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The Propagators of Quantum Field Theories as Green's Functions for Boundary Value Problems in Partial Differential Equations

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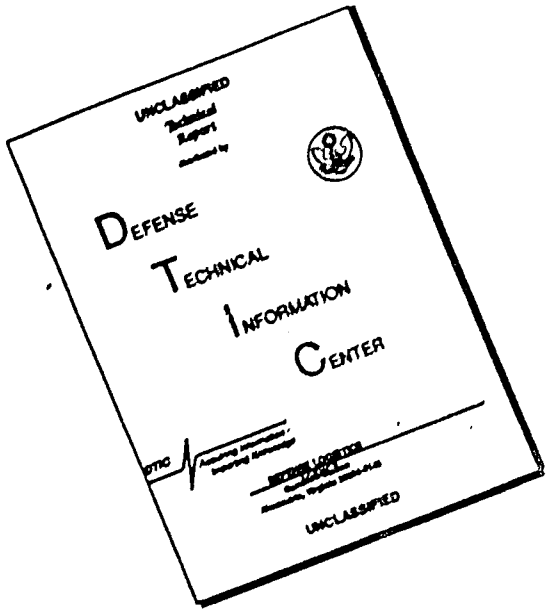

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FOREWORD

The material presented here is taken from lectures given by the author in an informal seminar on quantum field theories, held in the Radiation Division of the U.S. Naval Research Laboratory. The purpose of these lectures was to present a calculation of the propagators, or Green's functions, of the different types that appear in quantum field theories within the framework of the theory of boundary value problems for linear partial differential equations, thereby rendering the Green's functions more amenable to physical interpretation. Further, a classical setting of the propagators separates neatly those properties of the propagators which may be discussed without recourse to the procedures of second quantization from those properties which do require the latter methods.

A perusal of the table of contents will give the reader an idea of the scope of the subject matter and the direction that is followed. Chapter 1 is devoted primarily to the basic ideas that will be needed from the special theory of relativity and geometry, together with a presentation of our notation. Chapter 2 introduces the various boundary value problems that may be posed in conjunction with the Klein-Gordon equation and the auxiliary functions associated with these problems. It will become clear in the course of the development of these auxiliary functions, variously called propagators and Green's functions, that their physical interpretations reside in the formulation of the specific boundary value problems they enable us to solve. The propagators are determined explicitly, in this chapter, in terms of known higher transcendental functions, and are also presented in several integral representations that are useful in quantum field theories, or appear often in such theories. Chapter 2 is basic to the rest of the material of these lectures in that the formulation of the boundary value problem for the Klein-Gordon equation carries over to the wave equation for both scalar and vector fields virtually unchanged, and carries over, in substance, to the Dirac equation. In addition, the detailed results of this chapter are used in the calculations of the subsequent chapters. The reader for whom Chapter 2 has become a part of his own experience will find the subsequent chapters relatively simple fare. A summary of the results of Chapter 2 is presented for easy reference.

Chapter 3 contains a discussion of the boundary value problems of Chapter 2 but with respect to the wave equation. In applying the Green's functions of the wave equation to an integral formulation of the field equations of the four-potential for the electromagnetic field, we take proper account of the fact that the four-potential must satisfy the Lorentz condition. The boundary value problems of Chapter 2 vis-a-vis the Dirac equation are discussed in Chapter 4.

Chapter 5 is a simple introduction to scalar meson field theory with second quantization in order to show how a calculation of the propagators is rendered quite simple by the results of Chapter 2. Although analogous developments for the electromagnetic and electron fields are easy to carry through, they are not done here. Finally, a brief discussion is given, in this chapter, of a few of the mathematical problems that arise in quantum field theories. The discussion of mathematical rigor here is kept brief, for such a discussion in depth would carry us too far afield of our original purpose and requires volumes in itself. Finally, mathematical rigor in quantum field theory is still only little understood. The interested reader will find pertinent mathematical detail and development in, for example, Hille and Phillips, "Functional Analysis and Semi-groups," esp. Chapters I-V

Finally, we must mention the subject of references. The reader will find an occasional reference in footnotes scattered sparsely throughout the text. The author made no effort to systematically search the literature to be complete or to find original source material. The subject matter has become generally too well known for this to be necessary in a set of lectures; many textbooks will supply such a list of reference material. However the author wants to state his indebtedness in particular to the book "Field Theory," Vol. I, by Jan Rzewuski (Polish Academy of Science, Physical Monographs; Hofner Publishing Company, New York) and recommends it highly to the reader.

The Propagators of Quantum Field Theories as Green's Functions for Boundary Value Problems in Partial Differential Equations

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CHAPTER 1
RELATIVISTIC CONCEPTS; NOTATIONS

It is not our purpose to develop the special theory of relativity, but to present those ideas from that discipline that are pertinent to the subsequent work of this discussion. This short discussion also provides the opportunity of presenting the notation that will be used. On this latter point, the reader will no doubt be aware of the plethora of notations that are widely used; the choice that one makes, of course, is immaterial insofar as the physics is concerned, so that the selection that is made is based on personal tastes or is simply arbitrary. However, once having made a selection, we shall find little difficulty in comparing the final results with the conclusion of others using different conventions.

NOTATION

A point in space-time will be denoted by various symbols: x, (x_0, x_1, x_2, x_3), (x_0, x_i), (x_0, r), (x_μ); a space-time point will also, at times, be referred to as an event. The coordinates of a point in space-time, x_μ, will always be given in terms of the covariant components; on no occasion shall the contravariant components be used. If $a = (a_\mu)$ and $b = (b_\mu)$ are two four-vectors, their scalar or inner product will be denoted by $a \cdot b$ or $a_\mu b_\mu$ which will be a symbolic representation of the number $-a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3 = -a_0b_0 + \mathbf{a} \cdot \mathbf{b}$. The length of any four-vector a is $\sqrt{a \cdot a}$; since the inner product is clearly indefinite, the number $a^2 = a \cdot a$ may be positive, zero, or negative. If $a^2 < 0$, the vector is said to be a time-like vector; if $a^2 > 0$, the vector is said to be a space-like vector.

A set of four-points, $S = \{x, y, z, \ldots\}$ is said to be a space-like set if $(x - y)^2 > 0$ for every pair (x, y) of elements, each in S. In particular, if S constitutes a space-like three-dimensional "continuum" in four-space, S will be called a space-like hypersurface. (For example, the set of all space-time points for which x_0 is the same is the entire three-dimensional space we ordinarily perceive, and this constitutes a space-like hypersurface in the space-time continuum.) With the exception of the preceding parenthetic remark, meaningful definitions arise from this paragraph if the term "space-like" is replaced by "time-like" and $(x - y)^2 > 0$ is replaced by $(x - y)^2 < 0$.

The set of four-points $C_x = \{u, v, w, \ldots\}$ such that $(u - x)^2 = 0$ for all u belonging to C_x is said to be the light-cone associated with the point x; here, x may be any point of the space-time continuum. If each point of C_x is interpreted as a physical event, then C_x is that subset of all physical events whose occurrence coincides with the arrival of a light-signal from the event x or whose signals arrive at the event x.

The set of all time-like points $L_x^+ = \{u, v, w, \ldots\}$ such that $u_0 - x_0 > 0$ lie within the forward light-cone associated with point x, where again x is any point of the space-time continuum; similarly, the set $L_x^- = \{u, v, w, \ldots\}$ of time-like points such that $u_0 - x_0 < 0$ are said to lie within the backward light-cone of the event x.

A geometric representation of the above sets is obtained in the usual way: we suppress two of the space components of a four-point x in order that a point in the space-time continuum may be represented by a point in a plane; then a Cartesian representation of the remaining pair is used, with the remaining space component, say x_i, as the abscissa and x_0 as the ordinate. The union of the sets C_0^+ and C_0^- is the light-cone C_0 associated with the point $x = 0$; the shaded region marked by L_0^+ is within the forward light-cone associated with the origin, while the crosshatched area marked by

NRL Problem H02-22, Project RR-002-01-41-4908. This is a final report on one aspect of a continuing problem. Manuscript submitted October 14, 1965.

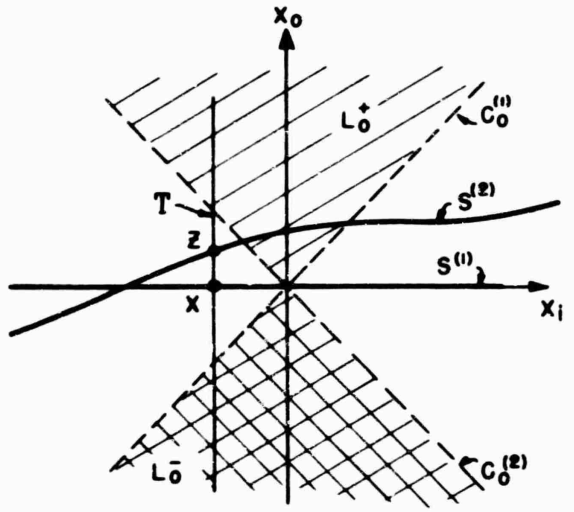

Figure 1

L_0^- is within the backward light-cone of the origin.* The axis $x_0 = 0$ is a special and important case of a space-like hypersurface $S^{(1)}$, while $S^{(2)}$ is a more general space-like hypersurface, always possessing the property that its slope nowhere acquires the value $+1$ or -1 and is always between these two members.

Let x be a point in the space-time continuum not on, say, the space-like hypersurface $S^{(2)}$, but otherwise arbitrary; with x, we associated a time-like hypersurface $T = \{u, v, w, \ldots\}$ such that $u_i = x_i$, all u in T. In our geometric representation T is a straight line through x parallel to the x_0 axis. T must intersect $S^{(2)}$ at some point z whose coordinates are finite. If $z_0 > x_0$, we shall say that x_0 precedes $S^{(2)}$, or is prior to $S^{(2)}$, or earlier than $S^{(2)}$; if $z_0 < x_0$, we shall say that x_0 is later than $S^{(2)}$. Note that x_0 being earlier than $S^{(2)}$ does not imply that all events on $S^{(2)}$ occur at a time later than the event x occurs, as may be seen from the example represented geometrically in Fig. 1. Of course, if the hypersurface in question is one for which $u_0 =$ constant, all $u \in S$, such as $S^{(1)}$ in Fig. 1, then x indeed is an event which occurs prior to all events on S. Similar observations for the case that x is later than $S^{(2)}$ may be made.

LORENTZ TRANSFORMATIONS

A Lorentz transformation is, by definition, a linear transformation on the components of a space-time which is, first, invertible, that is, the inverse of the transformation exists, and second, leaves the form $(x - y)^2$ unchanged in value and in form, that is, if x' is the transform of x, and y' of y, then $(x' - y')^2 = (x - y)^2$. A function on the space-time continuum $\varphi(x)$ with the property $\varphi(x') = \varphi(x)$ when x' is the Lorentz transform of x is called invariant. Thus, a Lorentz transformation is an invertible linear transformation which leaves the form $(x - y)^2$ invariant. It follows immediately then that under Lorentz transformations, space-like hypersurfaces transform into space-like hypersurfaces, time-like hypersurfaces transform into time-like hypersurfaces, and the light-cone of any point transforms into the light-cone of the transformed point. The forward and backward light-cones of a given point must be given more consideration, which will be done when more detailed study of Lorentz transformation is given.

Let x be a point of the space-time continuum whose coordinates are $(x_0, x_1, x_2, x_3) = (x_\mu)$. The point x', derived from performing a Lorentz transformation on x, has components x_0', x_1', x_2', x_3' which are related to those of x by the equation

$$x'_\mu = a_{\mu\lambda}\, x_\lambda + b_\mu \tag{1}$$

where $\mu = 0, 1, 2, 3$, and the Einstein summation convention is used. Equation (1) is linear by our definition of the preceding paragraph. The point y transforms to the point y' by equations of the same form as (1). The condition that $(x - y)^2 = (x' - y')^2$ leads to the condition

$$a_{\mu\nu}\, a_{\mu\lambda} = \delta_{\nu\lambda} \tag{2}$$

or

$$\det (a_{\mu\lambda}) = \pm 1. \tag{3}$$

Let $\alpha_{\lambda\mu}$ be the cofactor of $a_{\mu\lambda}$; then

$$\alpha_{\lambda\mu}\, a_{\mu\nu} = \delta_{\nu\lambda}\,. \tag{4}$$

Comparing Eqs. (4) and (3), we see that $\alpha_{\lambda\mu} = a_{\mu\lambda}$; Eq. (1) is now readily inverted, by multiplying by $\alpha_{\nu\mu} = a_{\mu\nu}$ and summing over μ:

$$a_{\mu\nu}\, x'_\mu = a_{\mu\nu} a_{\mu\lambda} + b_\mu\, a_{\mu\nu} = x_\nu - \beta_\nu$$

or

$$x_\nu = a_{\mu\nu}\, x'_\mu + \beta_\nu\,. \tag{5}$$

*Oftentimes, the "solid" set represented by the union of L_0^+ and C_0^+ will be referred to as the forward light-cone. The context will usually make clear whether one is speaking of points on C_0^+ or in L_0^+.

With Eq. (5) and the invariance of $(x - y)^2$, we conclude

$$a_{\nu\mu}\, a_{\lambda\mu} = \delta_{\nu\lambda} \,. \tag{6}$$

The definition of the Lorentz group given above admits a wider class of transformations than those encountered in the usual development of the theory of relativity; that is, in applying the principle of relativity to determine the transformations of the components of a given point in one inertial frame in terms of its components in another inertial frame, one obtains that subclass of the above transformation that may be developed in a continuous manner from the identity transformation and with the characteristic that $a_{00} > 0$; this class has the property also that $\det |a_{\mu\lambda}| = +1$; this subgroup of the full Lorentz group is called the proper orthochronous Lorentz group. (We are not attempting to prove the statements of this paragraph, but content ourselves here to accept their validity.) It is then clear that if x is a point in the forward light-cone of the origin, then $(x' - b)$ is also, where b is the image of the origin under the Lorentz transformation. Thus under Lorentz transformations that are proper and orthochronous, time-like intervals $(x - y)$ transform into time-like intervals, space-like into space-like, with the sign of the zero component preserved; here it follows that the forward and backward light-cones of a given point transform under proper orthochronous Lorentz transformations into the forward and backward light-cones of the transformed point, respectively. It becomes equally clear that if x precedes the surface S in one inertial frame, under a proper orthochronous Lorentz transformation, x' precedes S'. Finally, we observe that if S is, in one inertial frame, the hyperplane $x_0 =$ constant, then under a proper orthochronous transformation, S transforms into a hyperplane no longer parallel, in general, to any hyperplane of the form $x_0' =$ constant; and if one has a hyperplane of the latter type, there exists a Lorentz transformation which will transform the hyperplane into one parallel to $x_0 =$ constant in some (one) inertial frame. From this, it follows directly that if x precedes the hyperplane S in a given inertial frame, there exists another inertial frame wherein x', the image of x under the corresponding Lorentz transformation, not only precedes the transformed surface, but all events on the surface will have occurred at a time, in this reference frame, later than the event x'. (That this result is not true for more general hypersurfaces may be seen by considering a point x that precedes a nonplanar hypersurface that approaches the backward light-cone asymptotically. Since we are not concerned with such cases, we shall not dwell any further on this point.)

An example of a nonorthochronous, improper Lorentz transformation is

$$\left.\begin{aligned} x_0' &= -x_0 \\ x_1' &= x_1 \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}\right\} \,. \tag{7}$$

If x precedes the hypersurface S, it is clear that under the above transformation the image S' would precede x', the image of x under Eq. (7). Such transformations are of considerable interest in modern field theories but do not play any particularly important role for our purposes; therefore, their study will not be pursued further here.

REPRESENTATIONS OF THE LORENTZ GROUP; PARTIAL DIFFERENTIAL EQUATIONS

Let O be an observer in a given inertial frame studying a system which, he discovers, requires n functions $f_j(x) = f_j(\mathbf{r}, x_0)$ to describe it completely. According to the principle of relativity, an observer O' in a second inertial frame will also require n functions, $f_k'(x')$ to describe the system. The functional values at a point P as observed by O' will be related to the functional values at the point P as observed by O; if the coordinates of P are x' and x in the inertial frames of O' and O respectively, then with L denoting the Lorentz transformation parameters,

$$f_k'(x') = \Lambda_k^L \left(f_1(x), f_2(x), \ldots, f_n(x)\right) \tag{8}$$

where Λ_k^L is a general function of $f_1, f_2, \ldots, f_n$, but one such that the set $\{[\Lambda_k^L], [\Lambda_k^{L'}], \ldots\}$ form a continuous group; that is, Eq. (8) are required to be a realization of the Lorentz group. Hence,

$$\Lambda_k^{L^{-1}}$$

the inverse to Λ_k^L, exists:

$$f_j(x) = \Lambda_j^{L^{-1}}(f_1(x'), f_2(x'), \ldots, f_n(x')) \ . \quad (9)$$

Observer O will determine that his set $[\{f_i\}$ of functions are, in general, correlated with one another through some set of equations which we may denote by

$$M(f_1 f_2, \ldots, f_n) = 0 \ . \quad (10)$$

Again from the principle of relativity, Eq. (10) may be written so that observer O' arrives at the same equation except for primes in the appropriate places.

A very important class of fields will be those that obey some form of superposition principle; if O determines that $\{f_i\}$ and $\{g_i\}$ each satisfy (10) and that $\{f_j + g_j\}$ satisfies Eq. (10), then O' must observe that $f_k' + g_k'$ also satisfies his version of Eq. (10). Thus, $f + g \equiv \{f_i + g_i\}$ is an acceptable field configuration and transforms according to Eq. (8) also:

$$h_k'(x') = \Lambda_k^L\,(f_1(x) + g_1(x), f_2(x) + g_2(x), \ldots, f_n(x) + g_n(x)) \ . \quad (11)$$

In addition

$$h_k'(S) = f_k'(x') + g_k'(x') \quad (12)$$

where $f_k'(x')$ and $g_k'(x')$ are related to the functions $\{f_i\}$ and $\{g_i\}$ respectively by (8). From Eqs. (11), (12), and (8), it then follows that the functions Λ_k^L must have the property that they are linear in f:

$$\Lambda_k^L(\{f_i(S) + g_i(x)\}) = \Lambda_k^L(\{f_i(x)\}) + \Lambda_k^L(\{g_i(x)\}) \ . \quad (13)$$

If in the function space of the set of all acceptable vector functions $(\{f_i\})$ a metric is introduced, then the notion of the "nearness" of one function $\{f_i\}$ to another, say $\{g_i\}$, may be given definitively, and continuity of functions on this function space, such as the Λ_k^L, may be made precise also. Without going into detail, it is intuitively clear that if $\{f_i\}$ is near to $\{g_i\}$, in some sense, for observer O, then $\{f_i'\}$ must be near to $\{g_i'\}$ for observer O', which is equivalent to requiring that the functions Λ_k^L be continuous functions of the functions $\{f_i\}$; Eq. (13) requires them to be linear. It may be shown that the only continuous solutions of the functional equation (13) for Λ_k^L have the form

$$\Lambda_k^L(\{f_i(x)\}) = \sum_{i=1}^{n} \lambda_{ki}^L f_i(x) \ . \quad (14)$$

Thus, for fields described by n functions and obeying the principle of superposition, the set of functions must transform under Lorentz transformations according to (14), that is, according to some n-dimensional representation of the Lorentz group. The physical requirement that the functions Λ_k^L, for fixed L and k be continuous functions of the f's applies equally well to (8); that is to say, this requirement is not related to the superposition principle; hence if the system under study is a nonlinear system so that the superposition of two solutions to (16) is not a solution, then the set of functions that describe the system, if the description is to be Lorentz invariant, *need not* transform according to a representation of the Lorentz group but instead according to some (nonlinear) *realization* of the Lorentz group. Unfortunately, little is known about such systems, but for us, the linear problems constitute our main concern.

We here give a resumé of the equations of type (10) that we shall study. The first equation that will occupy our attention in considerable detail will be the Klein-Gordon equation. Let $\varphi'(x') = \varphi(x)$ obey the partial differential equation

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{m^2c^2}{h^2}\varphi = 0. \quad (15)$$

Henceforth, we shall take $h = c = 1$ and use the notation

$$\Box\varphi = \left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\varphi = \partial_\mu\partial_\mu\varphi \ . \quad (16)$$

Then (15) reads

$$\Box\varphi - m^2\varphi = 0 \ . \quad (17)$$

We shall, in Chapter 2, study this equation in considerable detail, showing how to extract from

a given solution of (17) the positive and negative frequency parts, and how to construct the Green's functions for the different boundary value problems associated with (17). In addition, we shall consider not only (17), but the inhomogeneous Klein-Gordon equation, and its Green's functions together with their associated boundary value problems. For all these functions, we shall develop several different and useful integral representations and also explicit representations in terms of known functions; further their asymptotic behavior will be made explicit. Considerable attention to detail is given for the Klein-Gordon equation because a thorough understanding of the work on that equation will greatly simplify the calculations to follow.

In Chapter 3, we shall study the wave equation

$$\Box\varphi = 0, \tag{18}$$

developing results analogous to those for the Klein-Gordon equation described above; it will become clear that the results of Chapter 2 will carry over to Chapter 3 by simply putting $m = 0$ or taking the limit as $m \to 0$. We shall then prove that the results for (18) may be applied directly to the wave equation for the four-potentials $\{A_\mu(x)\}$ of the electromagnetic field; in (18), the function φ is again a scalar, but in the equations for A_μ,

$$\Box A_\mu(x) = 0\,. \tag{19}$$

The $\{A_\mu(x)\}$ transform according to the vector transformation law and obey the subsidiary condition

$$\frac{\partial A_\mu}{\partial x_\mu} = 0\,. \tag{20}$$

We shall show that the integral formulation of (18) will carry over to (19) in spite of (20).

In Chapter 4, we shall consider the boundary value problems analogous to those considered in the two previous chapters for the Dirac equation

$$(\gamma_\mu\partial_\mu + m)\,\psi(x) = 0 \tag{21}$$

where $\{\gamma_\mu\}$ are the Dirac matrices and $\psi(x)$ is a four-component function which transforms under a Lorentz transformation according to a certain spinor representation of the Lorentz group, the details of which will not concern us here.

GAUSS' THEOREM; GREEN's THEOREM

Let Ω be a (four-dimensional) volume in the space-time continuum whose boundary is the space-like hypersurface S. To each point of S, we may associate a four-vector $(n_\mu(x))$ such that $n_\mu n_\mu = -1$ and such that $n_\mu(x)\delta x_\mu = 0$ where δx_μ is the μth component of an infinitesimal displacement from the point x in the surface S. The four-vector n will be called the normal to the surface S at x; that the requirement $n^2 = -1$ may be met is guaranteed by the condition that S be a space-like hypersurface. It becomes geometrically clear that if S is space-like at x, then n is time-like, so that $n_\mu n_\mu < 0$; thus, n_μ may always be normalized such that $n_\mu n_\mu = -1$. If at x, S has the tangent plane equal to $x_0 =$ constant, then it is clear that $n = (\pm 1, 0, 0, 0)$. We shall always select that choice of sign for n such that it points in the forward light-cone of the point x. Here in our special case, $n = (+1, 0, 0, 0)$. In addition to the normal $n(x)$ at the point x, we define the four-vector $n'(x)$ at the point x of S, calling it the outward normal, in the following way: let δx be a displacement from x on S along the direction of the normal at x, $n(x)$. If the point $x + \delta x$ does not belong to Ω for any such δx, then $n'(x) = n(x)$, by definition. If $x + \delta x$ belongs to the set Ω, then $n'(x) = -n(x)$, by definition. It is clear from this definition (and assuming Ω contains no points of S) that the outward normal points in the direction of n when Ω precedes the point x on S, *i.e.*, whenever *any* space-like hypersurface through Ω precedes x on S, and that $n'(x) = -n(x)$ when the opposite is true. The geometric interpretation is quite clear and is best illustrated by Fig. 2.

Analogous to ordinary geometry, in four-space we define the element of area on a surface as the pseudovector $d\sigma_\mu$ whose magnitude is that of the area of the element and whose direction is the outward normal n'_μ:

$$d\sigma_\mu = n'_\mu d\sigma \tag{22}$$

and, on a space-like surface,

$$d\sigma = -n'_\mu d\sigma_\mu, \quad n_\mu^2 = -1\,. \tag{23}$$

If we introduce n_4, it is related to n_0 by*

$$n_4 = n_0/i\,. \tag{24}$$

*C. Møller, "The Theory of Relativity," Oxford, 1952, page 129.

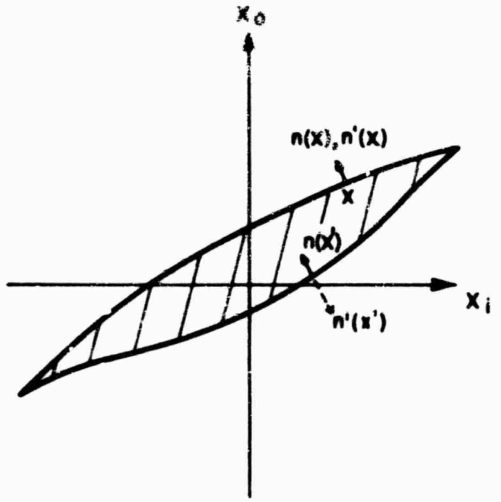

Figure 2

If f_μ is a four-vector, then, with $f_4 = if_0$ and $x_4 = ix_0$,

$$\frac{\partial f_\mu}{\partial x_\mu} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_0}{\partial x_0},$$

$$= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_4}{\partial x_4}. \qquad (25)$$

If Ω is a volume in space-time bounded by the hypersurface S and f_μ is a continuous differentiable function in Ω and all pertinent integrals exist, then

$$\int_\Omega d^4x \frac{\partial f_\mu}{\partial x_\mu} = \int_S d\sigma_\mu f_\mu = \int_S d\sigma\, n'_\mu f_\mu$$

$$(\text{sum } \mu = 1,2,3,4) \qquad (26)$$

where $d^4x = dx_0 dx_1\, dx_2\, dx_3$. Equation (26) is a statement of Gauss theorem. We are especially interested in the case $S = S_1 \cup S_2$,* where S_1 and S_2 are space-like hypersurfaces and S_1 is later than S_2; in this case Eq. (26) becomes

$$\int_\Omega d^4x \frac{df_\mu}{dx_\mu} = \int_{S_1} d\sigma_\mu f_\mu - \int_{S_2} d\sigma_\mu f_\mu \qquad (27)$$

where, now $d\sigma_\mu$ is a four-vector always pointing in the forward light-cone:

$$d\sigma_\mu = n_\mu\, d\sigma \qquad (\mu = 1,2,3,4 \text{ only}). \qquad (28)$$

*$S_1 \cup S_2$ stands for the set union of S_1 and S_2.

We shall often take the space case $S_1 = \{x | x_0 = \text{const.}\}$; then

$$\int_{S_1} d\sigma_\mu f_\mu = \int_{S_1} d\sigma n_\mu f_\mu$$

$$= \int d^3x \frac{n_0}{i} i f_0$$

$$= \int d^3x\, f_0\,. \qquad (29)$$

Equation (29) tells us that $d\sigma_0 = -d^3x$; we shall have frequent occasion to recall these results. Suppose next that $f_\mu = \varphi(x)\partial_\mu\psi - \psi\partial_\mu\varphi$; then

$$\int_\Omega d^4x\, (\varphi\Box\psi - \psi\Box\varphi) =$$

$$\int_{S_1} - \int_{S_2} d\sigma_\mu \left[\varphi(x) \frac{\partial\psi}{\partial x_\mu} - \psi(x) \frac{\partial\varphi}{\partial x_\mu}\right] \qquad (30)$$

which is Green's identity. This may also be written

$$\int_\Omega d^4x [\varphi(\Box - m^2)\psi - \psi(\Box - m^2)\varphi] =$$

$$\int_S d\sigma \left(\varphi \frac{\partial\psi}{\partial n'} - \psi \frac{\partial\varphi}{\partial n'}\right) \qquad (31)$$

where $\partial/\partial n' = n'_\mu\, \partial/\partial x_\mu$ is the derivative along the outward normal on S. Again, let $S = S_1 \cup S_2$ as defined above, and $S_1 = \{(x) | x_0 = \text{const.}\}$; then

$$\int_{S_1} d\sigma_\mu \varphi(x) \frac{\partial\psi}{\partial x_\mu} = -\int d^3x\, \varphi(x) \frac{\partial\psi}{\partial x_0} \qquad (32)$$

by (24). Thus, if we have an integral of the form of the right side of (32), it may be given a covariant generalization by replacing it by the left side of (32).

In the formulation of Gauss' theorem and the Green's identity, we required φ and ψ to be continuous and twice differentiable; the continuity requirement will now be dropped but the theorem retained; this is done to admit as solutions the generalized functions or the so-called distributions. Though distributions do not always possess desirable continuity properties, they are infinitely differentiable and always integrable; hence we can utilize them in our identities.

CHAPTER 2
THE KLEIN-GORDON EQUATION

THE HOMOGENEOUS KLEIN-GORDON EQUATION

We shall consider, in this section, the boundary value problem for a function $\varphi(x)$ which is a scalar function under Lorentz transformations that are proper and orthochronous and which satisfies the homogeneous Klein-Gordon equation:

$$(\Box - m^2)\,\varphi(x) = 0. \tag{1}$$

The pertinent boundary value problem is the determination of the function φ at x in terms of its values and the values of its derivatives on a prescribed space-like surface. Prior to this, we discuss the general solutions.

General Solutions; Positive and Negative Frequency Parts

Using the standard technique of separation of variables in a Cartesian coordinate system, we see immediately that $\varphi(x) = \exp ik\cdot x$ solves (1) if

$$k^2 + m^2 = 0 \tag{2}$$

or

$$k_0 = \pm\,\omega \tag{3}$$

where

$$\omega = +\sqrt{\mathbf{k}^2 + m^2}.$$

In general, one may obtain a solution of (1) by a superposition of such plane waves. Put

$$\varphi(x) = \frac{1}{(2\pi)^4}\int d^4k\; a(k)e^{ik\cdot x}. \tag{4}$$

Applying the differential operator $\Box - m^2$ to both sides of (4) and utilizing (1), a condition on $a(k)$ is seen to be

$$(k^2 + m^2)\,a(k) = 0. \tag{5}$$

Now $k^2 + m^2 = (-k_0^2 + \omega^2)$, which vanishes for those two values of k_0 given by (3) but not otherwise; therefore, in order that (5) be met for all values of k_0, $a(k)$ must vanish when (3) is not satisfied. This condition may be met if $a(k)$ vanishes identically; but then (4) vanishes identically also, and we have the trivial solution to (1). Thus, if $a(k)$ has the property

$$a(k) = 0,\; k_0 \neq \pm\,\omega$$

$$a(k) \neq 0,\; k_0 = \pm\,\omega$$

and the integral over k_0 of $a(k)$ is nonvanishing, (4) will acquire meaning. These conditions are met by

$$a(k) = \mathbf{a}(k)\;\delta(k^2 + m^2)$$

$$= \frac{\mathbf{a}(k)}{2\omega}\,[\delta(k_0 - \omega) + \delta(k_0 + \omega)] \tag{6}$$

where $\mathbf{a}(k)$ is, as yet, undetermined. With (6), (4) becomes

$$\varphi(x) = \frac{1}{(2\pi)^4}\int d^3k\,\frac{\mathbf{a}(\mathbf{k},\omega)}{2\omega}\,e^{i(\mathbf{k}\cdot\mathbf{r} - \omega x_0)}$$

$$+ \frac{1}{(2\pi)^4}\int d^3k\,\frac{\mathbf{a}(\mathbf{k},-\omega)}{2\omega}\,e^{i(\mathbf{k}\cdot\mathbf{r} + \omega x_0)}. \tag{7}$$

We define the two functions

$$\varphi^{(+)}(x) \equiv \frac{1}{(2\pi)^4}\int d^3k\,\frac{a(\mathbf{k},\omega)}{2\omega}\,e^{i(\mathbf{k}\cdot\mathbf{r} - \omega x_0)} \tag{8a}$$

$$\varphi^{(-)}(x) \equiv \frac{1}{(2\pi)^4}\int d^3k\,\frac{a(\mathbf{k},-\omega)}{2\omega}\,e^{i(\mathbf{k}\cdot\mathbf{r} + \omega x_0)} \tag{8b}$$

We shall call $\varphi^{(+)}(x)$ the positive frequency part of $\varphi(x)$ and $\varphi^{(-)}(x)$ the negative frequency part of $\varphi(x)$. The above discussion shows that any general solution of the Klein-Gordon equation in a given Lorentz frame may be decomposed into its positive and negative frequency parts:

$$\varphi(x) = \varphi^{(+)}(x) + \varphi^{(-)}(x). \tag{9}$$

We shall now show that this decomposition is invariant under proper orthochronous Lorentz

transformations. Consider a given Fourier component of $\varphi(x)$ characterized by the momentum vector $\mathbf{k}$; there are two terms in (7) affiliated with this momentum vector, one whose exponential part is characterized by the four-vector $\kappa_1 = (\omega, \mathbf{k})$ and the other by $\kappa_2 = (-\omega, \mathbf{k})$. Now $\kappa_1 \cdot \kappa_1 = \kappa_2 \cdot \kappa_2 = -m^2 < 0$; thus each of these four-vectors is a time-like vector, one (κ_1) lying in the forward light-cone and the other (κ_2) in the backward light-cone of the origin in momentum-energy four-space. Under proper orthochronous Lorentz transformations κ_1 will transform into a four-vector that lies within the forward light-cone (see Chapter 1) and κ_2 into one which lies in the backward light-cone. It is thus clear that if in a second inertial frame the transformed function $\varphi'(x')$ is decomposed into its positive and negative frequency parts $\varphi'^{(+)}(x')$ and $\varphi'^{(-)}(x')$, and if

$$\varphi'(x') = L\varphi(x), \tag{10}$$

then

$$\varphi'^{(\pm)}(x') = L\varphi^{(\pm)}(x) \tag{11}$$

and the Lorentz invariance of the decomposition (9) is established.

If, in Eq. (8b), $-\mathbf{k}$ replaces $\mathbf{k}$ as the integration variable, Eqs. (8) may be written as

$$\varphi^{(\pm)}(x) = \frac{1}{(2\pi)^4} \int d^3k \frac{a(\pm\kappa)}{2\omega} e^{\pm i\kappa \cdot x} \tag{12}$$

where

$$\kappa = (\omega, \mathbf{k}).$$

A very useful method for extracting the positive and negative frequency part of any function which shows clearly the invariant character of the decomposition is due to Schwinger. To develop this method, we first observe that

$$\frac{1}{2\pi i} \int_P \frac{d\tau}{\tau} e^{i\alpha\tau} = \begin{cases} 1, & \alpha > 0 \\ 0, & \alpha < 0 \end{cases}$$

where P is the contour in the complex τ-plane shown in Fig. 3.

Let n be a time-like four-vector pointing in the forward light-cone; using (8) and (7) we see that

$$\frac{1}{2\pi i} \int_P \frac{d\tau}{\tau} \varphi(x - \tau n) = \varphi^{(+)}(x) \tag{13}$$

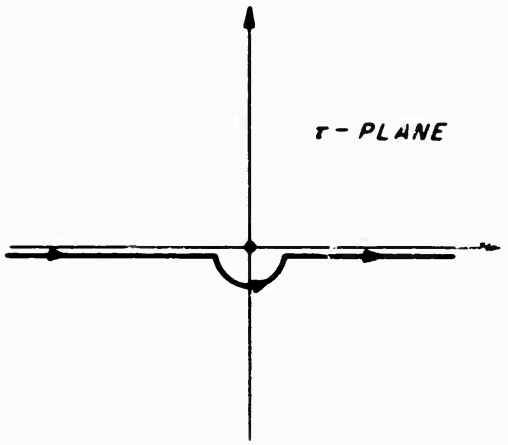

Figure 3

and

$$\frac{1}{2\pi i} \int_P \frac{d\tau}{\tau} \varphi(x + \tau n) = \varphi^{(-)}(x).$$

The calculation is facilitated by the choice $n = (+1, 0, 0, 0)$. We shall utilize (13) quite often.

The Boundary Value Problems and the Invariant Δ-Functions

The Invariant Function $\Delta(x)$ and Its Associated Boundary Value Problem

Let S be an arbitrary space-like hypersurface in the space-time continuum and let x be an arbitrary space-time point; x may precede S, lie on S, or be preceded by S. The boundary value problem we strive to solve here is the determination of the value of φ at x when φ and $\partial\varphi/\partial x_\mu$ ($\mu = 0,1,2,3$) are known at each point of S. (This may seem impossible for that case where x precedes S, because it would appear that we wish to determine the amplitude of the field at a given point in space and at a given time by its values (or events) that occur in the future, which is a violation of our intuitive notions of causality; but it must be pointed out that the Klein-Gordon equation does not contain in it anything that precludes such cases of boundary value problems. Said in another way, causality, however formulated, is a physical requirement imposed on those fields φ of interest quite distinctly from the mere solving of the equation, which is our purpose here. We shall discuss cases later that meet some of our intuitive notions of causal relations.)

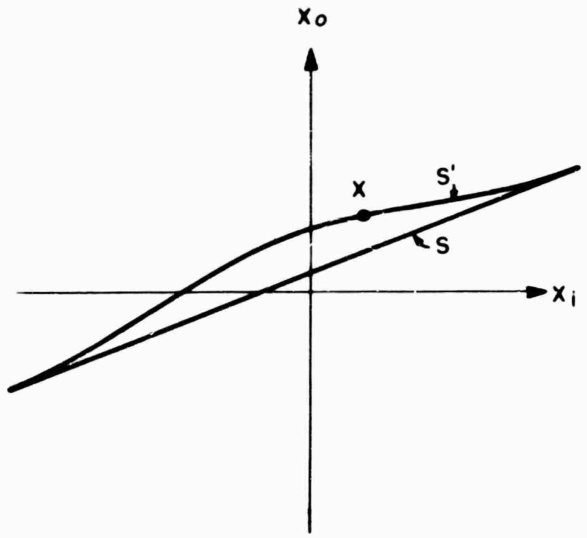

Figure 4

For the present discussion, assume x does not lie on S, let S' be a surface that is space-like and such that x lies on S', and let S and S' coincide everywhere except in the region of finite diameter; let Ω be the four-volume enclosed between S and S' (see Fig. 4, which shows x later than S, although the relations could as well be reversed).

Assuming $\varphi(x)$ and $\psi(x)$ both satisfy the Klein-Gordon equation for all x, Green's identity, Eq. (30) of Chapter 1, reduces to

$$\int_{S\cup S'} d\sigma_\mu' \left[\varphi(x') \frac{\partial \psi(x')}{\partial x_\mu'} - \psi(x') \frac{\partial \varphi(x')}{\partial x_\mu'} \right] = 0 \tag{14}$$

and because the volume integral vanishes for any Ω due to the assumption that φ and ψ solve the Klein-Gordon equation, (14) is independent of S and S'. Another way of writing Eq. (14) is

$$\int_{S'} d\sigma_\mu' \left[\varphi(x') \frac{\partial \psi(x')}{\partial x_\mu'} - \psi(x') \frac{\partial \varphi(x')}{\partial x_\mu'} \right]$$

$$= - \int_{S} d\sigma_\mu' \left[\psi(x') \frac{\partial \varphi(x')}{\partial x_\mu'} - \varphi(x') \frac{\partial \psi(x')}{\partial x_\mu'} \right]. \tag{15}$$

We shall impose conditions on S', S, and ψ in order to assist our evaluation of φ at x on S. Since (15) is independent of S' and S, choose S' to be the hyperplane $x_0' = x_0$ and S to be any space-like hypersurface preceding S or after it, but otherwise meeting the requirements that Ω be finite. Then the left side of (15) becomes

$$- \int_{x'_0 = x_0} d^3x_i \left[\varphi(x') \frac{\partial \psi(x')}{\partial x_0} - \psi(x') \frac{\partial \varphi(x')}{\partial x_0'} \right]. \tag{16}$$

We shall require of $\partial\psi(x')/\partial x_0$ that it be a three-dimensional delta function, $-\delta(\mathbf{r}' - \mathbf{r})$:

$$\frac{\partial \psi(x')}{\partial x_0'} = - \delta(\mathbf{r} - \mathbf{r}'). \tag{17}$$

Then (15) reduces to

$$\varphi(x) = - \int_{x_0' = x_0} d^3x_i' \psi(x') \frac{\partial \varphi(x')}{\partial x_0'}$$

$$- \int_S d\sigma_\mu' \left[\psi(x') \frac{\partial \varphi(x')}{\partial x_\mu'} - \varphi(x') \frac{\partial \psi(x')}{\partial x_\mu'} \right]. \tag{18}$$

Equation (18) involves integrals over two surfaces still; we wish to reduce it to only that integral over the surface S, which means we want the integral

$$\int d^3x_i \, \psi(x') \frac{\partial \varphi(x')}{\partial x_0'}$$

to vanish; this will be so if $\psi(x') \equiv 0$ on S'. This requirement may be made more general by noting that if we want (18) to be Lorentz invariant as it actually is, then $\psi(x')$ must vanish outside the light-cone of the point x. We see that this requirement is consistent with the above, because under a Lorentz transformation the surface $x_0' = x$ transforms into a hyperplane that is space-like and goes through x.

Characterizing the function $\psi(x')$ by x as well, the requirements we have placed on ψ are

$$\psi_x(x') = 0, \text{ for } (x' - x)^2 > 0 \tag{19a}$$

$$\frac{\partial \psi_x(x')}{\partial x_0'} = -\delta(\mathbf{r} - \mathbf{r}), \; x'_0 = x_0. \tag{19b}$$

If such a function $\psi_x(x')$ exists, then

$$\varphi(x) = - \int_S d\sigma_\mu' \left[\psi_x(x') \frac{\partial \varphi(x')}{\partial x_\mu'} - \varphi(x') \frac{\partial \psi_x(x')}{\partial x_\mu'} \right]. \tag{20}$$

We shall see that $\psi_x(x')$ does not exist as an ordinary function, but as a distribution instead. To this end, we decompose $\psi_x(x')$ into its positive and negative frequency parts just as we did the general solution in the preceding section:

$$\psi_x(x') = \frac{1}{(2\pi)^4}\int d^3k\,\frac{d(\mathbf{k},\omega;x)}{2\omega}\,e^{i(\mathbf{k}\cdot\mathbf{r}'-\omega x_0')} + \frac{1}{(2\pi)^4}\int d^3k\,\frac{d(\mathbf{k},-\omega;x)}{2\omega}\,e^{i(\mathbf{k}\cdot\mathbf{r}'+\omega x_0')}. \tag{21}$$

Impose on (21) the condition (19a) on $\psi_x(x')$ in the form that $\psi_x(x')$ vanish on $x_0' = x_0$; then because the Fourier transform of zero vanishes we get from (21) and (19a)

$$d(\mathbf{k},\omega;x) = -d(\mathbf{k},\omega;x)\,e^{-2i\omega x_0} \tag{22}$$

and (21) and (22) combine to yield

$$\psi_x(x') = -\frac{i}{(2\pi)^4}\int d^3k\,d(\mathbf{k},\omega;x)\,e^{-i\omega x_0+i\mathbf{k}\cdot\mathbf{r}'}\times \frac{\sin\omega(x_0'-x_0)}{\omega}. \tag{23}$$

Applying condition (19b) to (23) in order to determine $d(\mathbf{k},\omega;x)$, we get

$$\psi_x(x') = -\frac{1}{(2\pi)^3}\int d^3k\,e^{i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})}\times \frac{\sin\omega(x_0'-x_0)}{\omega}. \tag{24}$$

From (24), it is immediately evident that $\psi_x(x')$ is invariant under translations; hence we may write it as $\Delta(x'-x)$. It is *not* evidently invariant under more general proper orthochronous Lorentz transformations, although this will be established shortly. It is evident that the integral does not exist in the usual sense, since $d^3k/\omega \sim kdkd\Omega$; hence $\Delta(x'-x)$ must exist in the sense of a distribution (*i.e.*, it may be regarded as a linear functional on the linear space of the solutions of the Klein-Gordon equation.)

We have thus solved the boundary value problem posed:

$$\varphi(x) = -\int_S d\sigma_\mu'\left[\Delta(x'-x)\frac{\partial\varphi}{\partial x_\mu'} - \varphi(x')\frac{\partial\Delta(x'-x)}{\partial x_\mu'}\right] = \int_S d\sigma_\mu'\left[\Delta(x-x')\frac{\partial\varphi}{\partial x_\mu'} - \varphi(x')\frac{\partial\Delta(x-x')}{\partial x_\mu'}\right] \tag{25}$$

where

$$\Delta(x) = -\frac{1}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot\mathbf{r}}\frac{\sin\omega x_0}{\omega},$$

$$\omega = +\sqrt{\mathbf{k}^2+m^2} \tag{26}$$

and it can be readily seen that (25) reduces to an identity when S is chosen as $x_0' = x_0$. The Δ-function with the special value $m = 0$ was first introduced by Jordan and Pauli.*

Expression (26) for $\Delta(x)$ is an integral representation of this function. There are several others that are useful and important. Observe that

$$\frac{\sin\omega x_0}{\omega} = -\frac{1}{2\pi}\int_C dk_0\,\frac{e^{-ik_0x_0}}{k^2+m^2} \tag{27}$$

where the contour C in the k_0-plane is shown in Fig. 5. With (27) and (26), we obtain a second integral representation of $\Delta(x)$:

$$\Delta(x) = \frac{1}{(2\pi)^4}\int_C d^4k\,\frac{e^{ik\cdot x}}{k^2+m^2}. \tag{28}$$

To get a third, define

$$\epsilon(k) = \begin{cases} +1, \text{ if } k_0 > 0 \\ 0, \text{ if } k_0 = 0 \\ -1, \text{ if } k_0 < 0 \end{cases} \tag{29}$$

*P. Jordan and W. Pauli, *Z. Phys.* **47** 151 (1928)

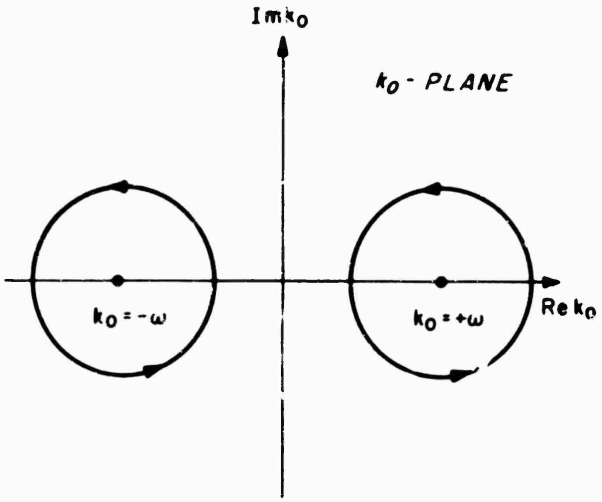

Figure 5

and note that

$$\frac{\sin \omega x_0}{\omega} = \frac{e^{i\omega x_0} - e^{-i\omega x_0}}{2\omega i}$$

$$= \frac{1}{2i\omega} \int dk_0 \, e^{-ik_0 x_0} \left[\delta(k_0 + \omega) - \delta(k_0 - \omega) \right]$$

$$= \frac{-1}{i} \int dk_0 \, \epsilon(k) e^{-ik_0 x_0} \left[\frac{\delta(k_0 + \omega) + \delta(k_0 - \omega)}{2\omega} \right]$$

$$= i \int dk_0 \, \epsilon(k) \, e^{-ik_0 x_0} \quad \delta(k^2 + m^2).$$

Then

$$\Delta(x) = -\frac{i}{(2\pi)^3} \int d^4 k \, e^{ik \cdot x} \, \epsilon(k) \, \delta(k^2 + m^2). \quad (30)$$

The invariance of $\Delta(x)$ under proper orthochronous Lorentz transformations, is now easy to prove; if $x'_\mu = a_{\mu\lambda} x_\lambda$, then

$$\Delta(x') = -\frac{i}{(2\pi)^3} \int d^4 k \, e^{ik_\mu a_{\mu\lambda} x_\lambda} \, \epsilon(k) \, \delta(k^2 + m^2) \, .$$

Make a change of integration variable from k_μ to $k'_\mu = a_{\mu\lambda} k_\mu$. Because the transformation is proper, the Jacobian is unity, and because it is orthochronous, $\epsilon(k')$ will have the same value as $\epsilon(k)$; because it is a Lorentz transformation, $k'^2 = k^2$; hence

$$\Delta(x') = -\frac{i}{(2\pi)^3} \int d^4 k' \, e^{ik' \cdot x} \, \epsilon(k') \, \delta(k'^2 + m^2)$$

$$= -\frac{i}{(2\pi)^3} \int d^4 k \, e^{ik \cdot x} \, \epsilon(k) \, \delta(k^2 + m^2)$$

or

$$\Delta(x') = \Delta(x) \, .$$

The following properties obtain, as is readily shown from (26), (28), and (30):

$$\Delta(x)^* = \Delta(x)$$

$$\Delta(-\mathbf{r}, x_0) = \Delta(\mathbf{r}, x_0)$$

$$\Delta(\mathbf{r}, -x_0) = -\Delta(\mathbf{r}, x_0)$$

$$\Delta(-x) = -\Delta(x) \quad (31)$$

i.e., $\Delta(x)$ is real, an even function of its space coordinates, and odd in its time coordinate. An explicit representation of $\Delta(x)$ in terms of better known functions will be derived in a later subsection.

The Invariant Functions $\Delta^{(+)}(x)$ and $\Delta^{(-)}(x)$ and their Associate Boundary Value Problems

We have seen, from Eq. (25), that the values of φ at x may be determined by the values of $\varphi(x)$ and $\varphi_\mu(x)$ on some space-like surface S. Knowing $\varphi(x)$ over all space, we may construct by direct computation or by Schwinger's method, the positive and negative frequence parts. Hence, one should be able to determine these functions directly in terms of φ and φ_μ on S. This is now quite straightforward; from (25)

$$\varphi(x \mp \tau n) = -\int_S d\sigma'_\mu \left[\Delta(x' \pm \tau n - x) \frac{\partial \varphi}{\partial x'_\mu} - \varphi(x') \frac{\partial \Delta}{\partial x'_\mu} (x' \pm \tau n - x) \right].$$

Performing the obvious integral to be done, according to Schwinger's prescription, we get

$$\varphi^{(\pm)}(x) = -\int_S d\sigma'_\mu \left[\Delta^{(\mp)}(x'-x)\,\frac{\partial\varphi}{\partial x'_\mu} - \varphi(x')\,\frac{\partial\Delta^{(\pm)}}{\partial x'_\mu}(x'-x)\right] \tag{32}$$

where

$$\Delta^{(\mp)}(x'-x) = \frac{1}{2\pi i}\int_P \frac{d\tau}{\tau}\,\Delta(x' \pm \tau n - x). \tag{33}$$

If we choose the second form of Eq. 25, *i.e.*,

$$\varphi(x) = +\int_S d\sigma'_\mu \left[\Delta(x-x')\,\frac{\partial\varphi}{\partial x'_\mu} - \frac{\partial\Delta(x-x')}{\partial x'_\mu}\,\varphi(x')\right] \tag{34}$$

then using Schwinger's integral again,

$$\varphi^{(\pm)}(x) = +\int_S d\sigma'_\mu \left[\Delta^{(\pm)}(x-x')\,\frac{\partial\varphi}{\partial x_\mu} - \frac{\partial\Delta^{(\pm)}(x-x')}{\partial x'_\mu}\,\varphi(x')\right]. \tag{35}$$

The physical interpretation of Eq. (35) is relatively straightforward, if we be lax in our terminology. The functions $\Delta^{(+)}$ and $\Delta^{(-)}$ determine directly the positive and negative frequency parts, respectively, of φ at x in terms of the values of φ and φ_μ on S. Eq. (35) rather than Eq. (32), will be the final form of the boundary value problems solved by $\Delta^{(+)}$ and $\Delta^{(-)}$. Comparing (35) and (32), we see

$$\Delta^{(\pm)}(x'-x) = -\Delta^{(\mp)}(x-x') \tag{36}$$

which also obtains from (31). From the definition of positive and negative frequency parts, we have immediately

$$\Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x). \tag{37}$$

Let us compute the integral representations of $\Delta^{(+)}(x)$ and $\Delta^{(-)}(x)$:

$$\Delta^{(+)}(x) = \frac{1}{2\pi i}\int_P \frac{d\tau}{\tau}\,\Delta(x-\tau n)$$

$$= -\frac{1}{(2\pi)^3}\int d^3k\,\frac{e^{ik\cdot r}}{\omega}\,\frac{1}{2\pi i}\int_P \frac{d\tau}{\tau}\sin\omega(x_0-\tau)$$

where we have taken $n=(1,0,0,0)$; this immediately yields

$$\Delta^{(+)}(x) = -\frac{i}{(2\pi)^3}\int d^3k\,\frac{e^{i(k\cdot r-\omega x_0)}}{2\omega} \tag{38}$$

and (38) and (36) together yield

$$\Delta^{(-)}(x) = \frac{i}{(2\pi)^3}\int d^3k\,\frac{e^{i(k\cdot r+\omega x)}}{2\omega}\,. \tag{39}$$

A second integral representation for these functions analogous to the second integral representation for $\Delta(x)$ as given by Eq. (28) is obtained in the manner that Eq. (28) was obtained:

$$\frac{e^{-i\omega x_0}}{2\omega} = +\frac{1}{2\pi i}\int_{C_+} dk_0\,\frac{e^{-ik_0x_0}}{(k_0+\omega)\,(k_0-\omega)}$$

$$= -\frac{1}{2\pi i}\int_{C_+} dk_0\,\frac{e^{-ik_0x_0}}{k^2+m^2} \tag{40}$$

where C_+ is shown in Fig. 6; also

$$\frac{e^{+i\omega x_0}}{2\omega} = -\frac{1}{2\pi i}\int_{C_-} dk_0\,\frac{e^{-ik_0x_0}}{(k_0+\omega)\,(k_0-\omega)}$$

$$= +\frac{1}{2\pi i}\int_{C_-} dk_0\,\frac{e^{-ik_0x_0}}{k^2+m^2} \tag{41}$$

where C_- is also shown in Fig. 6.

Applying (40) and (41) to (38) and (39) respectively, one obtains

$$\Delta^{(+)}(x) = \frac{1}{(2\pi)^4}\int_{C_+} d^4k\,\frac{e^{ik\cdot x}}{k^2+m^2} \tag{42}$$

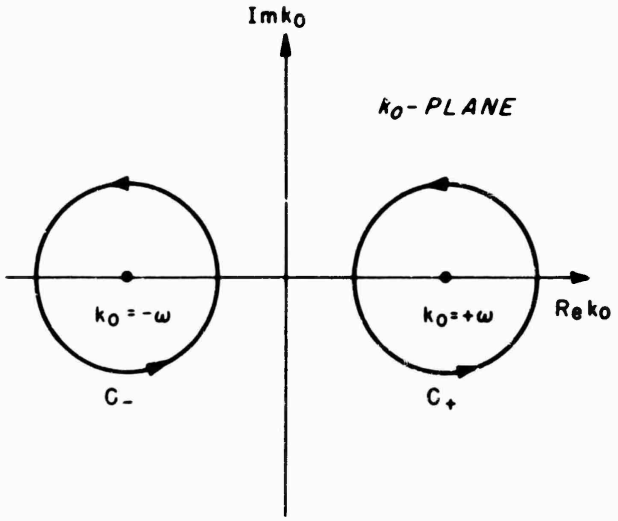

Figure 6

$$\Delta^{(-)}(x) = \frac{1}{(2\pi)^4} \int_{C_-} d^4k \, \frac{e^{ik\cdot x}}{k^2 + m^2}. \tag{43}$$

For a third integral representation, we use

$$\begin{aligned}\frac{e^{-i\omega_0 x_0}}{2\omega} &= \frac{1}{2\omega}\int_{-\infty}^{+\infty} dk_0 \, e^{-ik_0 x_0}\,\delta(k_0 - \omega) \\ &= \frac{1}{2\omega}\int_{-\infty}^{+\infty} dk_0 \, e^{-ik_0 x_0}\,\theta(k)[\delta(k_0 - \omega) \\ &\qquad + \delta(k_0 + \omega)] \\ &= \int_{-\infty}^{+\infty} dk_0 \, e^{-ik_0 x_0}\,\theta(k)\delta(k^2 + m^2) \end{aligned} \tag{44}$$

where

$$\theta(x) = \left.\begin{cases} 1, & x_0 > 0 \\ 1/2, & x_0 = 0 \\ 0, & x_0 < 0 \end{cases}\right\} \equiv \bar{\theta}(x_0). \tag{45}$$

In a similar way,

$$\frac{e^{i\omega_0 x_0}}{2\omega} = \int_{-\infty}^{+\infty} dk_0 \, e^{-ik_0 x_0}\theta(-k)\delta(k^2 + m^2) \; . \tag{46}$$

Using (38) and (39) with (44) and (40) respectively, we get

$$\Delta^{(+)}(x) = -\frac{i}{(2\pi)^3}\int_{-\infty}^{+\infty} d^4k \, e^{ik\cdot x}\theta(k)\delta(k^2 + m^2) \tag{47}$$

$$\Delta^{(-)}(x) = \frac{i}{(2\pi)^3}\int_{-\infty}^{+\infty} d^4k \, e^{ik\cdot x}\theta(-k)\delta(k^2 + m^2) \tag{48}$$

and it follows directly from (47) and (48) that

$$\Delta^{(+)}(x)^* = \Delta^{(-)}(x) \; . \tag{49}$$

Also, (47) and (48) exhibit the Lorentz invariance of these functions.

The Invariant Function $\Delta^{(1)}(x)$ and its Associated Boundary Value Problem

Define the function $\Delta^{(1)}(x)$ by

$$\Delta^{(1)}(x) \equiv \frac{1}{(2\pi)^3}\int d^3k \, e^{ik\cdot r}\,\frac{\cos \omega x_0}{\omega}. \tag{50}$$

It is clear that $\Delta^{(1)}(x)$ solves the Klein-Gordon equation. This function may be related to the function $\Delta(x)$ symbolically by

$$\Delta^{(1)}(x) = -\frac{\partial_0}{\sqrt{-\partial_i^2 + m^2}}\Delta(x) \tag{51}$$

where the symbolic operation $-\partial_0/\sqrt{-\partial_i^2 + m^2}$ is interpreted to mean, first, express $\Delta(x)$ (or any function the operation is applied to) in terms of a Fourier integral and, second apply the operation to each component; thus

$$\frac{-\partial_0}{\sqrt{-\partial_i^2 + m^2}}\Delta(x) = \frac{1}{(2\pi)^3}\int d^3k \, \frac{1}{\sqrt{-\partial_i^2 + m^2}} \times$$

$$e^{ik\cdot r}\,\frac{\partial}{\partial x_0}\,\frac{\sin \omega x_0}{\omega}$$

$$= \frac{1}{(2\pi)^3}\int d^3k \, \frac{1}{\omega}\, e^{ik\cdot r}\cos \omega x_0.$$

Suppose $\bar{\varphi}(x)$ and $\partial\bar{\varphi}/\partial x_\mu$ are known on S; let us compute the function

$$\int_S d\sigma'_\mu \left[\Delta^{(1)}(x-x')\,\partial'_\mu\,\bar{\varphi}(x')\right.$$

$$\left.\bar{\varphi}(x')\,\partial_\mu\Delta^{(1)}(x-x')\right] \equiv \Phi(x).$$

Here $\bar{\varphi}(x)$ may be any function whatsoever; now, with (51),

$$\Phi(x) = -\frac{\partial_0}{\sqrt{-\partial_i^2+m^2}}\int_S d\sigma'_\mu\left[\Delta(x-x')\,\frac{\partial\bar{\varphi}}{\partial x'_\mu}\right.$$

$$\left.-\bar{\varphi}(x')\,\frac{\partial\Delta(x-x')}{\partial x'_\mu}\right].$$

Suppose $\bar{\varphi}(x)$ is $\varphi^{(+)}(x)$, the positive frequency part of $\varphi(x)$, where $\varphi(x)$ solves the Klein-Gordon equation and φ and $\partial_\mu\varphi$ are known on S; the positive frequency part propagates independently of the negative frequency part. To see this, suppose $\varphi(x) = \varphi^{(+)}(x)$ with the negative frequency part identically vanishing for all x; then,

$$\varphi^{(+)}(x) = \int_S d\sigma'_\mu\left[\Delta(x-x')\,\frac{\partial\varphi^{(+)}(x')}{\partial x'_\mu}\right.$$

$$\left.-\varphi^{(+)}(x')\,\frac{\partial\Delta(x-x')}{\partial x_\mu}\right].$$

Analogously, if $\bar{\varphi}(x) = \varphi^{(-)}(x)$ with $\varphi^{(+)}(x) = 0$, then

$$\varphi^{(-)}(x) = \int_S d\sigma'_\mu\left[\Delta(x-x')\,\frac{\partial\varphi^{(-)}(x')}{\partial x'_\mu}\right.$$

$$\left.-\varphi^{(-)}(x')\,\frac{\partial\Delta(x-x')}{\partial x'_\mu}\right].$$

Now any function $\varphi(x)$ may be decomposed thusly. Hence we have

$$\int_S d\sigma'_\mu\left[\Delta^{(1)}(x-x')\,\partial'_\mu\varphi^{(\pm)}(x')\right.$$

$$\left.-\varphi^{(\pm)}(x')\,\partial'_\mu\Delta^{(1)}(x-x')\right]$$

$$= -\frac{\partial_0}{\sqrt{-\partial_i^2+m^2}}\,\varphi^{(\pm)}(x)\ .$$

But

$$-\frac{\partial_0}{\sqrt{-\partial_i^2+m^2}}\,\varphi^{(\pm)}(x) = -\frac{\partial_0}{\sqrt{-\partial_i^2+m^2}}\,\frac{1}{(2\pi)^4}$$

$$\int d^3k\,\frac{a(\mathbf{k},\pm\omega)}{2\omega}\,e^{i(\mathbf{k}\cdot\mathbf{r}\mp\omega x)}$$

$$= -\frac{1}{(2\pi)^4}\int d^3k\,\frac{a(\mathbf{k},\pm\omega)}{2\omega}\left(\frac{\mp i\omega}{\omega}\right)e^{i(\mathbf{k}\cdot\mathbf{r}\mp\omega x_0)}$$

$$= \pm\, i\varphi^{(\pm)}(x)\ .$$

Thus,

$$\int_S d\sigma'_\mu\left[\Delta^{(1)}(x-x')\,\frac{\partial\varphi^{(\pm)}(x')}{\partial x'_\mu}\right.$$

$$\left.-\varphi^{(\pm)}(x')\,\frac{\partial\Delta^{(1)}(x-x')}{\partial x'_\mu}\right] = \pm\, i\varphi^{(\pm)}(x)\ . \qquad (52)$$

The functions $\Delta^{(1)}(x)$ will "propagate" the positive and negative frequency parts of φ and $\partial\varphi/\partial x_\mu$ from S to the point x but with a change in phase, in contrast to $\Delta(x-x')$.

Next, we develop integral representations for $\Delta^{(1)}(x)$. Expanding $\cos\omega x_0$ in terms of exponential

$$\frac{\cos\omega x_0}{2\omega} = \int_{-\infty}^{+\infty} dk_0\,\delta(k^2+m^2)\,e^{-ik_0x_0} \qquad (53)$$

we get immediately from (50)

$$\Delta^{(1)}(x) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^4k\,\delta(k^2+m^2)\,e^{ik\cdot x} \qquad (54)$$

Utilizing (40) and (41) we get also

$$\frac{\cos \omega x_0}{2\omega} = \frac{1}{2\pi i}\int_{C_-} dk_0 \frac{e^{-ik_0x_0}}{k^2 + m^2} - \frac{1}{2\pi i}\int_{C_+} dk_0 \frac{e^{-ik_0x_0}}{k^2 + m^2}.$$

Let C'_+ be the contour C_+ traversed in the opposite sense and let $C^{(1)} = C_- UC'_+$; then

$$i\Delta^{(1)}(x) = \frac{1}{(2\pi)^4}\int_{C^{(1)}} d^4k \frac{e^{ik\cdot x}}{k^2 + m^2}. \quad (55)$$

Two equivalent contours $C^{(1)}$ are displayed in Fig. 7.

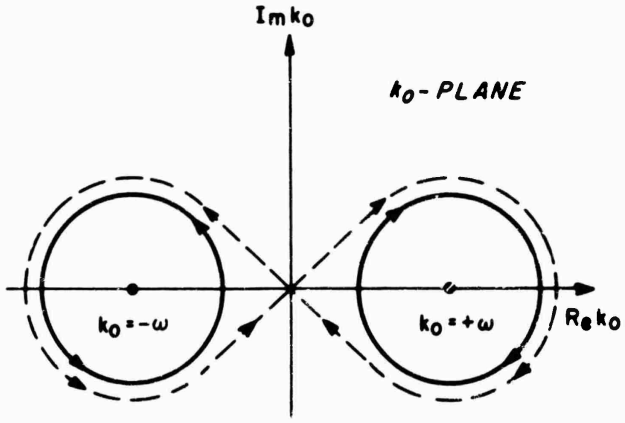

Figure 7

Relations Among the Invariant Δ-Functions

That the functions $\Delta(x)$, $\Delta^{(+)}(x)$, $\Delta^{(-)}(x)$, and $\Delta^{(1)}(x)$ are all invariant under proper orthochronous Lorentz transformations is evident from their integral representations involving integrations over the whole of the k-space; that they are not all independent of one another is evident from the integral representation over contours, if it were not evident before. The following relations are easy to verify:

$$\left.\begin{aligned}\Delta(x) &= \Delta^{(+)}(x) + \Delta^{(-)}(x)\\ i\Delta(x) &= \Delta^{(-)}(x) - \Delta^{(+)}(x)\end{aligned}\right\} \quad (56)$$

$$\left.\begin{aligned}\Delta^{(+)}(x) &= [\Delta(x) - i\Delta^{(1)}(x)]/2\\ \Delta^{(-)}(x) &= [\Delta(x) + i\Delta^{(1)}(x)]/2\end{aligned}\right\}. \quad (57)$$

It is seen from (56) and (57) that Δ and $\Delta^{(1)}$ play roles analogous to cos x and sin x functions, while $\Delta^{(+)}$ and $\Delta^{(-)}$ play roles analogous to the exponential functions exp $(-ix)$ and exp $(+ix)$, respectively.

Explicit Representations of the Invariant Delta Functions

In this section, we shall evaluate the integrals for $\Delta^{(+)}(x)$ and $\Delta^{(-)}(x)$ in terms of the higher transcendental functions and thereby obtain explicit representations not only for $\Delta^{(+)}$ and $\Delta^{(-)}$, but, through (56), also for Δ and $\Delta^{(1)}$.

In the integral representation (38), we transform from Cartesian coordinates in k-space to polar coordinates in k-space wherein the k_z-axis is made parallel to the vector $\mathbf{r}$. It is an easy matter to show that (38) reduces to

$$\Delta^{(+)}(x) = \frac{i}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{+\infty} dk \frac{e^{-i(kr+\omega x_0)}}{\omega} \quad (58)$$

where we place $|\mathbf{r}| = r$. Put $k = m \sinh \beta$; then $dk = m \cosh \beta\, d\beta$ and $\omega = m \cosh \beta$, and (58) may be rewritten as

$$\Delta^{(+)}(x) = \frac{1}{4\pi r}\frac{\partial}{\partial r} L^{(+)}(r,x_0) \quad (59)$$

where

$$L^{(+)}(r,x_0) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\beta \exp[-im(r \sinh \beta + x_0 \cosh \beta)]. \quad (60)$$

From the fact that $\Delta^{(-)}(x) = \Delta^{(+)}(x)^*$, we have

$$\left.\begin{aligned}\Delta^{(-)}(x) &= \frac{1}{4\pi r}\frac{\partial}{\partial r} L^{(-)}(r,x_0)\\ L^{(-)}(r,x_0) &= L^{(+)}(r,x_0)^*\end{aligned}\right\}. \quad (61)$$

We cannot derive all the pertinent results for all values of (r,x_0) with one development; instead we must consider certain regions of space-time separately. These are labeled in Fig. 8.

Region 1

Since

$$\frac{r}{\sqrt{x_0^2 - r^2}} < \frac{x_0}{\sqrt{x_0^2 - r^2}}$$

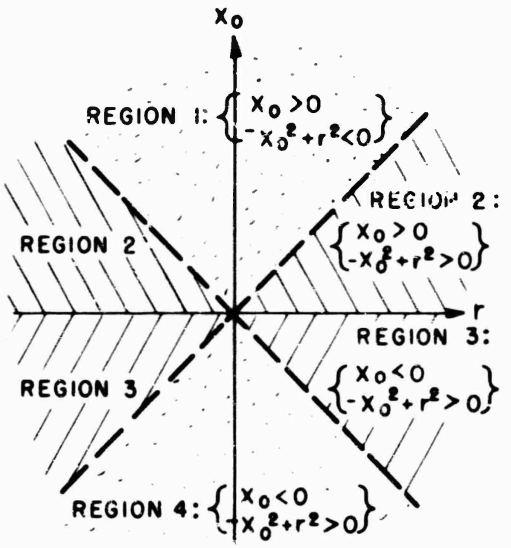

Figure 8

there exists a real β_0 such that

$$\sinh \beta_0 = \frac{r}{\sqrt{x_0^2 - r^2}}$$

$$\cosh \beta_0 = \frac{x_0}{\sqrt{x_0^2 - r^2}}$$

Then

$$r \sinh \beta + x_0 \cosh \beta = \sqrt{x_0^2 - r^2} \cosh (\beta + \beta_0)$$

and

$$L^{(+)}(r,x_0) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\beta \, e^{-im\sqrt{x_0^2 - r^2}\cosh(\beta+\beta_0)}$$

$$= \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\beta \, e^{-im\sqrt{x_0^2 - r^2}\cosh\beta}. \qquad (62)$$

We note that $L^{(+)}(r,x_0)$ does not converge in the usual sense; however, since $\cosh \beta$ is always positive, if $\lambda = \sqrt{x_0^2 - r^2}$ is regarded as a complex variable and λ assumes complex values with negative imaginary part, then $L^{(+)}\{\lambda\}$ converges off the real λ axis and below it. We may thus regard $L^{(+)}(r,x_0)$ as the boundary value of what is clearly an analytic function of λ.* Instead of the parameter λ, we put

$$m\sqrt{x_0^2 - r^2} = \zeta$$

where, in general, ζ may assume complex values whose real and imaginary parts we shall designate by ξ and η respectively. Thus $L^{(+)}(r,x_0)$ may be regarded as the boundary value of

$$\frac{i}{2\pi} \int_{-\infty}^{+\infty} d\beta \, e^{-i\zeta\cosh\beta}, \ \eta < 0. \qquad (64)$$

From the theory of the Hankel function†, we have

$$H_0^{(1)}(z) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} d\beta \, e^{iz\cosh\beta}, \ Im\, z > 0 \qquad (65)$$

$$H_0^{(2)}(z) = -\frac{1}{\pi i} \int_{-\infty}^{+\infty} d\beta \, e^{-iz\cosh\beta}, \ Im\, z < 0. \qquad (66)$$

Thus, by analytic continuation,

$$\left.\begin{aligned} L^{(+)}(r,x_0) &= +\,1/2\, H_0^{(2)}(m\sqrt{x_0^2 - r^2}) \\ L^{(-)}(r,x_0) &= +\,1/2\, H_0^{(1)}(m\sqrt{x_0^2 - r^2}) \end{aligned}\right\}, \text{ region 1.} \qquad (67)$$

The second of Eqs. (67) follows from the fact that $H_\lambda^{(1)}(x)^* = H_\lambda^{(2)}(x)$ when x and λ are real.

Region 2

Here, $r > x_0$, so we cannot put $r = \sqrt{r^2 - x_0^2}$ sinh β_0 and $x = \sqrt{r^2 - x_0^2}$ sinh β_0; instead we put

$$\frac{r}{\sqrt{r^2 - x_0^2}} = \cosh \beta_0$$

$$\frac{x_0}{\sqrt{r^2 + x_0^2}} = \sinh \beta_0 \ .$$

Then

$$m(r \sinh \beta + x_0 \cosh \beta) =$$

$$m\sqrt{r^2 - x_0^2} \ \sinh (\beta + \beta_0) \ .$$

Thus, in region 2,

*This result is a special case of a very general result derived by Wightmann (*Phys. Rev.* **101**:860 (1956). In this paper of Wightmann lies the foundation of a deep study of field theory on an axiomatic basis in which the Δ-function properties above come about in a very logical way, along with other important functions.

†See Courant and Hilbert, "Methods of Mathematical Physics," Interscience, New York, 1953, esp. Chapter VII, Vol. I.

$$\left.\begin{aligned} L^{(+)}(r,x_0) &= \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\beta\, e^{-im\sqrt{r^2-x_0^2}\sinh\beta} \\ L^{(-)}(r,x_0) &= -L^{(+)}(r,x_0) \end{aligned}\right\} . \tag{68}$$

In Eq. (68), replace $i\sinh\beta$ by its equivalent $\cosh(\beta + i(\pi/2))$ and then let $\gamma = \beta + i(\pi/2)$; then

$$L^{(+)}(r,x_0) = \frac{i}{2\pi}\int_{-\infty+i(\pi/2)}^{+\infty+i(\pi/2)} d\gamma\, e^{-m\sqrt{r^2-x_0^2}\cosh\gamma} . \tag{69}$$

In the step between $Im\gamma = 0$ and $Im\gamma = \pi/2$, the integrand in (69) has no poles, and it is easily seen that the countour integration above is then equivalent to an integration on the real γ-axis;

$$L^{(+)}(r,x_0) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\gamma\, e^{-m\sqrt{r^2-x_0^2}\cosh\gamma}$$

$$= \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\gamma\, e^{i(im\sqrt{r^2-x_0^2})\cosh\gamma}$$

or, using analytic continuation again,

$$\left.\begin{aligned} L^{(+)}(r,x_0) &= -1/2 H_0^{(1)}(im\sqrt{r^2-x_0^2}) \qquad (70) \\ &\text{while} \\ L^{(-)}(r,x_0) &= +1/2\, H_0^{(1)}(im\sqrt{r^2-x_0^2}) \qquad (71) \end{aligned}\right\} \text{, region 2.}$$

From (38) and (39), it follows that

$$\left.\begin{aligned} \Delta^{(+)}(-x_0, r) &= -\Delta^{(-)}(x_0, r) \\ \Delta^{(-)}(-x_0, r) &= -\Delta^{(+)}(x_0, r) \end{aligned}\right\} \tag{72}$$

from which it immediately obtains that

$$\left.\begin{aligned} L^{(+)}(r,x_0) &= -1/2\, H_0^{(1)}(im\sqrt{r^2-x_0^2}) \\ L^{(-)}(r,x_0) &= +1/2\, H_0^{(1)}(im\sqrt{r^2-x_0^2}) \end{aligned}\right\} \text{, region 3} \tag{73}$$

$$\left.\begin{aligned} L^{(+)}(r,x_0) &= -1/2\, H_0^{(1)}(m\sqrt{x_0^2-r^2}) \\ L^{(-)}(r,x_0) &= -1/2\, H_0^{(2)}(m\sqrt{x_0^2-r^2}) \end{aligned}\right\} \text{, region 4.} \tag{74}$$

We may combine the results (67) and (70) and (71) to get a representation of $L^{(+)}$ and $L^{(-)}$ in the union of regions 1 and 2 (with the light-cone itself omitted):

$$\left.\begin{aligned} L^{(+)}(r,x_0) &= \frac{1}{2}\left\{\bar\theta(-x^2)H_0^{(2)}(m\sqrt{-x^2}) - \bar\theta(x^2)H_0^{(1)}(m\sqrt{-x^2})\right\} \\ L^{(-)}(r,x_0) &= \frac{1}{2}H_0^{(1)}(m\sqrt{-x^2}) \end{aligned}\right\} \tag{75}*$$

(Region 1∪2)

and we may combine (73) and (74):

$$\left.\begin{aligned} L^{(+)}(r,x_0) &= -\frac{1}{2}H_0^{(1)}(m\sqrt{-x^2}) \\ L^{(-)}(r,x_0) &= -\frac{1}{2}\left\{\bar\theta(-x^2)H_0^{(2)}(m\sqrt{-x^2}) - \bar\theta(x^2)H_0^{(1)}(m\sqrt{-x^2})\right\} \end{aligned}\right\} \tag{76}$$

(Region 3∪4)

From (75) and (76), we get

$$\Delta(x) = \frac{1}{4\pi r}\frac{\partial}{\partial r}\left\{\epsilon(x)\left[L^{(+)}(r,x_0) + L^{(-)}(r,x_0)\right]\right\}$$

$$= \epsilon(x)\frac{1}{4\pi r}\frac{\partial}{\partial r}\left\{\bar\theta(-x^2)\times\left[\frac{H_0^{(2)}(m\sqrt{-x^2}) + H_0^{(1)}(m\sqrt{-x^2})}{2}\right]\right\}$$

$$= \epsilon(x)\frac{1}{4\pi r}\frac{\partial}{\partial r}\left\{\bar\theta(-x^2)J_0(m\sqrt{-x^2})\right\}. \tag{77}$$

*$\bar\theta(x^2)$ is the step function $\bar\theta(a) = 0$ for $a < 0$, $\bar\theta(0) = 1/2$, and $\bar\theta(a) = 1$ for $a > 0$; this is defined when a is a number. Contrast this with $\theta(x)$ where x is a four-vector; we see $\theta(x) = \bar\theta(x_0)$. In a similar manner we define $\epsilon(a)$. See Equation (45).

Using the fact that

$$\frac{1}{r}\frac{\partial\varphi(x^2)}{\partial r} = 2\frac{\partial\varphi(x^2)}{\partial r^2} = -2\frac{\partial\varphi(x^2)}{\partial(-x^2)},$$

when this operator operates on these functions of $(-x^2)$ we get finally

$$\Delta(x) = -\frac{\epsilon(x)}{2\pi}\left[\delta(x^2) - \frac{m^2\bar{\theta}(-x^2)}{2}\frac{J_1(m\sqrt{-x^2})}{m\sqrt{-x^2}}\right]. \tag{78}$$

Equation (78) applies everywhere including the light-cone: the derivative operation implied in (77) could not be performed for all points x in classical function theory; but $\theta(-x^2)J_0(m\sqrt{-x^2})$ exists as a regular generalized function and therefore possesses a derivative which, as (78) shows, is also a generalized function. From (78), it is manifest that $\Delta(x)$ vanishes outside the light-cone, and for $x_0 = 0$ (from the definition of $\epsilon(x)$). It is a delightful exercise to verify directly from (78) that $\Delta(x)$ satisfies the Klein-Gordon equation, and that

$$\left.\frac{\partial\Delta}{\partial x_0}\right|_{x_0=0} = -\delta(\mathbf{r}).$$

Turning to the function $\Delta^{(1)}(x)$, we see from Eq. (50) that

$$\Delta^{(1)}(-x_0, \mathbf{r}) = \Delta^{(1)}(x_0, \mathbf{r}) \tag{79}$$

so that it will only be necessary to obtain a representation of $\Delta^{(1)}$ for regions 1 and 2 and then it will be known for the whole of space-time. From (56)

$$i\Delta^{(1)}(x) = \frac{1}{4\pi r}\frac{\partial}{\partial r}[L^{(-)}(r,x_0) - L^{(+)}(r,x_0)]$$

$$= \frac{1}{4\pi r}\frac{\partial}{\partial r}\left[\frac{1+\bar{\theta}(x^2)}{2}H_0^{(1)}(m\sqrt{-x^2}) - \frac{\bar{\theta}(-x^2)}{2}H_0^{(2)}(m\sqrt{-x^2})\right]$$

$$= \frac{1}{4\pi r}\frac{\partial}{\partial r}\left[\frac{\bar{\epsilon}(x^2)H_0^{(1)}(m\sqrt{-x^2}) - \bar{\theta}(-x^2)H_0^{(2)}(m\sqrt{-x^2})}{2}\right]$$

$$= \frac{1}{4\pi r}\frac{\partial}{\partial r}\begin{cases} N_0(m\sqrt{-x^2}), & x^2 < 0 \\ -H_0^{(1)}(m\sqrt{-x^2}), & x^2 > 0 \end{cases}$$

$$= \begin{cases} \dfrac{mi}{4\pi\sqrt{-x^2}}N_1(m\sqrt{-x^2}), & x^2 < 0 \\ \dfrac{mi}{2\pi^2\sqrt{x^2}}K_1(m\sqrt{x^2}), & x^2 > 0 \end{cases}$$

or

$$\Delta^{(1)}(x) = \left\{\begin{array}{ll} \dfrac{m^2}{4\pi}\dfrac{N_1(m\sqrt{-x^2})}{m\sqrt{-x^2}}, & (x^2 < 0) \\ \dfrac{m^2}{2\pi^2}\dfrac{K_1(m\sqrt{x^2})}{m\sqrt{x^2}}, & (x^2 > 0) \end{array}\right\}. \tag{80}$$

From (80) it is quite clear that $\Delta^{(1)}(x)$ does not vanish identically outside the light-cone. The functions N_0 and N_1 in (80) are the Neumann functions of order zero and unity respectively, while $K_0(z)$ is given by

$$K_0(z) = \frac{\pi i}{2} H_0^{(1)}(iz).$$

For explicit representations of $\Delta^{(+)}(x)$ and $\Delta^{(-)}(x)$, one may use Eq. (57) together with (78) and (80):

$$\Delta^{(+)}(x) = \begin{cases} -\dfrac{\epsilon(x)\delta(x^2)}{2\pi} + \dfrac{m^2}{4\pi}\dfrac{H_1^{(2)}(m\sqrt{-x^2})}{m\sqrt{-x^2}}, \\ \qquad x^2 < 0,\ x_0 > 0 \\ -\dfrac{\epsilon(x)\,\delta(x^2)}{2\pi} - \dfrac{m^2}{4\pi}\dfrac{H_1^{(1)}(m\sqrt{-x^2})}{m\sqrt{-x^2}}, \\ \qquad x^2 < 0,\ x_0 < 0 \\ -\dfrac{\epsilon(x)\,\delta(x^2)}{2\pi} - \dfrac{im^2}{2\pi^2}\dfrac{K_1(m\sqrt{x^2})}{m\sqrt{x^2}}, \\ \qquad x^2 > 0 \end{cases} \tag{81}$$

and

$$\Delta^{(-)}(x) = \Delta^{(+)}(x)^*. \tag{82}$$

The behavior of $\Delta(x)$ and $\Delta^{(1)}(x)$ near the light-cone ($x^2 \sim 0$) and large distances away from it ($x^2 \sim \pm\infty$) and may be derived from the behavior of the function $J_1(z)$, $N_1(z)_1$, and $K_1(z)$ for $|z| \sim \pm\infty$ respectively.† Near the origin

$$J_1(z) \sim \frac{z}{2} + O(z^3)$$

$$N_1(z) \sim -\frac{2}{\pi z} + \frac{z}{\pi}\ln\frac{z}{2} + \frac{\gamma z}{\pi} - \frac{z}{2\pi} + \cdots$$

$$K_1(z) \sim \frac{1}{2} + \frac{z}{2}\ln\frac{z}{2} + \left(\gamma - \frac{1}{2}\right)\frac{z}{2} + \cdots$$

†Courant and Hilbert, *loc. cit.*

where γ is the Euler-Mascheroni constant. Thus

$$\Delta(x)\Big|_{x^2\sim 0} \sim -\frac{\epsilon(x)}{2\pi}\left\{\delta(x^2) - \frac{m^2}{4}\bar{\theta}(-x^2) + \cdots\right\} \tag{83}$$

$$\Delta^{(1)}(x)\Big|_{x^2\sim 0} \sim \frac{1}{2\pi^2 x^2} + \frac{m^2}{4\pi^2}\ln\frac{m\sqrt{|x^2|}}{2} + \left(\gamma - \frac{1}{2}\right)\frac{m^2}{4\pi^2} + \cdots. \tag{84}$$

Thus $\Delta(x)$ has a singularity of the light-cone with an additional jump discontinuity; $\Delta^{(1)}$ is likewise singular at the light-cone, but the singularity is much stronger than that of $\Delta(x)$.

For large values of $|z|$,

$$J_1(z) \sim \frac{2}{\sqrt{\pi z}}\cos z$$

$$N_1(z) \sim \frac{2}{\sqrt{\pi z}}\sin z$$

$$K_1(z) \sim \frac{2}{\sqrt{\pi z}}e^{-z}.$$

Thus, inside the light-cone

$$\Delta(x) \sim \frac{\epsilon(x)\cos m\sqrt{-x^2}}{2\pi^{3/2}(m\sqrt{-x^2})^{3/2}}, \quad -x^2 \sim +\infty \tag{85}$$

$$\Delta^{(1)}(x) \sim \frac{m^2 \sin m\sqrt{-x^2}}{2\pi^{3/2}(m\sqrt{-x^2})^{3/2}}, \quad -x^2 \sim +\infty \tag{86}$$

while outside the light-cone,

$$\Delta(x) = 0,\ x^2 \sim +\infty \tag{87}$$

$$\Delta^{(1)}(x) \sim \frac{m^2}{2^{1/2}\pi^{5/2}}\frac{e^{-m\sqrt{x^2}}}{(m\sqrt{x^2})^{3/2}},\ x^2 \sim +\infty. \tag{88}$$

The Boundary Value Problems and the Invariant Δ-Functions (Continued)

In the preceding section, we discussed the solution of several boundary value problems for

the Klein-Gordon equation which gave rise to four important Green's functions, all of which solved the homogeneous Klein-Gordon equation themselves. In this section, we solve several more boundary value problems which give rise to Green's functions different from those previously studied and which satisfy not the homogeneous Klein-Gordon equation, but a special version of the inhomogeneous Klein-Gordon equation, that is, Eq. (1) wherein the right side, rather than vanishing, is a prescribed function of space and time.

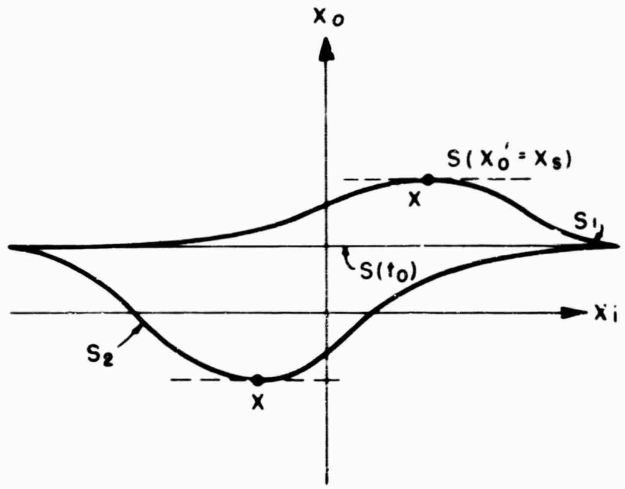

Figure 9

The Retarded Δ-Function, $\Delta_R(x)$

Consider a physical experiment wherein the experimenter sets up his field function φ at the time $t = t_0$ in such a way that the functional values of φ and all its first derivatives are known throughout all three-dimensional space at time $t = t_0$. Since the values of φ and its derivatives at previous times are immaterial, we may require that these vanish. Since the function φ develops in space and time according to the Klein-Gordon equation, we may expect to be able to compute the values of φ at any later point in space and at any time. We wish to construct an auxiliary function which we shall call the retarded Δ-function and shall designate by Δ_R to describe this situation, that is, a function which, when used in conjunction with Green's identity, will yield $\varphi(x)$ when x is later than the surface $S(t_0)$ and zero when x is earlier than the surface $S(t_0)$. Let us try to develop this function in a manner parallel to that used in the preceding section.

Consider first the case x later than $S(t_0)$; let S_1 be a space-like surface through x such that S_1 is tangent to the plane $x_0' = x_0$ at x (Fig. 9), and that the volume Ω interior to the union of these two surfaces is finite. From the first two subsections of the preceding section, the value of φ at x is given by

$$\varphi(x) = \int_{S(t_0)} d\sigma_\mu' \left[\Delta(x - x') \frac{\partial \varphi}{\partial x_\mu'} - \varphi(x') \frac{\partial \Delta(x - x)}{\partial x_\mu'} \right].$$

Thus, if

$$\Delta_R(x - x') = \Delta(x - x'), \; x_0 > x_0' \qquad (89)$$

we shall have achieved part of our goal.

Now suppose x lies earlier than $S(t_0)$. Construct S_2 through x in a manner analogous to the construction of S_1 (Fig. 9).

Let us assume again (as was tacitly done in (89)) that Δ_R obeys the Klein-Gordon equation:

$$(\Box - m^2)\, \Delta_R(x) = 0, \textit{ for } x \in \Omega_{S_1, S(t_0)}. \qquad (90)$$

Then in the Green's identity, the volume integral vanishes as before and we are left with

$$\int_{S_2} d\sigma_\mu' \left[\varphi(x') \frac{\partial \Delta_R(x - x')}{\partial x_\mu'} - \Delta_R(x - x') \frac{\partial \varphi}{\partial x_\mu'} \right]$$

$$= \int_{S(t_0)} d\sigma_\mu' \left[\varphi(x') \frac{\partial \Delta_R(x - x')}{\partial x_\mu'} - \Delta_R(x - x') \frac{\partial \varphi}{\partial x_\mu'} \right]. \qquad (91)$$

But our boundary conditions of φ stated $\varphi \equiv 0$ for all x prior to $S(t_0)$; hence $\varphi(x') \equiv 0 = \partial\varphi/\partial x_\mu'$ on S_2. Thus,

$$\int_{S(t_0)} d\sigma_\mu' \left\{ \varphi(x') \frac{\partial \Delta_R(x - x')}{\partial x_\mu'} - \Delta_R(x - x') \frac{\partial \varphi}{\partial x_\mu'} \right\} = 0$$

which can be satisfied only if

$$\Delta_R(x - x') \equiv 0, \; x_0 < x_0'. \tag{92}$$

Thus, if we choose $\Delta_R(x)$ for all x to be

$$\Delta_R(x) = \theta(x)\Delta(x), \text{ all } x, \tag{93}$$

a form which exhibits invariance under proper orthochronous Lorentz transformation, then the boundary value problem posed in the beginning of this section will be solved. While it is true that Δ_R satisfies the Klein-Gordon equation in $\Omega(S_1, S(t_0))$ and $\Omega(S_2, S(t_0))$ (the latter by virtue of (92)), Δ_R does not satisfy the Klein-Gordon equation everywhere. (Note that Δ_R as given by (93) is an extension of Δ_R outside the original domains Ω of definition.) Let us determine what equation Δ_R does satisfy in its extended domain. It is easy to show

$$\partial_\mu\partial_\mu\Delta_R(x) = \Delta(x)\;\partial_\mu\partial_\mu\theta(x) + \theta(x)\;\partial_\mu\partial_\mu\Delta(x)$$

$$+ 2\;\partial_\mu\theta(x)\;\partial_\mu\Delta(x).$$

Now

$$\partial_\mu\theta(x) = 0, \text{ if } \mu = 1,2,3$$

$$\partial_0\theta(x) = \delta(x_0).$$

Therefore

$$\partial_\mu\partial_\mu\theta(x) = -\,\partial_0\partial_0\theta(x)$$

$$= -\frac{\partial}{\partial x_0}\,\delta(x_0)$$

$$= \frac{\delta(x_0)}{x_0}.$$

Therefore

$$\Delta(x)\;\partial_\mu\partial_\mu\theta(x) = \delta(x_0)\,\frac{\Delta(x_0,\mathbf{r})}{x_0}.$$

At the point $x_0 = 0$, $\Delta(x)\Box\theta(x)$ is undefined; we shall define it by a limiting process; therefore

$$\lim \Delta(x)\;\partial_\mu\partial_\mu\theta(x) = \delta(x_0)\lim_{x_0\to 0}\frac{\Delta(x_0,\mathbf{r})}{x_0}$$

$$= \delta(x_0)\,\frac{\partial\Delta}{\partial x_0}\,(x_0,\mathbf{r})$$

$$= -\delta(x_0)\delta(\mathbf{r})$$

$$= -\,\delta(x)$$

where $\delta(x) = \delta(x_0)\delta(x_1)\delta(x_2)\delta(x_3)$. Since $\Delta\Box\theta$ vanishes elsewhere, we have

$$\Delta(x)\;\partial_\mu\partial_\mu\theta(x) = -\,\delta(x)$$

and similarly

$$2\;\partial_\mu\theta(x)\;\partial_\mu\Delta(x) = +\,2\;\delta(x).$$

Thus

$$\Box\Delta_R(x) = \theta(x)\;\Box\Delta(x) + \delta(x)$$

and utilizing the fact that $\Delta(x)$ obeys the homogeneous Klein-Gordon equation, we get

$$(\Box - m^2)\Delta_R(x) = \delta(x); \tag{94}$$

that is, the invariant function $\Delta_R(x)$ satisfies the inhomogeneous Klein-Gordon equation. With this fact, we may recast our treatment of the boundary value problem of this section in a manner different from above and in a way that, as we shall see, cannot possibly be applied to our previous Δ-functions.

Let Ω be a finite volume in space-time bounded by two space-like hypersurfaces S_1 and S_2, where S_1 is later than S_2. Let x be any point interior to Ω, *i.e.*, $x \in \Omega$ but $x \notin S_1 \cup S_2$. We shall try to find an auxiliary function $\psi_x(x')$ that satisfies

$$(\Box' - m^2)\;\psi_x(x') = \delta(x' - x) \tag{95}$$

and will solve the boundary value problem stated above; namely, the value of φ at x is determined solely by its values of S_2 alone. We shall use Green's identity in the form of Eq. (29) of Chapter 1:

$$\int_{\Omega} d^4x' [\varphi(x')(\Box' - m^2)\psi_x(x') - \psi_x(x')(\Box' - m^2)\varphi(x')]$$

$$= \int_{S_1 \cup S_2} d\sigma' \left[\varphi(x') \frac{\partial \psi_x(x')}{\partial n'} - \psi_x(x') \frac{\partial \varphi(x')}{\partial n'} \right]. \tag{96}$$

Using (95), the fact that S_1 is by hypothesis to make no contribution, we get, when $x \in \Omega$,

$$\varphi(x) = \int_{S_2} d\sigma' \left[\varphi(x') \frac{\partial \psi_x(x')}{\partial n'} - \psi_x(x') \frac{\partial \varphi(x')}{\partial n'} \right].$$

To assist us in casting the right side in the

$$\int d\sigma_\mu$$

form, assume, for the moment, S_2 is the hyperplane $x_0' =$ constant; then $d\sigma = dx_1 dx_2 dx_3 = d^3x = d\sigma_0$ and

$$\frac{\partial}{\partial n'} = -\frac{\partial}{\partial x_0'}$$

Then

$$\varphi(x) = -\int_{S_2} d\sigma_0' \left[\varphi(x') \frac{\partial \psi_x(x')}{\partial x_0'} - \psi_x(x') \frac{\partial \varphi(x')}{\partial x_0'} \right]$$

$$= \int_{S_2} d\sigma_\mu' \left[\varphi(x') \frac{\partial \psi_x(x')}{\partial x_\mu'} - \psi_x(x') \frac{\partial \varphi(x')}{\partial x_\mu'} \right]. \tag{97}$$

We have indicated all the steps in detail so that the signatures of various terms have their origins clearly delineated. Equation (97) is clearly covariant. When $x \notin \Omega$, the left side of (96) vanishes and

$$0 = \int_{S_2} d\sigma_\mu' \left[\psi_x(x') \frac{\partial \varphi(x')}{\partial x_\mu'} - \varphi(x') \frac{\partial \psi_x(x')}{\partial x_\mu'} \right]. \tag{98}$$

Comparing (97) with (34), we get

$$\psi_x(x') = \Delta(x - x'), \; x \in \Omega. \tag{99}$$

Since S_1 is arbitrary for this boundary value problem, (99) implies

$$\psi_x(x') = \Delta(x - x'), \quad \text{all } x \text{ later than } S_2. \tag{100}$$

Equation (98) implies

$$\psi_x(x') = 0, \quad \text{all } x \text{ earlier than } S_2. \tag{101}$$

Since $\Delta(x - x')$ vanishes outside the light-cone, (100) and (101) may be combined into

$$\psi_x(x') = \theta(x - x')\, \Delta(x - x') \tag{102}$$

so that $\psi_x(x')$ is identical with $\Delta_R(x - x')$. It remains to show that $\psi_x(x')$ solves (95):

$$(\partial_\mu' \partial_\mu' - m^2)\, \psi_x(x') = (\partial_\mu' \partial_\mu' - m^2)\, \Delta_R(x - x')$$

$$= [(-\partial_\mu)(-\partial_\mu) - m^2]\, \Delta_R(x - x')$$

$$= (\Box - m^2)\, \Delta_R(x - x')$$

$$= \delta(x - x')$$

where the last step follows from (94).

This method differs from the first method in two ways: first, we required the function $\psi_x(x')$ to solve the inhomogeneous Klein-Gordon Eq. (94) at the very outset, whereas in the first method, this was derived; second, the point x was not required to lie on the upper or lower surface as in the first method.

We conclude this subsection by noting that we shall derive integral representations for $\Delta_R(x)$

in a later subsection, and by summarizing the boundary value problem in the formula

$$\int_S d\sigma'_\mu \left[\Delta_R(x - x') \frac{\partial \varphi(x')}{\partial x'_\mu} - \varphi(x') \frac{\partial \Delta_R(x - x')}{\partial x'_\mu} \right] = \begin{cases} \varphi(x), & x \text{ later than } S \\ 0, & x \text{ earlier than } S . \end{cases} \tag{103}$$

The Advanced Δ-Function, $\Delta_A(x)$

Let us examine the above boundary value problem as it appears in a reference frame $\bar{O}$ which is the inversion of the reference frame in which the boundary value problem of the preceding subsection was formulated; *i.e.*, if P is a point of space-time whose coordinates in the above reference frame, called O, are x_0, x_1, x_2, x_3, then in $\bar{O}$ the coordinates are $\bar{x}_0$, $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, where $\bar{x}_\mu = -\bar{x}_\mu$. The surface S goes into $\bar{S}$ and

$$\int_S d\sigma'_\mu \to \int_{\bar{S}} d\bar{\sigma}'_\mu .$$

Transforming (103) we get

$$\int_{\bar{S}} d\sigma'_\mu \left[\Delta_R(-\bar{x} + \bar{x}') \frac{\partial \varphi(-\bar{x}')}{-\partial \bar{x}'_\mu} - \varphi(-\bar{x}') \frac{\partial \Delta_R(-\bar{x} + \bar{x}')}{-\partial \bar{x}'_\mu} \right] = \begin{cases} \varphi(-\bar{x}), & \bar{x} \text{ earlier than } \bar{S} \\ 0 , & \bar{x} \text{ later than } \bar{S}. \end{cases}$$

Denoting $\varphi(-\bar{x}')$ by $\bar{\varphi}(\bar{x}')$, the last equation reads

$$\int_{\bar{S}} d\bar{\sigma}'_\mu \left[\Delta_R(-\bar{x} + \bar{x}') \frac{\partial \bar{\varphi}(\bar{x}')}{\partial \bar{x}'_\mu} - \bar{\varphi}(\bar{x}') \frac{\partial \Delta_R(-\bar{x} + \bar{x}')}{\partial \bar{x}_\mu} \right] = \begin{cases} 0 & \bar{x} \text{ later than } \bar{S} \\ -\bar{\varphi}(\bar{x}) & \bar{x} \text{ earlier than } \bar{S}. \end{cases} \tag{104}$$

The function $\bar{\varphi}(\bar{x})$ satisfies the Klein-Gordon equation

$$(\bar{\partial}_\mu \bar{\partial}_\mu - m^2)\, \bar{\varphi}(\bar{x}) = 0.$$

Thus (104) represents the solution of a new boundary value problem in the reference frame of $\bar{O}$, had we solved the boundary value problem in the preceding subsection in $\bar{O}$'s reference frame and then transformed to the reference frame O, we would have derived (104) within the bars appearing there. Thus, we define

$$\Delta_A(x) = \Delta_R(-x) \tag{105}$$

which solves the boundary value problem summarized by

$$\int_S d\sigma'_\mu \left[\Delta_A(x - x') \frac{\partial \varphi(x')}{\partial x'_\mu} - \varphi(x') \frac{\partial \Delta_A(x - x')}{\partial x'_\mu} \right] = \begin{cases} 0 , & x \text{ later than } S \\ -\varphi(x), & x \text{ earlier than } S. \end{cases} \tag{106}$$

Utilizing (93) and the property that $\Delta(x)$ is an odd function in x, we have

$$\Delta_A(x) = -\theta(-x)\, \Delta(x). \tag{107}$$

Integral representations for $\Delta_A(x)$ will be developed in a subsequent subsection.

The Invariant Function $\bar{\Delta}(x)$

Define the function $\bar{\Delta}(x)$ by

$$\bar{\Delta}(x) = 1/2\, [\, \Delta_R(x) + \Delta_A(x)\,]. \tag{108}$$

It follows immediately that

$$\bar{\Delta}(x) = 1/2\, \epsilon(x) \Delta(x) \tag{109}$$

and

$$\int_S d\sigma'_\mu \left[\bar{\Delta}(x - x') \frac{\partial \varphi}{\partial x'_\mu} - \varphi(x') \frac{\partial \bar{\Delta}(x - x')}{\partial x'_\mu} \right] = \begin{cases} -\varphi(x), & x \text{ earlier than } S \\ +\varphi(x), & x \text{ later than } S \end{cases} \tag{110}$$

and that

$$(\Box - m^2)\,\bar{\Delta}(x) = \delta(x). \tag{111}$$

A subsequent subsection will give integral representations of $\bar{\Delta}(x)$.

The Feynman, or Causal, Propagator, $\Delta_F(x)$

We seek to construct the Green's function which will yield from the values of φ and $\partial\varphi(\partial x_\mu)$ on some space-like surface S the positive frequency part of φ at the point x when S precedes x. We could develop this boundary value problem as we did before directly from Green's identity, but this is not necessary, for we have enough developed with the functions $\Delta^{(+)}$ and $\Delta^{(-)}$ or $\bar{\Delta}$ to ease our path. For our purpose, we shall take (35) as our starting point and observe that the Green's function we want, which we label $\Delta_F(x)$, is given by

$$\Delta_F(x) = \begin{cases} \Delta^{(+)}(x), \text{ if } x_0 > 0 \\ -\Delta^{(-)}(x), \text{ if } x_0 < 0 \end{cases} \tag{112}$$

where for simplicity, we have taken S to be the hypersurface $x_0' = 0$. Equation (112) may be cast into a form explicitly covariant by use of the θ-function and thereby broaden its applicability to all space-like hypersurfaces:

$$\Delta_F(x) = \theta(x)\,\Delta^{(+)}(x) - \theta(-x)\,\Delta^{(-)}(x). \tag{113}$$

This function was introduced by Feynman in his theory of quantum electrodynamics and independently by Stückelberg and Rivier. The latter authors designated the function by $\Delta_c(x)$ and called it the causal propagator. Utilizing (15) and (57), we may also express $\Delta_F(x)$ by

$$\Delta_F(x) = \bar{\Delta}(x) - \frac{i}{2}\,\Delta^{(1)}(x) \tag{114}$$

and from (114), it is clear that

$$(\Box - m^2)\,\Delta_F(x) = \delta(x). \tag{115}$$

We summarize the boundary value problem by the relation

$$\int_S d\sigma_\mu' \left[\Delta_F(x - x')\,\frac{\partial\varphi(x')}{\partial x_\mu'} - \varphi(x')\,\frac{\partial\Delta_F(x - x')}{\partial x_\mu'} \right]$$

$$= \begin{cases} \varphi^{(+)}(x), & x \text{ later than } S \\ -\varphi^{(-)}(x), & x \text{ earlier than } S. \end{cases} \tag{116}$$

Integral representations of $\Delta_F(x)$ are derived in the next subsection.

Integral Representations of the Inhomogeneous Invariant Δ-Function

Let $\hat{\Delta}(x)$ be any one of the four functions Δ_R, Δ_A, $\bar{\Delta}$, and Δ_F. From the work of the preceding subsections we have seen

$$(\Box - m^2)\,\hat{\Delta}(x) = \delta(x)\,. \tag{117}$$

Because these functions all satisfy the inhomogeneous Klein-Gordon equation (117), we call these functions, collectively, the inhomogeneous invariant Δ-functions and the other four functions Δ, $\Delta^{(1)}$, $\Delta^{(+)}$, and $\Delta^{(-)}$ the homogeneous invariant Δ-functions. In contrast to the development of the integral representations of the latter class of functions from derived representations, we shall develop the contour integral representation of the function $\hat{\Delta}$ directly from (117), utilizing to the maximum our knowledge of the integral representations of the homogeneous functions.

Let us Fourier analyze the space part of $\hat{\Delta}(x)$ and $\delta(x)$, putting

$$\hat{\Delta}(x) = \frac{1}{(2\pi)^3}\int d^3k f(\mathbf{k},x_0)e^{i\mathbf{k}\cdot\mathbf{r}}.$$

Then (117) places as a condition on $f(\mathbf{k},x_0)$ that for each $\mathbf{k}$ it solves

$$\left(\frac{d^2}{dx_0^2} + \omega^2\right) f(\mathbf{k},x_0) = -\,\delta(x_0)$$

where $\omega^2 = \mathbf{k}^2 + m^2$. Equations such as these may be treated by the method of a contour integral*

*See E. L. Ince, "Ordinary Differential Equations," Dover Publications, 1956, esp. Chapter XVIII.

wherein the function $\bar{f}(x_0) = f(\mathbf{k},x_0)$ is represented as an integral of the general form

$$\bar{f}(x_0) = \int_C dk_0\, K(x_0,k_0)\, \bar{g}(k_0)$$

where $K(x_0,k_0)$ is chosen in a way convenient for the differential equation for $\bar{f}(x_0)$, and $\bar{g}(k_0)$ is determined subsequently by the choice of K and requiring that the differential equation be satisfied; C is a contour in the complex k_0 plane chosen so that $\bar{f}(x_0)$ not only satisfies the differential equation but so that the initial condition on $\bar{f}(x_0)$ also are satisfied. The function $\bar{g}(k_0)$ will have, in general, as many singularities in the complex plane as the order of the differential equation, and the contour integration must always be chosen so as to avoid these. One will be able to choose many distinct contours, but there will be only as many contours C as the order of the differential equation that yield linearly independent solutions.

For our problem, we naturally choose

$$K(x_0,k_0) = -\frac{e^{-ik_0x_0}}{2\pi}\,.$$

Then

$$\bar{g}(k_0) = \frac{1}{-k_0^2 + \omega^2}$$

$$= \frac{1}{k^2 + m^2}.$$

Rather than determine $\hat{C}$ so as to meet the boundary conditions on $\bar{f}(x_0)$, we shall go directly to the boundary conditions on $\hat{\Delta}$; that is, $\hat{\Delta}$ is now given by

$$\hat{\Delta}(x) = \frac{1}{(2\pi)^4}\int_{\hat{C}} d^4k\, \frac{e^{ik\cdot x}}{k^2 + m^2} \qquad (118)$$

where $\hat{C}$ is a contour in the complex k_0-plane to be chosen so as to yield the conditions placed on $\hat{\Delta}(x)$. The contours $\hat{C}$ are independent of x, of course.

We have seen already that the homogeneous invariant functions have integral representations analogous to (118) and that the contours involved there always lay in the finite k_0-plane, or if there were not so chosen, were always equivalent to contours in the finite k_0-plane. These contours, of course, were independent of x. We shall show that for the homogeneous invariant Δ-function, in particular, $\Delta(x)$, it is possible to choose an infinite contour, but that this choice will be dependent on x_0. The importance of this result will manifest itself in choosing contours for Δ_R and Δ_A.

The contour integral for $\Delta(x)$ is shown in Fig. 5. It is readily seen that two equivalent contour integrals are those shown in Figs. 10 and 11. In Fig. 10, it is readily seen that if L is allowed to go to infinity, the contour thus obtained will represent C only if $x_0 > 0$; for then the contribution from the semicircle vanishes, while for $x_0 < 0$ the contribution of the latter integral tends toward infinity in magnitude. Thus for $x_0 > 0$ a valid infinite representation of the contour C for $\Delta(x)$ is any line from $+\infty$ to $-\infty$ (note direction) above the real axis, or any contour equivalent to this, and clearly, for $x_0 < 0$ it is any line from $-\infty$ to $+\infty$ parallel to the real axis but a finite distance below it. Call the first of these contours C_R and the second $-C_A$.

Evaluate $\hat{\Delta}(x)$ for $\hat{C} = C_R$. We have just seen if $x_0 > 0$, $\Delta(x) = \Delta(x)$; if $x_0 < 0$, then $\Delta(x)$ may be evaluated by closing the contour above. But there are no poles of the integrand above C_R; hence $\Delta(x) = 0$ for $x_0 < 0$. Thus,

$$\frac{1}{(2\pi)^4}\int_{C_R} d^4k\, \frac{e^{ik\cdot x}}{k^2 + m^2} = \Delta_R(x) \qquad (119)$$

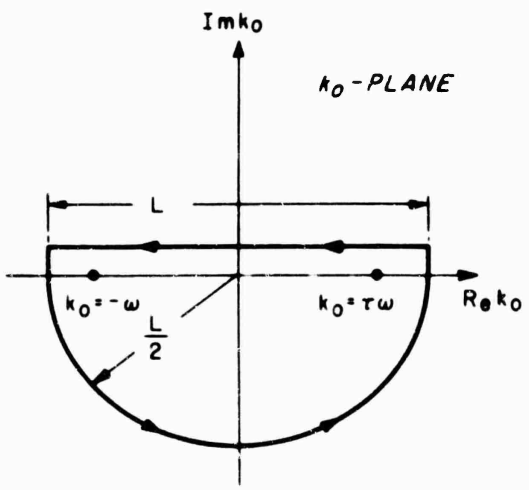

Figure 10

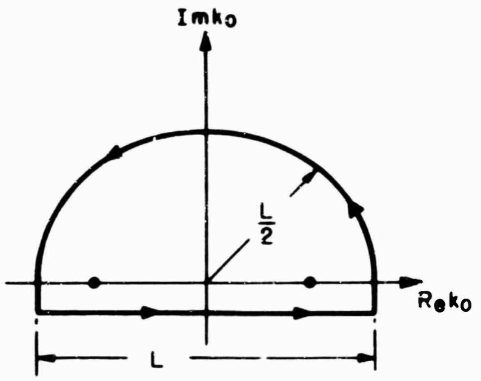

Figure 11

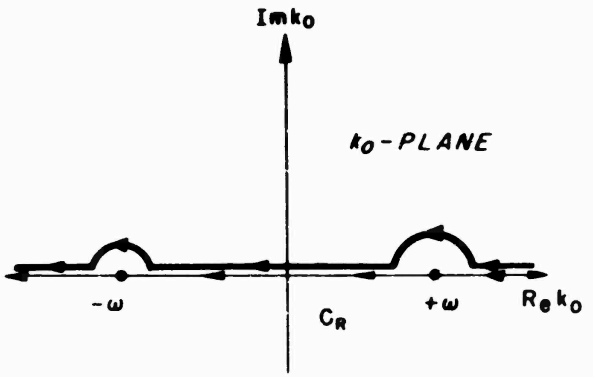

Figure 12

where C_R is shown in Fig. 12. The contour for C_R is clearly infinite, which we could have known before because all the finite contours were opted by the homogeneous functions and their linear combinations. Without further ado, it is clear that

$$\Delta_A(x) = \frac{1}{(2\pi)^4} \int_{C_A} d^4k \, \frac{e^{ik\cdot x}}{k^2 + m^2} \tag{120}$$

where C_A is shown in Fig. 13.

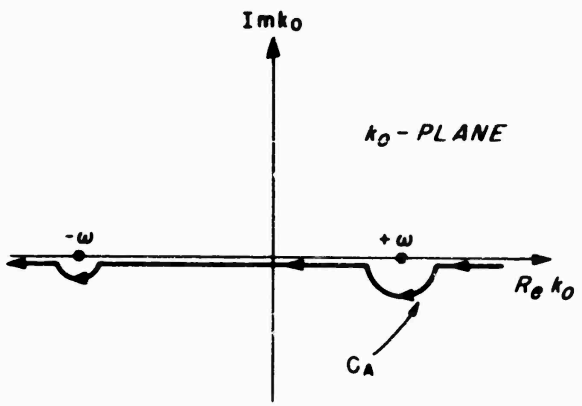

Figure 13

From the definition (108) of $\bar{\Delta}(x)$ and the contour integral representations of $\Delta_R(x)$ and $\Delta_A(x)$ we have

$$\bar{\Delta}(x) = \frac{1}{(2\pi)^4} \int_{\bar{C}} d^4k \, \frac{e^{ik\cdot x}}{k^2 + m^2} \tag{121}$$

$$= -\frac{1}{(2\pi)^4} P \int_{-\infty}^{+\infty} d^4k \, \frac{e^{ik\cdot x}}{k^2 + m^2} \tag{122}$$

where $\bar{C}$ is shown in Fig. 14. Finally, the Feynman contour is readily seen to be that shown in Fig. 15; hence

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int_{C_F} d^4k \, \frac{e^{ik\cdot x}}{k^2 + m^2}. \tag{123}$$

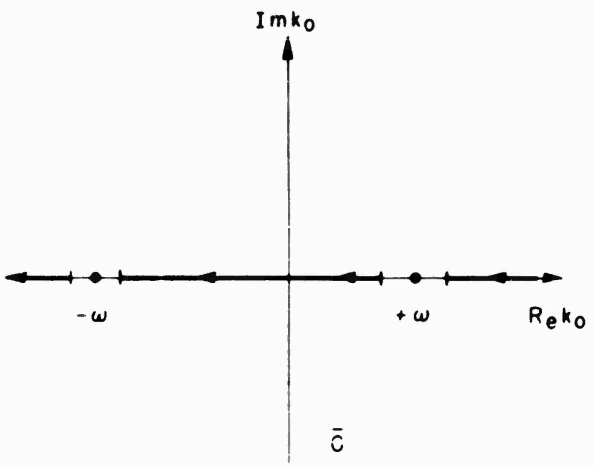

Figure 14

By actually performing these contour integrations, and then utilizing Dirac delta functions, we can get other useful representations. For example, we take the function $\Delta_R(x)$:

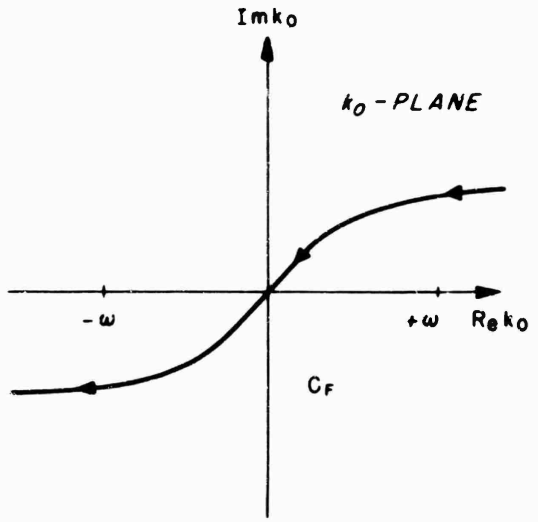

Figure 15

$$\Delta_R(x) = -\frac{1}{(2\pi)^4}\int d^3k \left(\int_{-\infty}^{-\omega-\epsilon} dk_0 + \int_{-\omega-\epsilon}^{-\omega+\epsilon} dk_0 \right.$$

$$\left. + \int_{-\omega+\epsilon}^{+\omega-\epsilon} dk_0 + \int_{\omega-\epsilon}^{\omega+\epsilon} dk_0 + \int_{\omega+\epsilon}^{\infty} dk_0 \frac{e^{ik\cdot x}}{\omega^2 - k_0^2} \right)$$

$$= -\frac{1}{(2\pi)^4} P \int_{-\infty}^{+\infty} d^4k \frac{e^{ik\cdot x}}{k^2 + m^2} - \frac{1}{(2\pi)^4}\int d^3k$$

$$\left(\int_{-\omega-\epsilon}^{-\omega+\epsilon} dk_0 \frac{e^{ik\cdot x}}{\omega^2 - k_0^2} + \int_{\omega-\epsilon}^{\omega+\epsilon} dk_0 \frac{e^{ik\cdot x}}{\omega^2 - k_0^2} \right)$$

where P stands for principal part (which can also be designated by applying P to the integrand instead of the integral, as Eq. (124) below), and where the last two integrals are clockwise loop integrals above the singularities. It is straightforward to show

$$-\frac{1}{(2\pi)^4}\int d^3k \int_{-\omega-\epsilon}^{-\omega+\epsilon} dk_0 \frac{e^{ik\cdot x}}{\omega^2 - k_0^2} = \frac{\pi i}{(2\pi)^4} \times$$

$$\int d^3k \frac{e^{ik\cdot x + i\omega x_0}}{2\omega} = \frac{\pi i}{(2\pi)^4}\int_{-\infty}^{+\infty} d^4k \frac{e^{ik\cdot x}\delta(k_0 + \omega)}{2\omega}$$

and

$$-\frac{1}{(2\pi)^4}\int d^3k \int_{\omega-\epsilon}^{\omega+\epsilon} dk_0 \frac{e^{ik\cdot x}}{\omega^2 - k_0^2} = -\frac{\pi i}{(2\pi)^4} \times$$

$$\int d^4k \frac{e^{ik\cdot x}\delta(k_0 - \omega)}{2\omega}$$

or

$$-\frac{1}{(2\pi)^4}\int d^3k \left\{ \int_{-\omega-\epsilon}^{-\omega+\epsilon} \cdots + \int_{\omega-\epsilon}^{\omega+\epsilon} \cdots \right\}$$

$$= \frac{\pi i}{(2\pi)^4}\int_{-\infty}^{+\infty} d^4k\, e^{ik\cdot x} \frac{\delta(k_0 + \omega) - \delta(k_0 - \omega)}{2\omega}$$

$$= -\frac{\pi i}{(2\pi)^4}\int_{-\infty}^{+\infty} d^4k\, \epsilon(k)\delta(k^2 + m^2)\, e^{ik\cdot x}.$$

Thus

$$\Delta_R(x) = \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} d^4k\, e^{ik\cdot x} \times$$

$$\left[P\frac{-1}{k^2 + m^2} - \pi i\, \epsilon(k)\delta(k^2 + m^2) \right]. \quad (124)$$

Since $\Delta_A(x) = \Delta_R(-x)$, in (124) change x to $-x$ and then k to $-k$; it follows immediately that

$$\Delta_A(x) = \frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty} d^4k\, e^{ik\cdot x}$$

$$\left[P\frac{-1}{k^2 + m^2} + \pi i\, \epsilon(k)\delta(k^2 + m^2) \right]. \quad (125)$$

From the definition of $\bar{\Delta}(x)$, its integral representation follows from (124) and (125) directly:

$$\bar{\Delta}(x) = \frac{1}{(2\pi)^4}\int d^4k\, e^{ik\cdot x} P\frac{-1}{k^2 + m^2} \quad (126)$$

while from (114), (126), and (54), we get

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int d^4k\, e^{ik\cdot x}\left[P\,\frac{-1}{k^2+m^2} - \pi i\,\delta(k^2+m^2)\right]. \tag{127}$$

Another form of these integral representations may be constructed by using the positive frequency and negative frequency parts of the Dirac delta function. With

$$\delta(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda\, e^{-i\alpha\lambda}$$

we have

$$\delta^{(+)}(\alpha) = \frac{1}{2}\int_0^{\infty} d\lambda\, e^{-i\alpha\lambda} = \frac{1}{2}\,\delta(\alpha) + \frac{1}{2\pi i}\,P\,\frac{1}{\alpha}$$

$$\delta^{(-)}(\alpha) = \frac{1}{2\pi}\int_0^{\infty} d\lambda\, e^{i\alpha\lambda} = \frac{1}{2}\,\delta(\alpha) - \frac{1}{2\pi i}\,P\,\frac{1}{\alpha}$$

and

$$\left.\begin{aligned}
&\Delta_R(x) = \\
&\frac{-i}{(2\pi)^3}\int d^4k\,\epsilon(k)\,\delta^{(\epsilon(k))}(k^2+m^2)\,e^{ik\cdot x} \\
&\Delta_A(x) = \\
&\frac{i}{(2\pi)^3}\int d^4k\,\epsilon(-k)\,\delta^{(\epsilon(-k))}(k^2+m^2)\,e^{ik\cdot x} \\
&\bar{\Delta}(x) = \\
&\frac{-i}{(2\pi)^3}\int d^4k\,\frac{\delta^{(+)}(k^2+m^2)-\delta^{(-)}(k^2+m^2)}{2}\,e^{ik\cdot x} \\
&\Delta_F(x) = \\
&\frac{-i}{(2\pi)^3}\int d^4k\,\delta^{(+)}(k^2+m^2)\,e^{ik\cdot x}
\end{aligned}\right\} \tag{128}$$

Explicit Representations of the Inhomogenous Invariant Functions; Relations

It is quite evident that there are several linear relations among the various homogeneous and inhomogeneous functions we have constructed. We list these without derivation, for they are easy to prove beginning with some that have already been established or defined:

$$\Delta_R(x) = \theta(x)\,\Delta(x) \tag{129a}$$

$$\Delta_A(x) = -\theta(-x)\,\Delta(x) \tag{129b}$$

$$\bar{\Delta}(x) = 1/2[\,\Delta_R(x) + \Delta_A(x)\,] \tag{129c}$$

$$\Delta_F(x) = \theta(x)\,\Delta^{(+)}(x) - \theta(-x)\,\Delta^{(-)}(x) \tag{129d}$$

These lead to

$$\Delta_A(x) = \Delta_R(-x) \tag{130a}$$

$$\bar{\Delta}(x) = 1/2\,\epsilon(x)\,\Delta(x) \tag{130b}$$

$$\Delta_F(x) = \bar{\Delta}(x) - \frac{i}{2}\,\Delta^{(1)}(x) \tag{130c}$$

$$\Delta_R(x) - \Delta_A(x) = \Delta(x). \tag{130d}$$

We have already obtained an explicit representation for $\Delta(x)$, *i.e.*, Eq. (78); from this and (129a) and (129b) we get, inside the light-cone,

$$\Delta_R(x) = -\frac{\epsilon(x)\theta(x)}{2\pi}\left[\,\delta(x^2) - \frac{m^2\bar{\theta}(-x^2)}{2}\,\frac{J_1(m\sqrt{-x^2})}{m\sqrt{-x^2}}\,\right] \tag{131}$$

$$\Delta_A(x) = \frac{\epsilon(x)\theta(-x)}{2\pi}\left[\,\delta(x^2) - \frac{m^2\bar{\theta}(-x^2)J_1(m\sqrt{-x^2})}{2m\sqrt{-x^2}}\,\right] \tag{132}$$

and $\Delta_R = \Delta_A = 0$ outside the light-cone.

From Eq. (130b) and Eq. (78), for $x_0 \neq 0$,

$$\bar{\Delta}(x) = \begin{cases} -\dfrac{\delta(x^2)}{4\pi} + \dfrac{m^2\bar{\theta}(-x^2)}{8\pi} \dfrac{J_1(m\sqrt{-x^2})}{(m\sqrt{-x^2})}, & x^2 < 0 \\ 0, & x^2 > 0. \end{cases} \tag{133}$$

By continuation, we define $\bar{\Delta}(x)$ by (133) everywhere, and we find that $\bar{\Delta}(x)$ thus defined obeys all the requirements placed on it. Finally Δ_F may be represented explicitly by (130c), (133), and (80). We shall not do so here.

THE INHOMOGENEOUS KLEIN-GORDON EQUATION; INTEGRAL RELATIONSHIPS OF THE Δ-FUNCTIONS

Consider the equation

$$(\Box - m^2)\,\varphi(x) = \rho(x). \tag{134}$$

If $\rho(x) \equiv 0$, this equation reduces to (1), the homogeneous Klein-Gordon equation; if $\rho(x) \not\equiv 0$, it is called, as we have already noted, the inhomogeneous Klein-Gordon equation. One must frequently solve (134) in both classical and quantum-field theory, subject to certain boundary conditions.

Let Ω be a space-time region bounded by two space-like surfaces S_1 and S_2, where S_1 is later than S_2, in such a way that Ω is finite in volume and all pertinent integrals are also finite. Let $\hat{\Delta}(x)$ be any of the inhomogeneous invariant functions; then applying Green's identity to $\varphi(x)$ and $\psi = \hat{\Delta}$, we get, when $x \in \Omega$,

$$\varphi(x) = -\int_{S_1} d\sigma'_\mu \left[\varphi(x') \frac{\partial \hat{\Delta}(x-x')}{\partial x'_\mu} - \hat{\Delta}(x'-x) \frac{\partial \varphi(x')}{\partial x'_\mu} \right] +$$

(continued next column)

$$\int_{S_2} d\sigma'_\mu \left[\varphi(x') \frac{\partial \Delta(x'-x)}{\partial x'_\mu} - \Delta(x'-x) \frac{\partial \varphi(x')}{\partial x'_\mu} \right] + \int_\Omega d^4x'\, \Delta(x'-x)\, \rho(x'). \tag{135}$$

Now we cannot specify $\varphi(x)$ and $\partial\varphi/\partial x_\mu$ on two separate surfaces, for we then impose, in general, too many restrictions on the problem. We consider, thus, the boundary value problems associated with Δ_R and Δ_A. If our values of φ, and its derivatives are specified on some surface prior to x, then

$$\varphi(x) = -\int d^4x'\, \Delta_R(x-x')\, \rho(x') + \int_{S_2} d\sigma'_\mu \left[\frac{\partial \varphi(x')}{\partial x'_\mu} \Delta_R(x-x') - \frac{\partial \Delta_R(x-x')}{\partial x'_\mu} \varphi(x') \right] \tag{136}$$

while if the surface is S_1, then

$$\varphi(x) = -\int d^4x'\, \Delta_A(x-x')\rho(x') - \int_{S_1} d\sigma'_\mu \left[\frac{\partial \varphi}{\partial x'_\mu} \Delta_A(x-x') - \frac{\partial \Delta_A(x-x')}{\partial x'_\mu} \varphi(x') \right]. \tag{137}$$

If we specialize (134) by taking $\rho = 0$ and φ to be any of the homogeneous invariant functions, which we indicate by $\tilde{\Delta}$, then

$$\tilde{\Delta}(x) = \int_{S_2} d\sigma'_\mu \left[\Delta_R(x-x') \frac{\partial \tilde{\Delta}(x')}{\partial x'_\mu} - \tilde{\Delta}(x') \frac{\partial \Delta_R(x-x')}{\partial x'_\mu} \right] \tag{138}$$

$$= -\int_S d\sigma_{\mu'}\left[\Delta_A(x-x')\,\frac{\partial\widetilde{\Delta}(x')}{\partial x'_\mu} - \widetilde{\Delta}(x')\,\frac{\partial\Delta_A(x-x')}{\partial x'_\mu}\right] \tag{139}$$

where

$$\widetilde{\Delta} = \Delta,\ \Delta^{(+)},\ \Delta^{(-)},\ \text{or}\ \Delta^{(1)}.$$

SUMMARY OF IMPORTANT FORMULAS

In this section, we bring together for easy reference all the pertinent formulas derived in the body of the text; we make no explanation of the symbols, since they should all be evident by now.

The Δ-Function

$$\varphi(x) = \int_S d\sigma'_\mu\left[\Delta(x-x')\,\frac{\partial\varphi(x')}{\partial x'_\mu} - \varphi(x')\,\frac{\partial\Delta(x-x')}{\partial x'_\mu}\right] \tag{140}$$

$$\left.\begin{aligned}&(\Box - m^2)\,\Delta(x) = 0\\ &\Delta(x) = 0,\ x^2 > 0\\ &\frac{\partial\Delta(x)}{\partial x_0}\bigg|_{x_0=0} = -\,\delta(\mathbf{r})\end{aligned}\right\} \tag{141}$$

$$\Delta(x) = -\frac{1}{(2\pi)^3}\int d^3k\, e^{i\mathbf{k}\cdot\mathbf{r}}\,\frac{\sin\omega x_0}{\omega}$$

$$= -\frac{i}{(2\pi)^3}\int d^4k\, e^{ik\cdot x}\,\epsilon(k)\delta(k^2+m^2) \tag{142}$$

$$= \frac{1}{(2\pi)^4}\int_C d^4k\,\frac{e^{ik\cdot x}}{k^2+m^2}$$

and

$$\Delta(x) = \begin{cases} 0, & x^2 > 0\\ -\dfrac{\epsilon(x)}{2\pi}\left[\delta(x^2) - \dfrac{m^2\bar{\theta}(-x^2)}{2}\times\dfrac{J_1(m\sqrt{-x^2})}{m\sqrt{-x^2}}\right], & x^2 < 0.\end{cases} \tag{143}$$

The $\Delta^{(+)}$ and $\Delta^{(-)}$ Functions

$$\int_S d\sigma'_\mu\left[\Delta^{(\pm)}(x-x')\,\frac{\partial\varphi}{\partial x'_\mu} - \varphi(x')\,\frac{\partial\Delta^{(\pm)}(x-x')}{\partial x'_\mu}\right] = \varphi^{(\pm)}(x) \tag{144}$$

$$\left.\begin{aligned}&(\Box - m^2)\,\Delta^{(\pm)}(x) = 0\\ &\Delta^{(+)}(x)^* = \Delta^{(-)}(x)\\ &\Delta^{(\pm)}(x) = 0,\ x^2 > 0\end{aligned}\right\} \tag{145}$$

$$\Delta^{(\pm)}(x) = \mp\frac{i}{(2\pi)^3}\int d^3k\,\frac{e^{i(\mathbf{k}\cdot\mathbf{r}\mp\omega x_0)}}{2\omega}$$

$$= \mp\frac{i}{(2\pi)^3}\int d^4k\, e^{ik\cdot x}\,\theta(\pm k)\,\delta(k^2+m^2)$$

$$= \frac{1}{(2\pi)^4}\int_{C_\pm} d^4k\,\frac{e^{ik\cdot x}}{k^2+m^2} \tag{146}$$

$$\Delta^{(+)}(x) = \begin{cases} -\dfrac{\epsilon(x)\,\delta(x^2)}{2\pi} - i\,\dfrac{m^2}{2\pi^2}\,\dfrac{K_1(m\sqrt{x^2})}{m\sqrt{x^2}}, & x^2 > 0\\ -\dfrac{\epsilon(x)\,\delta(x^2)}{2\pi} + \dfrac{m^2}{4\pi}\,\dfrac{H_1^{(2)}(m\sqrt{-x^2})}{m\sqrt{-x^2}}, & x^2 < 0,\ x_0 > 0\\ -\dfrac{\epsilon(x)\,\delta(x^2)}{2\pi} - \dfrac{m^2}{4\pi}\,\dfrac{H_1^{(1)}(m\sqrt{-x^2})}{m\sqrt{-x^2}}, & x^2 < 0,\ x_0 < 0.\end{cases} \tag{147}$$

The $\Delta^{(1)}$-Function

$$\int_S d\sigma'_\mu \left[\Delta^{(1)}(x-x') \frac{\partial \varphi^{(\pm)}}{\partial x'_\mu} - \varphi^{(\pm)}(x') \frac{\partial \Delta^{(1)}}{\partial x'_\mu}(x-x') \right] = \pm i \varphi^{(\pm)}(x) \tag{148}$$

$$(\Box - m^2)\,\Delta^{(1)}(x) = 0 \tag{149}$$

$$\Delta^{(1)}(x) = \frac{1}{(2\pi)^3} \int d^3k\, e^{ik\cdot x} \frac{\cos \omega x_0}{\omega} = \frac{1}{(2\pi)^3} \int d^4k\, \delta(k^2 + m^2)\, e^{ik\cdot x} \tag{150}$$

$$= -\frac{i}{(2\pi)^4} \int_{C^{(1)}} d^4k \frac{e^{ik\cdot x}}{k^2 + m^2}.$$

The Δ_R-Function

$$\int_S d\sigma'_\mu \left[\Delta_R(x-x') \frac{\partial \varphi}{\partial x'_\mu} - \varphi(x') \frac{\partial \Delta_R(x-x')}{\partial x'_\mu} \right] = \begin{cases} \varphi(x), & x \text{ later than } S \\ 0, & x \text{ earlier than } S \end{cases} \tag{151}$$

$$(\Box - m^2)\Delta_R(x) = \delta(x) \tag{152}$$

$$\Delta_R(x) = -\frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} d^4k\, e^{ik\cdot x} \left[P \frac{1}{k^2 + m^2} + \pi i \epsilon(k) \delta(k^2 + m^2) \right]$$

$$= -\frac{i}{(2\pi)^3} \int_{-\infty}^{+\infty} d^4k\, e^{ik\cdot x}\, \epsilon(k)\, \delta^{(\epsilon(k))}(k^2 + m^2)$$

$$= \frac{1}{(2\pi^4)} \int_{C_R} d^4k \frac{e^{ik\cdot x}}{k^2 + m^2} \tag{153}$$

$$\Delta_R(x) = -\frac{\epsilon(x)\theta(x)}{2\pi} \left[\delta(x^2) - \frac{m^2 \bar{\theta}(-x^2) J_1(m\sqrt{-x^2})}{2m\sqrt{-x^2}} \right]$$

$$= \theta(x)\Delta(x). \tag{154}$$

The Δ_A-Function

$$\int_S d\sigma'_\mu \left\{ \Delta_A(x-x') \frac{\partial \varphi}{\partial x'_\mu} - \varphi(x') \frac{\partial \Delta_A(x-x')}{\partial x'_\mu} \right\} = \begin{cases} 0, & x \text{ later than } S \\ -\varphi(x), & x \text{ earlier than } S \end{cases} \tag{155}$$

$$\left.\begin{aligned} (\Box - m^2)\Delta_A(x) &= \delta(x) \\ \Delta_A(x) &= -\theta(-x)\Delta(x) \\ &= \Delta_R(-x). \end{aligned}\right\} \tag{156}$$

$$\Delta_A(x) = -\frac{1}{(2\pi)^4} \int d^4k\, e^{ik\cdot x} \left[P \frac{1}{k^2 + m^2} - \pi i\, \epsilon(k)\, \delta(k^2 + m^2) \right]$$

$$= \frac{i}{(2\pi)^3} \int d^4k\, e^{ik\cdot x}\, \epsilon(-k)\, \delta^{(\epsilon(-k))}(k^2 + m^2)$$

$$= \frac{1}{(2\pi)^4} \int_{C_A} d^4k \frac{e^{ik\cdot x}}{k^2 + m^2} \tag{157}$$

$$\Delta_A(x) = \frac{\epsilon(x)\theta(-x)}{2\pi} \left[\delta(x^2) - \frac{m^2 \bar{\theta}(-x^2) J_1(m\sqrt{-x^2})}{2m\sqrt{-x^2}} \right]. \tag{158}$$

The $\bar{\Delta}$-Function

$$\int_S d\sigma'_\mu \left[\bar{\Delta}(x-x')\frac{\partial\varphi}{\partial x'_\mu} - \varphi(x')\frac{\partial\Delta(x-x')}{\partial x'_\mu} \right]$$

$$= \begin{cases} +\varphi(x), & x \text{ later than } S \\ -\varphi(x), & x \text{ earlier than } S \end{cases} \tag{159}$$

$$\left.\begin{aligned} &(\Box - m^2)\,\bar{\Delta}(x) = \delta(x) \\ &\bar{\Delta}(x) = \epsilon(x)\Delta(x)/2 \\ &\quad = [\,\Delta_R(x) + \Delta_A(x)\,]/2 \end{aligned}\right\} \tag{160}$$

$$\bar{\Delta}(x) = -\frac{1}{(2\pi)^4}\int d^4k\, e^{ik\cdot x}\, P\,\frac{1}{k^2+m^2}$$

$$= -\frac{i}{(2\pi)^3}\int d^4k\,\frac{\delta^{(+)}(k^2+m^2) - \delta^{(-)}(k^2+m^2)}{2}\, e^{ik\cdot x}$$

$$= \frac{1}{(2\pi)^4}\int_{\bar{C}} d^4k\,\frac{e^{ik\cdot x}}{k^2+m^2} \tag{161}$$

$$\bar{\Delta}(x) = \begin{cases} 0, & x^2 > 0 \\ -\dfrac{\delta(x^2)}{4\pi} + \dfrac{m^2\bar{\theta}(-x^2)}{8\pi}\,\dfrac{J_1(m\sqrt{-x^2})}{(m\sqrt{-x^2})}, & x^2 < 0. \end{cases} \tag{162}$$

The Δ_F-Function

$$\int_S d\sigma'_\mu \left[\Delta_F(x-x')\frac{\partial\varphi}{\partial x'_\mu} - \varphi(x')\frac{\partial\Delta_F(x-x')}{\partial x'_\mu} \right\}$$

$$= \begin{cases} \varphi^{(+)}(x), & x \text{ later than } S \\ \varphi^{(-)}(x), & x \text{ earlier than } S \end{cases} \tag{163}$$

$$\left.\begin{aligned} &(\Box - m^2)\Delta_F(x) = \delta(x) \\ &\Delta_F(x) = \theta(x)\Delta^{(+)}(x) - \theta(-x)\Delta^{(-)}(x) \\ &\quad = \bar{\Delta}(x) - \frac{i}{2}\Delta^{(1)}(x) \end{aligned}\right\} \tag{164}$$

$$\Delta_F(x) = -\frac{1}{(2\pi)^4}\int d^4k\, e^{ik\cdot x}\Big[\, P\,\frac{1}{k^2+m^2}$$

$$+ \pi i\,\delta(k^2+m^2)\,\Big]$$

$$= -\frac{i}{(2\pi)^3}\int d^4k\, e^{ik\cdot x}\mathfrak{S}^{(+)}(k^2+m^2)$$

$$= \frac{1}{(2\pi)^4}\int_{C_F} d^4k\,\frac{e^{ik\cdot x}}{k^2+m^2}. \tag{165}$$

Chapter 3
The Wave Equation and the Electromagnetic Field

THE WAVE EQUATION AND THE INVARIANT D-FUNCTIONS

By setting $m = 0$ in the Klein-Gordon equation we obtain the wave equation

$$\Box\varphi = 0. \tag{1}$$

The boundary value problems for (1) do not differ from those of the Klein-Gordon equation and will therefore not be discussed here; to each boundary value problem one may construct the appropriate Green's function by taking the appropriate invariant functions of the last chapter in the limit of vanishing mass. Designating the resultant function by D instead of Δ, one readily derives the following:

$$D(x) = -\frac{\epsilon(x)\delta(x^2)}{2\pi}, \quad \frac{\partial D}{\partial x_0}\bigg|_{x_0=0} = -\delta(\mathbf{r}).$$

$$D(x) = -\frac{1}{(2\pi)^3}\int d^3k\, e^{i\mathbf{k}\cdot\mathbf{r}}\,\frac{\sin\omega x_0}{\omega}$$

$$= -\frac{i}{(2\pi)^3}\int d^4k\,\epsilon(k_0)\delta(k^2)e^{ik\cdot x}$$

$$= \frac{1}{(2\pi)^4}\int_C d^4k\,\frac{e^{ik\cdot x}}{k^2}$$

$$= -\frac{1}{4\pi r}\{\delta(r-x_0)-\delta(r+x_0)\} \tag{2}$$

where $r = \sqrt{x_1^2+x_2^2+x_3^2}$ and $\omega = \sqrt{|\mathbf{k}|^2}$;

$$D^{(1)}(x) = \frac{1}{2\pi^2 x^2}$$

$$= \frac{1}{(2\pi)^3}\int d^4k\, e^{ik\cdot x}\delta(k^2)$$

$$= \frac{1}{(2\pi)^3}\int d^3k\, e^{i\mathbf{k}\cdot\mathbf{r}}\,\frac{\cos\omega x_0}{\omega},\ \omega = \sqrt{\mathbf{k}^2}$$

$$= -\frac{i}{(2\pi)^3}\int_{C^\omega} d^4k\,\frac{e^{ik\cdot x}}{k^2} \tag{3}$$

$$D^{(\pm)}(x) = -\frac{1}{4\pi}\left\{\epsilon(x)\delta(x^2) \mp \frac{1}{i\pi x^2}\right\}$$

$$= \mp\frac{i}{(2\pi)^3}\int d^4k\,\theta(\pm k)\,\delta(k^2)\,e^{ik\cdot x}$$

$$= \mp\frac{i}{(2\pi)^3}\int d^3k\,\frac{e^{i(\mathbf{k}\cdot\mathbf{r}\mp\omega x_0)}}{2\omega} -$$

$$= \frac{1}{(2\pi)^4}\int_{C^\pm} d^4k\,\frac{e^{ik\cdot x}}{k^2} \tag{4}$$

$$\bar{D}(x) = \frac{1}{2}\epsilon(x)D(x)$$

$$= -\frac{\delta(x^2)}{4\pi}$$

$$= -\frac{1}{(2\pi)^4}\int d^4k\, e^{ik\cdot x}\,P\,\frac{1}{k^2}$$

$$= -\frac{i}{(2\pi)^3}\int d^4k\,\frac{\delta^{(+)}(k^2)-\delta^{(-)}(k^2)}{2}\,e^{ik\cdot x}$$

$$= \frac{1}{(2\pi)^4}\int_{\bar{C}} d^4k\,\frac{e^{ik\cdot x}}{k^2} \tag{5}$$

$$D_R(x) = \theta(x)D(x)$$

$$= -\bar{\theta}(x_0)\,\frac{\delta(r-x_0)}{4\pi r}$$

$$= -\frac{1}{(2\pi)^4}\int d^4k\, e^{ik\cdot x}\left[P\,\frac{1}{k^2}+\pi i\epsilon(k)\delta(k^2)\right]$$

$$= -\frac{i}{(2\pi)^3}\int d^4k\, e^{ik\cdot x}\,\epsilon(k)\,\delta^{(\epsilon(k))}(k^2)$$

$$= \frac{1}{(2\pi)^4}\int d^4k\,\frac{e^{ik\cdot x}}{k^2} \tag{6}$$

$$
\begin{aligned}
D_A(x) &= -\theta(-x)D(x) \\
&= D_R(-x) \\
&= -\bar{\theta}(-x_0)\frac{\delta(r+x_0)}{4\pi r} \\
&= -\frac{1}{(2\pi)^4}\int d^4k\, e^{ik\cdot x}\left[P\frac{1}{k^2} - \pi i\epsilon(k)\delta(k^2)\right] \\
&= \frac{i}{(2\pi)^3}\int d^4k\, e^{ik\cdot x}\,\epsilon(-k)\delta^{(+)}(k^2) \\
&= \frac{1}{(2\pi)^4}\int_{C_A} d^4k\,\frac{e^{ik\cdot x}}{k^2}\,. \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
D_F(x) &= -\frac{\delta(x^2)}{4\pi} - \frac{i}{4\pi^2x^2} \\
&= \bar{D}(x) - \frac{i}{2}D^{(1)}(x) \\
&= -\frac{1}{(2\pi)^4}\int d^4k\, e^{ik\cdot x}\left[P\frac{1}{k^2} + \pi i\delta(k^2)\right] \\
&= -\frac{i}{(2\pi)^3}\int d^4k\, e^{ik\cdot x}\,\delta^{(+)}(k^2) \\
&= \frac{1}{(2\pi)^4}\int_{C_F} d^4k\,\frac{e^{ik\cdot x}}{k^2}. \qquad (8)
\end{aligned}
$$

If we compare these expressions with those of Källen*, we find that our homogeneous functions are identical with his, but the inhomogeneous functions differ from his by a sign change for D_R, D_A, $\bar{D}$ and by a factor $2i$ for D_F. Correspondingly, his D_R function, for example, satisfies $\Box D_R = -\delta(x)$, while $\Box D_F = 2i\delta(x)$, in Källen's notation.

APPLICATION TO THE ELECTROMAGNETIC FIELD

The free electromagnetic field is described by the six quantities $\mathbf{E}$, $\mathbf{H}$ which satisfy the partial differential equations

$$
\left.
\begin{aligned}
\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{H}}{\partial t} \\
\nabla\cdot\mathbf{H} &= 0 \\
\nabla\times\mathbf{H} &= \frac{\partial\mathbf{E}}{\partial t} \\
\nabla\cdot\mathbf{E} &= 0
\end{aligned}
\right\} \qquad (9)
$$

where Heaviside units have been used. It is well known from the theory of relativity that these equations may be combined into two equations:

$$
\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = 0 \qquad (10a)
$$

$$
\frac{\partial F_{\mu\nu}}{\partial x_\mu} = 0 \text{ (sum over } \mu) \qquad (10b)
$$

where the $F_{\mu\lambda}$ are the components of an antisymmetric tensor under Lorentz transformations and the $F_{\mu\lambda}$ are given, in the 1,2,3,4 notation, by

$$
(F_{\mu\lambda}) = \begin{pmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}
$$

Equation (10a) is solved identically by putting

$$
F_{\mu\lambda} = \frac{\partial A_\lambda}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\lambda} \qquad (11)
$$

while Eq. (10b) becomes

$$
\frac{\partial^2}{\partial x_\mu \partial x_\mu}A_\lambda - \frac{\partial}{\partial x_\lambda}\frac{\partial A_\mu}{\partial x_\mu} = 0. \qquad (12)
$$

We then say that the A_λ are determined up to a gauge function, and changing from A_λ to $A'_\lambda = A_\lambda + \partial_\lambda\chi$ is called a gauge transformation. We observe that (12) differs from the wave equation by the presence of the term $\partial_\lambda\partial_\mu A_\mu$; we shall eliminate this term by the choice of gauge. Suppose $\partial_\mu A_\mu$ does not vanish identically; then define A'_μ such that

$$
A'_\mu = A_\mu + \partial_\mu\chi \qquad (13)
$$

*G. Källen, "Handbuch der Physik," Vol. V, part 1, Springer Verlag, Berlin, 1958

where χ is to be chosen such that

$$\Box\chi = -\partial_\mu A_\mu. \tag{14}$$

Then computing $\partial_\mu A'_\mu$, we see it vanishes. Equation (12) becomes

$$\Box A_\mu - \partial_\mu \partial_\lambda A_\lambda = \Box(A'_\mu - \partial_\mu\chi) - \partial_\mu\partial_\lambda A'_\lambda + \partial_\mu\Box\chi$$

$$= \Box A'_\mu = 0.$$

Thus, with this choice of gauge, the equations for the four-vector $\{A_\mu\}$ read

$$\Box A_\mu = 0,\ \mu = 0,1,2,3. \tag{15}$$

A_μ is still undetermined up to a χ such that $\Box\chi$ and the A_μ must satisfy the Lorentz condition,

$$\frac{\partial A_\mu}{\partial x_\mu} = 0. \tag{16}$$

Equation (16) is a subsidiary condition that guarantees that the four-vector that solves (15) is also capable, through (11), of describing the electromagnetic field. This may be put another way: from the set of all solutions of the wave equation, construct the set of all ordered quadruplets of functions. From this set, select that subset of quadruplets of functions that transform, under Lorentz transformations, like four-vectors; from this subset, construct that subset of four-vectors which satisfies (10); this subset transforms into a like subset under Lorentz transformations because (16) is Lorentz invariant. Hence this selection procedure is covariant. This last subset is the subset of all four-vectors that describe the electromagnetic field through Eq. (11).

We now turn to the boundary value problem for the electromagnetic field. There would be no problem in applying the techniques of Chapter 2 and the functions D of the first section of this chapter directly to each component A_μ if it were not for the Lorentz condition. The Lorentz condition implies that the four-components are coupled. Thus we may expect that $A_\mu(x)$ must be expressed not only in terms of A_μ and $\partial_\lambda A_\mu$ on some surface S, but also in terms of the values of the other three components and their derivatives on S. However, we shall see that they may indeed be handled as if they were independent of one another. Consider the expression

$$\int_S d\sigma'_\mu \left[D(x-x')\frac{\partial A_\lambda(x')}{\partial x'_\mu} - A_\lambda(x')\frac{\partial D(x-x')}{\partial x'_\mu} \right]$$

$$= \Phi_\lambda(x). \tag{17}$$

The function $\Phi_\lambda(x)$ transforms under Lorentz transformations like a four-vector; furthermore, if $x \in S$, then

$$\Phi_\lambda(x) = A_\lambda(x) \tag{18}$$

which follows directly from the fact that

$$\left.\frac{\partial D}{\partial x_0}\right|_{x_0=0} = -\delta(\mathbf{r}). \tag{19}$$

The question now is whether or not $\Phi_\lambda(x)$ may be regarded as an extension of $A_\lambda(x)$ off the surface. To be so, it must solve the wave equation and satisfy the Lorentz condition. It is clear that $\Phi_\lambda(x)$ solves the wave equation, for $D(x-x')$ does so. Next we must show that $\partial_\lambda\Phi_\lambda(x)$ vanishes:

$$\frac{\partial\Phi_\lambda}{\partial x_\lambda} = \int_S d\sigma'_\mu \left[\frac{\partial D(x\mp x')}{\partial x_\lambda}\frac{\partial A_\lambda(x')}{\partial x'_\mu} - A_\lambda(x')\frac{\partial^2 D(\lambda - x')}{\partial x_\lambda \partial x'_\mu} \right]$$

$$= \int_S d\sigma'_\mu \left[D(x-x')\frac{\partial}{\partial x'_\mu}\frac{\partial A_\lambda(x')}{\partial x'_\lambda} - \frac{\partial A_\lambda}{\partial x'_\lambda}\frac{\partial D(x-x')}{\partial x'_\mu} \right]$$

$$- \int_S d\sigma'_\mu \frac{\partial}{\partial x'_\lambda}\left[D(x-x')\frac{\partial A_\lambda}{\partial x'_\mu} - A_\lambda(x')\frac{\partial D(x-x')}{\partial x'_\mu} \right].$$

The first surface integral vanishes because A_λ meets the Lorentz condition on S by hypothesis. The second vanishes also, but the arguments are

much different. We have seen that Φ_λ is independent of S; choose S to be the plane $x_0' =$ constant $\neq x_0$. Then

$$\partial_\lambda \Phi_\lambda = \int d^3x' \frac{\partial}{\partial x_\lambda'} \left[D(x-x') \frac{\partial A_\lambda}{\partial x_0'} - A_\lambda(x') \frac{\partial D(x-x')}{\partial x_0'} \right]$$

$$= \sum_{i=1}^{3} \int d^3x' \frac{\partial}{\partial x_i'} \left[D(x-x') \frac{\partial A_i}{\partial x_0'} - A_i(x') \frac{\partial D(x-x')}{\partial x_0'} \right]$$

$$- \int d^3x' \frac{\partial}{\partial x_0'} \left[D(x-x') \frac{\partial A_0}{\partial x_0'} - A_i(x') \frac{\partial D(x-x')}{\partial x_0'} \right].$$

The integrals of the form $\int d^3x' \partial[\ldots]/\partial x_i'$ may be integrated over x_i' from $x_i' = -\infty$ to $x_i' = +\infty$ directly and these vanish by boundary values on $D(x-x')$. The integrand of the last integral reduces to

$$D(x-x') \frac{\partial^2 A_0}{\partial x_0'^2} - A_0 \frac{\partial^2 D(x-x')}{\partial x_0'^2}$$

which becomes, in lieu of the fact that both A_0 and D solve the wave equation,

$$D(x-x')\, \partial_i' \partial_i' A_0(x') - A_0(x') \partial_i' \partial_i' D(x-x')$$

$$= \partial_i' \left[D(x-x') \partial' A_0(x') - A_0(x') \partial_i' D(x-x') \right]$$

and the integral of this vanishes by the same argument as above. Thus $\partial_\lambda \Phi_\lambda(x) = 0$.

Thus, $\Phi_\lambda(x)$ solves the wave equation, satisfies the Lorentz condition, and reduces to $A_\lambda(x)$ on S; hence, $\Phi_\lambda(x)$ may be regarded as an extension of $A_\lambda(x)$ off S, so we write (17) as

$$\int_S d\sigma_\mu' \left[D(x-x') \frac{\partial A_\lambda}{\partial x_\mu'} - A_\lambda(x') \frac{\partial D(x-x')}{\partial x_\mu'} \right] = A_\lambda(x). \tag{20}$$

It is an easy matter to show that $A_\lambda(x)$ as given by (20) is a unique extension; to do this, let the values of A_μ and $\partial_\lambda A_\mu$ on S be given by the functions U_μ and $V_{\lambda\mu}$ respectively; then (20) reads

$$A_\lambda(x) = \int d\sigma_\mu' \left[D(x-x') V_{\lambda\mu}(x') - U_\mu(x) \partial_\lambda' D(x-x') \right]. \tag{21}$$

Suppose $A_\lambda'(x)$ were another four-vector that solved the wave equation, satisfied the Lorentz condition, and reduced to U_μ on S while its derivatives reduced to $V_{\lambda\mu}$ on S. We have seen that A_λ and A_λ' must then be related by a gauge transformation,

$$A_\lambda'(x) = A_\lambda(x) + \frac{\partial \chi(x)}{\partial x_\lambda} \tag{22}$$

where

$$\Box \chi(x) = 0. \tag{23}$$

Since $A_\lambda'(x) = A_\lambda(x) = U_\lambda(x)$ when $x \in S$, we have

$$\left. \frac{\partial \chi}{\partial x_\lambda} \right|_{x \in S} = 0 \tag{24}$$

and similarly,

$$\left. \frac{\partial^2 \chi}{\partial x_\mu \partial x_\lambda} \right|_{x \in S} = 0. \tag{25}$$

Also,

$$\Box \partial_\lambda \chi = 0. \tag{26}$$

From (26), $\partial_\lambda \chi$ is a function of space-time whose values at x may be expressed in terms of its boundary values, *i.e.*,

$$\partial_\lambda \chi(x) = \int d\sigma_\mu' \left[D(x-x') \partial_\mu' \partial_\lambda' \chi - \partial_\lambda' \chi \partial_\mu' D(x-x') \right]. \tag{27}$$

But in light of (24) and (25) the right side of (27) vanishes, and hence $A_\lambda'(x) = A_\lambda(x)$ for all x.

Thus, Eq. (20) may be regarded as the unique solution to the boundary value problem for the electromagnetic field.

Chapter 4
The Dirac Equation

This chapter will be devoted principally to a study of boundary value problems of the Dirac equation. We begin by a brief study of notation, followed by plane wave and general solutions. We then formulate a "Green's identity" for the Dirac equation, followed by a study of the various Green's functions.

NOTATION; PLANE WAVES; GENERAL SOLUTIONS

In our notation, the Dirac equation reads

$$(\gamma_\mu \partial_\mu + m)\,\psi(x) = 0 \tag{1}$$

where the summation convention is used; the quantities γ_μ, $\mu = 1,2,3,4$, are elements of a noncommutative algebra characterized by

$$\gamma_\mu\,\gamma_\nu + \gamma_\nu\,\gamma_\mu = 2\delta_{\mu\nu}\ \mu,\nu = 1,2,3,4. \tag{2}$$

We define γ_0 by

$$\gamma_4 \equiv i\gamma_0 \tag{3}$$

so that, we note, $\gamma_\mu\,\partial_\mu = \boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + \gamma_4\,\partial_4 = \boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + \gamma_0\,\partial_0$. We take as a representation of the γ's,

$$\left.\begin{aligned}
\gamma_1 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
\gamma_3 &= \begin{pmatrix} 0 & 0 & +i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & +i & 0 & 0 \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}\right\} \tag{4}$$

We define $\bar{\psi}(x)$ by

$$\bar{\psi}(x) \equiv \psi(x)^*\gamma_4 \tag{5}$$

where ψ^* is the conjugate transpose of ψ; then from (1), (2), and (5), one can show

$$\frac{\partial\bar{\psi}\gamma_\mu}{\partial x_\mu} - m\bar{\psi}(x) = 0. \tag{6}$$

The functions ψ that solve (1) are four component quantities called spinors which transform according to a particular representation of the Lorentz group†. Although we shall not discuss the transformation properties, we do note that if $\bar{\varphi}(x)$ solves (6), then

$$\frac{\partial}{\partial x_\mu}\left[\,\bar{\varphi}(x)\,\gamma_\mu\,\psi(x)\,\right] = 0 \tag{7}$$

by virtue of (1) and (6), so that the quantity $\bar{\varphi}(x)\,\gamma_\mu\,\psi(x)$ transforms like a four-vector.

If we seek plane wave solutions to (1), *i.e.*, solutions of the form

$$\psi(x) = u(k)\,e^{ik\cdot x} \tag{8}$$

we find that there are four linearly independent solutions for a given space-part of the momentum vector k; *i.e.*, for given $\mathbf{k}$, there are four solutions. Without going into any details (see the Dirac reference) we give the results; we label the four solutions by

$$\psi^{(r)}(x) = u^{(r)}(\mathbf{k},\omega)\,e^{ik\cdot x}$$

where $k = (\mathbf{k},k_0)$ is a function of $\mathbf{k}$ and $\omega = \sqrt{\mathbf{k}^2 + m^2}$ will be defined below:

$$\psi^{(1)}(x) = \sqrt{\frac{m+\omega}{2\omega}}\begin{pmatrix} 1 \\ 0 \\ \dfrac{-k_3}{m+\omega} \\ -\dfrac{k_1+ik_2}{m+\omega} \end{pmatrix} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega x_0)} \tag{9a}$$

$$\equiv u^{(1)}(\mathbf{k},\omega)\,e^{i(\mathbf{k}\cdot\mathbf{r} - \omega x_0)}$$

†See P. A. M. Dirac, "The Principles of Quantum Mechanics," 3rd ed., Oxford University Press, 1947, p. 257.

$$\psi^{(2)}(x) = \sqrt{\frac{m+\omega}{2\omega}} \begin{pmatrix} 0 \\ 1 \\ -\dfrac{k_1 - ik_2}{m+\omega} \\ \dfrac{k_3}{m+\omega} \end{pmatrix} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega x_0)}$$

$$\equiv u^{(2)}(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega x_0)} \tag{9b}$$

$$\psi^{(3)}(x) = \sqrt{\frac{m+\omega}{2\omega}} \begin{pmatrix} \dfrac{k_3}{m+\omega} \\ \dfrac{k_1 + ik_2}{m+\omega} \\ 1 \\ 0 \end{pmatrix} e^{i(\mathbf{k}\cdot\mathbf{r} + \omega x_0)}$$

$$\equiv u^{(3)}(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega x_0)} \tag{9c}$$

$$\psi^{(4)}(x) = \sqrt{\frac{m+\omega}{2\omega}} \begin{pmatrix} \dfrac{k_1 - ik_2}{m+\omega} \\ -\dfrac{k_3}{m-\omega} \\ 0 \\ 1 \end{pmatrix} e^{i(\mathbf{k}\cdot\mathbf{r} + \omega x_0)}$$

$$\equiv u^{(4)}(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r} + \omega x_0)} \tag{9d}$$

These functions are normalized so that

$$\sum_{\alpha=1}^{4} u_\alpha^{(r)*} u_\alpha^{(s)} = \delta^{(rs)}. \tag{10}$$

$\psi^{(1)}$ and $\psi^{(2)}$ correspond to solutions with positive energy (positive frequency while $\psi^{(3)}$ and $\psi^{(4)}$ correspond to solutions with negative energy, $k_0 = -\omega < 0$ (negative frequency).

Since $\psi^{(1)}$, $\psi^{(2)}$, $\psi^{(3)}$, and $\psi^{(4)}$ each solve the Dirac equation, any superposition of them will also solve the Dirac equation; thus, for a given momentum vector $\mathbf{k}$, we can write the most general solution of (1) as

$$\psi_{\mathbf{k}}(x) = \sum_{j=1}^{2} C_j u^{(j)}(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega x_0)} + \sum_{j>3}^{4} C_j u^{(j)}(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r} + \omega x_0)}. \tag{11}$$

From Fourier analysis, any square integrable function in $L_{2,4}$ may be written as a superposition of such functions as (11); thus, for the most general solution we have

$$\psi(x) = \frac{1}{(2\pi)^3} \int d^3k \left[\sum_{j=1}^{2} C_j(\mathbf{k}) u^{(j)}(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r} + \omega x_0)} + \sum_{j=3}^{4} C_j(\mathbf{k}) u^{(j)}(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega x_0)} \right]. \tag{12}$$

We shall write this expression in a more compact form; define

$$b_j(\mathbf{k},\omega) \equiv 4\pi\omega C_j(\mathbf{k}) u_j(\mathbf{k},\omega), \quad j = 1,2$$

$$b_{j-2}(\mathbf{k},-\omega) \equiv 4\pi\omega C_j(\mathbf{k}) u_j(\mathbf{k},\omega), j = 3,4.$$

Then after some algebra which is by now quite familiar, we get

$$\psi(x) = \frac{1}{(2\pi)^4} \sum_{j=1}^{2} \int d^4k\, \delta(k^2 + m^2)\, b^{(j)}(k) e^{ik\cdot x} \tag{13}$$

for a plane wave expansion of the general solution to the Dirac equation for a free electron.

A GREEN'S IDENTITY FOR THE HOMOGENEOUS DIRAC EQUATION

Let $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$, and $\varphi_4(x)$ be four distinct solutions of the Dirac equation, Eq. (1); each is a four-component column vector; thus, the tetrad $\varphi(x) = (\varphi_1,\varphi_2,\varphi_3,\varphi_4)$ is a four-by-four matrix and has the property that

$$\left.\begin{aligned} (\gamma_\mu \partial_\mu + m)\, \varphi(x) = 0 \\ \partial_\mu \bar{\varphi} \gamma_\mu - m\varphi(x) = 0 \end{aligned}\right\} \tag{14}$$

The quantity $\bar{\varphi}(x)\gamma_\mu\psi(x)$ is a four-component quantity, a column, each of whose components transforms under Lorentz transformations like the μth component of a four-vector.

In Gauss' theorem, we put $f_\mu = \bar{\varphi}\,\gamma_\mu\psi$, where $\bar{\varphi}$ is described above. Then

$$\int_\Omega d^4x\,\frac{\partial}{\partial x'_\mu}\,\bar{\varphi}(x')\gamma_\mu\psi(x') = \int_{S_1} d\sigma'_\mu\,\bar{\varphi}(x')\gamma_\mu\psi(x') - \int_{S_2} d\sigma'_\mu\,\bar{\varphi}(x')\gamma_\mu\psi(x') \tag{15}$$

represents four separate equations which hold simultaneously. Performing the indicated differentiation in the integrand in the left side, we get a type of Green's identity for the Dirac equation:

$$\int_\Omega d^4x'\left[\frac{\partial\bar{\varphi}(x')}{\partial x'_\mu}\,\gamma_\mu\psi(x') + \bar{\varphi}(x')\gamma_\mu\frac{\partial\psi(x')}{\partial x'_\mu}\right] = -\int_{S_1} d\sigma'_\mu\,\bar{\varphi}(x')\gamma_\mu\psi(x') + \int_{S_2} d\sigma'_\mu\,\bar{\varphi}(x')\,\gamma_\mu\psi(x'). \tag{16}$$

Equation (10) will hold whether or not (14) does. Assuming (14) is valid, the left side of (16) vanishes; this may be seen by adding and subtracting $m\bar{\varphi}(x')\psi(x')$ to the integrand and using (1) and (14). Then (16) reduces to

$$\int_{S_1} d\sigma'_\mu\,\bar{\varphi}(x')\gamma_\mu\psi(x') = \int_{S_2} d\sigma'_\mu\,\bar{\varphi}(x')\gamma_\mu\psi(x'). \tag{17}$$

Equation (17) is independent of S_1 and S_2 because of (1) and (14). We shall use (17) to construct an auxiliary matrix $\bar{\varphi}_r(x')$ in order to formulate an integral representation of the Dirac equation that includes the boundary values of $\psi(x)$.

THE INVARIANT HOMOGENEOUS S-FUNCTIONS

Since (17) is independent of S_1, select S_1 to be the space-like hyperplane $x'_0 = x_0$, and label the auxiliary function (matrix) $\bar{\varphi}(x')$ with x as well: $\bar{\varphi}_r(x')$. Then (17) reads, noting $d\sigma'_\mu\bar{\varphi}\gamma_\mu\psi = -d\sigma'_0\bar{\varphi}\gamma_0\psi$ and $d\sigma'_0 = -d^3x'$,

$$\int_{-\infty}^{+\infty} d^3x'\,\bar{\varphi}_r(x')\gamma_0\psi(x') = \int_{S_2} d\sigma'_\mu\,\bar{\varphi}_r(x')\gamma_\mu\psi(x'). \tag{18}$$

Choose $\bar{\varphi}_r(x')$ such that

$$\int_{-\infty}^{+\infty} d^3x'\,\bar{\varphi}_r(x')\gamma_0\psi(x') = \psi(x),\ x'_0 = x_0 \tag{19}$$

i.e.,

$$\bar{\varphi}_r(x') = -\gamma_0\delta(\mathbf{r}' - \mathbf{r})$$

or

$$\varphi_r(x') = -i\delta(\mathbf{r}' - \mathbf{r}). \tag{20}$$

Then (18) reads

$$\psi(x) = \int_{S_2} d\sigma'_\mu\,\bar{\varphi}(x' - x)\,\gamma_\mu\psi(x') \tag{21}$$

where $\bar{\varphi}_r(x')$ was rewritten in the form $\bar{\varphi}(x' - x)$, taking advantage of the fact that φ must be invariant under translations. That $\psi(x)$ may be represented by (21) has yet to be shown; *i.e.*, we must show that $\bar{\varphi}(x' - x)$ exists. That it does is suggested by the fact that the Dirac equation is equivalent to eight coupled real first-order equations to which the Cauchy-Kowalewski theorem may be applied. Since the latter theorem exhibits solutions only locally, a global representation such as (21) is not yet guaranteed. We shall prove the existence of $\bar{\varphi}$ by construction. We could do this, as we constructed $\Delta(x)$, by using the Fourier expansion of $\bar{\varphi}$ as in Eq. (13). Instead we shall proceed more directly.

We seek a solution $\varphi(x' - x)$ in the form

$$\varphi(x) = \left(\gamma_\mu\frac{\partial}{\partial x_\mu} - m\right)\gamma_4\,\chi(x). \tag{23}$$

Since $\varphi(x)$ solves (1), $\chi(x)$ must satisfy

$$(\Box - m^2)\,\chi(x) = 0. \tag{24}$$

The boundary condition on φ becomes a condition on χ; putting $x = 0$ and $x' = x$, (20) becomes

$$\varphi(x)\Big|_{x_0=0} = -i\delta(\mathbf{r})$$

or

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)\gamma_4\chi\Big|_{x_0=0} = -i\delta(\mathbf{r}). \tag{25}$$

Equation (25) will be met if

$$\chi(x) \equiv 0, \; x^2 > 0 \tag{26}$$

and

$$\frac{\partial\chi}{\partial x_0}\Big|_{x_0=0} = \delta(\mathbf{r}) \tag{27}$$

because if (26) holds, $\partial\chi/\partial x_i \equiv 0$ on $x_0=0$ and (25) reduces to (27). Thus,

$$\chi(x) = -\Delta(x) \tag{28}$$

and

$$\varphi(x) = -\left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)\gamma_4\Delta(x)$$

or

$$\bar{\varphi}(x) = -\gamma_4^+\left(\gamma_\mu^+ \frac{\partial}{\partial x_\mu^*} - m\right)\gamma_4\Delta(x)$$

$$= \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)\Delta(x)$$

or

$$\bar{\varphi}(x'-x) = \left(\gamma_\mu \frac{\partial}{\partial x'_\mu} + m\right)\Delta(x'-x)$$

$$= \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)\Delta(x-x').$$

The function $\bar{\varphi}(x'-x)$ is generally written $S(x-x')$; thus, (21) becomes

$$\psi(x) = \int_S d\sigma'_\mu \, S(x-x')\,\gamma_\mu\,\psi(x') \tag{29}$$

$$S(x-x') = \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)\Delta(x-x') \tag{30}$$

From (29) and (30), follows immediately that

$$\bar{\psi}(x) = \int_S d\sigma'_\mu \, \bar{\psi}(x')\gamma_\mu \, S(x'-x). \tag{31}$$

By utilizing Schwinger's procedure to obtain the positive and negative frequency parts of $\psi(x)$, together with (29) and (30), we can readily obtain the propagators that give $\psi^{(+)}(x)$, $\psi^{(-)}$, $\bar{\psi}^{(-)}(x)$, and $\bar{\psi}^{(-)}(x)$ from the values of ψ on the surface S; these relations are easily shown to be

$$\left.\begin{aligned} \psi^{(+)}(x) &= \int_S d\sigma'_\mu \, S^{(+)}(x-x')\gamma_\mu\psi(x') \\ S^{(+)}(x) &\equiv (\gamma_\mu\partial_\mu - m)\Delta^{(+)}(x) \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} \psi^{(-)}(x) &= \int_S d\sigma'_\mu \, S^{(-)}(x-x')\gamma_\mu\psi(x') \\ S^{(-)}(x) &\equiv (\gamma_\mu\partial_\mu - m)\Delta^{(-)}(x) \end{aligned}\right\} \tag{33}$$

$$\bar{\psi}^{(+)}(x) = \int_S d\sigma'_\mu \, \bar{\psi}(x')\gamma_\mu S^{(-)}(x'-x) \tag{34}$$

$$\bar{\psi}^{(-)}(x) = \int_S d\sigma'_\mu \, \bar{\psi}(x')\gamma_\mu S^{(+)}(x'-x). \tag{35}$$

Next, define

$$S^{(1)}(x) \equiv (\gamma_\mu\partial_\mu - m)\Delta^{(1)}(x). \tag{36}$$

To seek the boundary value problem that $S^{(1)}(x)$ solves, instead of resorting to Green's theorem, we simply put

$$\int_S d\sigma'_\mu \, S^{(1)}(x-x')\gamma_\mu\psi(x') = \varphi(x) \tag{37}$$

and noting that

$$S^{(1)}(x) = (\gamma_\mu\partial_\mu - m)\frac{-\partial_0}{\sqrt{-\partial_i^2 + m^2}}\Delta(x)$$

$$= -\frac{\partial_0}{\sqrt{-\partial_i^2 + m^2}} S(x) \tag{38}$$

we get

$$\varphi(x) = -\frac{\partial_0}{\sqrt{-\partial_i^2 + m^2}}\psi(x).$$

Thus, if $\psi(x)$ has only positive frequency parts on S, then $\psi(x)$ has only positive frequency parts for any point x not on S, because all Fourier

components of $\psi(x)$ propagate independently for the solutions to the Dirac equation. Therefore

$$i\psi^{(+)}(x) = \int_S d\sigma'_\mu\, S^{(1)}(x-x')\gamma_\mu\psi^{(+)}(x') \qquad (39)$$

$$-i\psi^{(-)}(x) = \int_S d\sigma'_\mu\, S^{(1)}(x-x')\gamma_\mu\psi^{(-)}(x'). \qquad (40)$$

From (39) and (40) one can demonstrate, after some calculation, that $S^{(1)}$ solves the boundary value problem for $\bar{\psi}$ summarized in the equation below:

$$-i\bar{\psi}^{(+)}(x) = \int_S d\sigma'_\mu\, \bar{\psi}^{(+)}(x')\gamma_\mu S^{(1)}(x'-x) \qquad (41)$$

$$+i\bar{\psi}^{(-)}(x) = \int_S d\sigma'_\mu\, \bar{\psi}^{(-)}(x')\gamma_\mu S^{(1)}(x'-x) \qquad (42)$$

Equations (36) through (42) show that $S^{(1)}$ is the Green's function that enables one to construct the positive and negative frequency parts of a function in terms of its positive and negative frequency parts on the space-like surface S. The point x at which this is done may precede or be later then S, and indeed, in view of (27) and the derivation of $S(x)$, x may lie on S.

The results of this section show that the matrix functions $S(x)$, $S^{(+)}(x)$, $S^{(-)}(x)$, and $S^{(1)}(x)$ play roles quite analogous to their corresponding Δ-functions, even though the equations they solve are quite different. Further, the invariance of these functions under Lorentz transformations is a consequence of the invariance of the Δ-functions and of the Dirac equation, but we have not demonstrated this result. It is quite easy to do and is left to the reader.

The integral representation in terms of a Fourier analysis of the invariant S-functions is quite easy to come by. Let $\widetilde{S}(x)$ represent any one of the four invariant S-functions above and $\widetilde{\Delta}$ be the corresponding Δ-function; write

$$\widetilde{\Delta}(x) = \frac{1}{(2\pi)^4}\int d^4k\, \widetilde{\Delta}(k)e^{ik\cdot x}. \qquad (43)$$

Then it follows that

$$S(x) = -\frac{1}{(2\pi)^4}\int d^4k\, (ik\cdot\gamma - m)\, \widetilde{\Delta}(k)e^{ik\cdot x}. \qquad (44)$$

In general, if $\widetilde{\Sigma}(k)$ be the four-dimensional Fourier transform of $\widetilde{S}(x)$, then

$$\widetilde{\Sigma}(k) = (ik\cdot\gamma - m)\, \widetilde{\Delta}(k). \qquad (45)$$

THE INVARIANT INHOMOGENEOUS S-FUNCTIONS

Following the example of our study of the Klein-Gordon equation, it is quite natural to try to construct propagators which propagate asymmetrically about the space-like surface S. As a first case, let us construct a function S_R which expresses the value of ψ at x in terms of its values on S when S is prior to x, but gives zero otherwise. Such a function will be called the retarded S-function and its effect is summarized by

$$\int d\sigma'_\mu\, S_R(x-x')\gamma_\mu\psi(x') = \begin{cases} \psi(x), & x \text{ later than } S \\ 0, & x \text{ earlier than } S. \end{cases} \qquad (46)$$

Because the function $S(x)$ vanishes outside the light-cone, it is clear that

$$S_R(x) = \theta(x)S(x). \qquad (47)$$

A further property of $S_R(x)$ is readily proved, namely,

$$(\gamma_\mu\partial_\mu + m)\, S^R(x) = \delta(x) \qquad (48)$$

where, again, $\delta(x)$ is the four-dimensional delta function $\delta(x_0)\delta(\mathbf{r})$. Because $S_R(x)$ satisfies an inhomogeneous Dirac equation, it will be termed an inhomogeneous S-function; further, because both $S(x)$ and $\theta(x)$ are Lorentz invariant functions, $S_R(x)$ is Lorentz invariant. Rewriting (47) as

$$S_R(x) = \frac{1+\epsilon(x)}{2}\, S(x) \qquad (49)$$

it is then easy to show that

$$S_R(x) = (\gamma_\mu\partial_\mu - m)\, \Delta_R(x). \qquad (50)$$

In a similar fashion to that for defining the advanced Δ-function $\Delta_A(x)$, we introduce the

advanced S-function $S_A(x)$ whose defining properties are

$$\int_{S'} d\sigma'_\mu \, S_A(x-x') \, \gamma_\mu \psi(x') = \begin{cases} 0, & x \text{ later than } S' \\ -\psi(x), & x \text{ earlier than } S'. \end{cases} \tag{51}$$

Then

$$S_A(x) = -\theta(-x)S(x) = \frac{-1+\epsilon(x)}{2} S(x) \tag{52}$$

$$(\gamma_\mu \partial_\mu + m) S_A(x) = \delta(x) \tag{53}$$

and it can easily be shown that

$$S_A(x) = (\gamma_\mu \partial_\mu - m)\Delta_A(x). \tag{54}$$

For the Dirac equation, we may introduce a function $\bar{S}(x)$ that plays a role analogous to that of $\bar{\Delta}(x)$ for the Klein-Gordon equation; define $\bar{S}(x)$ such that

$$\int_{S'} d\sigma'_\mu \, \bar{S}(x-x') \, \gamma_\mu \psi(x') = \begin{cases} 1/2\psi(x), & x \text{ later than } S' \\ -1/2\psi(x), & x \text{ earlier than } S'. \end{cases} \tag{55}$$

We see immediately that $\bar{S}(x)$ must be given by

$$\bar{S}(x) = 1/2 \, \epsilon(x) \, S(x) \tag{56}$$

and that

$$(\gamma_\mu \partial_\mu + m) \, \bar{S}(x) = \delta(x) \tag{57}$$

while from (56) one can show that another expression for $\bar{S}(x)$ is

$$\bar{S}(x) = (\gamma_\mu \partial_\mu - m)\bar{\Delta}(x). \tag{58}$$

The steps involved in the proof of (58) are several in number and it is not obvious that (58) is the same as (56), but since it is not difficult to show this, we leave it to the reader.

Finally, we introduce the Feynman or causal propagator $S_F(x)$ by defining it as

$$S_F(x) = \bar{S}(x) - \frac{1}{2} S^{(1)}(x). \tag{59}$$

It is a straightforward demonstration to show

$$\int_{S'} d\sigma'_\mu \, S_F(x-x')\gamma_\mu \psi(x') = \begin{cases} \psi^{(+)}(x), & x \text{ later than } S' \\ -\psi^{(-)}(x), & x \text{ earlier than } S' \end{cases} \tag{60}$$

and it is obvious that

$$(\gamma_\mu \partial_\mu + m) \, S_F(x) = \delta(x). \tag{61}$$

From (58), (59), and (36) it follows immediately that

$$S_F(x) = (\gamma_\mu \partial_\mu - m)\Delta_F(x) \tag{62}$$

If $\bar{\bar{S}}(x)$ represents anyone of the four inhomogeneous invariant S-functions of this section and $\bar{\bar{\Delta}}(x)$ its analogue for the Klein-Gordon equation, and if the Fourier transform of $\bar{\bar{\Delta}}(x)$ is denoted simply by $\bar{\bar{\Lambda}}(k)$, then the integral representation of $\bar{\bar{S}}(x)$ (that is, its Fourier transform) is clearly given by

$$\bar{\bar{S}}(x) = \frac{1}{(2\pi)^4} \int d^4k \, (i\gamma \cdot k - m) \, \bar{\bar{\Lambda}}(k) e^{ik \cdot x}. \tag{63}$$

Finally, because all the invariant S-functions, homogeneous and inhomogeneous, are related to their analogues for the Klein-Gordon equation in the same way, *i.e.*,

$$S_i(x) = (\gamma \cdot \partial - m)\Delta_i(x) \tag{64}$$

all the relations between the Δ-functions also obtain for the S-functions and will therefore not be repeated here.

APPENDIX
Alternative Derivation of the S-Function

In this appendix, we give an alternative derivation of the function $S(x-x')$ of Chapter 4, Equation (30) along with an alternative derivation of Equation (29) of the same chapter.

Since each component of a Dirac spinor solves the Klein-Gordon equation, we may use Equation (25) to express $\psi_\alpha(x)$, a typical component, in terms of $\psi_\alpha(x')$ and $\partial\psi_\alpha(x')/\partial x'_0$ on a given surface. Of course, for the Dirac equation $\partial\psi_\alpha(x')/\partial x'_0$ cannot be specified independently of $\psi_\alpha(x')$ $(\alpha = 1, 2, 3, 4)$ on the surface, and that will prove the key to this development of equation (29). We have

$$\psi_\alpha(x) = \int_{S'} d\sigma'_\mu \left\{ \Delta(x-x') \frac{\partial\psi(x')}{\partial x'_\mu} - \frac{\partial\Delta(x-x')}{\partial x'_\mu}\psi_\alpha(x') \right\} (\alpha = 1, 2, 3, 4). \quad \text{(A1)}$$

The four equations represented by (A1) may be combined together in a matrix equation for the column vector $\psi(x)$:

$$\psi(x) = \int_{S'} d\sigma'_\mu \left\{ \Delta(x-x') \frac{\partial\psi}{\partial x'_\mu} - \frac{\partial\Delta(x-x')}{\partial x'_\mu}\psi(x') \right\}. \quad \text{(A2)}$$

Choose S' to be the surface $\{x' | x'_0 = const.\}$; then (A2) becomes

$$\psi(x) = -\int d^3x' \left\{ \Delta(x-x') \frac{\partial\psi}{\partial x'_0} - \frac{\partial\Delta(x-x')}{\partial x'_0}\psi(x') \right\}. \quad \text{(A3)}$$

The quantity $\partial\psi/\partial x'_0$ may be expressed in terms of ψ or S' and the spatial derivative of ψ (i.e., the derivatives of ψ parallel to the surface S', all of which are known when ψ is known on S') by Dirac equation:

$$\frac{\partial\psi(x')}{\partial x'_0} = -\alpha_i \frac{\partial\psi(x')}{\partial x'_i} - im\beta\psi(x'). \quad \text{(A4)}$$

Inserting (A4) into (A3) and integrating by parts, we obtain

$$\psi(x) = -\int d^3x' \left[\left(\alpha_i \frac{\partial}{\partial x'_i} - \frac{\partial}{\partial x_0} - im\right)\Delta(x-x') \right] \psi(x') + \int d^3x' \, \alpha_i \frac{\partial}{\partial x'_i} [\Delta(x-x')\psi(x')]. \quad \text{(A5)}$$

The last term of (A5) vanishes because of the boundary conditions on $\Delta(x-x')$; using this fact and the relations

$$\alpha_i = -i\gamma_i\gamma_4$$

$$\beta = \gamma_4$$

$$\gamma_4^2 = 1,$$

(A4) then becomes

$$\psi(x) = \int d^3x' i \left[\left(-\gamma_\mu \frac{\partial}{\partial x'_\mu} - m\right)\Delta(x-x') \right] \times \gamma_4 \psi(x')$$

$$= \int d\sigma'_4 \left[\left(+\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)\Delta(x-x') \right] \times \gamma_4 \psi(x') \quad \text{(A6)}$$

since $id^3x = d\sigma_4$ and $\partial/\partial x'_\mu \, \Delta(x-x') = -\partial/\partial x_\mu \, \Delta(x-x')$; finally a Lorentz transformation that alters S' to a more general surface yields a relation of the form

$$\psi(x) = \int d\sigma'_\mu \, S(x-x') \, \gamma_\mu \, \psi(x')$$

$$S(x-x') = (\gamma_\mu\partial_\mu - m) \, S(x-x'). \quad \text{(A7)}$$

The nicest part of the derivation of (A7) is that the method may be applied directly to any relativistic wave equation for free particles; the same technique may be used to develop a propagator $S^{(+)}$ and $S^{(-)}$ as well, and only minor modifications are necessary to develop in this manner the other propagators.

Chapter 5
THE ROLE OF THE PROPAGATORS IN QUANTUM FIELD THEORIES

The invariant functions derived in the previous three chapters are of particular significance in the quantum field theories of the pi mesons, photons, electrons, and nucleons. In particular, these functions arise in the commutators and anti-commutators of field operators for bosons and fermions respectively, and in the perturbation development of the S-matrix. While it is not our intention here to develop quantum field theories in full, or even the field theory of any one such field, we should like to do a few simple illustrative calculations to demonstrate how these functions enter the theory. For this purpose, we shall study, in part, a scalar meson field, which is about the simplest of the theories and yet is quite analogous in its development to all other fields. In its simplicity it avoids such complications as the need for the introduction of an indefinite metric; hence some modifications have to be made when extending the results to the electromagnetic field, but little or no changes in the general approach.

Our approach to the subject will be along fairly "classical" lines. We note first that the field equation for an unquantized scalar meson field is the Klein-Gordon equation:

$$(\Box - m^2)\varphi(x) = 0 \tag{1}$$

We should like to construct a Lagrangian, L, whose corresponding Euler-Lagrange equation is (1). This procedure is well treated in many books*, where one finds for a suitable Lagrangian

$$L = -\frac{1}{2}\left[\sum_{\mu=1}^{4}\left(\frac{\partial\varphi}{\partial x_\mu}\right)^2 + m^2\varphi^2\right] \tag{2}$$

as one may readily verify by calculating the Euler-Lagrange equation with (2). The momentum canonically conjugate to φ is

$$\pi(x) = \frac{\partial L}{\partial\dot{\varphi}} = \frac{\partial\varphi(\mathbf{r},t)}{\partial t} \tag{3}$$

Now in the first step of quantizing a field, we regard the field components and the canonically conjugate momenta no longer simply as functions, but also as elements, indeed, generators, of a noncommutative algebra wherein the fundamental relationship, for our case, is given by

$$[\pi(\mathbf{r},t),\ \psi(\mathbf{r}',t)] \equiv \pi(\mathbf{r},t)\varphi(\mathbf{r}',t) - \varphi(\mathbf{r}',t)\pi(\mathbf{r},t)$$

$$= \frac{\hbar}{i}\delta(\mathbf{r} - \mathbf{r}'). \tag{4}$$

We shall again take $\hbar = 1$; then with (3), (4) reads

$$\left[\frac{\partial\varphi(\mathbf{r}',t)}{\partial t},\ \varphi(\mathbf{r},t)\right] = i\delta(\mathbf{r} - \mathbf{r}') \tag{5}$$

Equation (4) or Eq. (5) provides the fundamental statement about the noncommutativity of the elements of algebra and has been used in this form in many treatments.* It suffers from a defect however in that the time coordinate is singled out in a manner different from the space coordinate. What we shall show is that this defect is simple to remove and that we can develop an expression for the commutator of φ at the space-time point x and for φ at x' in a Lorentz invariant manner.

Let S'' be a space-like hypersurface on which the classical function $\varphi(x)$ and its derivatives are defined. We have seen that from these data we may obtain φ at x' by Eq. (25) of Chapter 2, that is, by

$$\varphi(x') = \int_{S''} d\sigma''_\mu\left[\Delta(x' - x'')\frac{\partial\varphi}{\partial x''_\mu}\right.$$

$$\left. - \varphi(x'')\frac{\partial\Delta(x' - x'')}{\partial x''_\mu}\right]. \tag{6}$$

*See G. Wentzel, "Quantum Theory of Fields," Interscience, 1949

*Ibid

Now even though the φ's have been now considered as elements of a noncommutative algebra, they still satisfy (1), and hence (6) will remain valid. Thus,

$$[\,\varphi(x),\varphi(x')\,] = \int_{S''} d\sigma''_\mu\,\Delta(x'-x'')\left[\,\varphi(x),\frac{\partial\varphi(x'')}{\partial x''_\mu}\,\right] - \int_{S''} d\sigma''_\mu\,\frac{\partial\Delta(x'-x'')}{\partial x''_\mu}\,[\,\varphi(x),\varphi(x'')\,]. \quad (7)$$

We recognize that S'' is quite arbitrary except insofar as it is space-like and therefore utilize this freedom by choosing it so that it is the hyperplane $S'' = \{\, x'' |\, t'' = t\,\}$ going through the point $t'' = t$. Then (7) becomes, according to Eq. (32) of Chapter I,

$$[\,\varphi(x),\varphi(x')\,] = -\int d^3x''\,\Delta(\mathbf{r}'-\mathbf{r}'',t'-t)\left[\,\varphi(\mathbf{r},t),\frac{\partial\varphi(\mathbf{r}'',t)}{\partial t}\,\right] + \int d^3x''\,\frac{\partial\Delta(\mathbf{r}'-\mathbf{r}'',t'-t)}{\partial t}\,[\varphi(\mathbf{r},t),\varphi(\mathbf{r}'',t)]. \quad (8)$$

Putting (5) into (8), we get an integral equation for the commutator:

$$[\,\varphi(x),\varphi(x')\,] = -i\Delta(x-x') + \int_{t''=t} d^3x''\,\frac{\partial\Delta(\mathbf{r}'-\mathbf{r}'',t'-t'')}{\partial t''}\,[\varphi(\mathbf{r},t),\varphi(\mathbf{r}'',t'')]. \quad (9)$$

Utilizing the method of iteration and the fact that $\Delta(x-x')$ vanishes for space-like intervals $x-x'$, it is simple to see that the solution to (9) is simply

$$[\,\varphi(x),\varphi(x')\,] = -i\Delta(x-x') \quad (10)$$

which is what we were seeking.

From (10) it follows by simple substitution that

$$[\,\varphi(x-\tau n),\varphi(x')\,] = -i\Delta(x-x'-\tau n)$$

which leads to

$$[\varphi^{(+)}(x),\varphi(x')] = -i\Delta^{(+)}(x-x')$$

and also that

$$[\,\varphi^{(+)}(x),\varphi(x'+\tau n')\,] - i\Delta^{(+)}(x-x'-\tau n')$$

which leads finally to

$$[\,\varphi^{(+)}(x),\varphi^{(-)}(x')\,] = -i\Delta^{(+)}(x-x') \quad (11)$$

$$[\,\varphi^{(-)}(x),\varphi^{(+)}(x')\,] = -i\Delta^{(-)}(x-x'). \quad (12)$$

Before continuing with this development, it is interesting to examine some of the assumptions that have been tacitly made above with respect to the mathematical formalism that is used. It is not our purpose to go into a discussion of the mathematics on a rigorous basis; such a project would be, without doubt, of great value in understanding what we shall be doing and what we can do but would be too vast a subject to cover adequately here. We choose only to point out two or three of the major points in field theory that require some detailed mathematical study and to give an idea where one may find helpful information.

The first point comes up immediately upon applying the technique of second quantization. We have asserted that the classical field functions must no longer be regarded as ordinary functions but as elements of a noncommutative algebra. To understand the meaning of this, let us return to the definition of a function as we ordinarily encounter. If $D = \{\, x | a \leq x \leq b\,\}$, that is, if D be the set of all real members on the closed interval between a and b, and if $R = \{\, y | c \leq y \leq d\,\}$, and if there is a correspondence between D and R, that is, if to every element in D we associate one element of R, then we say that there is a mapping of D into R; D is called the domain and R the range of the map. The collection

of all ordered pairs (x,y) where y is the image of x under the given map is called the graph of the map. The map is also called a function. The usual prescription for specifying the function is to give a formula for computation of one or more of the ordered pairs (x,y); the formula is represented in general by the equation $y = f(x)$. Suppressing the independent variable x in the set of all the ordered pairs (x,y), we see then that the function is specificed by the totality of its values in the range R, and an essential feature of the function is the clear representation of what its domain is and what its range is. In our simple example, the domain and range were both subsets of the real line. In the theory of functions of n real variables, the domain will be a subset of E^n, the n-dimensional Euclidian space and the range the real line, or the domain might be an n-dimensional hypersurface in an $n+1$ dimensional space with the real line as the range.

But the range need not be restricted to the real line. If, for example the range of the function were E^n, while D were the real line, the function would be described as an n-dimensional vector function on the real line. In the case of our field theory, the domain of our functions is the space-time continuum, while the range is some subset of some noncommutative algebra, which has yet to be specified in greater detail. If φ symbolizes one particular such function, $D \to R$ and $x \in D$, then the image element of x under φ will be denoted simply by $\varphi(x)$, where $x = (\mathbf{r},t)$. If ψ is a second map of D into R, $D \to R$, then the image of x will be denoted by $\psi(x)$, and of course $\psi(x) \in R$.

It becomes necessary to define equality of two functions. Many definitions are readily available, but the two most useful are: (a) $\varphi = \psi$ if $\varphi(x) = \psi(x)$, all $x \in D$ and (b) $\varphi = \psi$ if $\varphi(x) = \psi(x)$, almost all $x \in D$, where "almost all" means $\varphi(x) = \psi(x)$ everywhere in D except on a set of measure zero. In the latter case, it is convenient to introduce the notion of equivalent classes as in the theory of measurable functions, but we shall not go into this in any more detail. We shall assume henceforth that some acceptable definition of equality of two maps or functions is given.

If φ and ψ are any two maps of $D \to R$, we may then define a third map π of $D \to R$, because of the fact that R is an algebra, by

$$\pi(x) = \varphi(x) + \psi(x)$$

and π will be symbolically denoted by $\varphi + \psi$.

Let $\Lambda = \{ \lambda,\mu,\ldots \}$ be the field over which the algebra R is defined; because R is an algebra, then such quantities as $\lambda\varphi(x) + \mu\psi(x')$ are defined and belong to R. Such quantities must be regarded as distinct from the sum of two functions, being merely the sum of two elements in the algebra.

Now an algebra R, though it may contain an infinite number of elements, when treated within the framework of algebra, is studied only by finite means; by this we mean only finite sums, differences, and products are considered. But we asserted that among the functions φ and ψ, etc., are those which satisfy the Klein-Gordon equation, or its equivalent integral equation with the desired boundary conditions. But it is clear that to give meaning to these latter concepts it is necessary to introduce concepts of analysis such as limit points and infinite sums. That such a procedure will work for our algebra R is intuitively clear, since it can readily be done in the space of all functions on D onto the real line and since these notions do not require any alteration on account of the noncommutativity of the algebra. Once having found a successful formulation of these analytical concepts for our algebra, one can then go on to introduce the analogues of derivative, Riemann integrals, and Lebesgue integrals, each case being a map of some subset of R into R.

One would then imagine that the next step in a mathematically rigorous discussion of our field theory would be to establish the existence of solutions to the field equations (1) or (6) consistent with the commutation rules. Indeed, if our algebra R were specified in detail beforehand, this would be the next step. On the other hand, as we have seen, R is not so specified; indeed, what we must do is to assert the existence of such solutions and use these as a basis for constructing, by the operations allowed in our algebra, the rest of the algebra.

Once having obtained the structure of our algebra, we may introduce mappings of the set of all mappings of D into R into itself. If φ and a be two maps of D into R such that φ is the image of a under the map F, say, we shall denote it by $\varphi = F(a)$ symbolically, which expresses a relationship between all the values $\varphi(x)$ and those of $a(y)$. One used often in field theory is

$$\varphi(x) = \frac{1}{(2\pi)^4} \int a(k) e^{ik \cdot x} d^4k$$

where we now assume that meaning has been given to the integral sign. For functions in $L_2(-\infty, +\infty)$ the Fourier integral transform is given by just such a formula and its inversion is well understood. However, the inversion of such a transformation when $\varphi(x)$ and $a(k)$ for each x and k respectively are elements of our algebra must be established anew.

The mathematical points touched upon in the discussion of the above few paragraphs are usually glossed over in most field theory studies, as we have done in our development. The development proceeds along the lines dictated by formalism and intuition. It is in this spirit that we proceed to derive one more result in field theory; namely, we assume the correctness of a theory of Fourier transforms for our field quantities which is formally identical to the theory of the Fourier transform for generalized functions. Thus we assume the existence of elements $a(k)$ such that

$$\left.\begin{aligned} \varphi(x) &= \frac{1}{(2\pi)^4}\int d^4k\ \bar{a}(k)e^{ik\cdot x} \\ \bar{a}(k) &= \int d^4x\ \varphi(x)e^{-ik\cdot x} \end{aligned}\right\}. \qquad (13)$$

Assuming that $\varphi(x)$ obeys the Klein-Gordon equation, these reduce to the form

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}}\int d^4k \frac{1}{2\omega}\left[a(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + a^*(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \qquad (14)$$

where here $a^*(\mathbf{k})$ does not mean anything other than $a^*(\mathbf{k})$ is different from $a(\mathbf{k})$. Equations (14) and (11) together yield a commutation result for the a's:

$$[\ a(\mathbf{k}), a^*(\mathbf{k})\ ] = \delta(\mathbf{k}-\mathbf{k}'). \qquad (15)$$

The physical interpretation of (14), (15), and $a(\mathbf{k})$ and $a(\mathbf{k})$ are too familiar to go into here. We could go on to show how the T and P products come about in the theory and are expressible in terms of the propagators derived in the previous chapters. This we leave to the interested reader.