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SACLANT ASW
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AN UNRESTRICTED LINEAR RANDOM WALK WITH NEGATIVE EXPONENTIALLY DISTRIBUTED STEP LENGTHS

by

B.W. CONOLLY

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VIALE SAN BARTOLOMEO, 92
LA SPEZIA, ITALY

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AN UNRESTRICTED LINEAR RANDOM WALK WITH NEGATIVE EXPONENTIALLY DISTRIBUTED STEP LENGTHS

By

B.W. Conolly

ABSTRACT

An account is given of the theory of a doubly infinite linear random walk in which step lengths have a negative exponential distribution and the direction of each step is not necessarily equiprobable. The problem of first passage time is also studied. The theory was developed in connection with a study of random linear anti-submarine patrols.

INTRODUCTION

The unrestricted linear random walk described in text books is usually a study of the motion of an imaginary particle that roams backwards and forwards along the doubly infinite x-axis according to definite probability laws. The axis is divided into equal intervals of length h, say, and the particle proceeds in leaps and bounds, the lengths of which are integral multiples of h. At the end of a leap, which, henceforward, to conform with convention, we shall call a 'step', the particle chooses the length and direction of its next step. The probability distribution associated with this choice is prescribed in advance and remains the same throughout the walk.

The most usual topics of investigation are, for a given starting point:

a. the probability that at the end of the n^{th} step the particle is at the point with x-coordinate mh, where m is a positive or negative integer;

b. the probability that at the end of, or during, the n^{th} step the particle reaches its starting point for the first time since the walk began. (This is the so-called 'first passage problem'.)

If the particle moves with constant speed, steps will occupy varying times proportional to their lengths. On the other hand, if it is supposed that a step occupies constant time, the particle has to move with variable velocity. In either case a plot of the particle's position as a function of time produces a sort of random noise record consisting of a sequence of straight line segments, and in this connection topic (b) above has an obvious relation to the problem of the interval of time between successive zeros of a random function.

The random walk described and investigated in this report arose from the study of a family of tactical problems. It differs from the discrete step random walk in that the lengths of steps are continuous. At the end of a step the particle is given a choice as to whether to move to the right with probability p, or to the left with probability q (= 1-p), and the length of the next step is governed by the negative exponential distribution. Thus, the probability that a step has length between s and s+ds is $\mu e^{-\mu s} ds$.

The tactical application of the first passage time aspect of this walk is described, together with the skeleton of the derivation of the relevant probabilities, in the classified Ref. 1. It is felt desirable to set down an unclassified account of the derivation of the formulae, not only for the first passage time problem, but also for the problem of where the particle is at the end of a given step. A similar, but in some aspects abbreviated, account has also been published by the author in Ref. 2.

1. POSITION OF THE PARTICLE AT THE END OF THE n^{th} STEP

A particle is considered that executes a random walk on a line in such a way that the probability that a step has a length between s and s+ds is $\mu e^{-\mu s} ds$. When the particle reaches the end of a step, it has probability p of moving to the right at the next step, and $q = 1-p$ of moving to the left.

We study here the joint probability and probability densities $R_n(x, y)$ and $L_n(x, y)$ that the particle, which begins its walk at the point X with coordinate x, finds itself at the end of the n^{th} step at a point Y in the small interval $(y, y+dy)$. When $y > x$, $R_n(x, y)$ is used to denote this probability; when $y \leq x$, $L_n(x, y)$ is the corresponding form.

The fundamental equations satisfied by $R_n(x, y)$ and $L_n(x, y)$ are as follows:

$$R_n(x, y) = q\mu \int_0^\infty R_{n-1}(x, y+s)\, e^{-\mu s} ds + p\mu \int_0^{y-x} R_{n-1}(x, y-s)\, e^{-\mu s} ds + p\mu \int_{y-x}^\infty L_{n-1}(x, y-s)\, e^{-\mu s} ds \qquad \text{(Eq. 1)}$$

$$L_n(x, y) = p\mu \int_0^\infty L_{n-1}(x, y-s)\, e^{-\mu s} ds + q\mu \int_0^{x-y} L_{n-1}(x, y+s)\, e^{-\mu s} ds + q\mu \int_{x-y}^\infty R_{n-1}(x, y+s)\, e^{-\mu s} ds \qquad \text{(Eq. 2)}$$

The first term in Eq. 1 deals with the case when the particle is to the right of Y at the end of the $(n-1)^{th}$ step. The second and third terms arise when the particle is to the left of Y at the end of the $(n-1)^{th}$ step, the integral from 0 to $y-x$ dealing with the case when it is to the right of X, and the final term when the end of the $(n-1)^{th}$ step is to the left of X. A similar explanation can be given for Eq. 2.

We can formulate similar equations by considering the particle's first step. Suppose this is to the left (probability q), and to the point within the small interval (x-s, x-s -ds). Then the particle must, during the next n-1 steps, march from x-s to y. The resulting integral is:

$$q\mu \int_0^\infty e^{-\mu s} R_{n-1}(x-s, y)\, ds.$$

If the first step is to the right, two cases arise according to whether it is to the right of Y or to the left. Thus, altogether

$$R_n(x, y) = q\mu \int_0^\infty e^{-\mu s} R_{n-1}(x-s, y)\, ds + p\mu \int_0^{y-x} e^{-\mu s} R_{n-1}(x+s, y)\, ds + \quad \text{(Eq. 3)}$$
$$+\ p\mu \int_{y-x}^\infty e^{-\mu s} L_{n-1}(x+s, y)\, ds$$

A comparison between Eqs. 1 & 3 shows that $R_n(x, y)$ is a function of y-x. Similarly, $L_n(x, y)$ is a function of x-y.

An evaluation by first principles of $R_1(x, y)$, $L_1(x, y)$, etc, gives:

$$R_1(x, y) = p\mu e^{-\mu(y-x)} , \quad \text{(Eq. 4)}$$

$$L_1(x, y) = q\mu e^{-\mu(x-y)} , \quad \text{(Eq. 5)}$$

$$R_2(x, y) = p\mu e^{-\mu(y-x)} \left[q + p\mu(y-x) \right] , \qquad \text{(Eq. 6)}$$

$$L_2(x, y) = q\mu e^{-\mu(x-y)} \left[p + q\mu(x-y) \right] , \qquad \text{(Eq. 7)}$$

and these lead us to consider the functions $\rho_n(y-x)$, $\lambda_n(x-y)$, defined by

$$R_n(x, y) = e^{-\mu(y-x)} \rho_n(y-x) , \qquad \text{(Eq. 8)}$$

$$L_n(x, y) = e^{-\mu(x-y)} \lambda_n(x-y) . \qquad \text{(Eq. 9)}$$

Evidently, either ρ_n or λ_n (and R_n and L_n) can be transformed into the other by interchanging p and q, x and y.

Finally we introduce the generating functions

$$\rho(\delta, t) = \sum_{n \geq 1} \rho_n(\delta) t^n , \qquad \text{(Eq. 10)}$$

$$\lambda(\delta, t) = \sum_{n \geq 1} \lambda_n(\delta) t^n , \qquad \text{(Eq. 11)}$$

and the Laplace transforms

$$r_n(z) = \int_0^{\infty} e^{-z\delta} \rho_n(\delta) \, d\delta , \qquad \text{(Eq. 12)}$$

$$\ell_n(z) = \int_0^\infty e^{-z\delta} \lambda_n(\delta)\, d\delta \quad , \qquad \text{(Eq. 13)}$$

$$r(z, t) = \int_0^\infty e^{-z\delta} \rho(\delta, t)\, d\delta \quad , \qquad \text{(Eq. 14)}$$

$$\ell(z, t) = \int_0 e^{-z\delta} \lambda(\delta, t)\, d\delta \quad . \qquad \text{(Eq. 15)}$$

Clearly

$$r(z, t) = \sum_{n \geq 1} r_n(z)\, t^n \quad , \qquad \text{(Eq. 16)}$$

with a similar definition for $\ell(z, t)$.

Since at the end of each step the particle must be somewhere, we have for each n :

$$\int_x^\infty R_n(x, y)\, dy + \int_{-\infty}^x L_n(x, y)\, dy = 1, \qquad \text{(Eq: 17)}$$

or

$$\int_0^\infty e^{-\mu\delta} \rho_n(\delta)\, d\delta + \int_0^\infty e^{-\mu\delta} \lambda_n(\delta)\, d\delta = 1, \qquad \text{(Eq. 18)}$$

or

$$r_n(\mu) + \ell_n(\mu) = 1 \quad . \qquad \text{(Eq. 19)}$$

Hence, for $|t| < 1$ and $\mathrm{Rl}\; z \geq \mu$, $r(z, t) + \ell(z, t)$ is analytic, and, in particular,

$$r(\mu, t) + \ell(\mu, t) = t/(1 - t) . \qquad \text{(Eq. 20)}$$

The integral equation for $\rho_n(\delta)$ deriving from Eq. 1 is

$$\rho_n(\delta) = q\mu \int_0^\infty e^{-2\mu s} \rho_{n-1}(\delta + s)ds + p\mu \int_0^\delta \rho_{n-1}(\delta - s)ds + + p\mu e^{2\mu\delta} \int_\delta^\infty e^{-2\mu s} \tau_{n-1}(s - \delta) ds . \qquad \text{(Eq. 21)}$$

Applying the Laplace transformation, we obtain

$$r_n(z) = q\mu \frac{\left[r_{n-1}(z) - r_{n-1}(2\mu)\right]}{(2\mu - z)} + \frac{p\mu}{z}\left[r_{n-1}(z) + 1_{n-1}(2\mu)\right] \qquad \text{(Eq. 22)}$$

and

$$\ell_n(z) = \frac{p\mu}{(2\mu - z)}\left[\ell_{n-1}(z) - \ell_{n-1}(2\mu)\right] + \frac{q\mu}{z}\left[\ell_{n-1}(z) + r_{n-1}(2\mu)\right] .$$

(Eq. 23)

These are simple difference equations, and it is easy to show, using Eq. 19, that

$$r_n(z) = \mu^n \left[\frac{q}{2\mu - z} + \frac{p}{z}\right]^n - \ell_n(2\mu - z) , \qquad \text{(Eq. 24)}$$

$$\ell_n(z) = \mu^n \left[\frac{p}{2\mu - z} + \frac{q}{z}\right]^n - r_n(2\mu - z) \quad . \qquad \text{(Eq. 25)}$$

The second term on the right-hand sides of Eqs. 24 & 25 are particular solutions of Eqs. 22 & 23, which have to be included to complete the general solution. Since $r_n(2\mu)$ is certainly not infinite (see above) one must expect that $\ell_n(2\mu - z)$ contains terms that annihilate the apparent poles, and it seems a reasonable conjecture that both $r_n(z)$ and $\ell_n(z)$ are n^{th} degree polynomials in descending powers of z.

Multiplying Eqs. 22 & 25 by t^n, adding, and using the results

$$r_1(z) = p\mu/z, \quad \ell_1(z) = q\mu/z,$$

we obtain

$$r(z,t) = \frac{q\mu t z r(2\mu, t) - p\mu t(2\mu - z)\ell(2\mu, t) - p\mu t(2\mu - z)}{z^2 - \mu z(2 - qt + pt) + 2p\mu^2 t}, \qquad \text{(Eq. 26)}$$

with a similar formula for $\ell(z, t)$, in which p and q are interchanged. We denote the zeros of the denominator of Eq. 26 by ω'_1 and ω'_2. Then

$$\left.\begin{aligned}
\omega_1 &= \mu\left[1 + \tfrac{1}{2}(p-q)t + R\right] \\
\omega_2 &= \mu\left[1 + \tfrac{1}{2}(p-q)t - R\right] \\
\omega'_1 &= \mu\left[1 + \tfrac{1}{2}(q-p)t + R\right] \\
\omega'_2 &= \mu\left[1 + \tfrac{1}{2}(q-p)t - R\right]
\end{aligned}\right\} \qquad \text{(Eq. 27)}$$

where

$$R^2 = (1 - \frac{1}{2} t)^2 - pqt^2 \; . \qquad \text{(Eq. 28)}$$

Now ω_1 and ω'_1 are not less than μ for all p, and $t < 1$. Hence, since $r(z, t)$ and $\ell(z, t)$ are analytic for $R1\, z \geq \mu$ and $t < 1$, we must arrange for the numerator of Eq. 26 and its companion expression for $\ell_n(z, t)$ to vanish respectively at $z = \omega_1$ and $z = \omega'_1$. Thus, noting that

$$\begin{aligned} 2\mu - \omega_1 &= \omega'_2 \; , \\ 2\mu - \omega'_1 &= \omega_2 \; , \end{aligned} \qquad \text{(Eq. 29)}$$

we obtain

$$\left. \begin{aligned} q\,\omega_1\, r(2\mu, t) - p\,\omega'_2\, \ell(2\mu, t) &= p\,\omega'_2 \\ q\,\omega_2\, r(2\mu, t) - p\,\omega'_1\, \ell(2\mu, t) &= -q\,\omega_2 \end{aligned} \right\} \qquad \text{(Eq. 30)}$$

and hence

$$r(2\mu, t) = \frac{p\,\omega'_1\, \omega'_2 + q\,\omega_2\, \omega'_2}{q(\omega'_1\, \omega_1 - \omega'_2\, \omega_2)} \; , \qquad \text{(Eq. 31)}$$

$$\ell(2\mu, t) = \frac{p\,\omega_2\, \omega'_2 + q\,\omega_1\, \omega_2}{p(\omega'_1\, \omega_1 - \omega'_2\, \omega_2)} \; . \qquad \text{(Eq. 32)}$$

Substituting in Eq. 26 we obtain

$$r(z, t) = \frac{t(p\,\omega'_1 + q\omega_2)}{2R} \cdot \frac{1}{(z - \omega_2)}, \qquad \text{(Eq. 33)}$$

$$\ell(z, t) = \frac{t(q\omega_1 + p\,\omega'_2)}{2R} \cdot \frac{1}{(z - \omega'_2)}, \qquad \text{(Eq. 34)}$$

These may be inverted easily and give

$$\rho(\delta, t) = \frac{t(p\omega'_1 + q\,\omega_2)}{2R}\, e^{\delta\omega 2}, \qquad \text{(Eq. 35)}$$

$$\lambda(\delta, t) = \frac{t(q\omega_1 + p\,\omega'_2)}{2R}\, e^{\delta\omega'_2}, \qquad \text{(Eq. 36)}$$

and, returning to the original generating functions,

$$R(x, y, t) = \sum_{n \geq 1} R_n(x, y)t^n = \frac{t(p\omega'_1 + q\omega_2)}{2R}\, e^{-(y-x)(\mu - \omega_2)}, \qquad \text{(Eq. 37)}$$

$$L(x, y, t) = \sum_{n \geq 1} L_n(x, y)t^n = \frac{t(q\omega_1 + p\omega'_2)}{2R}\, e^{-(x-y)(\mu - \omega'_2)}. \qquad \text{(Eq. 38)}$$

It is of particular interest to study the densities $R_n(x, y)$, $L_n(x, y)$ for fixed n and various y. The simplest case is that in which $p = q = \frac{1}{2}$, when the particle is as likely to move to the right as to the left during each step. $R(x, y, t)$ and $L(x, y t)$ are then identical, and, from Eqs. 37, 27 & 28,

$$R(x, y, t) = \frac{\mu t}{2\sqrt{1-t}} e^{-\mu\delta\sqrt{1-t}}, \qquad \text{(Eq. 39)}$$

where $\delta = y-x$. Now $R_n(x, y)$, being the coefficient of t^n in $R(x, y, t)$, is given by

$$R_n(x, y) = \frac{1}{n!}\left[\frac{d^n R}{dt^n}\right]_{t=0}$$

$$= \frac{1}{2\pi i}\int_C R(x, y, z)\frac{dz}{z^{n+1}}, \qquad \text{(Eq. 40)}$$

where C is a simple closed contour surrounding the origin, such that $R(x, y, z)$ is analytic in z, inside and on C.

We make the transformation

$$\sqrt{1-z} = 1-\zeta .$$

Then

$$R_n(x, y) = \frac{\mu}{2\pi i}\int_{C'} e^{-\mu\delta(1-\zeta)}\frac{d\zeta}{\zeta^n(2-\zeta)^n}, \qquad \text{(Eq. 41)}$$

where C' is a simple closed contour containing $\zeta = 0$ and excluding $\zeta = 2$. Evaluating the residue at $\zeta = 0$ we have

$$R_n(x, y) = \frac{\mu e^{-\mu\delta}}{2^{2n-1}} \sum_{r=0}^{n-1} \frac{(2\mu\delta)^r}{r!} \binom{2n-r-2}{n-1}, \qquad \text{(Eq. 42)}$$

where $\delta = y-x$.

The remaining paragraphs are intended to provide a check on Eq. 42.

Since

$$\int_x^\infty R_n(x, y)\, dy + \int_{-\infty}^x L_n(x, y)\, dy = 1$$

for every n, and since in this case (p = q) $L_n(x, y)$ and $R_n(x, y)$ are identical, except that in the case of the former, δ in Eq. 42 stands for x-y, we would expect each of the two terms on the left-hand side to be equal to one half. From Eq. 42,

$$\int_x^\infty R_n(x, y)\, dy = \frac{1}{2^n} \sum_{r=0}^{n-1} \binom{2n-r-2}{n-1} \frac{1}{2^{n-1-r}} \quad . \qquad \text{(Eq. 43)}$$

If we write

$$\frac{1}{(1-\zeta)(2-\zeta)^n} = \sum_{r=0}^{\infty} k_r^{(n)} \zeta^r \qquad \text{(Eq. 44)}$$

then it is easy to confirm that

$$\int_x^\infty R_n(x, y)\, dy = k_{n-1}^{(n)} \quad . \qquad \text{(Eq. 45)}$$

We note the identity

$$\frac{1}{(1-\zeta)(2-\zeta)^{n}} = \frac{1}{(2-\zeta)^{n+1}} + \frac{1}{(1-\zeta)(2-\zeta)^{n+1}} \quad . \qquad \text{(Eq. 46)}$$

Then

$$k_{n-1}^{(n)} = \binom{2n-1}{n} \frac{1}{2^{2n}} + k_{n-1}^{(n+1)} \quad . \qquad \text{(Eq. 47)}$$

We now proceed to show that the right-hand side of Eq. 47 is equal to $k_n^{(n+1)}$.

From Eq. 44

$$k_{n-1}^{(n+1)} = \frac{1}{2^{n+1}} \sum_{s=0}^{n} \binom{2n-s}{n} \frac{1}{2^{n-s}} \quad .$$

Then

$$\frac{1}{2^{2n}} \binom{2n-1}{n} + k_{n-1}^{(n+1)}$$

$$= \frac{1}{2^{2n-1}} \binom{2n}{n} + k_{n-1}^{(n+1)} \quad ,$$

$$= \frac{1}{2^{n+1}} \sum_{s=0}^{n} \binom{2n-s}{n} \frac{1}{2^{n-s}} \quad ,$$

$$= k_n^{(n+1)} \quad ;$$

i.e. $k_{n-1}^{(n)} = k_n^{(n+1)}$. (Eq. 48)

This common value is, in particular, true for n = 1, and evaluation of $k_o^{(1)}$ as the coefficient ζ^o in $\left[(1-\zeta)(2-\zeta) \right]^{-1}$ shows the common value to be 1/2. Thus

$$\int_x^\infty R_n(x, y)\ dy = \frac{1}{2}$$

as expected, and a check is provided on Eq. 42.

2. DISTANCE TRAVELLED DURING FIRST PASSAGE TO ORIGIN

In this section we consider the problem of the distance travelled by the particle from its initial point X (coordinate $x > 0$), until it reaches the origin for the first time. To be more precise, we consider the probability $u_n(x, y)\,dy$ $(y > x > 0)$ that simultaneously:

(a) the particle reaches or crosses the origin for the first time at the end of, or during its n^{th} step;

(b) the total distance traversed by the particle up to the moment when it first reaches the origin lies between y and $y + dy$.

It is convenient to introduce two new probabilities: $f_n(x, y)\,dy$ and $g_n(x, y)\,dy$. The probability $f_n(x, y)\,dy$ has a definition identical with that of $u_n(x, y)\,dy$, except that the particle's first step is to the right. The probability $g_n(x, y)\,dy$ corresponds to a first step to the left. Clearly $u_n(x, y)$, $f_n(x, y)$, and $g_n(x, y)$ are all zero for $y < x$ and $n = 1$, while, for $y > x$ and $n \geq 2$,

$$u_n(x, y) = p\, f_n(x, y) + q\, g_n(x, y) \quad . \tag{Eq. 49}$$

The case $y = x$ is a special one. The particle can return to the origin in n steps covering a total distance x only if it moves always to the left. We denote the corresponding probability by $v_n(x)$, and then

$$v_1(x) = q\, e^{-\mu x} , \tag{Eq. 50}$$

and generally

$$v_n(x) = q \int_0^x e^{-\mu G} v_{n-1}(x - G)\, dG, \qquad \text{(Eq. 51)}$$

which gives

$$v_n(x) = q\, e^{-\mu x} \frac{(q \mu x)^{n-1}}{(n-1)!} . \qquad \text{(Eq. 52)}$$

The generating function (g.f.) of $v_n(x)$ is given by

$$\left.\begin{aligned} V(x, t) &= \sum_{n \geq 1} t^n v_n(x) \\ &= q\, t\, e^{-\mu x(1-qt)} \end{aligned}\right\} \qquad \text{(Eq. 53)}$$

3. FORMULATION OF EQUATIONS

We consider first the case $x > 0$ and write down integro-difference equations for $f_n(x, y)$ and $g_n(x, y)$, for $y > x$. We have

$$f_n(x, y) = \frac{1}{2}\mu e^{-\frac{1}{2}(y-x)\mu} v_{n-1}\left[\frac{1}{2}(x+y)\right] + \mu\int_0^{\frac{1}{2}(y-x)} e^{-\mu G} u_{n-1}(x+G, y-G)\, dG \,.$$

(Eq. 54)

The first term in Eq. 54 arises because the particle's farthest possible first step to the right cannot carry it beyond the small interval at the point with coordinate $\frac{1}{2}(x + y)$ (a step length of $\frac{1}{2}(y - x)$), and a direct passage to or beyond the origin in n-1 steps must follow. The integral term arises to cover all the possible steps to a point in the interval $(x + G, x + G + dG)$, where $0 < G < \frac{1}{2}(y - x)$. Thence the particle starts another walk, the first step of which can be to the right or the left, and it must reach or overstep the origin for the first time at or during the $(n-1)^{th}$ step, covering a total distance to the origin of an amount between $y-G$ and $y - G + dy$. Similarly

$$g_n(x, y) = \int_0^{x} e^{-\mu G} u_{n-1}(x - G, y-G)\, dG \,. \qquad \text{(Eq. 55)}$$

By substituting Eqs. 54 & 55 in Eq. 49 we obtain a homogeneous integro-difference equation for $u_n(x, y)$.

An obvious way of converting integro-difference equations into integral equations is to introduce g.fs. Thus, let

$$\left.\begin{aligned} F(x, y, t) &= \sum_{n \geq 2} t^n f_n(x, y) \\ G(x, y, t) &= \sum_{n \geq 2} t^n g_n(x, y) \\ U(x, y, t) &= \sum_{n \geq 2} t^n u_n(x, y) \end{aligned}\right\} \qquad \text{(Eq. 56)}$$

The analogues to Eqs. 49, 54, & 55 are then

$$U(x, y, t) = p\ F(x, y, t) + q\ G(x, y, t)\ , \qquad \text{(Eq. 57)}$$

$$F(x, y, t) = \frac{1}{2}\mu t\ e^{-\frac{1}{2}\mu(y-x)} V\left[\frac{1}{2}(x+y),\ t\right] + \mu t \int_0^{\frac{1}{2}(y-x)} e^{-\mu G} U(x+G,\ y-G,\ t)\ dG,$$

(Eq. 58)

$$G(x, y, t) = \mu t \int_0^{x} e^{-\mu G}\ U(x-G,\ y-G\ ,\ t)\ dG\ . \qquad \text{(Eq. 59)}$$

In passing, it is worth mentioning that $F(x, y, t)$ and $G(x, y, t)$ satisfy the simultaneous differential equations

$$\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} - \mu F = -\mu t\ U, \qquad \text{(Eq. 60)}$$

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} + \mu G = +\mu t\ U, \qquad \text{(Eq. 61)}$$

which imply that all three of F, G, and U satisfy the telegrapher's equation

$$\Delta^2 F = 0 \quad ,$$

where

$$\Delta^2 = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \mu t(p - q) \frac{\partial}{\partial x} - \mu(2 - t) \frac{\partial}{\partial y} - \mu^2 (1 - t)$$

(Eq. 62)

and the omitted arguments x, y, and t are understood.

We now proceed to determine F, G and U from Eqs. 57, 58, & 59. We first introduce the function $\bar{F}(x, z, t)$, $\bar{G}(x, z, t)$, $\bar{U}(x, z, t)$, and $\bar{V}(z, t)$, where

$$\bar{F}(x, z, t) = \int_x^\infty e^{-zy} F(x, y, t)\, dy \qquad \text{(Eq. 63)}$$

$\bar{G}$ and $\bar{U}$ are similarly defined, and

$$\bar{V}(z, t) = \int_x^\infty e^{-zy} V(y, t)\, dy = \frac{qt}{a - q\mu t} e^{-x(a - q\mu t)}. \qquad \text{(Eq. 64)}$$

We then observe that

$$\int_0^\infty e^{-zy} dy \int_0^x e^{-\mu G} U(x-G, y-G, t) dG = e^{-ax} \int_0^x e^{as} \bar{U}(s, z, t)\, ds, \qquad \text{(Eq. 65)}$$

$$\int_x^\infty e^{-zy} dy \int_0^{\frac{1}{2}(y-x)} e^{-\mu G} U(x+G, y-G, t) dG = e^{ax} \int_x^\infty e^{-as} \bar{U}(s, z, t)\, ds,$$

(Eq. 66)

where

$$a = \mu + z \,. \qquad \text{(Eq. 67)}$$

Then

$$\bar{F}(x, z, t) = \mu t\, e^{ax}\, \bar{V}(a+z, t) + \mu t\, e^{ax} \int_x^\infty e^{-as}\, \bar{U}(s, z, t)\, ds, \qquad \text{(Eq. 68)}$$

$$\bar{G}(x, z, t) = \mu t e^{-ax} \int_0^x e^{as}\, \bar{U}(s, z, t)\, ds, \qquad \text{(Eq. 69)}$$

whence, using the transformed form of Eq. 57,

$$\bar{U}(x, z, t) = \mu p t e^{ax}\, \bar{V}(a+z, t) + \mu p t e^{ax} \int_x^\infty e^{-as}\, \bar{U}(s, z, t) ds + \mu q t e^{-ax} \int_0^x e^{as}\, \bar{U}(s, z, t)\, ds \,. \qquad \text{(Eq. 70)}$$

To solve Eq. 70 for $\bar{U}$ we introduce the further Laplace transform

$$\bar{\bar{U}}(\omega, z, t) = \int_0^\infty e^{-x\omega}\, \bar{U}(x, z, t)\, dx, \qquad \text{(Eq. 71)}$$

which, applied to Eq. 70, leads to

$$\bar{\bar{U}}(\omega, z, t) = \frac{\mu p t(\omega + a)}{\left\{\omega^2 + \mu t \omega (p-q) - (a^2 - a\mu t)\right\}} \left[\frac{\frac{1}{2} q t(\omega - a) + \bar{\bar{U}}(a, z, t)(a - \frac{1}{2}\mu q t)(a + \omega - \mu q t)}{(a - \frac{1}{2}\mu q t)\,(a + \omega - \mu q t)} \right] .$$

(Eq. 72)

Now $u_n(x, y) \rightarrow 0$ as $x \rightarrow \infty$ for any n, since it is clearly impossible for the particle to reach the origin in a finite number of steps from an infinite distance. It follows that $\bar{\bar{U}}(\omega, z, t)$ is an analytic function of ω, at least for $Rl\omega > 0$ and $0 < t < 1$.

The poles of the right hand side of Eq. 72 are at

$$\omega = \mu qt - a, \qquad \text{(Eq. 73)}$$

$$\omega = \omega_1 = -\frac{1}{2}\left[\mu t(p - q) + \sqrt{\mu^2 t^2 (p - q)^2 + 4\lambda^2}\right], \qquad \text{(Eq. 74)}$$

$$\omega = \omega_2 = -\frac{1}{2}\left[\mu t(p - q) - \sqrt{\mu^2 t^2 (p - q)^2 + 4\lambda^2}\right], \qquad \text{(Eq. 75)}$$

where

$$\lambda^2 = a^2 - a\mu t \ . \qquad \text{(Eq. 76)}$$

Taking $t < 1$ we find that

$$Rl\ (\mu qt - a) < 0,$$

so that the pole $\omega = (\mu qt - a)$ is admissible. Also $Rl\ \omega_1 < 0$ for all p and q while $R_1\ \omega_2 > 0$. Hence the numerator of the square bracket in Eq. 72 must vanish for $\omega = \omega_2$. This fact determines $\bar{\bar{U}}(a, z, t)$ and we obtain

$$\bar{\bar{U}}(\omega, z, t) = \frac{\mu p q t^2 (a + \omega)}{(\omega - \omega_1)(a + \omega_2 - \mu qt)(a + \omega - \mu qt)}$$

$$= \frac{\mu p q t^2}{(a+\omega_1 - \mu qt)(a+\omega_2 - \mu qt)} \left[\frac{\omega_1 + a}{\omega - \omega_1} - \frac{\mu qt}{\omega - q\mu t + a} \right] . \quad \text{(Eq. 77)}$$

Thus, on inversion, we have

$$\bar{U}(x, z, t) = \frac{1}{\mu} \left[(\omega_1 + a) \, e^{\omega_1 x} - \mu q t e^{-(a - \mu qt)x} \right] , \quad \text{(Eq. 78)}$$

and, upon substitution in Eqs. 68 & 69,

$$\bar{F}(x, z, t) = \left(\frac{a + \omega_1}{a - \omega_1} \right) t \, e^{\omega_1 x} , \quad \text{(Eq. 79)}$$

$$\bar{G}(x, z, t) = t \, e^{\omega_1 x} - t \, e^{-(a - \mu qt) x} . \quad \text{(Eq. 80)}$$

Since the Laplace transform $U^*(x, z, t)$ of $U(x, y, t)$ is given by

$$U^*(x, z, t) = e^{-zx} V(x, t) + \bar{U}(x, z, t) , \quad \text{(Eq. 81)}$$

we have

$$U^*(x, z, t) = \frac{(a + \omega_1)}{\mu} e^{\omega_1 x} . \quad \text{(Eq. 82)}$$

Similarly, the Laplace transforms of $F(x, y, t)$ and $G(x, y, t)$ are given by

$$F^*(x, z, t) = \left(\frac{a + \omega_1}{a - \omega_1}\right) t\, e^{\omega_1 x} , \qquad \text{(Eq. 83)}$$

$$G^*(x, z, t) = t\, e^{\omega_1 x} . \qquad \text{(Eq. 84)}$$

Recovery of $F(x, y, t)$, $G(x, y, t)$, and $U(x, y, t)$ can now be carried out by inverting the transforms. Thus we obtain for $y > x > 0$

$$G(x, y, t) = \frac{\sqrt{pq}\,\mu x t^2}{\sqrt{y^2 - x^2}}\, e^{-\mu y(1 - \frac{1}{2}t) - \frac{1}{2}\mu t(p-q)x}\, I_1\left[\mu t \sqrt{pq(y^2 - x^2)}\right] \qquad \text{(Eq. 85)}$$

and

$$U(x, y, t) = e^{-\mu y(1 - \frac{1}{2}t) - \frac{1}{2}\mu tx(p-q)}\, \frac{\mu pqt^2 x}{(y + x)} \left\{ I_o\left[\mu t \sqrt{pq(y^2 - x^2)}\right] + \right.$$

$$\left. + \frac{I_1\left[\mu t \sqrt{pq(y^2 - x^2)}\right]}{\mu t \sqrt{pq(y^2 - x^2)}} \left[\frac{y-x}{x} + \mu tq(y + x)\right] \right\} . \qquad \text{(Eq. 86)}$$

$F(x, y, t)$ can obviously be obtained immediately from U and G. If desired, the densities $u_n(x, y)$, etc. can be extracted from the expansion of Eq. 86 in powers of t.

In the case where the particle starts from the origin $(x = 0)$, the problem of first passage back to the origin concerns a path in which the first $n-1$ steps are either entirely to the right of the origin, or entirely to the left. In the former case, and with obvious notation,

$$f_n(0, y) = \frac{1}{2}\mu e^{-\frac{1}{2}\mu y} v_{n-1}(\tfrac{1}{2} y) + \mu\int_0^{\frac{1}{2}y} e^{-\mu G} u_{n-1}(G, y-G)\, dG . \quad \text{(Eq. 87)}$$

Corresponding to a walk to the left of the origin there is a probability density $f'_n(0, y)$ similar in all respects to $f_n(0, y)$, except that p and q are interchanged. We have

$$u_n(0, y) = pf_n(0, y) + q f'_n(0, y) . \quad \text{(Eq. 88)}$$

The same kind of treatment as for $x > 0$ yields the Laplace transform $\bar{U}(0, z, t)$ of the generating function $U(0, x, t)$. We obtain

$$\bar{U}(0, z, t) = (2a - \mu t - s) , \quad \text{(Eq. 89)}$$

where

$$s^2 = \mu^2 t^2 (p - q)^2 + 4\lambda^2 ,$$

$$\lambda^2 = a^2 - a\mu t ,$$

$$a = \mu + z .$$

On inversion this gives

$$U(0, y, t) = \frac{2t\sqrt{pq}}{y} e^{-\frac{1}{2}\mu y(2 - t)} I_1(\mu t y \sqrt{pq}). \qquad \text{(Eq. 90)}$$

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