the two expressions is similar.

For $t = t' = t''$, S is the kind of triple correlation which, in actual turbulence, is associated with energy-transfer. It was pointed out earlier in this Section that when direct-interaction is applied to actual turbulence, relaxation effects associated with energy-range convection account for the difference between (2.1) and (4.1). These effects are precisely analogous to the spurious decay of $S(\underline{k},\underline{p},\underline{q};t,t,t)$ given by (4.8) in the present example. The exact nature of the interaction between high and low wavenumbers in actual high Reynolds number flows cannot be inferred from the present considerations. However, it would be miraculous if the spurious effects associated with direct-interaction here were somehow exactly compensated in actual flows. Hence there is no justification for considering the result (4.1) as evidence against the Kolmogorov hypotheses.

5. A HIGHER EULERIAN APPROXIMATION

The direct-interaction approximation closes the Eulerian statistical equations at the lowest nontrivial level. It provides a formula for expressing triple moments in terms of the velocity covariance $\langle \underline{u}(\underline{k},t)\cdot\underline{u}^*(\underline{k},t')\rangle$ and the averaged response function G(k;t,t') of an isotropic flow field. The final equations contain only the latter two quantities. For the problem of Sec. 3, this closure gives a qualitatively acceptable approximation to the double correlation R(k;t,t') but a seriously faulty representation of convection effects on the triple correlation S. The question naturally arises of whether this situation can be remedied by a higher closure approximation in which the final equations are at the level of triple moments. An apparently self-consistent approximation of this kind was introduced in a previous paper.[4] When formulated for isotropic turbulence, the final equations involve the following three kinds of higher statistical quantities:

$$\langle u_i(\underline{k},t)u_j(\underline{p},t')u_m(\underline{q},t'')\rangle, \quad \langle u_i(\underline{k},t)\delta u_j(\underline{p},t')/\delta f_m^*(\underline{q},t'')\rangle,$$
$$\langle \delta^2 u_i(\underline{k},t)/\delta f_j^*(\underline{p},t')\delta f_m^*(\underline{q},t'')\rangle,$$

where $\underline{k} + \underline{p} + \underline{q} = 0$ and f is an arbitrary infinitesimal forcing term added on the right-hand-side of the Navier-Stokes equation. An extended treatment of the higher approximation is in preparation. Some results obtained for the idealized convection problem of Sec. 3 will be stated now and compared with the direct-interaction results.

The exact equation for G(k;t,t') in the idealized problem is

$$\partial G(k;t,t')/\partial t = H(k;t,t') \qquad (t \geq t'), \quad G(k;t,t) = 1, \quad (5.1)$$

where

$$H(k;t,t') = -i\langle \underline{k}\cdot\underline{v}\delta u_j(\underline{k},t)/\delta f_j(\underline{k},t')\rangle \quad \text{(not summed on } j) \quad (5.2)$$

if $\underline{v}$ is constant in space, or

$$H(k;t,t') = -i\sum_{\underline{k}'}\langle \underline{k}\cdot v(\underline{k}')\delta u_j(\underline{k}-\underline{k}',t)/\delta f_j(\underline{k},t')\rangle \qquad (5.3)$$

if $\underline{v}$ contains very small nonzero wavenumbers. Here f is an arbitrary forcing term added to the right-hand-side of (3.1) or (3.10). It is convenient to introduce the Laplace transform

$$\hat{G}(k,a) = \int_0^\infty e^{-a(t-t')} G(k;t,t')d(t-t'). \qquad (5.4)$$

The transform of (5.1) is

$$a\hat{G}(k,a) = 1 + \hat{H}(k,a). \qquad (5.5)$$

The direct-interaction closure formula for $\hat{H}(k,a)$ is

$$\hat{H}(k,a) = -v_0^2 k^2 [\hat{G}(k,a)]^2. \qquad (5.6)$$

If (5.6) is substituted into (5.5), there results the transform of (4.6). The higher approximation yields the closure equation

$$\hat{H}(k,a) = -(v_0 k)^2 [\hat{G}(k,a)]^2 - (v_0 k)^{-2} [\hat{G}(k,a)]^{-2} [\hat{H}(k,a)]^3. \qquad (5.7)$$

Note that (5.6) expresses $\hat{H}$ as a function of $\hat{G}$ while (5.7) contains $\hat{H}$ on both sides of the equation. Equations (5.5) and (5.7) may be solved together to yield $\hat{G}(k,a)$. The result has been obtained and discussed previously.[4] The higher approximation for $R(k;t,t')$ satisfies (4.7), in agreement with both the exact and direct-interaction results.

Let

$$\hat{S}(\underline{k},\underline{p},\underline{q};a,b,c) = \iiint_0^\infty e^{-at-bt'-ct''} S(\underline{k},\underline{p},\underline{q};t,t',t'')dtdt'dt''. \qquad (5.8)$$

The direct-interaction closure formula for $\hat{S}$ is

$$\hat{S}(\underline{k},\underline{p},\underline{q};a,b,c) = \hat{G}(k,a)\hat{R}(p,b)\hat{R}(q,c), \qquad (5.9)$$

the transform of (4.8). The higher approximation for $\hat{S}$ satisfies the somewhat complicated equation

$$\hat{S}(\underline{k},\underline{p},\underline{q};a,b,c) = \hat{G}(k,a)\hat{R}(p,b)\hat{R}(q,c) + \hat{D}(\underline{k},\underline{p},\underline{q};a,b,c)\hat{S}(\underline{k},\underline{p},\underline{q};a,b,c), \qquad (5.10)$$

where

$$\hat{D}(\underline{k},\underline{p},\underline{q};a,b,c) = x\hat{Y}(p,b)\hat{Y}(q,c)+y\hat{Y}(q,c)\hat{Y}(k,a)+z\hat{Y}(k,a)\hat{Y}(p,b), \tag{5.11}$$

$$\hat{Y}(k,a) = (v_0k)^{-1}\hat{H}(k,a)/\hat{G}(k,a) = (v_0k)^{-1}\{[\hat{G}(k,a)]^{-1}-a\}. \tag{5.12}$$

The quantities x, y, z are the cosines of the interior angles opposite k, p, q, respectively, in a triangle whose sides are k, p, q. Equation (5.9) yields

$$\hat{S}(\underline{k},\underline{p},\underline{q};a,b,c) = \frac{\hat{G}(k,a)\hat{R}(p,b)\hat{R}(q,c)}{1 - \hat{D}(\underline{k},\underline{p},\underline{q};a,b,c)}. \tag{5.13}$$

It will be noted that the higher approximations for $\hat{H}$ and $\hat{S}$ consist of the direct-interaction contributions plus terms which involve $\hat{H}$ and $\hat{S}$ themselves. The added terms correspond to the inclusion of additional infinite subclasses of terms from the expansions of the exact $H(k;t,t')$ and $S(\underline{k},\underline{p},\underline{q};t,t',t")$ in powers of the time-arguments. In the diagram language of Ref. 4, the added terms constitute iterated vertex corrections.

To what extent is the higher approximation an improvement? The new values for G and R turn out to be very close approximations to the exact values and represent a marked quantitative improvement over the direct-interaction approximation (see Ref. 4, Fig. 13). The question of prime interest here is the behavior of S on the diagonal $t = t' = t"$, which may be explored by taking the inverse transform of (5.13). The analysis is cumbersome and leads to numerical evaluations. However, an estimate of the improvement in the behavior on the diagonal may be obtained easily. In all three cases, exact, direct-interaction, and higher approximation, $S(\underline{k},\underline{p},\underline{q};t,t',t")$ decays away from the diagonal in more or less the same way. Thus the behavior on the diagonal may be crudely estimated from the value of the integral $\hat{S}(\underline{k},\underline{p},\underline{q};0,0,0)$. For the exact S, this quantity is infinite because there is zero decay on the diagonal. In the direct-interaction approximation,

$$v_0^3 kpq\hat{S}(\underline{k},\underline{p},\underline{q};0,0,0) = 1. \qquad (5.14)$$

In the higher approximation, $\hat{S}(\underline{k},\underline{p},\underline{q};0,0,0)$ remains finite, which shows that decay on the diagonal still occurs. However, there is an increase over the direct-interaction value, which indicates that the decay is slower. The factor of increase depends on the shape of the triangle formed by k, p, and q.

Table I shows the values of $\hat{S}(\underline{k},\underline{p},\underline{q};0,0,0)$ given by (5.13) for several choices of wavenumber ratios. The increase of $\hat{S}(\underline{k},\underline{p},\underline{q};0,0,0)$ comes from two factors. First, $\hat{G}(k,0)$ is bigger in the higher approximation [the direct-interaction, higher-approximation, and exact values of $v_0 k\hat{G}(k,0)$ are, respectively 1, $[(1+\sqrt{5})/2]^{1/2} = 1.272$, and $(\pi/2)^{1/2} = 1.253$]. The second factor is the denominator in (5.13). This factor alone is actually a more valid measure of the decay of $S(\underline{k},\underline{p},\underline{q};t,t,t)$ than is $\hat{S}(\underline{k},\underline{p},\underline{q};0,0,0)$ itself. The lat_ter quantity reflects, in part, changes in rate-of-decay perpendicular to the diagonal.

TABLE I. Values of $\hat{S}(\underline{k},\underline{p},\underline{q};0,0,0)$ in the higher approximation.

k:p:q	$1/[1-\hat{D}(k,p,q;0,0,0)]$	$v_0^3 kpq\hat{S}(\underline{k},\underline{p},\underline{q};0,0,0)$
1:1:1	13.71	28.21
1:2:2	6.66	13.70
1:4:4	4.05	8.34
1:∞:∞	2.62	5.39
3:4:5	7.42	15.27
1:1:$\sqrt{2}$	7.94	16.34

Another measure of the improvement in the higher approximation is the behavior of $S(\underline{k},\underline{p},\underline{q};t,t,t)$ at t = 0. In the direct-interaction approximation the second derivative with respect to t is negative. In the higher approximation, it is not hard to show that the first three derivatives vanish and the fourth derivative is negative.

In the present context, the fact of principal interest is simply that the spurious relaxation effects of the direct-interaction approximation are reduced, but not eliminated. Direct-interaction provides closure at the covariance level and gives a qualitatively satisfactory double correlation $R(k;t,t')$. It might be thought, by analogy, that the higher approximation, which gives closure at the triple moment level, would give a qualitatively satisfactory triple correlation $S(\underline{k},\underline{p},\underline{q};t,t',t'')$. But clearly this is not the case.

6. MODIFIED NAVIER-STOKES EQUATION

The Navier-Stokes equation can be altered so as to remove consistently the convection effects of low wavenumbers on high wavenumbers. The modified equation serves to give a precise statement of Kolmogorov's basic hypothesis and to confirm the role of convection effects in producing the discrepancy between the direct-interaction and Kolmogorov spectra.

The modified equation may be written

$$(\partial/\partial t+\nu k^2)\underline{u}(\underline{k},t) = -i\sum_{\underline{k}'}^{\alpha}[\underline{u}(\underline{k}-\underline{k}')\cdot\underline{k}']\underline{u}(\underline{k}',t)-i\underline{k}\varpi(\underline{k},t), \quad \underline{k}\cdot\underline{u}(\underline{k},t) = 0, \tag{6.1}$$

where $\varpi(\underline{k},t)$ is the Fourier amplitude of kinematic pressure and $\sum_{\underline{k}'}^{\alpha}$ omits all terms such that

$$|\underline{k}-\underline{k}'| < k/\alpha \quad \text{or} \quad |\underline{k}-\underline{k}'| < k'/\alpha. \tag{6.2}$$

The parameter α is an arbitrary cut-off ratio >1. The case $\alpha = \infty$ gives the unmodified Navier-Stokes equation.

$\sum_{\underline{k}'}^{\alpha} \underline{u}(\underline{k}-\underline{k}',t)\cdot\underline{k}'$ is the Fourier transform of the convection operator $\underline{u}\cdot\underline{\nabla}$. To see the significance of truncating this sum, consider a given wave-vector triad $\pm\underline{k}$, $\pm\underline{p}$, $\pm\underline{q}$ such that $\underline{k}+\underline{p}+\underline{q} = 0$ and either $q < k/\alpha$ or $q < p/\alpha$. The interaction of this triad is represented by a set of terms in the Navier-Stokes equations for $\underline{u}(\underline{k},t)$, $\underline{u}(\underline{p},t)$, and $\underline{u}(\underline{q},t)$. The terms in this set which are omitted and retained in the modified equations are shown in Table II. The omitted and retained terms separately conserve the energy of the triad. Thus the modified equations are energetically consistent. The terms omitted correspond to the convection and straining of small spatial scales by large scales. It is clear from Table II that these terms are not associated with energy transfer out of $\underline{q}$ into $\underline{k}$ and $\underline{p}$, since the terms do not contribute to the change of $\underline{u}(\underline{q},t)$ in the unmodified equations. However, these terms do represent a straining action which can yield energy transfer <u>between</u> $\underline{k}$ and $\underline{p}$. The terms retained represent a transport of the momentum of the large-scale motions by convective action of the small scales (small and large here are relative terms). These terms comprise the eddy-viscosity mechanism (u altered from the original Navier-Stokes equation) by which energy is transfered from $\underline{q}$ to $\underline{k}$ and $\underline{p}$.

TABLE II. Terms omitted and retained in the modified Navier-Stokes equation when $q < k/\alpha$ or $q < p/\alpha$.

	Omitted	Retained
In $\underline{k}$ Eq.	$i[\underline{u}(-\underline{q})\cdot\underline{p}]\underline{u}(-\underline{p})$	$i[\underline{u}(-\underline{p})\cdot\underline{q}]\underline{u}(-\underline{q})$
In $\underline{p}$ Eq.	$i[\underline{u}(-\underline{q})\cdot\underline{k}]\underline{u}(-\underline{k})$	$i[\underline{u}(-\underline{k})\cdot\underline{q}]\underline{u}(-\underline{q})$
In $\underline{q}$ Eq.	None	$i[\underline{u}(-\underline{k})\cdot\underline{p}]\underline{u}(-\underline{p})+i[\underline{u}(-\underline{p})\cdot\underline{k}]\underline{u}(-\underline{k})$

Kolmogorov's basic assumption is that the action of large-scale motions on sufficiently smaller-scale motions is a simple convection process that does not affect appreciably the energy dynamics of the smaller scales. This hypothesis can be given a precise statement in terms of the modified Navier-Stokes equation: If the cut-off ratio α is taken large enough, the difference in the energy spectrum $E(k)$ for the modified and unmodified equations can be made as small as desired, when the spectrum is normalized in terms of the Kolmogorov similarity parameters. For given α, this difference is independent of R_λ as $R_\lambda \to \infty$.

Since the modified equation removes the convection effects of large scales on small scales, it represents, in effect, a transformation to the quasi-Lagrangian coordinates called for in the original statement of Kolmogorov's theory.[1] In this regard, the modified equation has the advantage of being self-consistent for all t. It is difficult to construct a well-defined transformation to locally co-moving coordinates in x space which is similarly consistent.

The direct-interaction approximation for (6.1) goes through just as for the unmodified Navier-Stokes equation. The only difference in the final statistical equations is that the geometrical coefficients which occur now have altered values. The new values are given in the Appendix. When α is finite, the direct-interaction approximation yields an asymptotic spectrum of the Kolmogorov form (2.1) rather than (4.1). This comes about as follows: The removal of convection effects of the energy-range on high wavenumbers has the result that $R(k;t,t')$ and $G(k;t,t')$ for large k have decay times determined by local excitation levels rather than by the energy-range excitation level v_0. In consequence, the relaxation of triple correlations among the high wavenumbers is independent of v_0. The Kolmogorov spectrum then results inescapably because it turns out that the energy-transfer is local, as in the unmodified case. $R(k;t,t')$ and $G(k;t,t')$ may now be interpreted as correlation and response functions measured in quasi-Lagrangian coordinates.

Figure 1 shows some results of numerical solution of the direct-interaction equations for several values of α, in the range $R_\lambda \sim 600$-800. The curves were obtained as follows. Random stirring forces, confined to wavenumbers below those shown, were invoked in order to give a statistically steady state.[3] The response and time-correlation functions were fitted to Gaussian functions of difference time, thereby yielding closed equations which contain only $E(k)$ and the characteristic correlation and response times $\hat{R}(k,0)$ and $\hat{G}(k,0)$.[9] These equations were discretized in 1/4 octave k bands by a previously tested scheme[7] and solved by iteration. The overall numerical error, due to the fitting of the response and correlation functions, the discretizing in k, and truncation in k, is estimated as less than 10% of local value.

The function plotted is the one-dimensional spectrum

$$\phi_1(k) = \frac{1}{2}\int_k^\infty (1 - k^2/p^2)p^{-1}E(p)dp, \tag{6.3}$$

normalized according to the Kolmogorov similarity scaling. Numerical investigation indicates that the finite α curves differ inappreciably from the asymptotic curves for $R_\lambda = \infty$, in the range of k/k_s shown. The $\alpha = \infty$ curve (unmodified equation) differs inappreciably from the $R_\lambda = \infty$ values, if it is normalized according to the k_d scaling associated with (4.1). It follows from (4.2) that the position of this curve on the present Kolmogorov-scaled among the high wavenumbers is independent of v_0. The Kolmogorov spectrum then results inescapably because it turns out that the energy-transfer is local, as in the unmodified case. $R(k;t,t')$ and $G(k;t,t')$ may now be interpreted as correlation and response functions measured in quasi-Lagrangian coordinates.

Figure 1 shows some results of numerical solution of the direct-interaction equations for several values of α, in the range $R_\lambda \sim 600$-800. The curves were obtained as follows. Random stirring forces, confined to wavenumbers below those shown, were invoked in order to give a statistically steady state.[3] The response and time-correlation functions were fitted to Gaussian functions of difference time, thereby yielding closed equations which contain only $E(k)$ and the characteristic correlation and response times $\hat{R}(k,0)$ and $\hat{G}(k,0)$.[9] These equations were discretized in 1/8 octave k bands by a previously tested scheme[7] and solved by iteration. The overall numerical error, due to the fitting of the response and correlation functions, the discretizing in k, and truncation in k, is estimated as less than 10% of local value.

The function plotted is the one-dimensional spectrum

$$\phi_1(k) = \frac{1}{2}\int_k^\infty (1 - k^2/p^2)p^{-1}E(p)dp, \tag{6.3}$$

normalized according to the Kolmogorov similarity scaling. Numerical investigation indicates that the finite α curves differ inappreciably from the asymptotic curves for $R_\lambda = \infty$, in the range of k/k_s shown. The $\alpha = \infty$ curve (unmodified equation) differs inappreciably from the $R_\lambda = \infty$ values, if it is normalized according to the k_d scaling associated with (4.1). It follows from (4.2) that the position of this curve on the present Kolmogorov-scaled estimates of C(0) from different sources show a substantial spread.[11] Grant, Stewart, and Moilliet[10] have obtained $C(0) \sim 1.44$ from measurements in a tidal channel at $R_\lambda > 3000$.

The higher approximation discussed in Sec. 5 does not remove completely the spurious relaxation of high-wavenumber triple correlations due to energy-range convection effects. Consequently, it leads, like the direct-interaction approximation, to an asymptotic $k^{-3/2}$ inertial range law. However, the spurious effects are sufficiently reduced that the higher approximation may yield a rather accurate prediction of the Kolmogorov constant and of the Kolmogorov dissipation spectrum if it is applied to the modified Navier-Stokes equation. For this to be so, there must be values of the cut-off ratio α such that: 1) The exact normalized $E(k)$ associated with modified and unmodified equations differ very little. This requires that α be large enough that the cut-off removes very little of the energy-cascade of the unmodified equation; it is presupposed here that Kolmogorov's hypotheses are correct. 2) The errors due to spurious relaxation are sufficiently small that the higher approximation gives an accurate representation of the dynamics of the modified equation; this requires that α be small enough that the residual spurious effects are small compared to the real relaxation effects associated with energy-transfer and internal distortion among local wavenumbers.

An investigation now in progress indicates that values of α satisfying these two opposing conditions should exist and indicates also a way of estimating the residual errors, on the assumption that Kolmogorov is correct. The difficulties of numerical solution posed by the final equations are severe.

7. MOTIVATION FOR LAGRANGIAN APPROACHES

The preceding analysis has made clear that direct-interaction gives spurious relaxation effects by low wavenumber flow-components on triple correlations among high-wavenumber components. It is pertinent to ask why this particular kind of trouble should arise in an approximation which is energetically consistent and which appears to provide a reasonably faithful description of turbulence at Reynolds numbers low enough that a wide range of wavenumbers is not strongly excited. In the author's judgment, the underlying trouble is elementary and can be expected to afflict any closure approximation which expresses triple correlations as a function of $E(k)$, $R(k;t,t')$, and $G(k;t,t')$.

In the case of turbulent interaction among wavenumbers similar in magnitude, the function $R(k;t,t')$ expresses the loss of correlation associated with disordered internal distortion of the flow patterns. The numerical success at low Reynolds numbers[7] suggests that this internal distortion produces relaxation of triple correlations in a way that is reasonably well described by the direct-interaction closure formula. On the other hand, the energy-range convection at high Reynolds numbers produces a loss of correlation in the amplitude of a high-wavenumber mode which is not associated with significant internal distortion. The difficulty here is that it is impossible, given only $R(k;t,t')$, to say how much of the correlation loss is due to distortion and how much to simple convection. In order to distinguish the two effects, it is necessary to specify the higher statistical structure of the velocity field. In particular, it is necessary to say something about the cumulants of form (3.6) which describe the joint-distribution of low and high wavenumbers. This higher statistical information has no way of entering the direct-interaction scheme. In effect, the closure formula makes the decision that the loss of correlation in individual mode amplitudes is entirely associated with intrinsic, internal disordering of the flow patterns.

Since direct-interaction gives qualitatively correct behavior for the double correlation $R(k;t,t')$ even when convection effects are strong, it might be hoped that closure of the Eulerian statistical equations at one step higher would give qualitatively correct triple correlations. The results presented in Sec. 5 do not support this hope. The failure might have been anticipated. In order to specify that the $\underline{u}$ field is suffering pure convection over finite times of evolution, it is necessary to give proper values to the entire infinite set of cumulants of the form (3.6). It is not too surprising that a finite-order truncation of the statistical equations fails to accomplish this.

The analysis in this paper has been based on the many-time distribution of the velocity amplitudes. The cumulants (3.6) describing interdependence of low and high wavenumbers all can vanish if the amplitudes are measured simultaneously. Why not then seek a closure approximation involving only simultaneous amplitudes and thereby avoid all the difficulty? The quasi-normality approximation[12] is an energetically conservative closure of this kind. If this approximation is applied to the idealized convection problem of Sec. 3, it is easy to see that spurious relaxation of triple moments of the $\underline{u}$ field in fact does not occur. The trouble is that, when applied to actual turbulence, the approximation also gives no relaxation of triple correlations produced by interaction of similar spatial scales. Since only simultaneous amplitudes enter the equations, the approximation has no way of describing loss of correlation and consequently embodies no mechanism at all for dynamical relaxation of dynamically induced correlations. The consequence is an unphysical overshoot of initially built-up energy-transfer and an eventual prediction of negative energies in the spectrum regions which have strongest initial excitation.[13,14] Because of the failure to embody relaxation effects in interactions of nearby wavenumbers, the approximation does not yield Kolmogorov's spectrum.

In contrast to $R(k;t,t')$, an appropriately constructed Lagrangian double correlation appears to permit a clear specification of whether loss of correlation in a single Fourier mode is due to internal distortion of the flow structures or to simple convection of these structures. Let $\underline{u}(\underline{x},t|s)$ be defined as the velocity at time s of a fluid element which arrives at the point $\underline{x}$ at time t. This will be called the Lagrangian velocity in what follows, because it is measured along the trajectory of a fluid particle. If t is fixed, $\underline{u}(\underline{x},t|s)$ may be identified with the Lagrangian velocity as more customarily defined,[15] with $\underline{x}$ serving to label the particular particle whose path is being followed. The full function $\underline{u}(\underline{x},t|s)$ may be regarded as giving an infinite set of Lagrangian velocity fields (according to the restricted definition) with each time t specifying a different labeling of the particle trajectories.

The statistical structure of $\underline{u}(\underline{x},t|s)$ may be specified by its moments. If the field has zero mean, the simplest nonvanishing moment is the covariance

$$U_{ij}(\underline{x},t;\underline{x}',t'|s,s') = \langle u_i(\underline{x},t|s)u_j(\underline{x}',t'|s')\rangle. \tag{7.1}$$

This function has four time arguments instead of the two which appear in the Eulerian covariance $\langle u_i(\underline{x},t)u_j(\underline{x}',t')\rangle$. Consequently, it conveys more information. Consider again the idealized convection problem of Sec. 3. Under the assumption of spatially constant $\underset{\sim}{v}$ field and negligible terms bilinear in $\underset{\sim}{u}$, the Lagrangian velocity is

$$\underset{\sim}{u}(\underset{\sim}{x},t|s) = \underset{\sim}{u}(\underset{\sim}{x},t), \tag{7.2}$$

since the uniform translation by $\underset{\sim}{v}$ does not change the velocity of any particle. Let the Lagrangian time-correlation be defined by

$$R(k;t,t'|s,s') = \langle \underset{\sim}{u}(\underset{\sim}{k},t|s)\cdot\underset{\sim}{u}^*(\underset{\sim}{k},t'|s')\rangle / [\langle|\underset{\sim}{u}(\underset{\sim}{k},t|s)|^2\rangle\langle|\underset{\sim}{u}(\underset{\sim}{k},t'|s')|^2\rangle]^{1/2}, \tag{7.3}$$

where $\underset{\sim}{u}(\underset{\sim}{k},t|s)$ is the Fourier coefficient of $\underset{\sim}{u}(\underset{\sim}{x},t|s)$. By (7.2),

$$R(k;t,t'|s,s') = R(k;t,t'). \tag{7.4}$$

According to (7.4), $R(k;t,t'|s,s')$ exhibits no dependence on s and s', the times at which the velocity field is measured. It varies only with the times t and t' which specify the particle labeling. On the other hand, if the $\underset{\sim}{u}$ field exhibited internal distortion also (change with time in a frame moving with $\underset{\sim}{v}$) there clearly would be a dependence on t and t'. Thus it appears possible that the lowest-order Lagrangian time-correlation may convey enough information to decide how much of the change in given Fourier components of the velocity field is due to simple convection and how much to internal distortion. This suggests that it is worthwhile to search for a consistent Lagrangian closure approximation at the level of $U_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t'|s,s')$.

Unfortunately, the equation obeyed by $\underset{\sim}{u}(\underset{\sim}{x},t|s)$ is much more complicated than the Navier-Stokes equation. When the pressure is eliminated, the Navier-Stokes equation may be written

$$(\partial/\partial t + \underset{\sim}{u}\cdot\underset{\sim}{\nabla})u_i(\underset{\sim}{x},t) = \nu\nabla^2 u_i(x,t) + \Pi_{imn}(\nabla)[u_m(\underset{\sim}{x},t)u_n(\underset{\sim}{x},t)], \tag{7.5}$$

where

$$\Pi_{imn}(\nabla) = -\nabla^{-2}\partial^3/\partial x_i\partial x_m\partial x_n,$$

$$\nabla^{-2}w(x) \equiv -(4\pi)^{-1}\int|\underset{\sim}{x}-\underset{\sim}{x}'|^{-1}w(\underset{\sim}{x}')d^3x', \tag{7.6}$$

for any suitably behaved function $w(\underset{\sim}{x})$. The field $\underset{\sim}{u}(\underset{\sim}{x},t|s)$ satisfies

$$u_i(\underset{\sim}{x},t|s) = \exp_+[-\underset{\sim}{\xi}(\underset{\sim}{x},t|0)\cdot\underset{\sim}{\nabla}]u_i(\underset{\sim}{x},0|0) + \int_0^s dr\, \exp_+[-\underset{\sim}{\xi}(\underset{\sim}{x},t|r)\cdot\underset{\sim}{\nabla}]$$

$$\times \{\nu\nabla^2 u_i(\underset{\sim}{x},r|r) + \Pi_{imn}(\nabla)[u_m(\underset{\sim}{x},r|r)u_n(\underset{\sim}{x},r|r)]\}, \tag{7.7}$$

where

$$\underset{\sim}{\xi}(\underset{\sim}{x},t|r) = \int_r^t \underset{\sim}{u}(\underset{\sim}{x},t|r')dr' \tag{7.8}$$

measures particle displacement in the interval (r,t) and $\underset{\sim}{u}(\underset{\sim}{x},0|0)$ is the prescribed initial field. Note that $\underset{\sim}{u}(\underset{\sim}{x},t|t) = \underset{\sim}{u}(\underset{\sim}{x},t)$ by definition. The notation $\exp_+$ signifies that in the expansion of the exponential all $\underset{\sim}{\nabla}$ factors appear to the right of all $\underset{\sim}{\xi}$ factors; that is, $\underset{\sim}{\nabla}$ does not operate on $\underset{\sim}{\xi}$. The derivation and interpretation of (7.7) is simple. The operator $\exp_+[-\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}]$ is a displacement operator; e.g.,

$$\exp_+[-\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}]\underset{\sim}{u}(\underset{\sim}{x},0) = \underset{\sim}{u}(\underset{\sim}{x}-\underset{\sim}{\xi},0). \tag{7.9}$$

Thus (7.7) states that the velocity field at later times is the result of self-convection of the initial velocity field, together with convection of all the velocity increments induced at later times by viscous and pressure forces.

It has so far proved difficult to carry out consistent closures of the statistical equations generated by (7.7). The nonlinearity is transcendental, in contrast to (7.5). A related fact is that the incompressibility condition for the full field $\underset{\sim}{u}(\underset{\sim}{x},t|s)$ has an awkward form. Incompressibility for all times is automatically guaranteed by (7.7), if $\underset{\sim}{u}(\underset{\sim}{x},0|0)$ is divergenceless, but it is difficult to make statistical approximations which preserve this property.

A preliminary application of the $\underset{\sim}{u}(\underset{\sim}{x},t|s)$ formalism to dispersion of a scalar field by homogeneous turbulence has been reported.[16] Closure was obtained by the crude approximation that $\underset{\sim}{\xi}(\underset{\sim}{x},t|s)$ is multivariate-Gaussian. This gives some reasonable predictions for the evolution of the scalar spectrum. (The results are quite different from those of the Eulerian quasi-normality approximation.) In particular, the result for the straining effect of large eddies on small scalar-field structures seems qualitatively plausible. In contrast, the direct-interaction approximation for this problem,[17] exhibits spurious relaxation effects on the scalar field very similar to those discussed in the present paper.

The assumption of multivariate-Gaussian $\underset{\sim}{\xi}$ is exactly valid for the idealized problem of Sec. 3 and yields both (3.4) and (3.9). The reason is that under the conditions imposed there

$$\underset{\sim}{\xi}(\underset{\sim}{x},t|s) = (t - s)\underset{\sim}{v},$$

and $\underset{\sim}{v}$ is Gaussian. However, this closure approximation can violate incompressibility and thereby lead to serious troubles if it is applied to (7.7) under more physically interesting conditions. Other closure approximations should be sought, and alternative Lagrangian formulations should be explored.

ACKNOWLEDGEMENTS

I am grateful to Dr. M. M. Gibson for supplying tabulations of his original data. The value of C'(0) reported in Sec. 6 corroborates a calculation of this constant made several years ago by Mr. Robert Wernick, using a different method.

This work was supported by the Fluid Dynamics Branch of the Office of Naval Research under Contract Nonr 4307(00). Facilities for the numerical calculations were kindly furnished by the Goddard Institute for Space Studies.

APPENDIX

The final isotropic direct-interaction equations are given in Sec. 3 of Ref. 3, for the statistically steady case, and in Sec. 2 of Ref. 7, for free decay. When the Navier-Stokes equation is modified to remove nonlocal convection effects, the geometrical coefficients $a(k,p,q)$ and $b(k,p,q)$ which occur in the final equations have altered values if any pair of the wavenumbers k, p, q has a ratio which exceeds the cut-off parameter α. There are no other changes. The new values are

$$a(k,p,q) = a(k,q,p) = \frac{1}{4}(q^2/k^2)(1 - x^2)(1 + y^2) \qquad (k > \alpha q \text{ or } p > \alpha q),$$

$$a(k,p,q) = a(k,q,p) = \frac{1}{2}(1 - xyz - 2y^2z^2) \qquad (p > \alpha k \text{ or } q > \alpha k),$$

$$b(k,p,q) = \frac{1}{2}(q^2/k^2)(xyz + x^2y^2) \qquad (k > \alpha q \text{ or } p > \alpha q),$$

$$b(k,p,q) = 2a(k,p,q) - b(k,q,p) \qquad (k > \alpha p \text{ or } q > \alpha p),$$

$$b(k,p,q) = a(k,p,q) + \frac{1}{2}(z^2 - y^2) \qquad (p > \alpha k \text{ or } q > \alpha k),$$

where x, y, and z are the cosines defined in the Refs. and the present text. The elimination of v_0 from the asymptotic inertial-range functions $R(k;t,t')$ and $G(k;t,t')$ can be traced to the factors (q^2/k^2) in the expressions above.

There are also other ways of removing nonlocal relaxation effects from the final statistical equations. Let the direct-interaction equations be written as separate equations for $E(k)$, $R(k;t,t')$, and $G(k;t,t')$ [Eqs. (4.2), (5.1), and (5.4) of Ref. 3]. Let the $E(k)$ equation be unaltered, but in the R and G equations eliminate nonlocal convection effects. This can be done by excluding from the integrations in (5.1) and (5.4) all regions of the (p,q) plane such that $k > \alpha p$ or $k > \alpha q$. The result is to eliminate spurious relaxation effects on triple correlations but to retain in the energy-balance equation the transfer effects associated with straining of small scales by large scales. Detailed energy conservation is exactly preserved.

REFERENCES

1. A. N. Kolmogorov, C. R. Acad. Sci. U.S.S.R. 30,301 (1941); 32, 16 (1941).

2. G. K. Batchelor, The Theory of Homogeneous Turbulence (Cambridge University Press, Cambridge, England, 1953).

3. R. H. Kraichnan, J. Fluid Mech. 5, 497 (1959). Some confusing differences in notation between this paper and the present one should be noted.

4. R. H. Kraichnan, J. Math. Phys. 2, 124 (1961).

5. A full report on the application of the higher approximation to isotropic turbulence is in preparation.

6. R. H. Kraichnan, Phys. Fluids 6, 1603 (1963).

7. R. H. Kraichnan, Phys. Fluids 7, xxx (1964).

8. R. H. Kraichnan, Phys. Fluids 7, xxx (1964).

9. R. H. Kraichnan, Phys. Fluids 7, xxx (1964).

10. H. L. Grant, R. W. Stewart, and A. Moilliet, J. Fluid Mech. 12, 241 (1962).

11. M. M. Gibson, J. Fluid Mech. 15, 161 (1963).

12. I. Proudman and W. H. Reid, Phil. Trans. Roy. Soc. London, A247, 163 (1954).

13. Y. Ogura, J. Fluid Mech. 16, 33 (1963).

14. R. H. Kraichnan, in Mécanique de la Turbulence (Centre National de la Recherche Scientifique, Paris, France, 1962).

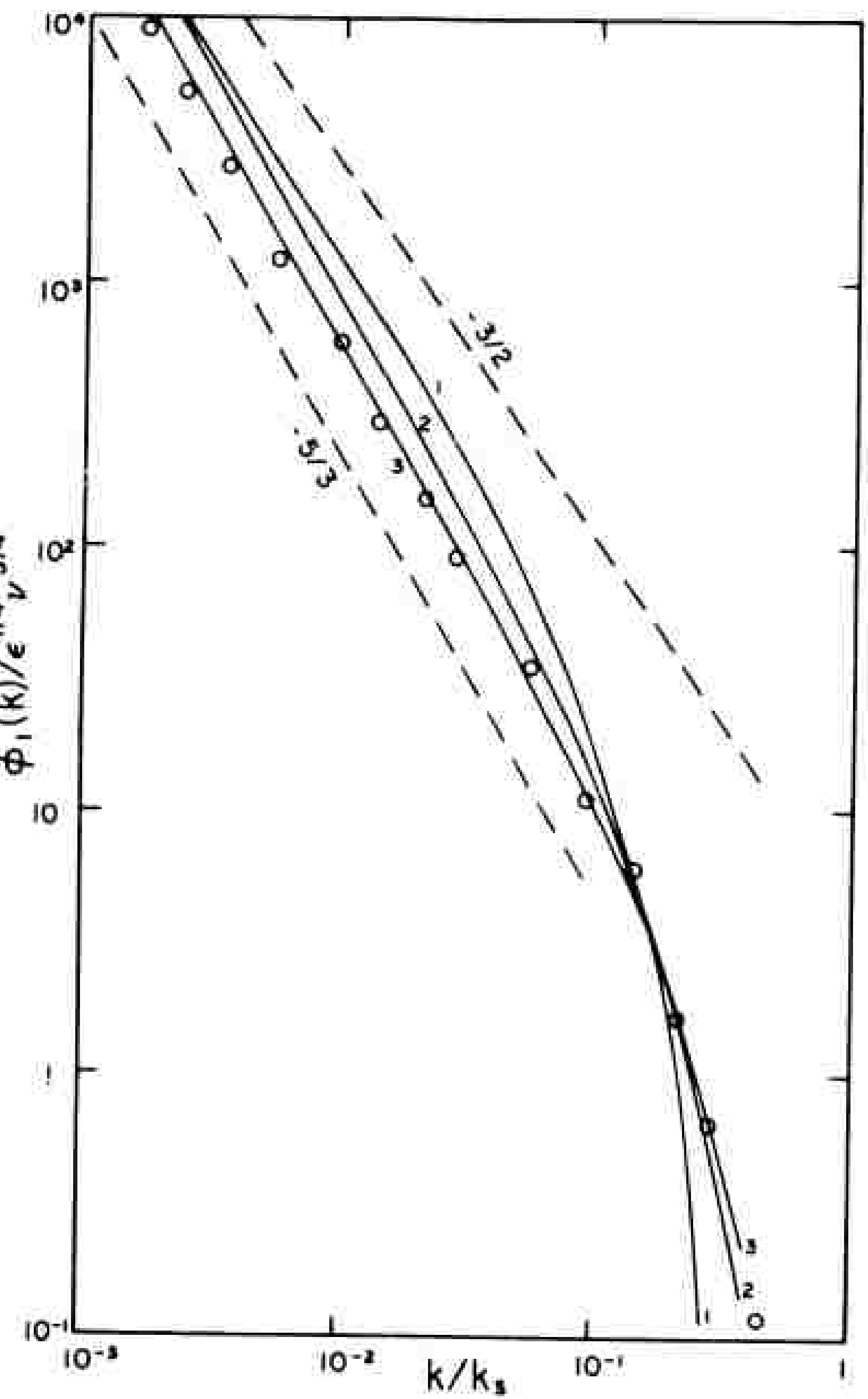
15. S. Corrsin, ibid.

16. R. H. Kraichnan, Phys. Fluids, to be published.

17. P. H. Roberts, J. Fluid Mech. 11, 257 (1961).

FIGURE CAPTION

FIG. 1. One-dimensional energy spectrum at high wavenumbers according to the direct-interaction approximation. Curve 1, unmodified Navier-Stokes equation ($\alpha = \infty$), $R_\lambda \sim 820$. Curve 2, modified Navier-Stokes equation, $\alpha = 4\sqrt{2}$, $R_\lambda \sim 800$. Curve 3, modified equation, $\alpha = 2\sqrt{2}$, $R_\lambda \sim 610$. The dashed lines show slopes of -3/2 and -5/3 for comparison. The circles represent data points measured by M. M. Gibson on the axis of an air jet at $R_\lambda \sim 800$.



10^4
10^3
10^2
10
1
10^{-1}
$\phi_1(k)/\epsilon^{1/4}\nu^{5/4}$
10^{-3}
10^{-2}
10^{-1}
1
k/k_s
-3/2
-5/3
1
2
3